Quantum topology via modified traces

From semisimple to non-semisimple.

Solutions of exercices

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Solutions to Chapter 1: Monoidal categories

Solution of Exercise 1.2.6. Consider the map $\mathrm{Hom}_{\mathscr{C}}(U\otimes V, W) \to \mathrm{Hom}_{\mathscr{C}}(U, W\otimes V^*)$ given by

$$f \mapsto (f\otimes \mathrm{Id}_{V^*})\circ(\mathrm{Id}_U \otimes \overleftarrow{\mathrm{coev}}_V).$$

Equation (1.2.2) implies this map is an isomorphism with inverse $\mathrm{Hom}_{\mathscr{C}}(U, W\otimes V^*) \to \mathrm{Hom}_{\mathscr{C}}(U\otimes V, W)$ given by

$$g \mapsto (\mathrm{Id}_W \otimes \overleftarrow{\mathrm{ev}}_V)\circ(g\otimes \mathrm{Id}_V).$$

The other isomorphism is similar. □

Solution of Exercise 1.2.8. Let $(V', \overleftarrow{\mathrm{ev}}'_V, \overleftarrow{\mathrm{coev}}'_V)$ be another left duality of V. Then, the zigzag relations show that the morphism

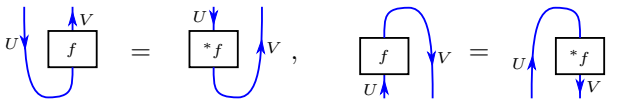
$$V' \xrightarrow{\mathrm{Id}_{V'}\otimes\overleftarrow{\mathrm{coev}}_V} V'\otimes V\otimes V^* \xrightarrow{\overleftarrow{\mathrm{ev}}'_V\otimes\mathrm{Id}_{V^*}} V^*$$

is an isomorphism whose inverse is given by

$$V^* \xrightarrow{\mathrm{Id}_{V^*}\otimes\overleftarrow{\mathrm{coev}}'_V} V^*\otimes V\otimes V' \xrightarrow{\overleftarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_{V'}} V'.$$

□

Solution of Exercise 1.2.11. The zigzag relation for the left duality implies that the equalities in Equation (1.2.7) are equivalent to the equality of Equation (1.2.6). One can make an analogous statement for right duals using



□

Solution of Exercise 1.2.15. We have

$$\begin{aligned}
(\overleftarrow{\mathrm{coev}}_V)^* &= (\overleftarrow{\mathrm{ev}}_{V\otimes V^*}\otimes \mathrm{Id}_{\mathbb{1}^*})\circ(\mathrm{Id}_{(V\otimes V^*)^*}\otimes\overleftarrow{\mathrm{coev}}_V\otimes\mathrm{Id}_{\mathbb{1}^*})\circ(\mathrm{Id}_{(V\otimes V^*)^*}\otimes\overleftarrow{\mathrm{coev}}_{\mathbb{1}})\\
&= (\overleftarrow{\mathrm{ev}}_{V\otimes V^*})\circ(\mathrm{Id}_{(V\otimes V^*)^*}\otimes\overleftarrow{\mathrm{coev}}_V)\\
&= \big(\overleftarrow{\mathrm{ev}}_{V^*}\circ(\mathrm{Id}_{V^{**}}\otimes\overleftarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_{V^*})\big)\circ\big(\mathrm{Id}_{V^{**}}\otimes\mathrm{Id}_{V^*}\otimes\overleftarrow{\mathrm{coev}}_V\big)\\
&= \overleftarrow{\mathrm{ev}}_{V^*}\circ\Big(\mathrm{Id}_{V^{**}}\otimes\big((\overleftarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_{V^*})\circ(\mathrm{Id}_{V^*}\otimes\overleftarrow{\mathrm{coev}}_V)\big)\Big)\\
&= \overleftarrow{\mathrm{ev}}_{V^*}\circ(\mathrm{Id}_{V^{**}}\otimes\mathrm{Id}_{V^*}).
\end{aligned}$$

where the second equality follows from the requirement that $\overleftarrow{\mathrm{coev}}_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}^*}$ and the third equality follows from the compatibility of the duality with the tensor product. The other equality of the exercise is proven similarly. □

Solution of Exercise 1.2.18. Suppose $V \in \mathrm{Vect}_{\mathbb{K}}$ has a left duality. Consider $\overleftarrow{\mathrm{coev}}^{\mathbb{k}}_V(1) = \sum_i v_i\otimes v'_i \in V\otimes V^*$. Let W be the vector subspace of V generated by $\{v_i\}_i$. Then, the image $\mathrm{Im}(\overleftarrow{\mathrm{coev}}^{\mathbb{k}}_V)$ is in $W\otimes V^*$ and

$$V = \mathrm{Im}(\mathrm{Id}_V) = \mathrm{Im}((\mathrm{Id}_V\otimes\overleftarrow{\mathrm{ev}}^{\mathbb{k}}_V)\circ(\overleftarrow{\mathrm{coev}}^{\mathbb{k}}_V\otimes\mathrm{Id}_V)) \subset W.$$

But W is finite dimensional and so is V.

Conversely, if $V \in \mathrm{Vect}_{\mathbb{K}}$ is finite dimensional then as explained in Example 1.2.17 the usual left and right evaluation and coevaluation morphisms define left and right dualities. □

Solution of Exercise 1.2.19. Let $\{v_i\}_i$ and $\{v_i^*\}_i$ be a basis of V and its dual basis of V^*, respectively. Write $f(v_j) = \sum_i f_{ij} v_i$ then

$$f^*(v_k^*) = \sum_{i,j} \otimes f_{ij} v_k^*(v_i) \otimes v_j^* = \sum_j f_{kj} v_j^*.$$

So $f^*(v_k^*)(v_j) = f_{kj}$. On the other hand,

$${}^t f(v_k^*)(v_j) = (v_k^* \circ f)(v_j) = \sum_i f_{ij} v_k^*(v_i) = f_{kj}.$$

□

Solution of Exercise 1.2.20. Applying the functor F to Equations (1.2.2) and (1.2.3) and using the fact that F preserves the tensor product and the identity we see that $F(\overleftarrow{\mathrm{ev}}_V)$ and $F(\overleftarrow{\mathrm{coev}}_V)$ are left duality morphisms of $F(V)$. The statement about right duality is similar. The last statement of the exercise follows from Exercise 1.2.18. □

Solution of Exercise 1.2.21. The morphism

$$(\mathrm{Id}_{{}^*(V^*)} \otimes \overleftarrow{\mathrm{ev}}_V) \circ (\overrightarrow{\mathrm{coev}}_{V^*} \otimes \mathrm{Id}_V) : V \to {}^*(V^*)$$

is an isomorphism with inverse

$$(\mathrm{Id}_V \otimes \overrightarrow{\mathrm{ev}}_{V^*}) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_{{}^*(V^*)}) : {}^*(V^*) \to V.$$

The other isomorphism is constructed similarly. □

Solution of Exercise 1.2.24. If $\mathscr{C}$ is a pivotal category then Proposition 1.2.23 says that ϕ is the desired natural monoidal isomorphism. Conversely, suppose $\mathscr{C}$ is a monoidal category with a left duality compatible with the tensor product, together with a natural monoidal isomorphism $\phi : \mathrm{Id}_{\mathscr{C}} \to (-)^{**}$. First, define the right dual of an object $V \in \mathscr{C}$ as ${}^*V = V^*$. Then, the assignments

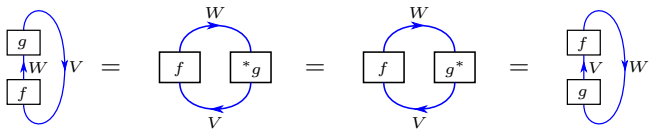
$$\overrightarrow{\mathrm{ev}}_V = \overleftarrow{\mathrm{ev}}_{V^*} \circ (\phi_V \otimes \mathrm{Id}_{V^*}) \text{ and } \overrightarrow{\mathrm{coev}}_V = (\mathrm{Id}_{V^*} \otimes \phi_V^{-1}) \circ \overleftarrow{\mathrm{coev}}_{V^*}$$

define a right duality which is compatible with the tensor product. Given a morphism $f : V \to W$ we have

$$\begin{aligned} {}^*f &= (\mathrm{Id}_{V^*} \otimes \overrightarrow{\mathrm{ev}}_W) \circ (\mathrm{Id}_{V^*} \otimes f \otimes \mathrm{Id}_{W^*}) \circ (\overrightarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_{W^*}) \\ &= (\mathrm{Id}_{V^*} \otimes \overleftarrow{\mathrm{ev}}_{W^*}) \circ (\mathrm{Id}_{V^*} \otimes (\phi_W \circ f \circ \phi_V^{-1}) \otimes \mathrm{Id}_{W^*}) \circ (\overleftarrow{\mathrm{coev}}_{V^*} \otimes \mathrm{Id}_{W^*}) \\ &= (\mathrm{Id}_{V^*} \otimes \overleftarrow{\mathrm{ev}}_{W^*}) \circ (\mathrm{Id}_{V^*} \otimes f^{**} \otimes \mathrm{Id}_{W^*}) \circ (\overleftarrow{\mathrm{coev}}_{V^*} \otimes \mathrm{Id}_{W^*}) = f^* \end{aligned}$$

Thus, $\mathscr{C}$ is pivotal. □

Solution of Exercise 1.2.28. We use the graphical calculus and Exercise 1.2.11 to prove the first statement:



Suppose $f : V \to W$ is an isomorphism. Then,

$$\dim^R_{\mathscr{C}}(V) = \mathrm{tr}^R_V(\mathrm{Id}_V) = \mathrm{tr}^R_V(f^{-1} \circ f) = \mathrm{tr}^R_W(f \circ f^{-1}) = \mathrm{tr}^R_W(\mathrm{Id}_W) = \dim^R_{\mathscr{C}}(W).$$

□

Solution of Exercise 1.2.29. Use that $(\mathrm{Id}_{W'} \otimes \overrightarrow{\mathrm{ev}}_V) \circ (h \otimes \mathrm{Id}_{V \otimes V^*}) = h \circ (\mathrm{Id}_W \otimes \overrightarrow{\mathrm{ev}}_V)$. □

Solution of Exercise 1.2.30. Use Exercise 1.2.11 and pivotality. □

Solution of Exercise 1.2.31. Use the compatibility of the duality with the tensor product.

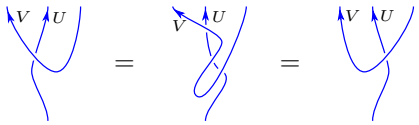
□

Solution of Exercise 1.2.35. Set $V = W = \mathbb{1}$ then Equation (1.2.10) becomes

$$c_{U,\mathbb{1}\otimes\mathbb{1}} = (\mathrm{Id}_{\mathbb{1}} \otimes c_{U,\mathbb{1}}) \circ (c_{U,\mathbb{1}} \otimes \mathrm{Id}_{\mathbb{1}})$$

but since $U \otimes \mathbb{1} = U$ this equality implies $c_{U,\mathbb{1}} = \mathrm{Id}_U$. □

Solution of Exercise 1.2.38. We give the following graphical proof:

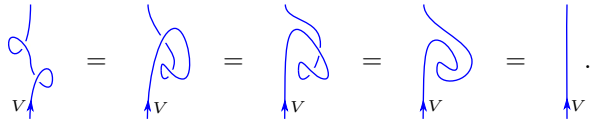


where the first equality comes from the naturality of the braiding (Equation (1.2.11)) for $f \otimes g = \mathrm{Id}_U \otimes \overleftarrow{\mathrm{coev}}_V$ and the second equality comes from applying the braiding and its inverse. The other statements follow similarly. □

Solution of Exercise 1.2.39. To simplify notation in this solution we will suppress the V when writing the morphisms in the category, e.g., we write c and Id for $c_{V,V}$ and Id_V, respectively. We have

$$\begin{aligned}
\mathrm{ptr}^L_V(c^{-1}) \circ \mathrm{ptr}^R_V(c) &= \mathrm{ptr}^R_V\left(\mathrm{ptr}^L_V((c^{-1} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes c))\right) \\
&= \mathrm{ptr}^R_V\left(c \circ (\mathrm{Id}_V \otimes \mathrm{ptr}^L_V(c^{-1}))\right) \\
&= (\overrightarrow{\mathrm{ev}} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes c^{-1}_{V^*,V}) \circ \Big(\mathrm{Id} \otimes \big((\mathrm{ptr}^L_V(c^{-1}) \otimes \mathrm{Id}_{V^*}) \circ \overleftarrow{\mathrm{coev}}\big)\Big) \\
&= (\overrightarrow{\mathrm{ev}} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes c^{-1}_{V^*,V}) \circ \big(\mathrm{Id} \otimes (c_{V^*,V} \circ \overrightarrow{\mathrm{coev}})\big) \\
&= (\overrightarrow{\mathrm{ev}} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \overrightarrow{\mathrm{coev}})\big) = \mathrm{Id}
\end{aligned}$$

