Algebraic & Geometric Topology **[Logo here]**
Volume X (20XX) 1–XXX
Published: XX Xxxember 20XX

Quantum Link Invariant from the Lie Superalgebra $\mathfrak{D}_{2\,1,\alpha}$

Bertrand Patureau-Mirand

Abstract The usual construction of link invariants from quantum groups applied to the superalgebra $\mathfrak{D}_{2\,1,\alpha}$ is shown to be trivial. One can modify this construction to get a two variable invariant. Unusually, this invariant is additive with respect to connected sum or disjoint union. This invariant contains an infinity of Vassiliev invariants that are not seen by the quantum invariants coming from Lie algebras (so neither by the colored HOMFLY-PT nor by the colored Kauffman polynomials).

AMS Classification 57M25; 57M27; 17B37

Keywords Finite type invariants, Quantum groups, Lie superalgebra

Introduction

In his classification of finite dimensional Lie superalgebras, V.G. Kac introduces (cf [11]) a family of simple Lie superalgebras $\mathfrak{D}_{2\,1,\alpha}$ depending of the parameter $\alpha \in \mathbb{C} \setminus \{0, -1\}$. The notation evokes a deformation of the Lie superalgebra $\mathfrak{osp}(4, 2)$ which is obtained for $\alpha \in \{-2, -\frac{1}{2}, 1\}$.

There is a method to construct framed link invariants with a deformation of an enveloping Lie algebras. It follows from work of Drinfeld (cf [6]) that for a fixed simple Lie algebra, all deformations give the same link invariant. This is not clear[1] for the simple Lie superalgebra $\mathfrak{D}_{2\,1,\alpha}$. So we explore here two possibly different deformations of $U\mathfrak{D}_{2\,1,\alpha}$: the Kontsevich-Drinfel'd deformation and the quantum group $U_h\mathfrak{D}_{2\,1}$ described by Y. M. Zou and H. Thys (cf [27] and [24]). The two corresponding link invariants will be denoted by the letters Z and Q.

First we will see that the quantum link invariants $Z_{\mathfrak{D}_{2\,1,\alpha},V}$ obtained from any representation V of the Lie superalgebra $\mathfrak{D}_{2\,1,\alpha}$ is determined by the linking matrix (see section 3.2.4 and remark 3.4). Similarly the quantum link invariants

[1] At the time of writing this paper; but see [7] and [8] for new results.

$Q_{\mathfrak{D}_{2\,1,\alpha},L}$ obtained from the adjoint representation of $\mathfrak{D}_{2\,1,\alpha}$ is constant equal to 1. (see section 4.1.2).

A similar problem was encountered by J. Murakami [17], Kashaev [12] and Degushi [5]: the quantum invariants they considered factor by the zero quantum dimension (the invariant of the unknot). The remedy is to "divide" by this quantum dimension by considering $(1,1)$-tangles instead of links. Here the invariant of the planar trivalent tangle Θ is zero and we give a construction to "divide" trivalent tangles and links by this Θ.

Hence we construct a map $\widetilde{\widetilde{Z}}$ in section 3.3 (resp. $\widetilde{Q}$ in section 4.2.2) that associates to each framed link an element of the ring $\mathbb{Q}[[a_1,a_2,a_3]]^{\mathfrak{S}_3}_{/(a_1+a_2+a_3)}$ (quotient of the ring of symmetric series in three variables) (resp.

$\mathbb{Z}[\frac{1}{2},q_1,q_2,q_3,{[4]_1}^{-1},{[4]_2}^{-1},{[4]_3}^{-1}]/(q_1q_2q_3=1))$.

The existence and the invariance of $\widetilde{\widetilde{Z}}$ is easy to proof but one can hardly compute it. On the other side, it takes much more work to proof that $\widetilde{Q}$ is well defined but it can be computed with an R-matrix.

It is natural to conjecture that the two deformations of $U\mathfrak{D}_{2\,1,\alpha}$ are equivalent. Knowing this would implies that the two maps $\widetilde{Q}$ and $\widetilde{\widetilde{Z}}$ would essentially be the same (setting $q_i=\mathrm{e}^{\frac{a_i}{2}}$ etc) and then their value would be in the intersection of these two different rings.

The author thank Y. M. Zou for sending its papers and C. Blanchet for reading the first version of this manuscript. The author also wishes to thank the referee for numerous helpful comments.

1 Statement of the results

We work with framed trivalent tangles and knotted trivalent graphs which are generalizations of framed tangles and links (they are embeddings of 1-3-valent graphs in S^3). Here are an example of a trivalent tangle and of a knotted trivalent graph (see section 3.1 for precise definition).

We will call a framed knotted trivalent graph "proper" if it has at least one trivalent vertex.

The "adjoint" Kontsevich integral (cf section 3.2.2) associate to each trivalent tangle a series of 1-3-valent Jacobi diagrams. When composed with the weight function $\Phi_{\mathfrak{D}_{2\,1}}$ associated to the Lie superalgebra $\mathfrak{D}_{2\,1}$ (which is a generalization of $\mathfrak{D}_{2\,1,\alpha}$), it gives an functor $Z_{\mathfrak{D}_{2\,1},L}$ from the category of trivalent tangles to a completion of the category of representation of $\mathfrak{D}_{2\,1}$ (here L denotes the adjoint representation of $\mathfrak{D}_{2\,1}$).

Proposition 1.1 *On trivalent tangles,* $Z_{\mathfrak{D}_{2\,1},L}$ *does not depend of the framing. Furthermore, if* N *is a knotted trivalent graph then*

$$Z_{\mathfrak{D}_{2\,1},L}(N) = \begin{cases} 0 & \text{if } N \text{ is a proper knotted trivalent graph,} \\ 1 & \text{if } N \text{ is a link} \end{cases}$$

The "adjoint" Kontsevich integral of a knotted trivalent graph lives in the space of closed 3-valent Jacobi diagrams. This space has a summand isomorphic to the algebra Λ defined by P. Vogel in [26] on which $\Phi_{\mathfrak{D}_{2\,1},L}$ is not trivial. Using this map one can construct an invariant of knotted trivalent graph $\widetilde{Z}$ with values in the ring $\mathbb{Q}[[a_1, a_2, a_3]]^{\mathfrak{S}_3}_{/(a_1+a_2+a_3)}$.

Proposition 1.2 $Z_{\mathfrak{D}_{2\,1},L}$ *and* $\widetilde{Z}$ *are related by*

$$Z_{\mathfrak{D}_{2\,1},L}\left(\equiv\boxed{\mathrm{T}}\right) = \widetilde{Z}\left(\in\boxed{\mathrm{T}}\right).Z_{\mathfrak{D}_{2\,1},L}\left(\ni\right)$$

for any trivalent tangle T *with* 3 1*-valents vertices.*

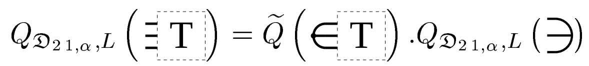
We state similar results for $Q_{\mathfrak{D}_{2\,1,\alpha},L}$: The quantum group $U_h\mathfrak{D}_{2\,1}$ has an unique topologically free representation L whose classical limit is the adjoint representation of $\mathfrak{D}_{2\,1,\alpha}$. This module is autodual and there is an unique map (up to a scalar) $L \otimes L \longrightarrow L$ whose classical limit is the Lie bracket. As usual, coloring a trivalent tangle with L gives a functor $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ from the category of trivalent tangles to the category of representation of the quantum group $U_h\mathfrak{D}_{2\,1}$ and, in particular, a knotted trivalent graph invariant.

