Éléments de correction des exercices maison n°3

Exercice 1

On a $\det A = \begin{vmatrix} i & 0 & -i \\ 0 & 1 & 0 \\ 1 & i & 1 \end{vmatrix} = \begin{vmatrix} i & -i \\ 1 & 1 \end{vmatrix} = 2i \neq 0$

donc $A \in GL_3(\mathbb{C})$.

Si l'on note $v_1 := \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$, $v_2 := \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ et $v_3 := \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$,

la famille $\{v_1, v_2, v_3\}$ est donc une base de $\mathbb{C}^3$, à laquelle on applique le procédé d'orthonormalisation de Gram-Schmidt : on pose

$$\varepsilon_1 := v_1 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix},$$

$$\varepsilon_2 := v_2 - \frac{\overline{\langle v_2, \varepsilon_1 \rangle}}{\|\varepsilon_1\|^2} \varepsilon_1 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} - \frac{i}{2} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix},$$

$$\varepsilon_2' := \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix},$$

$$\varepsilon_3' := v_3 - \frac{\overline{\langle v_3, \varepsilon_1 \rangle}}{\|\varepsilon_1\|^2} \varepsilon_1 - \frac{\overline{\langle v_3, \varepsilon_2' \rangle}}{\|\varepsilon_2'\|^2} \varepsilon_2'$$

$$= \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} \qquad - \frac{-2i}{6} \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} i \\ \frac{2}{3} i \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3} \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix},$$

$$\varepsilon_3' := \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix},$$

$$e_1 := \frac{1}{\|\varepsilon_1\|} \varepsilon_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix},$$

$$e_2 := \frac{1}{\|\varepsilon_2'\|} \varepsilon_2' = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix},$$

$$e_3 := \frac{1}{\|\varepsilon_3'\|} \varepsilon_3' = \frac{1}{\sqrt{3}} \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix}.$$

Si l'on note alors $Q := \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \in M_3(\mathbb{C})$,

$Q \in U_3(\mathbb{C})$ et, si l'on note

$$R := Q^{-1} A = {}^t\overline{Q} A = \begin{pmatrix} \sqrt{2} & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{6}} & -\frac{2i}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix},$$

la décomposition QR de A est

$$A = QR = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{6}} & -\frac{2i}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

Exercice 2

1. On a $M = P + iQ$ donc

$${}^t\overline{M} = M \text{ ssi } \cancel{{}^tP + {}^tQ =} \ {}^t\overline{(P+iQ)} = P + iQ$$

$$\text{ssi } {}^tP - i\,{}^tQ = P + iQ \quad (P, Q \in M_n(\mathbb{R}))$$

$$\text{ssi } \begin{cases} {}^tP = P \\ -{}^tQ = P \end{cases}$$

Or ${}^tN = N$ ssi ${}^t\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} = \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$

$$\text{ssi } \begin{pmatrix} {}^tP & {}^tQ \\ -{}^tQ & {}^tP \end{pmatrix} = \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$$

$$\text{ssi } \begin{cases} {}^tP = P \\ -{}^tQ = Q \end{cases}$$

Ainsi ${}^t\overline{M} = M$ ssi ${}^tNN$ i.e. M est hermitienne si et seulement si N est orthogonale.

2. On a

$$ {}^t\bar{M}M = I_n \text{ ssi } ({}^tP - i\,{}^tQ)(P + iQ) = I_n $$

$$ \text{ssi } {}^tPP + {}^tQQ + i({}^tPQ - {}^tQP) = I_n $$

$$ \text{ssi } \begin{cases} {}^tPP + {}^tQQ = I_n \\ {}^tPQ - {}^tQP = O_n \end{cases} $$

Or ${}^tNN = I_{2n}$ ssi $\begin{pmatrix} {}^tP & {}^tQ \\ -{}^tQ & {}^tP \end{pmatrix}\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} = I_{2n}$

$$ \text{ssi } \begin{pmatrix} {}^tPP + {}^tQQ & -{}^tPQ + {}^tQP \\ -{}^tQP + {}^tPQ & {}^tQQ + {}^tPP \end{pmatrix} = \begin{pmatrix} I_n & O_n \\ O_n & I_n \end{pmatrix} $$

$$ \text{ssi } \begin{cases} {}^tPP + {}^tQQ = I_n \\ {}^tPQ - {}^tQP = O_n \end{cases} $$

Ainsi ${}^t\bar{M}M = I_n$ ssi ${}^tNN = I_{2n}$ i.e. M est unitaire si et seulement si N est orthogonale.