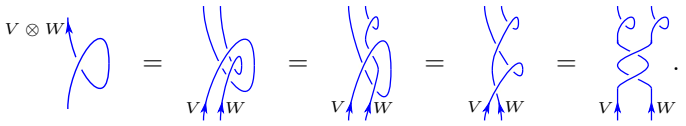
where the second equality follows from the naturality of the braiding and the third and fourth equalities follow from Exercise 1.2.38. To emphasize its simplicity, we use graphical calculus to give a second proof:



□

Solution of Exercise 1.2.40. From Exercise 1.2.39 we have that θ^R_V is invertible. To see the family is natural, use the naturality of the braiding and Exercise 1.2.11. □

SOLUTION OF EXERCISE 1.2.41. We give the following graphical proof:



□

SOLUTION OF EXERCISE 1.2.44. We provide the following hints: 1) the right evaluation has to be defined by

$$\overrightarrow{\mathrm{ev}}_V = \overleftarrow{\mathrm{ev}}_V \circ c_{V,V^*} \circ (\theta'_V \otimes \mathrm{Id}_{V^*}).$$

2) Similarly, define a right coevaluation which satisfies the zigzag relation. 3) Show that ${}^*f = f^*$. 4) Show that the right duality is compatible with the tensor product. Conclude that $\mathscr{C}$ is pivotal. 5) Show $\mathrm{ptr}^L_V(c^{-1}_{V,V})$ is the inverse of θ'_V. Use this to conclude that $\theta^R_V = \mathrm{ptr}^R_V(c_{V,V})$ satisfies Equation (1.2.15). □

SOLUTION OF EXERCISE 1.2.50. This follows directly from Exercise 1.2.41. □

SOLUTION OF EXERCISE 1.2.53. This follows from the zigzag relations and G preserving the evaluation and monoidal structures. □

SOLUTION OF EXERCISE 1.2.54. For $V \in \mathscr{C}$, we have $\eta(\mathbb{1}) \circ G(\overleftarrow{\mathrm{ev}}_V) = G'(\overleftarrow{\mathrm{ev}}_V) \circ \eta(V^* \otimes V)$ which implies

$$\overleftarrow{\mathrm{ev}}_{G(V)} = \overleftarrow{\mathrm{ev}}_{G'(V)} \circ (\eta(V^*) \otimes \eta(V)) = \overleftarrow{\mathrm{ev}}_{G(V)} \circ \big((\eta(V)^* \circ \eta(V^*)) \otimes \mathrm{Id}_{G(V)}\big),$$

where the last equality comes from Equation (1.2.7). Then, using the zigzag relation $\curvearrowright\curvearrowleft = \downarrow$, we get $\eta(V)^* \circ \eta(V^*) = \mathrm{Id}_{G(V)}$. □

SOLUTION OF EXERCISE 1.2.56. The definition of a ribbon functor and Exercise 1.2.53 imply that the functor preserves the duality and the braiding. Then, since $\theta_V = \mathrm{ptr}^L_V(c_{V,V}) = \mathrm{ptr}^R_V(c_{V,V})$, the result follows. □

SOLUTION OF EXERCISE 1.3.5. Let p_1, p_2 be two epimorphisms and suppose $f \circ (p_1 \otimes p_2) = g \circ (p_1 \otimes p_2)$. Precomposing this morphism with $\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}}$ then Exercise 1.2.11 implies

$$(f \otimes p_2^*) \circ (\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}}) \circ p_1 = (g \otimes p_2^*) \circ (\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}}) \circ p_1.$$

Since p_1 is an epimorphism, we have $(f \otimes p_2^*) \circ (\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}}) = (g \otimes p_2^*) \circ (\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}})$. An analogous argument shows that the last equality is equivalent to

$$(\mathrm{Id} \otimes f) \circ (\overrightarrow{\mathrm{coev}} \otimes \mathrm{Id}) \circ p_2 = (\mathrm{Id} \otimes g) \circ (\overrightarrow{\mathrm{coev}} \otimes \mathrm{Id}) \circ p_2$$

where we also use the zigzag relation. Since p_2 is an epimorphism, we have $(\mathrm{Id} \otimes f) \circ (\overrightarrow{\mathrm{coev}} \otimes \mathrm{Id}) = (\mathrm{Id} \otimes g) \circ (\overrightarrow{\mathrm{coev}} \otimes \mathrm{Id})$ then the zigzag relation implies $f = g$. □

SOLUTION OF EXERCISE 1.3.15. If an object is not simple, it is a non-trivial direct sum and the associated idempotents are linearly independent. Hence, its endomorphism space has at least rank 2 and the object is not absolutely simple. □

Solution of Exercise 1.3.19. Let $f, g : W \to X$ be any morphisms. Then, the equation $\mathrm{Id}_V \otimes f = \mathrm{Id}_V \otimes g$ holds if and only if

$$(\overleftarrow{\mathrm{ev}}_V \otimes f \otimes \mathrm{Id}_{W^*}) \circ (\mathrm{Id}_{V^* \otimes V} \otimes \overleftarrow{\mathrm{coev}}_W) = (\overleftarrow{\mathrm{ev}}_V \otimes g \otimes \mathrm{Id}_{W^*}) \circ (\mathrm{Id}_{V^* \otimes V} \otimes \overleftarrow{\mathrm{coev}}_W)$$

if and only if

$$\big((f \otimes \mathrm{Id}_{W^*}) \circ \overleftarrow{\mathrm{coev}}_W\big) \circ \overleftarrow{\mathrm{ev}}_V = \big((g \otimes \mathrm{Id}_{W^*}) \circ \overleftarrow{\mathrm{coev}}_W\big) \circ \overleftarrow{\mathrm{ev}}_V. \tag{1}$$

Now, if V is left sharp then

$$(f \otimes \mathrm{Id}_{W^*}) \circ \overleftarrow{\mathrm{coev}}_W = (g \otimes \mathrm{Id}_{W^*}) \circ \overleftarrow{\mathrm{coev}}_W.$$

Tensoring this equation on the right by Id_W and then composing with $\mathrm{Id}_X \otimes \overleftarrow{\mathrm{ev}}_W$ one can use the zigzag relation to show $f = g$.

Conversely, suppose $f, g : \mathbb{1} \to X$ are morphisms such that

$$f \circ \overleftarrow{\mathrm{ev}}_V = g \circ \overleftarrow{\mathrm{ev}}_V.$$

After setting $W = \mathbb{1}$, the last equation is equal to Equation (1) since $\overleftarrow{\mathrm{coev}}_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}}$. Thus, from the above equivalences we have $\mathrm{Id}_V \otimes f = \mathrm{Id}_V \otimes g$ and the hypotheses imply $f = g$. □

Solution of Exercise 1.3.22. Since S is absolutely simple, $\mathrm{End}_{\mathscr{C}}(S)$ is free of rank one and is generated by the identity Id_S. Then, by Exercise 1.2.6 there are isomorphisms from $\mathrm{End}_{\mathscr{C}}(S)$ to the four Hom-spaces listed in the exercise which send the identity Id_S to the four duality morphisms of S, respectively. □

Solution of Exercise 1.3.25. If $\Phi : V \times W \to \Bbbk$ is a perfect pairing then there exists a bijection $V \to V^{**}$. This is only possible if V has finite rank because otherwise the cardinality of a basis of V^* would be strictly larger than that of a basis of V. For the second statement, if $\Phi : V \times W \to \mathbb{K}$ is a non-degenerate pairing of finite dimensional vector spaces then it induces injective linear maps $V \to W^*$ and $W \to V^*$ between vector spaces of the same dimension and so they are isomorphisms. □

Solution of Exercise 1.3.27. Let $f : W \to V^*$ and $g : V \to W^*$ be the maps induced by Φ. If x is in the kernel of f then $x = \sum_i \Phi(e^i, x)e_i = 0$. Similarly, the kernel of g is zero. Then, for any $\varphi \in W^*$ and any $x \in W$, we have $\varphi(x) = \varphi(\sum_i \Phi(e^i, x)e_i) = \Phi(\sum_i \varphi(e_i)e^i, x)$, thus, $\varphi = f(\sum_i \varphi(e_i)e^i)$ and f is surjective. Similarly, for any $\psi \in V^*$, $\psi = g(\sum_i \psi(e^i)e_i)$ and g is surjective. □

Solution of Exercise 1.3.28. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a $\Bbbk$-additive monoidal category which is absolutely semisimple and assume that $\mathbb{1}$ is absolutely simple. Then, by Exercise 1.3.15, $\mathbb{1}$ is simple. Furthermore, if $\{S_i\}_i$ is a finite set of pairwise non-isomorphic simple objects of $\mathscr{C}$, and if $\{m_i\}_i$, $\{n_i\}_i$ are sets of non-negative integers, then

$$\mathrm{Hom}_{\mathscr{C}}(\bigoplus_i S_i^{\oplus m_i}, \bigoplus_i S_i^{\oplus n_i}) \simeq \bigoplus_i \mathcal{M}at_{n_i, m_i}(\Bbbk)$$

is a finite rank free $\Bbbk$-module, where $\mathcal{M}at_{n_i,m_i}(\Bbbk)$ is the free $\Bbbk$-module of $n_i \times m_i$ matrices. Since any Hom-set of $\mathscr{C}$ has this form, $\mathscr{C}$ is sharp. □

SOLUTION OF EXERCISE 1.3.36. Let $A = \bigoplus_i A_i$ be a direct sum of a finite family of algebras. If S_j is a simple A_j-module, then it is a simple $\bigoplus_i A_i$-module where the other summands act by zero. Hence if each A_i is semisimple, so is A.

The standard $\mathcal{M}at_{n_i}(\mathbb{K})$-module $\mathbb{K}^{n_i}$ is simple since it is generated by any non-zero vector. Now, the map that associates to a matrix its n_i columns is a $\mathcal{M}at_{n_i}(\mathbb{K})$-module morphism from the regular module to $(\mathbb{K}^{n_i})^{\oplus n_i}$. □

SOLUTION OF EXERCISE 1.3.43. If e is an idempotent of a $\Bbbk$-module V then the image of e is the sub $\Bbbk$-module of V defined by $I = \{e(v) : v \in V\}$ with complement the kernel K of e. Then, $V = I \oplus K$ and the retract is the pair of the inclusion and projection on I. Finally, the direct sum of $\Bbbk$-modules is a $\Bbbk$-module. □

SOLUTION OF EXERCISE 1.3.54. In this proof we use the fact that if N is a null object and V is any object then $N \oplus V$ is isomorphic to V. Write $\mathbb{1} = \bigoplus_{g\in\mathsf{G}} \mathbb{1}_g$. Then, for each $g \in \mathsf{G}$ we have

$$\mathbb{1}_g = \mathbb{1} \otimes \mathbb{1}_g \xrightarrow{\sim} \bigoplus_{h\in\mathsf{G}} \mathbb{1}_h \otimes \mathbb{1}_g.$$

Hence, the identity of $\mathbb{1}_h \otimes \mathbb{1}_g$ factors through $\mathbb{1} \otimes \mathbb{1}_g$:

$$\mathbb{1}_h \otimes \mathbb{1}_g \to \mathbb{1} \otimes \mathbb{1}_g \to \mathbb{1}_h \otimes \mathbb{1}_g$$

and when $h \neq 1$ this morphism is zero since $hg \neq g$. Thus, if $h \neq 1$ then $\mathbb{1}_h = \mathbb{1}_h \otimes \mathbb{1} \simeq \bigoplus_{g\in\mathsf{G}} \mathbb{1}_h \otimes \mathbb{1}_g$ is a null object and it follows that $\mathbb{1} = \bigoplus_{g\in\mathsf{G}} \mathbb{1}_g \xrightarrow{\sim} \mathbb{1}_1 \in \mathscr{C}_1$. □