Proposition 1.3 *On trivalent tangles,* $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ *does not depend of the framing. Furthermore, if* N *is a knotted trivalent graph then*

$$Q_{\mathfrak{D}_{2\,1,\alpha},L}(N) = \begin{cases} 0 & \text{if } N \text{ is a proper knotted trivalent graph,} \\ 1 & \text{if } N \text{ is a link} \end{cases}$$

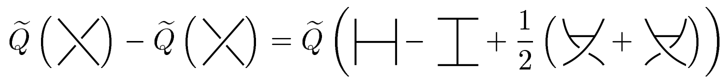
We modify this invariant to the following :

Theorem 1.4 *There is an unique invariant of proper knotted trivalent graph* $\widetilde{Q}$*, with values in the ring* $\mathbb{Z}[\frac{1}{2}, q_1, q_2, q_3, [4]_1{}^{-1}, [4]_2{}^{-1}, [4]_3{}^{-1}]/(q_1 q_2 q_3 = 1)$ *defined by the property :*



for any trivalent tangle T *with* 3 *univalents vertices.*

Theorem 1.5 *There exists an invariant of framed links uniquely determined by :*



$$\text{and} \qquad \widetilde{Q}\,(Unlink) = 0$$

Conjecture 1.6 $\widetilde{Q}$ *takes value in the polynomial algebra* $\mathbb{Z}[\sigma_+, \sigma_-]$

where $\sigma_+ = (q_1^2 + q_2^2 + q_3^2)$ *and* $\sigma_- = (q_1^{-2} + q_2^{-2} + q_3^{-2})$.

2 The Superalgebra $\mathfrak{D}_{2\,1,\alpha}$

2.1 The Cartan Matrix

Let $\alpha \in \mathbb{C} \setminus \{0, -1\}$. The simple Lie superalgebra $\mathfrak{D}_{2\,1,\alpha}$ introduced by V.G. Kac (*cf.* [11]) has the following Cartan matrix

$$A_\alpha = (a_{ij})_{1 \le i,j \le 3} = \begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix},$$

Where the first simple root is odd and the two others are even. So the superalgebra is generated by the nine elements e_i, f_i, h_i $(i = 1 \cdots 3)$ with the following relations :

$$[h_i, h_j] = 0 \quad , \quad [e_i, f_j] = \delta_{i,j} h_i \quad , \quad [h_i, e_j] = a_{ij} e_j \quad , \quad [h_i, f_j] = -a_{ij} f_j \quad ,$$

$$[e_2, e_3] = [f_2, f_3] = [e_1, e_1] = [f_1, f_1] = [e_i, [e_i, e_1]] = [f_i, [f_i, f_1]] = 0 \text{ where } i = 1, 2$$

Its even part is isomorphic to the Lie algebra $L = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and the bracket makes the odd part an L-module isomorphic to the tensor product of the three standard representations of $\mathfrak{sl}_2$. So we can identify the set of weights with a subset of $\mathbb{Z}^3$ such that the three simple roots are :

$$(1,-1,-1) \quad (0,2,0) \quad (0,0,2)$$

Then the set of positive roots is

$$\{\beta_1=(0,0,2),\,\beta_2=(1,-1,1),\,\beta_3=(1,1,1),\,\beta_4=(2,0,0),$$
$$\beta_5=(1,-1,-1),\,\beta_6=(1,1,-1),\,\beta_7=(0,2,0)\}$$

2.2 The Superalgebra $\mathfrak{D}_{2\,1}$

We use in section 3.2.3 the following construction of $\mathfrak{D}_{2\,1}$ (see [26] and [28]) :

Let $R=\mathbb{Q}[a_1^{\pm},a_2^{\pm},a_3^{\pm}]_{/(a_1+a_2+a_3)}$ the quotient of a Laurent polynomial algebra in three variables. Let E_1, E_2 and E_3 be three two dimensional free R-modules equipped with an non degenerate antisymmetric form : $\cdot\wedge\cdot : \Lambda^2 E_i \xrightarrow{\sim} R$. We can see E_i as a supermodule concentrated in odd degree equipped with a supersymmetric form.

The superalgebra $\mathfrak{D}_{2\,1}$ is defined as a supermodule by

$$\mathfrak{D}_{2\,1}=\mathfrak{sp}(E_1)\oplus\mathfrak{sp}(E_2)\oplus\mathfrak{sp}(E_3)\oplus(E_1\otimes E_2\otimes E_3)$$

The bracket is defined by the Lie algebra structure on the even part, by the standard representation of the even part on the odd part and for the tensor product of two odd elements by the formula :

$$\begin{aligned}[e_1\otimes e_2\otimes e_3, e_1'\otimes e_2'\otimes e_3'] = {}& \tfrac{1}{2}\,(a_1\, e_2\wedge e_2'\, e_3\wedge e_3'\,(e_1.e_1') \\ &+a_2\, e_1\wedge e_1'\, e_3\wedge e_3'\,(e_2.e_2')+a_3\, e_1\wedge e_1'\, e_2\wedge e_2'\,(e_3.e_3'))\end{aligned}$$

where $(e_i.e_i')\in\mathfrak{sp}(E_i)$ sends $x\in E_i$ on $e_i\wedge x\, e_i' + e_i'\wedge x\, e_i$.

The non degenerate supersymmetric bilinear form on $\mathfrak{D}_{2\,1}$ is up to multiplication by a scalar the orthogonal sum of $-\frac{4}{a_i}$ times the killing form of $\mathfrak{sp}(E_i)$ plus the tensor product of the three antisymmetric forms on $E_1\otimes E_2\otimes E_3$.

If $\phi : R\longrightarrow\mathbb{C}$ is a ring map, then the complex Lie superalgebra $\mathfrak{D}_{2\,1}\otimes_\phi\mathbb{C}$ is isomorphic to $\mathfrak{D}_{2\,1,\alpha}$ where $\alpha=\frac{\phi(a_3)}{\phi(a_2)}$.

3 The Kontsevich-Drinfel'd invariant

In the following, if $n\in\mathbb{N}$ we will denote by $[n]$ the set $\{1,\,2,\cdots,n\}$.

3.1 The Category of trivalent tangles

Let X be a finite set. A X-diagram is a finite graph, whose vertices are either 1-valent or 3-valent and oriented, with the data of an isomorphism between X and the set of 1-valent vertices. An orientation at a 3-valent vertex v is a cyclic ordering on the set of the three edges going to x.

Following [3] we define a trivalent tangle on $\phi : X \hookrightarrow \mathbb{R}^3$ as an embedding of an X-diagram in $\mathbb{R}^3$, with image $N \subset [0,1] \times \mathbb{R}^2$ together with a vector field along N such that the points of N lying in the planes $\{0\} \times \mathbb{R}^2$ and $\{1\} \times \mathbb{R}^2$ are exactly $\phi(X)$. Additionally, we require that the normal plane of N at an univalent vertex v is parallel to the plane $\{0\} \times \mathbb{R}^2$, the vector field assigned to N at v is $(0,0,1)$, and at each 3-valent vertex, the orientation of the plane tangent to N given by the vector field agree with the orientation of the 3-valent vertex of the underlying X-diagram. When represented by planar graphs the framing is obtained by taking the vector field pointing upward.

Two trivalent tangles T_1 and T_2 are equivalent if one can be deformed into the other (within the class of trivalent tangles) by a smooth one parameter family of diffeomorphisms of $\mathbb{R}^3$.