SOLUTION OF EXERCISE 1.3.55. If $V \in \mathscr{C}_g \cap \mathscr{C}_h$ then $\mathrm{Id}_V = 0$ by the fourth property of G-graded category. Hence, V is a null object. Reciprocally, since $\mathscr{C}_g$ is additive, it contains a null object which is isomorphic to any null object of $\mathscr{C}$ via the null maps. Thus, any null object is in $\mathscr{C}_g$. □

SOLUTION OF EXERCISE 1.3.56. Suppose that $V = \bigoplus_{h\in\mathsf{G}} V_h$ is a retract of $W \in \mathscr{C}_g$, then the identity of V_h factors through W thus, if $h \neq g$, it is zero and V_h is a null object. Hence $V \simeq V_g \in \mathscr{C}_g$. □

SOLUTION OF EXERCISE 1.3.57. Suppose that $V^* = \bigoplus_{h\in\mathsf{G}} V_h^*$. Exercise 1.3.54 implies $\mathrm{Hom}_{\mathscr{C}}(V_h^* \otimes V, \mathbb{1}) = 0$ for $h \neq g^{-1}$. Thus, $\overleftarrow{\mathrm{ev}}_V$ restricted to $\bigoplus_{h\neq g^{-1}} V_h^* \otimes V$ is zero and $\overleftarrow{\mathrm{ev}}_V$ factors through $V^* \otimes V \to V_{g^{-1}}^* \otimes V \to V^* \otimes V$. Now, we have $\mathrm{Id}_{V^*} = \left(\overleftarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_{V^*}\right) \circ \left(\mathrm{Id}_{V^*} \otimes \overleftarrow{\mathrm{coev}}_V\right)$ factors through $V_{g^{-1}}^*$ thus, it is a retract of an object of $\mathscr{C}_{g^{-1}}$ and Exercise 1.3.56 implies it belongs to $\mathscr{C}_{g^{-1}}$. □

Solutions to Chapter 2: Hopf algebras

Solution of Exercise 2.2.2. The compatibility axioms can be interpreted as saying ϵ and Δ are algebra morphisms: if $x, y \in B$ then $\Delta(1_B) = \Delta(\eta(1)) = \eta(1) \otimes \eta(1) = 1_B \otimes 1_B$ and

$$\begin{aligned}\Delta(xy) = \Delta(m(x,y)) &= \big(((m \otimes m) \circ (\mathrm{Id}_B \otimes \tau \otimes \mathrm{Id}_B)) \circ (\Delta \otimes \Delta)\big)(x,y) \\ &= x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} = \Delta(x)\Delta(y)\end{aligned}$$

where the second equality follows from the compatibility axioms of a bialgebra. Similarly,

$$\epsilon(1_B) = \epsilon(\eta(1)) = 1; \quad \epsilon(xy) = \epsilon(m(x,y)) = \epsilon(x)\epsilon(y).$$

The proof of the second statement is similar. □

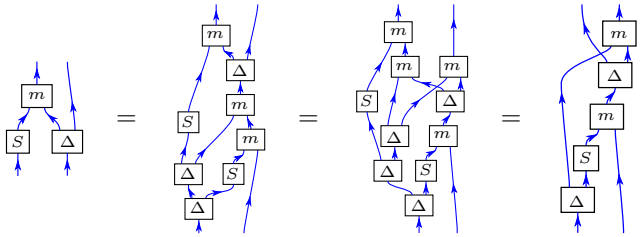
Solution of Exercise 2.2.7. Suppose that S and S' are two antipodes for the same bialgebra H. Apply Proposition 2.2.6 to $\mathrm{Id}_H : (H, m, \eta, \Delta, \varepsilon, S) \to (H, m, \eta, \Delta, \varepsilon, S')$. □

Solution of Exercise 2.2.8. From the defining property of the antipode and the counital coalgebra axiom we have

$$x_{(1)} \otimes x_{(2)}S(x_{(3)}) = x_{(1)} \otimes \big(m \circ (\mathrm{Id}_H \otimes S)(\Delta(x_{(2)}))\big) = x_{(1)} \otimes (\eta \circ \varepsilon)(x_{(2)}) = x \otimes 1_H.$$

Applying the linear map $f : H \otimes H \to H$ given by $a \otimes b \mapsto ybza$ to the previous equality leads to the second statement. □

Solution of Exercise 2.2.9. We have



where the first equality follows from the property of the antipode applied to the right side of the second graph, the second equality is the compatibility of the top Δ with m and the third equality is the property of the antipode applied to the left side of the third graph. □

Solution of Exercise 2.2.11. The solution to this exercise is similar to that of Exercise 2.2.8. We have

$$S(y_{(1)})S(x_{(1)})x_{(2)}y_{(2)}S(x_{(3)}y_{(3)}) = S(y_{(1)})y_{(2)}S(xy_{(3)}) = S(xy),$$

and

$$\begin{aligned}S(y_{(1)})S(x_{(1)})x_{(2)}y_{(2)}S(x_{(3)}y_{(3)}) &= S(y_{(1)})S(x_{(1)})(x_{(2)}y_{(2)})_{(1)}S((x_{(2)}y_{(2)})_{(2)}) \\ &= S(y_{(1)})S(x_{(1)})\varepsilon(x_{(2)}y_{(2)}) \\ &= S(y_{(1)})S(x_{(1)})\varepsilon(x_{(2)})\varepsilon(y_{(2)}) \\ &= S(y_{(1)}\varepsilon(y_{(2)}))S(x_{(1)}\varepsilon(x_{(2)})) \\ &= S(y)S(x).\end{aligned}$$

□

Solution of Exercise 2.2.20. See also [**Mol77**, Theorem 2.13].
With omitted summation, the associativity of the product is given by identifying the two triple products with

$$(u \otimes h)(v \otimes k)(w \otimes l) = u(h_{(1)}.v)(h_{(2)}k_{(1)}.w) \otimes h_{(3)}k_{(2)}l.$$

The coproduct Δ is an algebra morphism because

$$\Delta((u \otimes h)(v \otimes k)) = (u_{(1)}(h_{(1)}.v_{(1)}) \otimes h_{(3)}k_{(1)}) \otimes (u_{(2)}(h_{(2)}.v_{(2)}) \otimes h_{(4)}k_{(2)})$$

$$= (u_{(1)}(h_{(1)}.v_{(1)}) \otimes h_{(2)}k_{(1)}) \otimes (u_{(2)}(h_{(3)}.v_{(2)}) \otimes h_{(4)}k_{(2)}) = \Delta(u \otimes h)\Delta(v \otimes k)$$

where the second equality holds because H is cocommutative. To check that the formula given for S defines an antipode, we compute for $(u \otimes h) \in U \rtimes H$:

$$\begin{aligned} m \circ (S \otimes \mathrm{Id}_{U\rtimes H}) \circ \Delta(u \otimes h) &= S(u_{(1)} \otimes h_{(1)})(u_{(2)} \otimes h_{(2)}) \\ &= \big(S(h_{(1)}).S(u_{(1)}) \otimes S(h_{(2)})\big)(u_{(2)} \otimes h_{(3)}) \\ &= \big(S(h_{(1)}).S(u_{(1)})\big)\big(S(h_{(2)}).u_{(2)}\big) \otimes S(h_{(3)})h_{(4)}) \\ &= \big(S(h_{(1)})_{(1)}.S(u_{(1)})\big)\big(S(h_{(1)})_{(2)}.u_{(2)}\big) \otimes \varepsilon(h_{(2)})1_H \\ &= S(h).\big(S(u_{(1)})u_{(2)}\big) \otimes 1_H = \varepsilon(u)S(h).1_U \otimes 1_H \\ &= \varepsilon(u)\varepsilon(h)1_U \otimes 1_H = \varepsilon(u \otimes h)1_{U\rtimes H}, \end{aligned}$$

where in the fourth equality we used that $S(h_{(1)}) \otimes S(h_{(2)}) \otimes h_{(3)} = S(h_{(1)})_{(1)} \otimes S(h_{(1)})_{(2)} \otimes h_{(2)}$ which comes from the cocommutativity of H. Similarly, $m \circ (\mathrm{Id}_{U\rtimes H} \otimes S) \circ \Delta(u \otimes h) = \varepsilon(u \otimes h)1_{U\rtimes H}$, so S is an antipode. □

Solution of Exercise 2.2.21. Compare the product law in $\mathsf{G}_2 \rtimes \mathsf{G}_1$ given by $(g_2, g_1)(g_2', g_1') = (g_2\,\varphi(g_1)(g_2'), g_1g_1')$ with the product in $\Bbbk[\mathsf{G}_2] \rtimes \Bbbk[\mathsf{G}_1]$ given by $(g_2 \otimes g_1)(g_2' \otimes g_1') = g_2(g_{1\,(1)}.g_2') \otimes g_{1\,(2)}g_1' = g_2\,\varphi(g_1)(g_2') \otimes g_1g_1'$. □

Solution of Exercise 2.2.22. Since $S = S^{-1}$, we have $V^* = {}^{\bar{*}}V$ and the exercise is a consequence of Propositions 2.2.17 and 2.2.18. □

Solution of Exercise 2.2.26. The element K is grouplike and we have seen that $S^2(E) = KEK^{-1}$. Then, we have $S^2(K) = S(K^{-1}) = K = K\,K\,K^{-1}$ so the two automorphisms of algebra S^2 and $x \mapsto KxK^{-1}$ are equal since they coincide on generators. □

Solution of Exercise 2.2.27. First, since g and x are grouplike then

$$\Delta(xg) = \Delta(x)\Delta(g) = (x \otimes x)(g \otimes g) = xg \otimes xg.$$

Moreover, since x is central then for $y \in H$ we have

$$S^2(y) = gyg^{-1} = xgyg^{-1}x^{-1} = xgy(xg)^{-1}$$

and xg is a pivot for H. To see the equality of sets let g' be a pivot in H. Then, we have $S^2(x) = g'x(g')^{-1}$ for any $x \in H$. Since g is also a pivot, we have $g'x(g')^{-1} = gxg^{-1}$. Thus,

$$x = (g')^{-1}gxg^{-1}g' = (g^{-1}g')^{-1}xg^{-1}g'$$

so $g^{-1}g'$ is central and $g' = g(g^{-1}g')$ which completes the exercise. □

Solution of Exercise 2.2.28. First, note

$$\mathrm{Ad} = m \circ (\mathrm{Id}_H \otimes (m \circ \tau)) \circ ((\mathrm{Id}_H \otimes S) \circ \Delta) \otimes \mathrm{Id}_H)$$

where τ is the flip map. Now, let $x, y \in H$ and $v \in H$ then the definition of the adjoint representation of x acting on v is $\rho_H(x, v) = \mathrm{Ad}(x \otimes v) = x_{(1)} v S(x_{(2)})$. Using this twice we have

$$\rho_H(x, \rho_H(y, v)) = x_{(1)} y_{(1)} v S(y_{(2)}) S(x_{(2)}) = x_{(1)} y_{(1)} v S(x_{(2)} y_{(2)})$$

where the last expression in this equation is equal to

$$m \circ \big(\mathrm{Id}_H \otimes (m \circ \tau)\big) \circ \big(((\mathrm{Id}_H \otimes S) \circ \Delta(x)\Delta(y)) \otimes v\big).$$

Thus, replacing $\Delta(x)\Delta(y)$ with $\Delta(xy)$ in the last formula implies the result. □

Solution of Exercise 2.2.33. Recall that $\overleftarrow{\mathrm{ev}}_V = \overleftarrow{\mathrm{ev}}^{\Bbbk}_V : \varphi \otimes v \mapsto \varphi(v)$. For any $v \in V$ and any $\varphi \in V^*$, the right-hand side of Equation (2.2.4) (resp. Equation (2.2.6)) applied to $\varphi \otimes v$ (resp. $v \otimes \varphi$) gives $\varphi(S(S^{-1}(x))) = \varphi(x)$ which is the image of the left-hand side. Similarly, the right-hand side of Equation (2.2.8) applied to $v \otimes \varphi$ gives $\varphi(S^2(x)g.v) = \varphi(gx.v)$ which is the image of the left-hand side.
Now, from Equation (2.2.4) we get $\mathrm{Id}_V \otimes \Big(\overleftarrow{\mathrm{ev}}_V \circ (\mathrm{Id}_{V^*} \otimes \widetilde{\rho}_V(x))\Big) \otimes \mathrm{Id}_{V^*} = \mathrm{Id}_V \otimes \Big(\overleftarrow{\mathrm{ev}}_V \circ \big(\widetilde{\rho}_{V^*}(S^{-1}(x)) \otimes \mathrm{Id}_V\big)\Big) \otimes \mathrm{Id}_{V^*}$. Precomposing both sides with $\overleftarrow{\mathrm{coev}}_V \otimes \overleftarrow{\mathrm{coev}}_V$ then applying the zigzag relation $\curvearrowleft\curvearrowright = \uparrow$ on the left-hand side, (resp. $\curvearrowright\curvearrowleft = \downarrow$ on the right-hand side) shows that Equality (2.2.5) holds. Similarly, Equation (2.2.8) implies that Equality (2.2.9) holds. The proof is the same for Equation (2.2.7) but with coevaluations and zigzag relations in $\Bbbk$-mod$^{\mathrm{f}}$. □

Solution of Exercise 2.2.37. By Proposition 2.2.36, since H is quasitriangular, H-Mod is braided. We now show the converse (see also [**KRT97**, Chapter 2, Theorem 4.1]): Suppose H-Mod is braided and consider the H-module H with the left regular representation given by $\widetilde{\rho}_H(h)(v) = hv$ for all $h, v \in H$. Consider the element

$$R = \tau \circ c_{H,H}(1_H \otimes 1_H) \in H \otimes H.$$

We claim that R is a universal R-matrix. The fact that $c_{H,H}$ is a morphism of H-modules implies it satisfies Equation (2.2.10). Now, for any $(v, w) \in V \times W$ where V, W are H-modules define H-module morphisms $f_v : H \to V$ and $f_w : H \to W$ by $f_v(x) = xv$ and $f_w(x) = xw$. Then, the naturality of the braiding (see Equation (1.2.11)) implies that

$$\begin{aligned} c_{V,W}(v \otimes w) &= c_{V,W} \circ (f_v \otimes f_w)(1_H \otimes 1_H) \\ &= (f_w \otimes f_v)(R_{21}) \\ &= \tau \circ (\widetilde{\rho}_V \otimes \widetilde{\rho}_W)(R) \circ (f_v \otimes f_w)(1_H \otimes 1_H) \\ &= \tau \circ (\widetilde{\rho}_V \otimes \widetilde{\rho}_W)(R)(v \otimes w). \end{aligned}$$

Then, the Equations (2.2.11) and (2.2.12) for R follow from the equalities in Equation (1.2.10) applied to $1_H \otimes 1_H \otimes 1_H \in U \otimes V \otimes W = H^{\otimes 3}$. □

Solution of Exercise 2.3.14. We prove the equality by induction, for $n \in \mathbb{Z}_{>0}$:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n-1} q^{k(n-1-k)} \begin{bmatrix} n-1 \\ k \end{bmatrix} (x+y)x^k y^{n-1-k} \\ &= \sum_{k=0}^{n-1} q^{k(n-1-k)} \begin{bmatrix} n-1 \\ k \end{bmatrix} (x^{k+1}y^{n-1-k} + q^{2k}x^k y^{n-k}) \\ &= x^n + y^n + \sum_{k=1}^{n-1} q^{k(n-k)} \left(q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \right) x^k y^{n-k}.\end{aligned}$$

Then, the result follows from Lemma 2.3.13.

□

Solution of Exercise 2.3.21. Similar to the proof of Lemma 2.3.19, we first prove the statements over the ring $\mathbb{K} = \mathbb{Q}(q)$. In this case, we have $\exp_q(x) = \exp_q^{<r}(x)$ modulo $x^r \Bbbk_q[[x]]$ and $\exp_{q^{-1}}(-x) = \exp_{q^{-1}}^{<r}(-x)$ modulo $x^r \Bbbk_q[[x]]$. Thus, $\exp_q^{<r}(x)\exp_{q^{-1}}^{<r}(-x)-1 = \exp_q(x)\exp_{q^{-1}}(-x)-1 = 0$ modulo $x^r \Bbbk_q[[x]]$. Then, the first statement holds because $\exp_q^{<r}(x)\exp_{q^{-1}}^{<r}(-x)-1$ belongs to $(x^r\mathbb{K}[[x]])\cap\mathbb{K}[x] = x^r\mathbb{K}[x]$. Similarly, Lemma 2.3.19 implies the second statement is true modulo $I_{\geq r}\mathbb{K}_q[[x,y]]$ so

$$\exp_q^{<r}(x+y) - \exp_q^{<r}(x)\exp_q^{<r}(y) \in I_{\geq r}\mathbb{K}_q[[x,y]] \cap \mathbb{K}_q[x,y] = I_{\geq r}.$$

Now, consider the subring $\Bbbk'$ of $\mathbb{K}$ formed by fractions of elements of $\mathbb{Z}[q^{\pm 1}]$ whose denominators are products of elements of the form $\frac{1-q^n}{1-q}$ for some $n \in \{1, \dots, r-1\}$. Then, $\Bbbk'_q[[x,y]]$ is a subring of $\Bbbk'_q[[x,y]]$ that contains both sides of the expressions. The ring morphism $\Bbbk' \to \Bbbk$ sending $q \in \Bbbk'$ to $q \in \Bbbk$ induces a morphism of algebra $\Bbbk'_q[x,y] \to \Bbbk_q[x,y]$ that proves the two statements hold in $\Bbbk[x]$ and $\Bbbk_q[x,y]$, respectively. □

Solution of Exercise 2.3.31. Replace R by $q^{H\otimes H/2}\check{R}$ in Equations (2.2.10), (2.2.11) and (2.2.12) then commute the factor $q^{H\otimes H/2}$ using the definition of $\mathcal{H}_{ad}$. For example, for any $x \in U_h(\mathfrak{sl}_2)$,

$$\begin{aligned}(2.2.10) &\iff \tau(\Delta x) = q^{H\otimes H/2}\check{R}(\Delta x)\check{R}^{-1}q^{-H\otimes H/2} \\ &\iff q^{-H\otimes H/2}\tau(\Delta x)q^{H\otimes H/2} = \check{R}(\Delta x)\check{R}^{-1} \\ &\overset{(i)}{\iff} \tau(q^{-H\otimes H/2}(\Delta x)q^{H\otimes H/2}) = \check{R}(\Delta x)\check{R}^{-1} \\ &\iff \tau(\mathcal{H}_{ad}^{-1}(\Delta x)) = \check{R}(\Delta x)\check{R}^{-1} \iff (2.3.12).\end{aligned}$$

□

Solution of Exercise 2.3.32. First, the element $g = K$ is grouplike. Next, we prove $S^2(x) = gxg^{-1}$ for any $x \in U_q(\mathfrak{sl}_2)$. Since S^2 is a morphism of algebras, it is enough to show this when x is a generator:

$$\begin{aligned}S^2(E) &= -S(K^{-1})S(E) = KEK^{-1}, \\ S^2(F) &= -S(F)S(K) = KFK^{-1}, \\ S^2(K) &= S(K^{-1}) = K = KKK^{-1}.\end{aligned}$$

□

Solution of Exercise 2.3.34. A direct computation shows

$$(E\otimes K)(1\otimes E)=q^2(1\otimes E)(E\otimes K) \text{ and } (K^{-1}\otimes F)(F\otimes 1)=q^2(F\otimes 1)(K^{-1}\otimes F).$$

Thus, the elements $E\otimes K$ and $1\otimes E$ satisfy the relation of Exercise 2.3.14 and we have

$$\Delta(E)^n=((1\otimes E)+(E\otimes K))^n=\sum_{k=0}^{n} q^{k(n-k)}\left[\begin{array}{c} n \\ k \end{array}\right](1\otimes E)^k(E\otimes K)^{n-k}.$$

Similarly, $K^{-1}\otimes F$ and $F\otimes 1$ satisfy the relation of Exercise 2.3.14 and the resulting formula implies the exercise. □

Solution of Exercise 2.3.45. By Exercise 2.3.32, $U_q(\mathfrak{sl}_2)$ is a pivotal Hopf $\mathbb{K}$-algebra with pivot $g=K$. Then, by Proposition 2.2.29 the category of finite dimensional weight $U_q(\mathfrak{sl}_2)$-modules is a pivotal $\mathbb{K}$-additive category.

The structure of this pivotal category restricts to a pivotal structure on its full subcategory $U_q(\mathfrak{sl}_2)$-mod$_+$: first, since $\Delta(K)=K\otimes K$ then the tensor product of two objects of $U_q(\mathfrak{sl}_2)$-mod$_+$ is an object of $U_q(\mathfrak{sl}_2)$-mod$_+$. Second, since $S(K)=K^{-1}$ then the dual of an object in $U_q(\mathfrak{sl}_2)$-mod$_+$ is an object in $U_q(\mathfrak{sl}_2)$-mod$_+$. □

Solution of Exercise 2.3.46. Let $\check{R}=\sum_{n=0}^{\infty}\frac{\{1\}^{2n}}{\{n\}!}q^{n(n-1)/2}E^n\otimes F^n$ be the quasi R-matrix. Since E^2 acts by 0 in S_1^+, the quasi R-matrix reduces to

$$\check{R}(v_i\otimes v_j)=(1+\{1\}E\otimes F)v_i\otimes v_j=\begin{cases} v_1\otimes v_0+\{1\}v_0\otimes v_1 \text{ if } (i,j)=(1,0),\\ v_i\otimes v_j \text{ otherwise.}\end{cases}$$

Also, $q^{\frac{1}{2}H_{S_1^+}\otimes H_{S_1^+}}(v_i\otimes v_j)=q^{\frac{(1-2i)(1-2j)}{2}}v_i\otimes v_j$. The trace (resp. right partial trace) in $\mathscr{C}$ is the trace (resp. right partial trace) in $\mathrm{vect}_{\mathbb{Q}(q^{\frac{1}{2}})}$ of the endomorphism composed with the action of K (resp of $1\otimes K$). For the partial trace, this can be computed in the basis by taking the trace of the 2×2 blocks composing the matrix. □

Solution of Exercise 2.3.48. If a $U_\xi(\mathfrak{sl}_2)$-module V has a basis consisting of weight vectors, then as a Z_0-module, it is a direct sum of one dimensional (thus simple) Z_0-modules. Conversely, since Z_0 is commutative, simple Z_0-modules have dimension 1 and an element $z\in Z_0$ acts by a scalar $\chi(z)\in\mathbb{C}$ in such a module. So if V is a weight $U_\xi(\mathfrak{sl}_2)$-module, we have $V=\bigoplus_{\chi\in\mathsf{G}}V_\chi$ where $\mathsf{G}=\mathrm{Hom}_{\mathrm{alg}}(Z_0,\mathbb{C})$ is the set of algebra morphisms from Z_0 to $\mathbb{C}$ and $V_\chi=\{v\in V:\forall z\in Z_0, zv=\chi(z)v\}$. Since K commutes with elements of Z_0, for any $\chi\in\mathsf{G}$, V_χ is stable by K. Furthermore, the minimal polynomial of the action of K in V_χ divides the polynomial $X^r-\chi(K^r)$ which has r distinct roots (since $\chi(K^r)\neq 0$). So the action of K in V_χ is diagonalizable and V has a basis consisting of weight vectors. □

Solution of Exercise 2.3.49. To see V is a $U_\xi(\mathfrak{sl}_2)$-module, the only non-trivial relation to check is

$$\rho_V(E)\circ\rho_V(F)-\rho_V(F)\circ\rho_V(E)-\frac{\rho_V(K-K^{-1})}{\{1\}}=0.$$

We can show this by evaluating it on v_k:

$$-\frac{\{k\}\{k+1\}}{\{1\}^2}v_k+\frac{\{k-1\}\{k\}}{\{1\}^2}v_k-\frac{\{-2k\}}{\{1\}}v_k= \\ =\frac{\{k\}}{\{1\}}\left(\frac{\{k-1\}-\{k+1\}+\{1\}(\xi^k+\xi^{-k})}{\{1\}}\right)v_k=0.$$

If f is any morphism with source V, then for $k=0,\dots,r-1$, we have $f(v_k)=f(F^kv_0)=F^kf(v_0)$. Thus, f is determined by $f(v_0)$. In particular, if $f\in\mathrm{End}_{\mathscr{C}}(V)$, since v_0 generates the weight space of weight 1 of V, there exists $\lambda\in\mathbb{C}$ such that $f(v_0)=\lambda v_0$. Then, for $k=0,\dots,r-1$, we have $f(v_k)=F^k(\lambda v_0)=\lambda v_k$ and $f=\lambda\,\mathrm{Id}_V$. Hence V is absolutely simple.