A framed knotted trivalent graph is a trivalent tangle with no univalent vertices. We will call a framed knotted trivalent graph "proper" if it has at least one trivalent vertex.

Let M be the non-associative monoid freely generated by one letter noted "$\circ$". If $m \in M$, the length of m is the number of letter in m. This gives a partition $M = \bigsqcup_{n \in \mathbb{N}} M_n$.

The category $\mathcal{T}_p$ (resp. $\mathcal{T}$) is the $\mathbb{Q}$-linear monoidal category whose set of object is M (resp. $\mathbb{N}$). If $(m,m') \in M_n \times M_{n'}$ (resp. if $(n,n') \in \mathbb{N}^2$), the set of morphisms $\mathcal{T}_p(m,m') \simeq \mathcal{T}(n,n')$ is the vector space with bases the set of equivalence classes of trivalent tangles on the map $(\phi : [n] \sqcup [n'] \simeq (\{0\} \times [n] \times \{0\}) \cup (\{1\} \times [n'] \times \{0\}) \subset \mathbb{R}^3$.

The composition is just given by gluing the corresponding univalent vertices of the tangles. The tensor product of morphism is given by the juxtaposition of tangles.

3.2 The Kontsevich integral for trivalent tangles and the functor $Z_{\mathfrak{D}_{2\,1},L}$

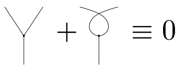
3.2.1 Category of Jacobi diagrams

We represent an X-diagram (or "Jacobi diagram") by a graph immersed in the plane in such a way that the cyclic order at each vertex is given by the orientation of the plane.

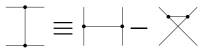
We define the degree of an X-diagram to be half the number of the vertices.

Let $\mathcal{A}(X)$ (resp. $\mathcal{A}_c(X)$) denotes the completion (with respect to the degree) of the quotient of the $\mathbb{Q}$-vector space with basis the X-diagrams (resp. connected X-diagrams) by the relations (AS) and (IHX):

(1) If two Jacobi diagrams are the same except for the cyclic order of one of their vertices, then one is minus the other (relation called (AS) for antisymmetry).



(2) The relation (IHX) (or Jacobi) deals with three diagrams which differ only in the neighborhood of an edge.



As we want to work with $\mathfrak{D}_{2\,1}$ (which has superdimension 1), we add the relation that identify the Jacobi diagram with only one circle with 1 (this will mean that the superdimension of $\mathfrak{D}_{2\,1}$ is 1). So we can remove or add some circle to a Jacobi diagram without changing its value in $\mathcal{A}$.

Let $\mathcal{D}$ denote the $\mathbb{Q}$-linear monoidal category defined by

(1) $\mathrm{Obj}(\mathcal{D}) = \{[n], n \in \mathbb{N}\}$

(2) $\mathcal{D}([p],[q]) = \mathcal{A}([p] \amalg [q])$

(3) The composition of a Jacobi diagram from $[p]$ to $[q]$ with a Jacobi diagram from $[q]$ to $[r]$ is given by gluing the two diagrams along $[q]$.

(4) The tensor product of two object is $[p] \otimes [q] = [p+q]$ and the tensor product of two Jacobi diagrams is given by their disjoint union.

Remark 3.1 The composition map $\mathcal{D}([p],[q]) \otimes \mathcal{D}([q],[r]) \longrightarrow \mathcal{D}([p],[r])$ has degree $-q$.

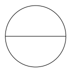
The algebra Λ is the sub-vector space of $\mathcal{A}_c([3])$ formed by totally antisymmetric elements for the action of the permutation group $\mathfrak{S}_3$.

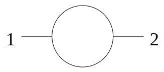
Λ has a natural commutative algebra structure and acts on each space $\mathcal{A}_c(X)$: If u lies in Λ and $K \in \mathcal{A}_c(X)$ is a X-diagram, a Jacobi diagram representative for $u.K$ is obtained by inserting u at a 3-valent vertex of K. Exceptionally in Λ, the degree will be defined by half the number of vertices minus two so that the unit of Λ has degree 0.

In the following, we will denote by $\mathcal{A}^+$ and $\mathcal{A}_c^+$ (resp. by $\mathcal{A}^0$ and $\mathcal{A}_c^0$) the subspaces generated by Jacobi diagrams having at least one (resp. having no) 3-valent vertices.

For small n one can describe $\mathcal{A}_c([n])$ (cf [26]) :

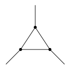
$\mathcal{A}_c([1])$ is zero; $\mathcal{A}_c^+([0])$ and $\mathcal{A}_c^+([2])$ are free Λ-modules with rank one, generated by the following elements :





Let Θ be this generator of $\mathcal{A}_c^+(\emptyset)$. Furthermore, we don't know any counterexample to the following conjecture : $\mathcal{A}_c([3]) = \Lambda$.

Λ is generated in degree one by the element t :

$t =$  $= \frac{1}{2}$ 

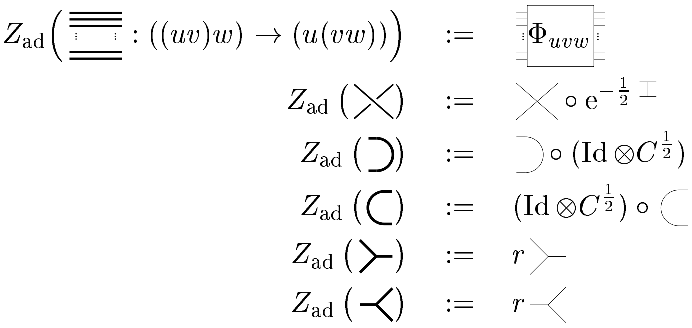
3.2.2 The Kontsevich-adjoint functor Z_{ad}

We follow here A.B. Berger and I. Stassen ([3] Definition and Theorem 2.8) who have defined a *unoriented universal Vassiliev-Kontsevich invariant* generalized for trivalent tangles (cf also [18]). We just consider it for the "adjoint" representation so we compose their functor (whose values are bicolored graph) with the functor that forget the coloring of the edges.

Theorem 3.2 *(cf [3])*

There is an unique monoidal functor $Z_{\mathrm{ad}} : \mathcal{T}_p \longrightarrow \mathcal{D}$ (the universal adjoint Vassiliev-Kontsevich invariant) defined by the following assignments :

$Z_{\mathrm{ad}}(m) := [n]$, *where* $m \in M_n$ *is a non-associative word of length* n.



where

- $e^{\pm\frac{1}{2}} := \sum_{n=0}^{\infty} (\pm\frac{1}{2})^n \frac{1}{n!}\ {}^{\circ n}$
- *The elements* Φ_{uvw} *are constructed from an even rational horizontal associator* Φ *with* $\Phi^{321} = \Phi^{-1}$ *as in [16].*
- $C := \boxed{\Phi} = Id + \gamma$ *with* $\gamma \in \Lambda$
- r *can be any element of* Λ. *We make the following normalization :* $r = 1$ *so that* $Z_{\mathrm{ad}}(\Theta \in \mathcal{T}_p(\emptyset, \emptyset)) = (1 + 2t\gamma)\Theta \in \mathcal{D}(0,0)$

The difference with [3] for the image of the elementary trivalent tangles with one 3-valent vertex is because they use the Knizhnik-Zamolodchikov associator which has not the good property for cabling (see [16]).

The Kontsevich integral of the unknot has an explicit expression (see [2]) but it seems difficult to give an explicit expression of γ. Nevertheless it allows to say that γ lives in odd degree and starts with $\gamma = \frac{1}{24}t + \cdots$.

3.2.3 Weight functions

Let $< .,. >_{\mathfrak{D}_{2\,1}}$ denotes the supersymmetric invariant non degenerate bilinear form on $\mathfrak{D}_{2\,1}$) and let $\Omega \in \mathfrak{D}_{2\,1} \otimes \mathfrak{D}_{2\,1}$ be the associated Casimir element.

Theorem 3.3 *(cf [26]). There exists an unique* $\mathbb{Q}$*-linear monoidal functor* $\Phi_{\mathfrak{D}_{2\,1}}$ *from* $\mathcal{D}$ *to the category* $\mathrm{Mod}_{\mathfrak{D}_{2\,1}}$ *of representations of* $\mathfrak{D}_{2\,1}$ *such that:*

(1) $\Phi_{\mathfrak{D}_{2\,1}}([1]) = \mathfrak{D}_{2\,1}$ *(the adjoint representation).*

(2) *Its values on the elementary morphisms*

are given by:

(a) *The Casimir of* $\mathfrak{D}_{2\,1}$*:* $\Omega \in {\mathfrak{D}_{2\,1}}^{\otimes 2} \hookrightarrow \mathrm{Mod}_{\mathfrak{D}_{2\,1}}(R, (\mathfrak{D}_{2\,1})^{\otimes 2})$

(b) *The bilinear form* $< .,. >_{\mathfrak{D}_{2\,1}}: {\mathfrak{D}_{2\,1}}^{\otimes 2} \longrightarrow R$

(c) *The Lie bracket seen as a map in* $\mathrm{Mod}_{\mathfrak{D}_{2\,1}}((\mathfrak{D}_{2\,1})^{\otimes 2}, (\mathfrak{D}_{2\,1}))$

(d) *The symmetry operators:* $\begin{array}{ccc} {\mathfrak{D}_{2\,1}}^{\otimes 2} & \longrightarrow & {\mathfrak{D}_{2\,1}}^{\otimes 2} \\ x \otimes y & \mapsto & (-1)^{|x||y|} y \otimes x \end{array}$

Furthermore, there exists a graded character with value in $\widetilde{R} = \mathbb{Q}[[\sigma_2, \sigma_3]]$ *(here we set* $\sigma_2 = a_1a_2 + a_2a_3 + a_3a_1$ *and* $\sigma_3 = a_1a_2a_3$*) :* $\chi_{\mathfrak{D}_{2\,1}} : \Lambda \longrightarrow \widetilde{R}$ *such that :*

$$\forall u \in \Lambda,\ \forall K \in \mathcal{A}_c^+([p] \amalg [q]) \subset \mathcal{D}([p],[q]),\ \Phi_{\mathfrak{D}_{2\,1}}(uK) = \chi_{\mathfrak{D}_{2\,1}}(u)\Phi_{\mathfrak{D}_{2\,1},\Omega}(K)$$

One has $\chi_{\mathfrak{D}_{2\,1}}(t) = 0$ *and the functor* $\Phi_{\mathfrak{D}_{2\,1}}$ *is zero on the generators of* $\mathcal{A}_c^+(\emptyset)$ *and* $\mathcal{A}_c^+([2])$*.*

3.2.4 $Z_{\mathfrak{D}_{2\,1},L}$ and the quantum Jacobi relation

Composing the adjoint-Kontsevich invariant with the weight function associated with $\mathfrak{D}_{2\,1}$, one get a functor $Z_{\mathfrak{D}_{2\,1},L} : \mathcal{T}_p \longrightarrow \mathrm{Mod}_{\mathfrak{D}_{2\,1}}$. For a simple Lie algebra, Drinfel'd equivalence results for quasi-triangular quasi-Hopf algebra would imply for the two constructions to give equivalent representations of $\mathcal{T}_p$ but this is not clear for $\mathfrak{D}_{2\,1,\alpha}$. So we do the same work for Z :

The functor $Z_{\mathfrak{D}_{2\,1},L}$ produces an invariant of framed knotted trivalent graphs with values in $\widetilde{R}$.

Proof *of proposition 1.1 :*

This is a consequence of the fact that $\Phi_{\mathfrak{D}_{2\,1}}$ is zero on any closed Jacobi diagram having at least one trivalent vertex because $\Phi_{\mathfrak{D}_{2\,1}}$ is 0 on the generator of $\mathcal{A}_c^+(\emptyset)$. □

Remark 3.4 This argument can be adapted for other choices of representations of $\mathfrak{D}_{2\,1}$ to prove that the corresponding invariant is determined by the linking matrix. In particular this is the case for the $\alpha = s$ specialization of the Kauffman polynomial.

$Z_{\mathfrak{D}_{2\,1},L}$ also verify the relations satisfied by $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ in theorem 4.1. In particular, as for any simple quadratic Lie superalgebra, it satisfies the quantum Jacobi relation for $\phi = -\frac{1}{2r^2}$ as we will show in a following paper.

3.3 Renormalization of $Z_{\mathfrak{D}_{2\,1},L}$

The adjoint Kontsevich integral of a knot K is of the form $Z_{\mathrm{ad}}(K) = 1 + \lambda.\Theta$ for some $\lambda \in \Lambda$. If we apply the $\mathfrak{D}_{2\,1}$ weight system, we just get 1 since Θ goes to zero. But we can "divide by zero" defining $\widetilde{Z}(K)$ as the weight system applied to λ. In the following, we generalize this construction for links and knotted trivalent graphs.

Let us define $\mathcal{A}'$ to be the quotient of $\mathcal{A} \otimes \Lambda$ by the relation :

If a Jacobi diagram $K = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_n$ represents an element of $\mathcal{A}$ where K_1 is a connected Jacobi diagram such that $K_1 = u.K_1'$ for $u \in \Lambda$ then $K \otimes v = (K_1' \sqcup K_2 \sqcup \cdots \sqcup K_n) \otimes u.v$. (This extends the action of Λ to disconnected Jacobi diagrams.)

As before, we define $\mathcal{D}'$ as the Λ-linear monoidal category with the modules $\mathcal{A}'$ as morphisms.

Proposition 3.5 *The quotient algebra $\mathcal{A}'(\emptyset)$ is isomorphic to the subalgebra $\mathbb{Q} \oplus \Theta\Lambda[\Theta] \subset \Lambda[\Theta]$.*

The functor $\Phi_{\mathfrak{D}_{2\,1}}$ factor through $p : \mathcal{D} \longrightarrow \mathcal{D}'$.

Proof Just see that $\mathcal{A}(\emptyset)$ is the symmetric algebra on the vector space $\mathcal{A}_c(\emptyset) \simeq \mathbb{Q} \oplus \Lambda\Theta$.

$\Phi_{\mathfrak{D}_{2\,1}}$ naturally satisfy the additional relations of $\mathcal{D}'$ as by lemma 3.3, it sends via $\chi_{\mathfrak{D}_{2\,1}}$ the elements of Λ on scalars. □

We will use the following map on $\mathcal{A}'(\emptyset)$ to get a new invariant :

$$\begin{aligned} \Phi'_{\mathfrak{D}_{2\,1}} : \mathcal{A}'(\emptyset) \simeq \mathbb{Q} \oplus \Theta\Lambda[\Theta] &\longrightarrow \mathbb{Q}[[\sigma_2, \sigma_3]] \\ z + \Theta\lambda + \Theta^2 x &\mapsto \chi_{\mathfrak{D}_{2\,1}}(\lambda) \end{aligned}$$

Where $z \in \mathbb{Q}$, $\lambda \in \Lambda$ and $x \in \Lambda[\Theta]$.