The vector v_1 is a highest weight vector that generates a proper submodule of V with basis $\{v_k : k=1,\dots,r-1\}$, so by Proposition 1.3.33, V is not simple. Finally, if V was semisimple, we would have $\dim_{\mathbb{C}}(\mathrm{End}_{\mathscr{C}}(V))\geq 2$ which is not the case so V is not semisimple. □

Solution of Exercise 2.3.52. Here, we use the notion of a minimal polynomial, see Subsection 1.3.2. To prove the exercise, it is enough to consider a homogeneous module $V\in\mathscr{C}_\chi$. We choose $\alpha\in\mathbb{C}$ such that $\chi(K^r)=\xi^{r\alpha}$. Then, the minimal polynomial of $\widetilde{\rho}(K)$ divides $X^r-\xi^{r\alpha}=\prod_{k=1}^r(X-\xi^{\alpha+2k})$. Since this polynomial only has simple roots, $\widetilde{\rho}(K)$ is diagonalizable. □

Solution of Exercise 2.3.54. The first property follows from the fact that $\mathsf{G}\setminus X$ is a non-empty Zariski open set of G. To see the second, we compute

$$M_\chi^*=\begin{pmatrix}\kappa & -\varepsilon\\ \varphi & \kappa^{-1}-\kappa^{-1}\varphi\varepsilon\end{pmatrix}.$$

Then, notice that $(M_\chi^*)^{-1}$ and $M_{\chi^{-1}}^*$ are conjugate in the general linear group $\mathsf{GL}_2(\mathbb{C})$ and remark that a $\mathsf{SL}_2(\mathbb{C})$ matrix is conjugate to its inverse. Hence, M_χ^* and $M_{\chi^{-1}}^*$ have the same trace, which implies that X is symmetric. Finally, the third property follows from the second property and the fact that a $\mathsf{SL}_2(\mathbb{C})$ matrix has distinct eigenvalues if and only if its trace is not ±2. □

Solution of Exercise 2.3.61. Show that $v_{r-1}=F^{r-1}v_0$ is non-zero and conclude that $\dim_{\mathbb{C}}(V)=r$. For the last statement, show that $v_2=F^2v_0$ is a highest weight vector and conclude that $E^{r-1}F^{r-1}$ acts by 0.

□

Solution of Exercise 2.3.62. For a 1-dimensional weight module $V\in\mathscr{C}_\chi$, we must have $\widetilde{\rho}(E)=\widetilde{\rho}(F)=\widetilde{\rho}(K-K^{-1})=0$. Hence, its weight is ±1, which implies $\widetilde{\rho}(K^r)=1$, $\widetilde{\rho}(E^r)=\widetilde{\rho}(F^r)=0$ and thus $\chi(\delta)=-1$ (see Definition 2.3.53). Then, the exercise follows from Lemma 2.3.60. □

Solution of Exercise 2.3.77. First, the element $g=K^{1-r}$ is grouplike. Since S^2 is a morphism of algebras, to show $S^2(x)=gxg^{-1}$ for any $x\in U_q(\mathfrak{sl}_2)$, it is enough to show this when x is a generator which can be done as in the solution of Exercise 2.3.32. Thus, from Proposition 2.2.29 we have $\mathscr{C}$ is a pivotal $\mathbb{C}$-additive category. □

Solution of Exercise 2.3.78. First, for homogeneous elements $x, x', y, y' \in U_\xi^H(\mathfrak{sl}_2)$, we have

$$\begin{aligned}\mathcal{H}_{ad}^{-1}(x\otimes y)\mathcal{H}_{ad}^{-1}(x'\otimes y') &= (K^{-\frac{|y|}{2}}xK^{-\frac{|y'|}{2}}x')\otimes(yK^{-\frac{|x|}{2}}y'K^{-\frac{|x'|}{2}})\\ &= (\xi^{\frac{|x||y'|}{2}}K^{-\frac{|yy'|}{2}}xx')\otimes(\xi^{-\frac{|x||y'|}{2}}yy'K^{-\frac{|xx'|}{2}})\\ &= \mathcal{H}_{ad}^{-1}(xx'\otimes yy'),\end{aligned}$$

where in the second equality, we used the commutation relations of K and the additivity of weights, i.e., $|xx'| = |x| + |x'|$.

For the second equality, it is enough to look at the action on weight vectors $v \in V$, $w \in W$ of weight α and β, respectively. Then,

$$\begin{aligned}&\mathcal{H}_{V,W}^{-1}\big((x\otimes y)\mathcal{H}_{V,W}(v\otimes w)\big)\\ &\qquad= \mathcal{H}_{V,W}^{-1}\Big(\xi^{\frac{\alpha\beta}{2}}\,xv\otimes yw\Big) = \xi^{\frac{\alpha\beta}{2}}\xi^{-\frac{(\alpha+|x|)(\beta+|y|)}{2}}\,xv\otimes yw\\ &\qquad= \xi^{-\frac{(\alpha+|x|)|y|}{2}}xv\otimes\xi^{-\frac{|x|\beta}{2}}yw = K^{-\frac{|y|}{2}}xv\otimes yK^{-\frac{|x|}{2}}w\\ &\qquad= \mathcal{H}_{ad}^{-1}(x\otimes y)(v\otimes w).\end{aligned}$$

□

Solution of Exercise 2.3.85. By Part 1 of Exercise 2.3.84 we have $k_i v = \delta_{im} v$. Therefore,

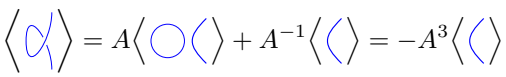
$$\overline{\mathcal{H}}(v\otimes w) = \sum_{i,j=0}^{r-1}\xi^{2ij}k_i\otimes k_j(v\otimes w) = \sum_{i,j=0}^{r-1}\xi^{2ij}\delta_{im}\delta_{jm}(v\otimes w) = \xi^{2mn}v\otimes w.$$

□

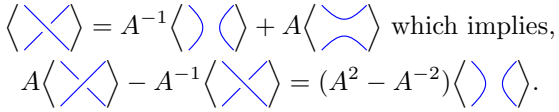
Solutions to Chapter 3: Ribbon graphs and the Reshetikhin-Turaev functor

SOLUTION OF EXERCISE 3.4.8. The solution to this exercise is analogous to the proof of the second statement of Lemma 3.4.5. In particular, one can replace $[\ell_j]$ in the proof of Lemma 3.4.5 with $[\ell]$ here. The above proof applies because here L is a circle component and in the proof of Lemma 3.4.5 all other strands only involve a local crossing with L. □

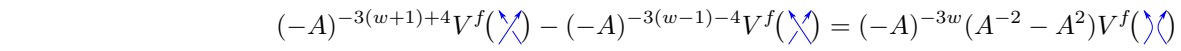
SOLUTION OF EXERCISE 3.4.13. For the first part, we have



then



For the second part, remark that if the total writhe of the link in the right-hand side of Equation (3.4.4) is w, then the two links in the left-hand side have total writhe $w+1$ and $w-1$, respectively. Then, the Jones skein relation implies that V^f defines a ribbon link invariant which satisfies



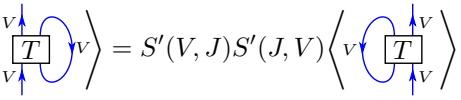
which is equivalent to (3.4.4). Equation (3.4.5) is a relation satisfied by the total writhe.

Finally, since both are oriented ribbon link invariants, Lemma 3.4.11 implies they coincide. □

Solutions to Chapter 4: M-traces and the renormalized Reshetikhin-Turaev invariant

SOLUTION OF EXERCISE 4.2.6. As in the proof of Theorem 4.2.2 and Equation (4.2.9) with $V = W$ we have:

$$S'(V,J)S'(J,V)\left\langle \right\rangle = S'(V,J)S'(J,V)\left\langle \right\rangle$$



for all $(2,2)$-ribbon $\mathscr{C}$-graphs T whose open components are colored with V. □

SOLUTION OF EXERCISE 4.2.7. The first claim follows Exercise 4.2.6. For the second claim assume $J \in B$; then notice the definitions imply $B \subset A(J)$. Conversely, let $V \in A(J)$. Then, by definition of $A(J)$ we have V is absolutely simple and $S'(J,V) \neq 0$. Since $J \in B$, we have $V \in B$ otherwise completeness of B would imply $S'(J,V) = 0$. Thus, $A(J) = B$.

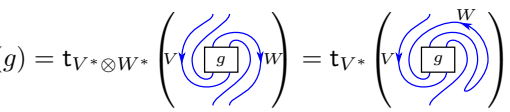
Lemma 4.2.5 implies that $\mathsf{d}_{J'}$ is proportional to d_J. More precisely, the proof of this lemma implies $\mathsf{d}_J = \frac{\mathsf{d}_J(J')}{d'_0}\,\mathsf{d}_{J'}$. Thus,

$$\mathsf{F}'_J(\widehat{T}) = \mathsf{d}_J(V)\langle T_V\rangle = \frac{\mathsf{d}_J(J')}{d'_0}\,\mathsf{d}_{J'}(V)\langle T_V\rangle = \frac{\mathsf{d}_J(J')}{d'_0}\,\mathsf{F}'_{J'}(\widehat{T})$$

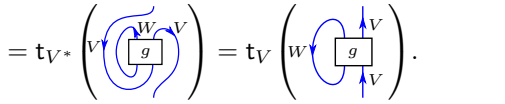
and this completes the proof. □

SOLUTION OF EXERCISE 4.2.18. We need to show that the family t satisfies the left partial trace property. Let $V \in \mathcal{I}$, $W \in \mathscr{C}$ and $g \in \mathrm{End}_{\mathscr{C}}(W \otimes V)$. Then, we have

$$\mathsf{t}_{W\otimes V}(g) = \mathsf{t}_{V^*\otimes W^*}\left(\right) = \mathsf{t}_{V^*}\left(\right)$$

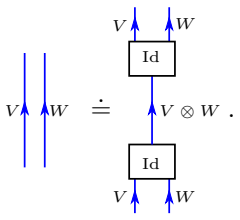


$$= \mathsf{t}_{V^*}\left(\right) = \mathsf{t}_V\left(\right).$$



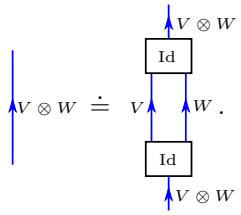
□

SOLUTION OF EXERCISE 4.2.24. Choose a box in $(T\backslash K)\cup K_V \cup K_W$ whose pre-image is two small vertical edges which are portions of K_V and K_W, respectively. Do a skein relation in this box which replaces the two small vertical edges with two coupons filled with $\mathrm{Id}_{V\otimes W}$ and its inverse is also equal to $\mathrm{Id}_{V\otimes W}$:



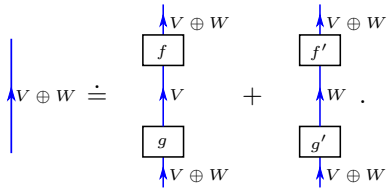
Similarly, the same box in T has a pre-image which is a small vertical portion of K and do a skein relation which replaces this portion of K with two coupons filled

with $\mathrm{Id}_{V\otimes W}$ and its inverse also equal to $\mathrm{Id}_{V\otimes W}$:

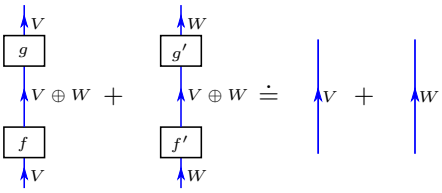


The resulting graphs are isotopic where the isotopy moves one of the coupons in one of the graphs all the way around the component K. □

Solution of Exercise 4.2.25. The direct sum gives morphisms $V \xrightarrow{f} V \oplus W$, $V \oplus W \xrightarrow{g} V$, $W \xrightarrow{f'} V \oplus W$, $V \oplus W \xrightarrow{g'} W$ such that $g \circ f = \mathrm{Id}_V$, $g' \circ f' = \mathrm{Id}_W$, and $\mathrm{Id}_{V\oplus W} = f \circ g + f' \circ g'$. Then, we have a skein equivalence given by:

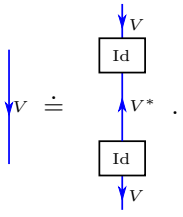


Now, move by an isotopy the coupons colored by f and f' all the way around the distinguished closed component and then use the following skein equivalence:



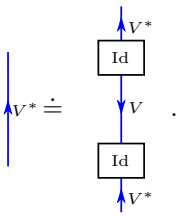
which finishes the proof of the exercise. □

Solution of Exercise 4.2.26. Choose a box that intersects T in a small vertical portion of K oriented down (in the box) and colored with V. Do a skein relation in this box which replaces this portion of K with two coupons filled with Id_{V^*} and an edge colored by V^*:



The same box intersects T' in a small vertical portion of K oriented up and colored with V^* and we do a skein relation which replaces this portion of K with two

coupons both filled with Id_{V^*} and an edge colored with V oriented down:



The resulting graphs are isotopic where the isotopy moves one of the coupons in one of the graphs all the way around the component K. □

Solution of Exercise 4.2.29. See the last part of the proof of Theorem 5.4.41 where such a connected sum appears. □

Solution of Exercise 4.3.1. Let $\mathcal{I}$ be a full subcategory which is stable under retracts and $V \otimes U \in \mathcal{I}$ for any $U \in \mathscr{C}$ and $V \in \mathcal{I}$. The braiding $c_{U,V} : U \otimes V \to V \otimes U$ is invertible and so $U \otimes V$ is a retract of $V \otimes U \in \mathcal{I}$ thus $U \otimes V \in \mathcal{I}$. The converse is true by definition. Finally, the last statement follows from the first statement and the definition of $\mathcal{I}_V$. □

Solution of Exercise 4.3.2.

(1) Since $U \in \mathcal{I}_V$, there exists a retract $\mathrm{Id}_U = (U \xrightarrow{\alpha} W_1 \otimes V \otimes W_1' \xrightarrow{\beta} U)$. If $W \in \mathcal{I}_U$ then there exists a retract $\mathrm{Id}_W = (W \xrightarrow{\alpha'} W_2 \otimes U \otimes W_2' \xrightarrow{\beta'} W)$ then there is a retract $\mathrm{Id}_W = (W \to W_2 \otimes W_1 \otimes V \otimes W_1' \otimes W_2' \to W)$ thus, $W \in \mathcal{I}_V$.

(2) Notice, $\mathrm{Id}_V = (V \xrightarrow{\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{Id}_V \otimes \overleftarrow{\mathrm{ev}}_V} V)$ is a retract implying $V \in \mathcal{I}_{V^*}$. Then, the first property of the exercise implies $\mathcal{I}_V \subset \mathcal{I}_{V^*}$. Similarly, $\mathrm{Id}_{V^*} = (V^* \xrightarrow{\mathrm{Id}_{V^*} \otimes \overleftarrow{\mathrm{coev}}_V} V^* \otimes V \otimes V^* \xrightarrow{\overleftarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_{V^*}} V^*)$ is a retract implying $V^* \in \mathcal{I}_V$ and so $\mathcal{I}_{V^*} \subset \mathcal{I}_V$. See also Proposition 4.2.15.

(3) Given $\mathrm{Id}_U = (U \xrightarrow{\alpha} W_1 \otimes V \otimes W_2 \xrightarrow{\beta} U)$ and $\mathrm{Id}_{U'} = (U' \xrightarrow{\alpha'} W_1' \otimes V \otimes W_2' \xrightarrow{\beta'} U')$ let $W_1'' = W_1 \oplus W_1'$ and $W_2'' = W_2 \oplus W_2'$. Then, $W_1'' \otimes V \otimes W_2'' \simeq (W_1 \otimes V \otimes W_2) \oplus (W_1' \otimes V \otimes W_2') \oplus \cdots$. In particular, $U \oplus U'$ is a retract of $W_1'' \otimes V \otimes W_2''$.

□

Solution of Exercise 4.3.4. Applying the definition of projective to $f : V \to P$ and $g = \mathrm{Id}_P$ we get a morphism $h : P \to V$ such that $\mathrm{Id}_P = f \circ h$. □

Solution of Exercise 4.3.9. Let $f \in \mathrm{End}_{\mathscr{C}}(J \otimes J)$. By Proposition 1.2.47, we have

$$\mathrm{ptr}_J^R(c_{J,J}^{-1} \circ f \circ c_{J,J}) = \mathrm{ptr}_J^L(f).$$

But $c_{J,J}$ commutes with $\mathrm{End}_{\mathscr{C}}(J \otimes J)$ and so $c_{J,J}^{-1} \circ f \circ c_{J,J} = f$. □

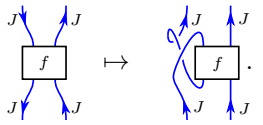
Solution of Exercise 4.3.15. This is a corollary of Lemma 4.3.13 but we give a direct proof here. First, since V is projective then by Proposition 4.2.15, V^* is also projective. Then, by Lemma 4.3.7 the ideal Proj is generated by V^*, i.e.,

$\mathsf{Proj} = \mathcal{I}_{V^*}$. So there exists an object W in $\mathscr{C}$ and morphisms $\alpha : P \to V^* \otimes W$ and $\beta : V^* \otimes W \to P$ such that $\beta \circ \alpha = \mathrm{Id}_P$. Define

$$f = (\mathrm{Id}_V \otimes \beta) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_W) \circ (\overrightarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_V \otimes \alpha).$$

□

Solution of Exercise 4.3.19. Hint: first show that one can define a bijection $\mathrm{End}_{\mathscr{C}}(J^* \otimes J) \to \mathrm{End}_{\mathscr{C}}(J \otimes J)$ by



□

Solution of Exercise 4.3.28. If $U \in \mathcal{I}^R_{\eta,\varepsilon}$, then $V \otimes U \in \mathcal{I}^R_V$ and $\mathrm{Id}_U = (U \xrightarrow{\eta \otimes \mathrm{Id}_U} V \otimes U \xrightarrow{\sigma_V} U)$ is a retract. □

Solution of Exercise 4.3.36. Since P and Q are projective, there exists morphisms $i' : P \to Q$ and $p : Q \to P$ such that $\pi = g \circ i'$ and $g = \pi \circ p$. Therefore, $\pi = \pi \circ (p \circ i')$ implying $\pi = \pi \circ (p \circ i')^n$ for all $n \in \mathbb{N}$. Thus, $p \circ i'$ is not nilpotent and so by the Fitting Lemma 1.3.46, $p \circ i'$ is invertible with inverse f (note by definition $\mathrm{End}_{\mathscr{C}}(P)$ is an indecomposable algebra). Set $i = i' \circ f$ then $\mathrm{Id}_P = (P \xrightarrow{i} Q \xrightarrow{p} P)$ is a retract. Moreover, we have

$$g \circ i = \pi \circ p \circ i = \pi \circ p \circ i' \circ f = \pi.$$

For the second statement, assume $\mathscr{C}$ is idempotent complete. Then, i can be completed to an isomorphism $h = i \oplus j : P \oplus Q' \to Q$ with $p \circ j = 0$. Finally, $g \circ h = \pi \circ p \circ (i \oplus j) = \pi \oplus 0$. □

Solution of Exercise 4.3.50.

(1) The map $f \mapsto f(1_H)$ gives a linear bijection $\mathrm{Hom}_{H\text{-mod}}(H_L, V) \xrightarrow{\sim} V$. Hence, f factors through any epimorphism $W \to V$ by choosing a preimage in W of $f(1_H)$.
(2) The map is a morphism since $f \circ g(1_H) = f(g(1_H)1_H) = g(1_H)f(1_H)$. The right regular action $R : H^{op} \to \mathrm{End}_{H\text{-mod}}(H_L)$ is its inverse.
(3) As above $\mathrm{Hom}(H_L, \mathbb{1}) \simeq \mathbb{K}$ and ε is clearly a generator. By H-linearity, a map $\mathbb{1} \to H$ sends 1 to a left cointegral which is unique up to a scalar.

□

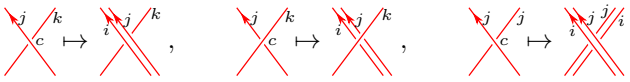
Solution of Exercise 4.3.55. If $\overrightarrow{\mathcal{N}}(\mathbb{1}, \mathbb{1}) = \{0\}$ then $\overrightarrow{\mathcal{N}}$ is proper. Conversely, suppose $\overrightarrow{\mathcal{N}}$ is proper. Then, there exists objects V, W in $\mathscr{C}$ and a morphism $f \in \mathrm{Hom}_{\mathscr{C}}(V, W)$ such that $f \notin \overrightarrow{\mathcal{N}}(V, W)$. Then, $f \otimes \mathrm{Id}_{\mathbb{1}} = f \notin \overrightarrow{\mathcal{N}}(V, W)$ implies $\mathrm{Id}_{\mathbb{1}} \notin \overrightarrow{\mathcal{N}}$. Since $\mathscr{C}$ is sharp, $\mathrm{End}_{\mathscr{C}}(\mathbb{1}) = \mathbb{K}\,\mathrm{Id}_{\mathbb{1}}$ and thus $\overrightarrow{\mathcal{N}}(\mathbb{1}, \mathbb{1}) = 0$. □

Solution of Exercise 4.3.62. Hint: for this last exercise, remark that for an indecomposable non $\overrightarrow{\mathcal{N}}$-negligible object $V \in \mathscr{C}$, we have $\overrightarrow{\mathcal{N}}(V, V) \subset J(V)$ where $J(V)$ is the Jacobson ideal of $\mathrm{End}_{\mathscr{C}}(V)$.

□

Solutions to Chapter5: Ribbon 3-manifold invariants

Solution of Exercise 5.3.3. Using the diagrammatic definition of Subsection 3.4.1 we can compute the change of the linking matrix. A diagram of L' is obtained from a diagram of L by modifying it near every crossing c of L_j as follows (up to rotation of the picture):



where $k \neq j$. Hence, $\mathrm{lk}(L') = f(\mathrm{lk}(L))$ where f is the linear endomorphism of the space of matrices sending

$$E_{j,k} \mapsto E_{j,k} + E_{i,k}\ , \qquad E_{k,j} \mapsto E_{k,j} + E_{k,i}\ , \qquad E_{j,j} \mapsto E_{j,j} + E_{i,j} + E_{j,i} + E_{i,i},$$

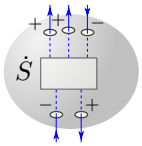
and f leaves the other elementary matrices $E_{k,\ell}$ unchanged. Finally, one checks in each case that f is indeed the linear map given by $M \mapsto (I_n + E_{ij})M(I_n + E_{ji})$. Since f corresponds to a change of basis for the quadratic form represented by a symmetric matrix, Sylvester's law of inertia implies that the signature (r, s) of the linking matrix is not affected by a Kirby II move. □

Solution of Exercise 5.4.7. We have

$$\Sigma\overline{\Sigma} = \sum_{j,k=0}^{r-1} \left(\xi^2\right)^{j^2-k^2+j-k} = \sum_{j,k=0}^{r-1} \left(\xi^2\right)^{(j-k)(j+k+1)}.$$

The map $(j,k) \mapsto (\frac{(r+1)(j+k)}{2}, \frac{(r+1)(j-k)}{2})$ is the inverse bijection. The sum can then be rewritten $\Sigma\overline{\Sigma} = \sum_{j,k=0}^{r-1} \left(\xi^2\right)^{j(k+1)} = \sum_{k=0}^{r-1} \sum_{j=0}^{r-1} \left(\xi^{2(k+1)}\right)^j$ and the sum over j is zero unless $k = r-1$. □

Solution of Exercise 5.4.11. Let v be a vertex of T, with incident edges $e_1, \ldots, e_n$. For $i \in \{1, \ldots n\}$, let $\varepsilon_i = 1$ if e_i is incoming to v and $\varepsilon_i = -1$ otherwise. Then, the cycle condition at v is $\sum_i \varepsilon_i \omega_T(e_i) = 0$. Let S be the boundary of a small 3-ball which is a neighborhood of v oriented with the "outward first" rule (see Example 7.2.4). We can assume that S intersects each edge e_i once transversally. Let $\dot{S}$ be S minus a tubular neighborhood of the edges $e_1, \ldots, e_n$. The following picture is a representation of $\dot{S}$:



Then, the boundary of $\dot{S}$ is the signed sum of the meridians of the edges $\sum_{i=1}^n \varepsilon_i [m_i]$ which is then 0 in $H_1(M \setminus T, \mathbb{Z})$. Hence, $\sum_i \varepsilon_i \omega([m_i]) = 0$ which is equivalent to the boundary condition at v. □

Solution of Exercise 5.4.24. From Proposition 4.3.77 and Corollary 4.3.22, we have that any generic simple S is right ambidextrous and $\mathsf{d}(S)$ is non-zero (note that in generic degree, the simples are absolutely simple). Then, Theorem 4.3.78 implies that the m-trace is unique and non-degenerate (note here a right m-trace is an m-trace because $\mathscr{C}$ is ribbon, see Proposition 4.3.23). □

SOLUTION OF EXERCISE 5.4.25. For the first equality, compose $c^{-1}_{\sigma^k,V}$ on both sides of Equation (5.4.3). The other equality is analogous. □

SOLUTION OF EXERCISE 5.4.37. To see this one needs to replace the component colored with $W \otimes \sigma^k$ with two parallel copies colored with W and σ^k, respectively. Then, make two crossing changes (one with the strand colored with W and the other with V, see Equation (5.4.4)) to pull the component colored with σ^k above the whole diagram. Then, you will see an unknot with framing ± 1 colored with σ^k which is equal to $\dim_{\mathscr{C}}(\sigma^k)$, since the twist is trivial on σ^k. □

SOLUTION OF EXERCISE 5.4.45. We show the first equality, the second is similar. Since E acts as zero in σ^k, the action of the quasi R-matrix on $\sigma^k \otimes V$ is trivial. Then, the braiding is given by $\tau \circ \xi^{\frac{1}{2}\widetilde{\rho}_{\sigma^k}(H)\otimes\widetilde{\rho}_V(H)} = \tau \circ \xi^{\frac{rk}{1+\varepsilon_\ell}\operatorname{Id}_{\sigma^k}\otimes\widetilde{\rho}_V(H)} = \tau \circ (\operatorname{Id}_{\sigma^k} \otimes \widetilde{\rho}_V(K^{\frac{rk}{1+\varepsilon_\ell}}))$.

□

SOLUTION OF EXERCISE 5.4.47. Since $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$, it is enough and easy to check the action of H is the same on $\sigma^j \otimes \sigma^k$ and σ^{j+k}. For the twist, Exercise 5.4.45 gives that the braiding on $\sigma^j \otimes \sigma^k$ is $\xi^{\frac{2r^2jk}{(1+\varepsilon_\ell)^2}} \circ \tau$ hence $\sigma^k \otimes \sigma^k = (-1)^{\varepsilon_\ell k}\tau$ (check this last equality by separating the cases $\varepsilon_\ell = 0$ and $\varepsilon_\ell = 1$). The pivot acts on σ^k by multiplication by $\widetilde{\rho}_{\sigma^k}(K^{1-r}) = \xi^{\frac{2r(1-r)k}{1+\varepsilon_\ell}} = (-1)^{\varepsilon_\ell k}$ so the twist on σ^k is the partial trace in $\mathbb{C}$-vect of τ which is the identity.

Then, for the module V_α of Lemma 4.2.8, $V_\alpha \otimes \sigma^k$ is simple with highest weight $\alpha + \frac{2rk}{1+\varepsilon_\ell}$. This implies $V_\alpha \otimes \sigma^k \simeq V_{\alpha+\frac{2rk}{1+\varepsilon_\ell}}$ which is not isomorphic to V_α if $k \neq 0$.

Finally, if $V \in \mathscr{C}_{\overline{\alpha}}$ has a highest weight in $\overline{\alpha} = \alpha + 2\mathbb{Z}$, from Exercise 5.4.45 we get $c_{\sigma^k,V} \circ c_{V,\sigma^k} = \widetilde{\rho}_V(K^r)^{\frac{2k}{1+\varepsilon_\ell}} \otimes \operatorname{Id}_{\sigma^k} = \xi^{\frac{2rk}{1+\varepsilon_\ell}\alpha} \operatorname{Id}_{V\otimes\sigma^k}$.

□

SOLUTION OF EXERCISE 5.4.48.

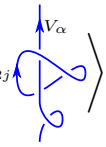
$$\begin{aligned}\Sigma\overline{\Sigma} &= \sum_{j,k=0}^{r-1} \left(\xi^2\right)^{j^2-k^2-j+k} = \sum_{j,k=0}^{r-1} \left(\xi^2\right)^{(j-k)(j+k-1)} \\ &= \sum_{m,k=0}^{r-1} \left(\xi^2\right)^{m(m+2k-1)} = \sum_{m=0}^{r-1} \left(\xi^2\right)^{m^2-m} \sum_{k=0}^{r-1} \left(\xi^{2m}\right)^k\end{aligned}$$

This last sum over k is zero unless $\xi^{2m} = 1$ which only happens for $m = 0$ if r is odd. If r is even, then $\ell \in 4\mathbb{Z}$ and $\xi^{2m} = 1$ for $m = 0$ and for $m = r/2$ so the sum becomes $r(1 + \left(\xi^2\right)^{r^2/4-r/2}) = r(1 + -(-1)^{r/2})$ which is 0 if $r \in 4\mathbb{Z}$ and $2r$ if $r \in 4\mathbb{Z} + 2$. □

SOLUTION OF EXERCISE 5.4.49. Recall from Proposition 4.2.10, $\mathsf{d}(V_\alpha) = d_0 \frac{r\{\alpha\}}{\{r\alpha\}}$, from Equation (4.2.10), $S'(V_\alpha, V_\beta) = \xi^{\alpha\beta}\frac{\{r\beta\}}{\{\beta\}}$, and from the proof of Theorem

2.3.79, $\theta_{V_\alpha} = \xi^{\frac{\alpha^2-c}{2}}$ where $c = (r-1)^2$.

$$\Delta_- = \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \mathsf{d}(V_{\alpha+2j}) \left\langle V_{\alpha+2j} \right.$$



$$\left. \right\rangle = \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \mathsf{d}(V_{\alpha+2j}) S'(V_{\alpha+2j}, V_\alpha) \theta_{V_\alpha}^{-1} \theta_{V_{\alpha+2j}}^{-1}$$

$$= \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} r\,\mathsf{d}_0 \frac{\{\alpha+2j\}}{\{\alpha\}} \xi^{\alpha(\alpha+2j)} \xi^{-\frac{\alpha^2-c}{2}} \xi^{-\frac{(\alpha+2j)^2-c}{2}} = r\,\mathsf{d}_0\,\xi^c \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \frac{\{\alpha+2j\}}{\{\alpha\}} \xi^{-2j^2}$$

$$= r\,\mathsf{d}_0 \frac{\xi^c}{\{\alpha\}} \left(\xi^\alpha \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \xi^{2j-2j^2} + \xi^{-\alpha} \sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \xi^{-2j-2j^2} \right)$$

But if $\varepsilon_\ell = 1$ then $\xi^{2(j+\frac{r}{2})-2(j+\frac{r}{2})^2} = \xi^{2j-2j^2}$ so in any case $\sum_{j=1}^{\frac{r}{1+\varepsilon_\ell}} \xi^{-2j-2j^2} = \frac{1}{1+\varepsilon_\ell} \sum_{j=1}^{r} \left(\xi^2\right)^{j-j^2}$ where ξ^2 is a primitive r^{th} root of unity. Also $\sum_{j=1}^{r} \left(\xi^2\right)^{-j-j^2} = \sum_{j=1}^{r} \left(\xi^2\right)^{j-j^2}$, thus

$$\Delta_- = \frac{r\,\mathsf{d}_0\,\xi^c}{1+\varepsilon_\ell} \sum_{j=1}^{r} \left(\xi^2\right)^{j-j^2}.$$

This is a generalized quadratic Gauss sum which is computed for example in [**BEW98**, Chapter 1] and satisfies (see Exercise 5.4.48)

$$\left| \sum_{j=1}^{r} \left(\xi^2\right)^{j-j^2} \right| = \begin{cases} 0 & \text{if } \ell \in 8\mathbb{Z}, \\ \sqrt{2r} & \text{if } \ell \in 4+8\mathbb{Z}, \\ \sqrt{r} & \text{otherwise.} \end{cases}$$

Finally, a similar computation shows that $\mathsf{d}_0^{-1}\,\Delta_+$ is the complex conjugate of $\mathsf{d}_0^{-1}\,\Delta_-$.

□

Solution of Exercise 5.4.51. (1-3) Let $b_0 = p$, $b_1 = q$, then for $i \geq 1$, a_i is the smallest integer bigger or equal to $\frac{b_{i-1}}{b_i} > 1$ and $b_{i+1} = a_i b_i - b_{i-1} \in \{0, \dots, b_i - 1\}$ is coprime with b_i. The strictly decreasing sequence b_i ends with $b_n = 1$ and we set $a_n = b_{n-1}$. By induction, for $i = 1, \dots, n$, we have $\frac{p}{q} = \left\langle a_1, a_2, \dots, a_{i-1}, \frac{b_{i-1}}{b_i} \right\rangle$.

(2) L is symmetric thus diagonalisable over $\mathbb{R}$. If $L\vec{v} = \lambda\vec{v}$ is an eigenvector and if $|v_i|$ is its biggest coordinate in absolute value, then $\lambda v_i = (L\vec{v})_i = (a_i - 2)v_i + (v_i + v_{i-1}) + (v_i + v_{i+1})$ where the three terms have the same sign as v_i, hence $\lambda > 0$.

(4) Since L is non-singular, ω is unique. Then we can check for $i = 2, \dots, n-1$, $(L\vec{\omega})_i = \frac{2h}{p}(-1)^i(-b_{i-1} + a_i b_i - b_{i+1}) = 0$, $(L\vec{\omega})_n = \frac{2h}{p}(-1)^n(-b_{n-1} + a_n b_n) = 0$ and $(L\vec{\omega})_1 = \frac{2h}{p}(-a_1 b_1 + b_2) = 2h$.

(5) Removing the last circle of the chain link, we have for $i = 2, \dots, n$, $\mathsf{F}'(C^{\alpha_1,\dots,\alpha_i}_{a_1,\dots,a_i}) = \theta^{a_i}_{\alpha_i} S'(V_{\alpha_i}, V_{\alpha_{i-1}})\,\mathsf{F}'(C^{\alpha_1,\dots,\alpha_{i-1}}_{a_1,\dots,a_{i-1}})$ and $\mathsf{F}'(C^{\alpha_1}_{a_1}) = \theta^{a_1}_{\alpha_1}\,\mathsf{d}(V_{\alpha_1})$. Then

$$\mathsf{F}'(C^{\underline{\alpha}}_{\underline{a}}) = \mathsf{d}(V_{\alpha_1}) \xi^{a_1 \frac{\alpha_1^2-c}{2}} \prod_{i=2}^{n} \xi^{a_i \frac{\alpha_i^2-c}{2}} \xi^{\alpha_{i-1}\alpha_i} \frac{\{r\alpha_{i-1}\}}{\{\alpha_{i-1}\}},$$

which is equivalent to the given formula.