Definition 3.6 For a knotted trivalent graph L set

$$\widetilde{Z}(L) = \Phi'_{\mathfrak{D}_{2\,1}}(p(Z_{\mathrm{ad}}(L)))$$

The planar knotted trivalent graph Θ is sent by Z_{ad} on $(1 + 2t\gamma)\Theta \in \mathcal{A}'(\emptyset)$ so $\widetilde{Z}(\Theta) = 1$.

Remark that the decomposition $\mathcal{A}' \simeq \mathcal{A}'^0 \oplus \mathcal{A}'^+$ is still valid.

Lemma 3.7 *Let* $K = \circ K'$ *for some* $K' \in \mathcal{D}([0],[3])$ *then*

$$\Phi_{\mathfrak{D}_{2\,1}}(K') = \Phi'_{\mathfrak{D}_{2\,1}}(K).\Phi_{\mathfrak{D}_{2\,1}}(\ )$$

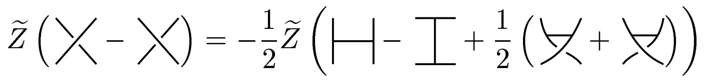
Furthermore, if $K \in \mathcal{A}([n]) = \mathcal{D}([0],[n])$ *is sent on zero by* $\Phi_{\mathfrak{D}_{2\,1}}$ *then for any* $K' \in \mathcal{A}'^{+}([n]) \subset \mathcal{D}'([n],[0])$, *one has* $\Phi'_{\mathfrak{D}_{2\,1}}(K' \circ K) = 0$.

Remark that the second assertion is false for $K' \in \mathcal{A}'^{0}([n])$!

Theorem 3.8 *Let* $T = \circ T'$ *for some* $T' \in \mathcal{T}_p(\emptyset, (\circ\circ)\circ)$ *then*

$$Z_{\mathfrak{D}_{2\,1},L}(T') = \widetilde{Z}(T).\Phi_{\mathfrak{D}_{2\,1}}(\ )$$

Thus on proper knotted trivalent graph, $\widetilde{Z}$ *can be computed using* $Z_{\mathfrak{D}_{2\,1},L}$. *For links, one can compute the variation of* $\widetilde{Z}$ *when one changes a crossing with :*



Furthermore, if L_1 *and* L_2 *are links,* L'_1 *and* L'_2 *are proper knotted trivalent graphs,*

$\widetilde{Z}(L_1 \sqcup L_2) = \widetilde{Z}(L_1) + \widetilde{Z}(L_1)$, $\widetilde{Z}(L'_1 \sqcup L'_2) = 0$ *and* $\widetilde{Z}(L_1 \sqcup L'_1) = \widetilde{Z}(L'_1)$.

Conjecture 3.9 *The value of* $\widetilde{Z}$ *on the unframed unknot is obtained by removing the term with degree* -1 *in* $\frac{\sigma_+ + \sigma_- - 6}{4(\sigma_+ - \sigma_-)}$ *where* $\sigma_\pm$ *is defined as in conjecture 1.6 with* $q_i = \mathrm{e}^{\frac{a_i}{2}}$.

4 The quantum group invariant

4.1 The quantum group $U_h\mathfrak{D}_{2\,1}$ and the functor $Q_{\mathfrak{D}_{2\,1,\alpha},L}$

4.1.1 The quantum group $U_h\mathfrak{D}_{2\,1}$

Remark that there is three simple root systems for $\mathfrak{D}_{2\,1,\alpha}$. Here and in section 2.1, the presentations of the algebra are based on the distinguish simple root system (with the smallest number of odd simple roots) of $\mathfrak{D}_{2\,1,\alpha}$. Unfortunately, this simple root system (contrary to the simple root system with three odd simple roots) breaks the symmetry that appear in section 2.2. This symmetry, hidden in the presentation of $\mathfrak{D}_{2\,1,\alpha}$ section 2.1 seem totally lost with the deformation $U_h\mathfrak{D}_{2\,1}$ of Y.M. Zou(cf [27]).

An universal R-matrix for $\mathfrak{D}_{2\,1,\alpha}$ has been computed by H. Thys (cf [24]). It depends of the three parameters q_1, q_2 and q_3 where $q_3 = q_2^\alpha$ and $q_1 q_2 q_3 = 1$.

In the following, we will adopt the following notation for $i = 1 \cdots 3$:

$$[n]_i = q_i^n - q_i^{-n}$$

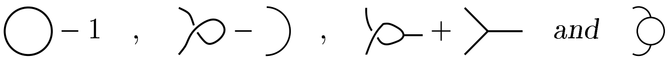
There exists an unique 17-dimensional irreducible representation ρ of $U_h\mathfrak{D}_{2\,1}$. Its classical limit is the adjoint representation of the Lie superalgebras $\mathfrak{D}_{2\,1,\alpha}$. This $U_h\mathfrak{D}_{2\,1}$-module L is autodual (there is a (unique up to a scalar) bilinear map $\beta : L \otimes L \longrightarrow \mathbb{C}[[h]]$), has the set of roots for set of weights, and has a (unique up to a scalar) bilinear map $\gamma : L \otimes L \longrightarrow L$ (whose classical limit is the Lie bracket).

We have computed using Maple the $17^2 \times 17^2$ R-matrix for L, the tensor realizing the duality β, its dual and the tensor γ. For a good choice of a basis of L, and a good normalization of β and γ, all the coordinates of these tensors are in the ring $\mathbb{Z}[\frac{1}{2}, q_1, q_2, q_3, {[4]_1}^{-1}, {[4]_2}^{-1}, {[4]_3}^{-1}]/(q_1q_2q_3 = 1)$. The computations with Maple are available on the author's web page.

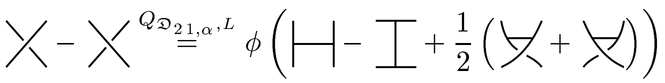
4.1.2 $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ and the quantum Jacobi relation

As usual one has a functor $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ from $\mathcal{T}$ to $\mathrm{Mod}_{U_h\mathfrak{D}_{2\,1}}$ sending $[1] \in \mathrm{Obj}(\mathcal{T})$ to the $U_h\mathfrak{D}_{2\,1}$-module L. This give an invariant of framed knotted trivalent graphs with values in $\mathbb{Z}[\frac{1}{2}, q_1, q_2, q_3, {[4]_1}^{-1}, {[4]_2}^{-1}, {[4]_3}^{-1}]/(q_1q_2q_3 = 1)$.

Theorem 4.1 *The following elements are in the kernel of* $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ *:*



Furthermore, $\mathrm{Mod}_{U_h\mathfrak{D}_{2\,1}}(L^{\otimes 2}, L^{\otimes 2})$ *has dimension 6 and is generated by the images by* $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ *of the powers of the half twist. The "quantum Jacobi relation" is satisfied by* $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ *:*



where ϕ *depends of the normalization chosen for* $\beta : L \otimes L \longrightarrow \mathbb{C}[[h]]$ *and* $\gamma : L \otimes L \longrightarrow L$.

Proof This is a computation made with Maple. □

Proof *of proposition 1.3 :*

Every proper knotted trivalent graph T can be written $T_1 \circ T_2$ with $T_1 \in \mathcal{T}(0,3)$ and $T_2 \in \mathcal{T}(3,0)$. But the spaces $\mathrm{Mod}_{U_h\mathfrak{D}_{2\,1}}(L^{\otimes p}, L^{\otimes q})$ with $p+q=3$ are all isomorphic with dimension 1 so $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T)$ is proportional to the image by $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ of the knotted trivalent graph Θ which is 0. As a consequence, the "quantum Jacobi relation" implies that $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ is unchanged when one change the crossings of a link and so is constant equal to its value on the unlink which is 1. □

We will need the following :

Corollary 4.2 *If $r_\pi : \mathcal{T}(2,2) \longrightarrow \mathcal{T}(2,2)$ is the map induced by the rotation around the line $\{\frac{1}{2}\} \times \{\frac{3}{2}\} \times \mathbb{R}$ by π then one has $Q_{\mathfrak{D}_{2\,1,\alpha},L} \circ r_\pi = Q_{\mathfrak{D}_{2\,1,\alpha},L}$ on $\mathcal{T}(2,2)$.*

4.2 Renormalization of $Q_{\mathfrak{D}_{2\,1,\alpha},L}$

The idea is to define a renormalization $\widetilde{Q}$ of $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ using some relation between the two as in theorem 3.8. The demonstration of the invariance is then not trivial[2] but it is made by analogy to some demonstrations for weight functions. We give the steps of the demonstration :

- $\widetilde{Q}$ is well defined on proper knotted trivalent graphs.
- Using the quantum Jacobi relation, it can be extended to an invariant of singular links with one double point.
- This invariant can be integrated to a link invariant $\widetilde{Q}$.

4.2.1 $\widetilde{Q}$ for Proper knotted trivalent graphs and Singular Link

Theorem 4.3 *Let $L = \circ\, T$ for some $T \in \mathcal{T}(0,3)$ then the scalar $\widetilde{Q}(L)$ defined by $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T) = \widetilde{Q}(L).Q_{\mathfrak{D}_{2\,1,\alpha},L}(\Theta)$ is independent of T.*

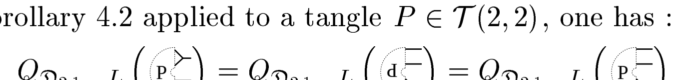
Proof First remark that by theorem 4.1, $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T)$ does not depend of the framing of T and that the braid group $B_3 \subset \mathcal{T}(3,3)$ acts on $Q_{\mathfrak{D}_{2\,1,\alpha},L}(\mathcal{T}(0,3))$ by multiplication by the signature (a braid b with projection $\sigma \in \mathfrak{S}_3$ act by

[2] Remark that according to the new results of [8], the invariance of $\widetilde{Z}$ implies the existence and the invariance of $\widetilde{Q}$

the multiplication by the signature of $\sigma \in \mathfrak{S}_3$ (cf the third relation of theorem 4.1)). So it is easily seen that $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T)$ depends only of the choice of the trivalent vertex of L that is removed in T.

Second, if L is a disjoint union of knotted trivalent graph then clearly, $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T) = 0$.

Third, by corollary 4.2 applied to a tangle $P \in \mathcal{T}(2,2)$, one has :



So $\widetilde{Q}(L)$ is unchanged when one choose any 3-vertex in the same connected component of L and the theorem is proved for connected knotted trivalent graph.

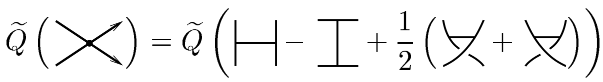
Last, consider two trivalent tangles T and T' in $\mathcal{T}(0,3)$ giving the same knotted trivalent graph. One can find a trivalent tangle $\overline{T} \in \mathcal{T}(0,6)$ such that

$$T = (\mathrm{Id}^{\otimes 3} \otimes \supset) \circ \overline{T} \text{ and } T' = (\supset \otimes \mathrm{Id}^{\otimes 3}) \circ \overline{T}$$

Now one can use the quantum Jacobi relation to change the crossing in $\overline{T}$ and find a sum of trivalent tangles $\widetilde{T}$ such that $Q_{\mathfrak{D}_{2\,1,\alpha},L}(\overline{T}) = Q_{\mathfrak{D}_{2\,1,\alpha},L}(\widetilde{T})$ and the trivalent tangles that appear in $\widetilde{T}$ are either the tensor product of two trivalent tangles in $\mathcal{T}(0,3)$ (so do not contribute to $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T)$ nor to $Q_{\mathfrak{D}_{2\,1,\alpha},L}(T')$) or are trivalent tangles with at least one component intersecting both the sets of univalent vertices $\{1,2,3\}$ and $\{4,5,6\}$ (so they contribute for the same as they give connected knotted trivalent graph). □

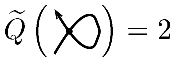
Remark 4.4 By theorem 4.1, $\widetilde{Q}$ is independent of the framing.

Definition 4.5 If L is a framed oriented link with one double point, we define $\widetilde{Q}(L)$ by the following substitution :

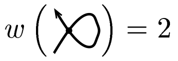


Remark that the orientation of L is forgotten in the right hand side.

Proposition 4.6 *For framed oriented links with one double point as follow, one has*



For a framed link L, let $w(L)$ denotes the diagonal writhe of L (i.e. the trace of the linking matrix of any orientation of L). Then w extends to an invariant of framed oriented links with one double point which also satisfies :

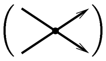
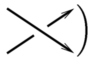
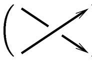


4.2.2 Integration of $\widetilde{Q}$

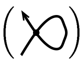
In ([23], Th. 1), T. Stanford gives local conditions for a singular link invariant to be integrable to a link invariant: specialized in our context, it gives:

Theorem 4.7 (Stanford) *Let $\mathcal{L}^{(1)}$ be the set of isotopy classes of singular links in $\mathbb{R}^3$ with one double point and let k be a field of characteristic zero.*

Then, for any finite type singular link invariant $f : \mathcal{L}^{(1)} \longrightarrow k$, there exists a link invariant $F : \mathcal{L} \longrightarrow R$, such that

f () $=$ F () $-$ F ()

iff

f () $= 0$ *and* $f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times})$ (1)

(where L_{**} denotes some desingularisations of a any singular link $L_{\times\times}$ with two double points).

Theorem 4.8 *$\widetilde{Q} - w$ satisfies the two conditions (1) of Proposition 4.7 and so $\widetilde{Q}$ extends in an unique way to a framed link invariant which takes value zero on the unlink.*

Furthermore, $\widetilde{Q}$ takes value in the ring $\mathbb{Q}[q_1, q_2, q_3, [4]_1{}^{-1}, [4]_2{}^{-1}, [4]_3{}^{-1}]_{/q_1q_2q_3=1}$ and satisfy $\widetilde{Q}(L\#L') = \widetilde{Q}(L) + \widetilde{Q}(L')$ where $L\#L'$ denotes a connected sum of L and L' along one of their components.

Proof For a singular link $L_{\times\times}$ with two double points, the two terms $\widetilde{Q}(L_{\times +} - L_{\times -})$ and $\widetilde{Q}(L_{+\times} - L_{-\times})$ are equal to $\widetilde{Q}(K')$ where K' is the sum of knotted trivalent graphs obtained by replacing the two singular points of $L_{\times\times}$ as in definition 4.5.