□

SOLUTION OF EXERCISE 5.6.4. Since $\mathscr{C}$ is unimodular, there exists a projective cover (P_0, ε) of $\mathbb{1}$. Then, since Id_{P_0} factors through G, we have $\mathrm{Hom}_{\mathscr{C}}(G, \mathbb{1}) \neq \{0\}$. Since $\mathbb{1}$ is simple, any non-zero morphism ε_G in $\mathrm{Hom}_{\mathscr{C}}(G, \mathbb{1})$ is an epimorphism. Now, $P \otimes P^*$ is projective so $\overrightarrow{\mathrm{ev}}_P$ factors through ε_G and there exists $P \otimes P^* \xrightarrow{\widetilde{e}} G$ such that $\overrightarrow{\mathrm{ev}}_P = \varepsilon_G \circ \widetilde{e}$. Then, $e_{P,G} = (\widetilde{e} \otimes \mathrm{Id}_P)(\mathrm{Id}_P \otimes \overrightarrow{\mathrm{coev}}_P)$ is the required map. □

SOLUTION OF EXERCISE 5.6.22. Exercise 5.6.4 implies $T \sqcup$  is Proj-skein equivalent to a Proj-admissible ribbon bichrome $\mathscr{C}$-graph $T' \sqcup$  which contains a coupon filled with a morphism ε_G. Up to isotopy we can pull this coupon close to the $+1$ framed red unknot. Then, from the definition of twist non-degenerate and since F is invariant under isotopy and skein relations we have

$$\mathsf{F}\left(T \sqcup \text{}\right) = \mathsf{F}\left(T' \sqcup \text{}\right) = \Delta_+ \mathsf{F}(T') = \Delta_+ \mathsf{F}(T).$$

□

Solutions to Chapter 6: Spherical 3-manifold invariants

SOLUTION OF EXERCISE 6.2.10. If $Q \in \mathsf{Proj}$, by Lemma 5.6.14, we have $\overleftarrow{\mathrm{coev}}_Q \circ \overrightarrow{\mathrm{ev}}_P \circ (\mathrm{ptr}_G^L(\mathsf{c}_P) \otimes \mathrm{Id}) = (\mathrm{ptr}_G^L(\mathsf{c}_Q) \otimes \mathrm{Id}) \circ \overleftarrow{\mathrm{coev}}_Q \circ \overrightarrow{\mathrm{ev}}_P$ which implies $\mathrm{ptr}_G^L(\mathsf{c}_Q) = D\,\mathrm{Id}_Q$. □

SOLUTION OF EXERCISE 6.2.11. Hint: If $P_0 \xrightarrow{\varepsilon} \mathbb{1}$ is the projective cover of $\mathbb{1}$, define D as the unique scalar such that $\varepsilon \circ \mathrm{ptr}_G^L(\mathsf{c}_{P_0}) = D\varepsilon$. □

Solutions to Chapter 7: Topological Quantum Field Theories and decorations

SOLUTION OF EXERCISE 7.3.8. For $\Sigma \in \mathbf{Man}_n$, there is a natural choice of dual for Σ given by $-\Sigma$. Then, the duality morphisms are all given by the same cylinder $\Sigma \times [0,1]$ seen as cobordisms $\overleftarrow{\mathrm{ev}}_\Sigma : (-\Sigma) \sqcup \Sigma \to \emptyset$, $\overrightarrow{\mathrm{ev}}_\Sigma : \Sigma \sqcup (-\Sigma) \to \emptyset$, $\overleftarrow{\mathrm{coev}}_\Sigma : \emptyset \to \Sigma \sqcup (-\Sigma)$ and $\overrightarrow{\mathrm{coev}}_\Sigma : \emptyset \to (-\Sigma) \sqcup \Sigma$. □

SOLUTION OF EXERCISE 7.3.9. By Proposition 7.3.6 and Exercise 7.3.8 we have $\mathbf{Cob}_{n+1}$ is a braided symmetric category which is also pivotal. The exercise follows from noticing that the twist in this category is trivial:

$$\mathrm{ptr}^L_\Sigma(c_{\Sigma,\Sigma}) = \mathrm{ptr}^R_\Sigma(c_{\Sigma,\Sigma}) = \mathrm{Id}_\Sigma$$

for any n-manifold Σ. □

SOLUTION OF EXERCISE 7.6.4.
(1) $\mathrm{Id}_{\Sigma_2} \triangleleft f = h_*(f \triangleright \mathrm{Id}_{\Sigma_1})$ where $h = (f \times \mathrm{Id}_{[0,1]}) : \Sigma_1 \times [0,1] \to \Sigma_2 \times [0,1]$.
(2) $C_f \circ C_g = (C_{\Sigma_3} \triangleleft f) \circ (C_{\Sigma_2} \triangleleft g) \simeq (C_{\Sigma_3} \triangleleft f) \circ (g \triangleright C_{\Sigma_1}) = C_{\Sigma_3} \triangleleft (f \circ g)$.
(3) Since $f_0 = f$ and $f_1 = g$ are isotopic, they are smoothly isotopic. Choose such a smooth isotopy f_t with $f_0 = f$ and $f_1 = g$. Then, $h : (x,t) \mapsto (f_{1-t} \circ f^{-1}(x), t)$ is a diffeomorphism such that $C_g = h_* C_f$.

□

SOLUTION OF EXERCISE 7.6.10.
(1) The map $f^{-1} \triangleright - \triangleleft f : \mathrm{End}_{\widetilde{\mathbf{Cob}}_{n+1}}(\widetilde{\Sigma}_2) \to \mathrm{End}_{\widetilde{\mathbf{Cob}}_{n+1}}(\widetilde{\Sigma}_1)$ is an isomorphism of monoids (with inverse $f \triangleright - \triangleleft f^{-1}$) thus, it sends the identity to the identity. Hence $f^{-1} \triangleright \widetilde{C}_{\widetilde{\Sigma}_2} \triangleleft f \simeq \widetilde{C}_{\widetilde{\Sigma}_1}$ so that $\widetilde{C}_{\widetilde{\Sigma}_2} \triangleleft f \simeq f \triangleright \widetilde{C}_{\widetilde{\Sigma}_1}$.
(2) If $\widetilde{\Sigma}_1 \xrightarrow{\widetilde{f}} \widetilde{\Sigma}_2$, then $\widetilde{C}_{\widetilde{f}} \circ \widetilde{C}_{\widetilde{g}} = (\widetilde{C}_{\widetilde{\Sigma}_2} \triangleleft |\widetilde{f}|) \circ (\widetilde{C}_{\widetilde{\Sigma}_1} \triangleleft |\widetilde{g}|) \simeq \widetilde{C}_{\widetilde{\Sigma}_2} \circ (|\widetilde{f}| \triangleright \widetilde{C}_{\widetilde{\Sigma}_1} \triangleleft |\widetilde{g}|) \simeq \widetilde{C}_{\widetilde{\Sigma}_2} \circ (\widetilde{C}_{\widetilde{\Sigma}_2} \triangleleft |\widetilde{f}|) \triangleleft |\widetilde{g}| \simeq \widetilde{C}_{\widetilde{\Sigma}_2} \triangleleft (|\widetilde{f}| \circ |\widetilde{g}|) = \widetilde{C}_{\widetilde{f} \circ \widetilde{g}}$.

□

SOLUTION OF EXERCISE 7.7.7. Let $(p,q) \in \partial M \times N$, let $(v_1, v_2, \ldots, v_m)$ be a direct basis of T_pM and $(w_1, w_2, \ldots, w_n)$ be a direct basis of T_qN where v_1 is an outgoing normal to ∂M. Then, by definition of the orientation of boundaries, $(v_2, \ldots, v_m)$ is a direct basis of $T_p\partial M$. The orientation c is the one for which the basis $b = (v_2, \ldots, v_m, w_1, \ldots, w_n)$ of $T_p\partial M \oplus T_qN \cong T_{(p,q)}(\partial M \times N)$ is direct and the orientation c' is the one for which the basis $b' = (w_1, \ldots, w_n, v_2, \ldots, v_m)$ of $T_qN \oplus T_p\partial M \cong T_{(q,p)}(N \times \partial M)$ is direct.

(1) To see if $b = (v_2, \ldots, v_m, w_1, \ldots, w_n)$ is a direct basis of the boundary of $M \times N$, we add in first position the vector $v_1 \in T_{(p,q)}(M \times \partial N)$ which is an outgoing normal vector to $\partial M \times \partial N$. The result is a basis $(v_1, v_2, \ldots, v_m, w_1, \ldots, w_n)$ which is a direct basis of $M \times N$. So c is the orientation of $\partial M \times N$ as the boundary of $M \times N$.
(2) To see if b' is a direct basis of the boundary of $N \times \partial M$, we add in first position the vector $v_1 \in T_{(q,p)}(N \times \partial M)$ which is an outgoing normal vector to $N \times \partial M$. The result is a basis $(v_1, w_1, \ldots, w_n, v_2, \ldots, v_m)$ of $N \times M$. The orientation c' of $N \times M$ is given by the direct basis $(w_1, \ldots, w_n v_1, \ldots, v_m,)$. These two orientations differ if and only if n is odd.

□

Solution of Exercise 7.7.16. Use Item 4 of Definition 7.5.5. □

Solution of Exercise 7.7.20. Since V is bordism generated, it is generated by vectors $\pi(\widetilde{C}) \in \mathsf{V}_C(\widetilde{\Sigma})$ for decorated cobordisms $\widetilde{C} : \widetilde{\emptyset} \to \widetilde{\Sigma}$ with $|\widetilde{C}| = C$. Since V has the index 1 surgery property, $\mathsf{V}_C(\widetilde{\Sigma}) \subset \mathsf{V}_{C^\#}(\widetilde{\Sigma})$ where $C^\#$ is obtained from C by doing the connected sums of its components (see Lemma 7.7.17). □

Solution of Exercise 7.8.1. From Theorem 5.6.23 we have B' is an invariant of Proj-admissible ribbon $\mathscr{C}$-graphs in a closed connected 3-manifold. Then, it follows that $\alpha^{b_0(M)}\beta^{b_1(M)}\gamma^n\mathsf{V}_{\text{chrom}}(M,T,n)$ is an invariant of the tuple (M,T,n) since b_0 and b_1 are invariants of M. Moreover, the definition of $\mathsf{V}_{\text{chrom}}$ is a multiplicative extension of B' to non-connected manifolds and the Betti numbers are additive under disjoint union as is the signature weight thus, the formula $\alpha^{b_0(M)}\beta^{b_1(M)}\gamma^n\mathsf{V}_{\text{chrom}}(M,T,n)$ defines a multiplicative invariant. □

Solution of Exercise 7.8.2. The first homology of M is isomorphic to the cokernel of the linking matrix. Hence, the rank of the linking matrix is equal to $m - b_1(M)$. On the other hand, The rank of the linking matrix is $r + s$ which implies $m - b_1(M) = r + s$. Also, a direct calculation shows $\Delta_+^r\Delta_-^s = \mathscr{D}^{r+s}\delta^{r-s}$. Using all these equalities we have

$$\frac{\mathscr{D}^{-b_0(M)-b_1(M)}\delta^n}{\Delta_+^r\Delta_-^s} = \mathscr{D}^{-1-(m-r-s)-(r+s)}\delta^{n-(r-s)} = \mathscr{D}^{-1-m}\delta^{n-\sigma},$$

Thus, Equation (7.8.2) follows from the definition on B' (see Equation (5.6.10)). □

Solution of Exercise 7.9.2. The solution is analogous to the solution of Exercise 7.8.2. □

Solutions to Chapter 8: Topological Quantum Field Theories from skein modules

SOLUTION OF EXERCISE 8.3.5. This is Proposition 2.4 of [**CGP23a**]. □

SOLUTION OF EXERCISE 8.4.6. This is clear for the filling map $\chi_{M,\mathbb{S}^n}$ which is directly computed by taking the m-trace of a cutting presentation. For the cutting map $\chi_{M,\mathbb{S}^{n-1}}$, this follows from the fact that the copairing associated to t' is $\frac{1}{\lambda}$ times the copairing associated to t. Finally, for the red meridian maps $\chi_{M,\mathbb{S}^{n-2}}$, the scalar λ appears because we introduce a chromatic map for each chromatic move. Indeed, the defining identity for the chromatic map (see Equation 5.6.1) implies that a chromatic map associated to t' is λ times a chromatic map associated to t. □

SOLUTION OF EXERCISE 8.6.5. If $\mathscr{C}$ is chromatic compact, then $\Delta_0^{P_\mathbb{1}} = \zeta\Lambda_{P_\mathbb{1}}$ thus $\Lambda_{P_\mathbb{1}} \circ \mathsf{g} = \mathsf{g} \circ \Lambda_{P_\mathbb{1}} = \zeta^{-1}\mathsf{g} \circ \Delta_0^{P_\mathbb{1}} = \zeta^{-1}\Lambda_{P_\mathbb{1}}$ and by Lemma 8.6.1(2), g is not nilpotent thus, it is invertible.

If g is invertible, then there exists $\zeta \in \mathbb{K}$ such that $\Delta_0^{P_\mathbb{1}} = \mathsf{g}^{-1} \circ \Lambda_{P_\mathbb{1}} = (\mathsf{g}^{-1} \circ \eta) \circ \varepsilon = \zeta\Lambda_{P_\mathbb{1}}$ (since η generates $\mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, P_\mathbb{1})$) and $\zeta \neq 0$ since $\mathscr{C}$ is chromatic non-degenerate. □

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