□

Conjecture 4.9 *Relation between $\widetilde{Z}$ and $\widetilde{Q}$*[3]

(1) *For any proper knotted trivalent graph L one has $\widetilde{Q}(L) = \widetilde{Z}(L)$.*

(2) *For any framed link L with n component,*

$$\widetilde{Q}(L) = 2\widetilde{Z}(L) - n\frac{\sigma_+ + \sigma_- - 6}{2(\sigma_+ - \sigma_-)}^{4}$$

[3]The first part of this conjecture is now proven by N. Geer in [8].
[4]after removing the term of degree -1

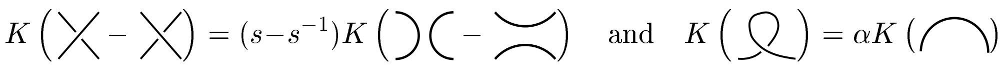
(3) *$\widetilde{Q}$ takes value in the polynomial algebra $\mathbb{Z}[\sigma_+,\sigma_-]$ where $\sigma_+ = (q_1^2+q_2^2+q_3^2)$ and $\sigma_- = (q_1^{-2}+q_2^{-2}+q_3^{-2})$.*

Remark that 1 implies 2 (with conjecture 3.9), and the fact that the values of $\widetilde{Q}$ are symetrics in the three variables.

5 Properties of the Invariants

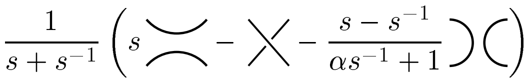
5.1 The Common Specialisation with the Kauffman Polynomial

Remember that the specialisation $\alpha \in \{-2, -\frac{1}{2}, 1\}$ of $\mathfrak{D}_{2\,1,\alpha}$ give a Lie superalgebra isomorphic with $\mathfrak{osp}(4,2)$. So in this case, $U_h\mathfrak{D}_{2\,1}$ admit a 6-dimensional representation which satisfies the skein relations of the Kauffman polynomial :



for $\alpha = s$.

As these skein relations determine the tangle invariant, the specialisations of the functor $Z_{\mathfrak{D}_{2\,1}}$ obtained by setting $a_1 = a_2 = a$, $a_3 = -2a$ (or any permutation of this) and the specialisations of the functor $Q_{\mathfrak{D}_{2\,1,\alpha},L}$ obtained by setting $\alpha \in \{-2, -\frac{1}{2}, 1\}$ (i.e. $q_1 = q_2 = s^{-1}$, $q_3 = s^2$) are both equivalent to the $\alpha = s$ specialisation of the "adjoint" Kauffman skein quotient which is obtained by cabling each component of a tangle with the following projector of $\mathcal{T}([2],[2])$:



and imposing the Kauffman skein relations.

Let K_{ad} be the framed link invariant obtained by cabling each component of a framed link with the previous projector and computing its Kauffman polynomial then:

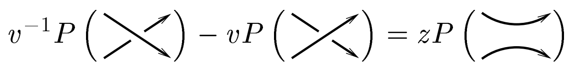
Theorem 5.1 *Let ϕ be the specialisation $\phi(q_1) = \phi(q_2) = s^{-1}$, $\phi(q_3) = s^2$ (so that $\phi(\sigma_+) = 2s^{-2}+s^4$ and $\phi(\sigma_-) = 2s^2+s^{-4}$). Then for any framed link L, $\phi(\widetilde{Q}(L))$ and $\phi(\widetilde{Z}(L))$ are related as in conjecture 4.9 and*

$$K_{\mathrm{ad}}(L)|_{\alpha=s} = 1$$
$$\left.\left(\frac{K_{\mathrm{ad}}(L)-1}{\alpha - s}\right)\right|_{\alpha=s} = \frac{2}{s}\phi(\widetilde{Q}(L))$$

5.2 The Common Specialisation with the HOMFLY-PT Polynomial

It would be more difficult to make appear the common specialisation of $\widetilde{Q}$ with the HOMFLY-PT polynomial. This should appear for the degenerate specialisation $\alpha \in \{0, -1, +\infty\}$ of $\mathfrak{D}_{2\,1,\alpha}$. We state the existing relation between $\widetilde{Z}$ and HOMFLY-PT and we just state a conjecture for the relation between $\widetilde{Q}$ and HOMFLY-PT[5].

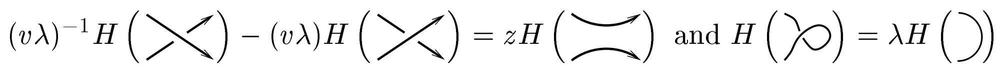
The HOMFLY-PT polynomial of an oriented link $L \neq \emptyset$ is an element $P(v, z) \in \mathbb{Z}[v^{\pm}, z^{\pm}]$ which is equal to 1 for the unknot and satisfy the skein relation :



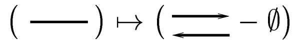
If W denotes the total writhe of an oriented framed link L (i.e. the total algebraic number of crossing of L) then we get an oriented framed link invariant of L : $H(\lambda, v, z) \in \mathbb{Z}[\lambda^{\pm}, v^{\pm}, z^{\pm}]$ by the formula

$$H(L) = \left| \begin{array}{l} 1 \text{ if } L = \emptyset \\ \lambda^{W(L)} P(L) \times \frac{v^{-1}-v}{z} \text{ else} \end{array} \right.$$

H satisfy the skein relations :



We define the adjoint HOMFLY-PT polynomial H_{ad} of a framed unoriented link L as the H polynomial of the framed oriented link obtained by cabling each component of L with the following :



Remark that the cabled link has total writhe 0, so $H_{\mathrm{ad}}(L)$ is proportional to $P(v, z)$ of the cabled link on L and so lies in $\mathbb{Z}[v^{\pm}, z^{\pm}]$. One can also compute $H_{\mathrm{ad}}(q^n, q^{-1} - q)$ by the way of a quantum group $U_q(\mathfrak{sl}_n)$ and its "adjoint" (n^2-1)-dimensional representation (here "adjoint" mean the quantum analogue of the adjoint representation of $\mathfrak{sl}_n$) or equivalently by composing Z_{ad} with the $\mathfrak{sl}_n$ weight system.

In fact $f \circ H = \Phi_{\mathfrak{sl}} \circ Z$ where the weight function $\Phi_{\mathfrak{sl}}$ takes values in the ring $\mathbb{Q}[\delta^{\pm}, h]$ and f is the ring morphism such that $f(v) = \mathrm{e}^{-\frac{h}{2}\delta}$, $f(z) = 2\sinh(h/2) = \mathrm{e}^{\frac{1}{2}h} - \mathrm{e}^{-\frac{1}{2}h}$ and $f(\lambda) = \mathrm{e}^{\frac{h}{2}(\delta - \frac{1}{\delta})}$. Remark that there exists a character $\chi_{\mathfrak{sl}}$ on Λ associated with $\Phi_{\mathfrak{sl}}$ and whose values belong to $\mathbb{Q}[[\delta h, h^2]]$.

[5] This conjecture now results of the work of N. Geer (see [8]).

We show in [19] that on Λ, the map $\chi_{\mathfrak{sl}}$ modulo δ and $\chi_{\mathfrak{D}_{2\,1}}$ modulo σ_3 were both zero and that $\chi_{\mathfrak{sl}}$ modulo δ^2 and $\chi_{\mathfrak{D}_{2\,1}}$ modulo σ_3^2 were equal up to renormalization :

For $\lambda \in \Lambda$ of degree $2p+1 \geq 3$, if $\chi_{\mathfrak{D}_{2\,1}}(\lambda) = \mu(\lambda)\sigma_3(\sigma_2)^{p-1} + O(\sigma_3^2)$

then $\Phi_{\mathfrak{sl}}(\lambda) = (-1)^p\mu(\lambda)\delta h^{2p} + O(\delta^2)$. (Here and after, $O(x)$ denotes an element of the ideal generated by x).

We call ψ the specialization defined by $\psi(\sigma_3) = -\delta h^2$, $\psi(\sigma_2) = -h^2$ (So that $\psi(\sigma_\pm) = 1 + \mathrm{e}^h + \mathrm{e}^{-h} \mp \frac{\delta}{2}(\mathrm{e}^h + \mathrm{e}^{-h} - 2) + O(\delta^2) = f(z^2 + 3 \mp \frac{\delta}{2}z^2) + O(\delta^2)$).

For a knot K closure of $T \in \mathcal{T}([1],[1])$, we have

$Z_{\mathrm{ad}}(T) = 1 + \frac{w(K)}{2}\,{-\!\bigcirc\!-} + t(\frac{w(K)}{2})^2\,{-\!\bigcirc\!-} - V_2(K)t\,{-\!\bigcirc\!-} + \lambda.\,{-\!\bigcirc\!-} \in \mathcal{D}([1],[1])$

where $w(K)$ is the writhe of K, $V_2(K)$ is the (standardly normalized) type 2 Vassiliev invariant of the knot K (i.e. the coefficient of z^2 in $P(1,z)$) and $\lambda \in \Lambda$, λ nul in degree ≤ 1.

So $\Phi_{\mathfrak{sl}}(Z_{\mathrm{ad}}(T)) = 1 + w(K)\delta h + \left(\frac{w(K)^2}{2} - 2V_2(K)\right)\delta^2 h^2 + 2\delta\chi_{\mathfrak{sl}}(\lambda)$

whereas $\widetilde{Z}(K) = \widetilde{Z}(U_0) + \frac{w(K)}{2} + \chi_{\mathfrak{D}_{2\,1}}(\lambda)$ (with U_0 the unknot).

So we get

$$\frac{f(H_{\mathrm{ad}}(K))}{f(H_{\mathrm{ad}}(U_0))} = 1 + \delta^2\left(\frac{w(K)^2}{2} - 2V_2(K)\right) + 2\delta\psi(\widetilde{Z}(K) - \widetilde{Z}(U_0)) + O(\delta^3)$$

So using 1) and 3) of conjecture 4.9 one has

Conjecture 5.2 [6] *For K a 0-framed knot,*

$$\left.\frac{\frac{f(H_{\mathrm{ad}}(K))}{f(H_{\mathrm{ad}}(U_0))} - 1}{\left(v - \frac{1}{v}\right)^2}\right|_{v=1} = -2V_2(K) - \frac{1}{z^2}\left.\frac{\widetilde{Q}(K)}{\sigma_+ - \sigma_-}\right|_{\sigma_+=\sigma_-=z^2+3}$$

5.3 Example of computation

With some computations on Maple, we found that in the base of $\mathrm{Mod}_{U_h\mathfrak{D}_{2\,1}}(L^{\otimes 2}, L^{\otimes 2})$ given by $(U, T^{-2}, T^{-1}, Id, T, T^2)$ where $T = Q_{\mathfrak{D}_{2\,1,\alpha},L}\left(\times\right)$ is the positive half

[6] This conjecture now results of the work of N. Geer (see [8]).

twist and $U = Q_{\mathfrak{D}_{2\,1,\alpha},L}$ , one has

$Q_{\mathfrak{D}_{2\,1,\alpha},L}$ 

$$= \begin{bmatrix} 1-2(\sigma_+ - \sigma_-) \\ 0 \\ -2+\sigma_- \\ -1+2(\sigma_+ - \sigma_-) \\ 2-\sigma_- \\ 1 \end{bmatrix} \tag{2}$$

Furthermore,

$$Q_{\mathfrak{D}_{2\,1,\alpha},L}\left(T^3\right) = \begin{bmatrix} 4\left(\sigma_+ - \sigma_-\right) \\ 1 \\ 2-\sigma_+ \\ 1-2\sigma_+ + \sigma_- \\ -1-\sigma_+ + 2\sigma_- \\ -2+\sigma_- \end{bmatrix} \tag{3}$$

Let K_{2n+1} be the knot obtained as the closure of T^{2n+1} and I_n be the knotted trivalent graph obtained as the closure of $I \circ T^n$ where $I =$ .

So by theorem 1.5 one has :

$$\begin{aligned} \widetilde{Q}(K_{2n+1}) - \widetilde{Q}(K_{2n-1}) &= \left((-1)^{2n} - \widetilde{Q}(I_{2n}) + \frac{1}{2}(-\widetilde{Q}(I_{2n-1}) - \widetilde{Q}(I_{2n+1}))\right) \\ &= 1 - \frac{1}{2}\left(\widetilde{Q}(I_{2n-1}) + 2\widetilde{Q}(I_{2n}) + \widetilde{Q}(I_{2n+1})\right) \end{aligned}$$

with

$$\begin{aligned} &\widetilde{Q}(K_1) = 1 = -\widetilde{Q}(K_{-1}), \quad \widetilde{Q}(I_0) = 0, \quad \widetilde{Q}(I_{\pm 1}) = -1 \\ &\widetilde{Q}(I_2) = 2(\sigma_+ - \sigma_-), \quad \widetilde{Q}(I_{-2}) = -2(\sigma_+ - \sigma_-) \quad \text{by equation (2)} \\ &\widetilde{Q}(I_{n+3}) = 4(\sigma_+ - \sigma_-) + \widetilde{Q}(I_{n-2}) + (2-\sigma_+)\widetilde{Q}(I_{n-1}) + (1-2\sigma_+ + \sigma_-)\widetilde{Q}(I_n) \\ &\qquad + (-1-\sigma_+ + 2\sigma_-)\widetilde{Q}(I_{n+1}) + (-2+\sigma_-)\widetilde{Q}(I_{n+2}) \quad \text{by equation (3)} \end{aligned}$$

Thus one can compute :

$$\begin{aligned} \widetilde{Q}(K_3) &= 3 - (\sigma_+ - \sigma_-)(2+\sigma_-) \\ \widetilde{Q}(K_5) &= 5 + (\sigma_+ - \sigma_-)(-6 + 2\sigma_+ - 4\sigma_- + 2\sigma_+\sigma_- - 2\sigma_-^2 - \sigma_-^3) \\ &\ \ \vdots \end{aligned}$$

The same method gives $\widetilde{Q}(\text{Hopf link}) = \sigma_- - \sigma_+$.

And to compare with theorem 5.1, (with U_0 the unframed unknot) see that

$$\begin{aligned}
K_{\mathrm{ad}}(U_0) &= \frac{\left(\alpha^2-1\right)\left(s^3+\alpha\right)(s\alpha-1)\,s}{\alpha^2\,(s^4-1)\,(s^2-1)} \\
&= 1+\frac{(\alpha-s)\left(\alpha^3 s^2+\alpha^2(s^5+s^3-s)+\alpha(s^4-s^2-1)-s^3\right)}{\alpha^2\,(s^4-1)\,(s^2-1)} \\
\left.\frac{K_{\mathrm{ad}}(U_0)-1}{\alpha-s}\right|_{\alpha=s} &= \frac{s^4+4\,s^2+1}{s\,(s^4-1)} \\
\frac{K_{\mathrm{ad}}(K_3)}{K_{\mathrm{ad}}(U_0)} &= (\alpha^2-s^2)\left(\frac{s^{12}+s^8+s^6+1}{s^{10}}+\frac{\left(s^4-1\right)\left(s^6+1\right)}{s^7\alpha}-\frac{\left(s^4-1\right)\left(s^2-1\right)}{\alpha^4}\right. \\
&\qquad \left.-\frac{s^{12}-s^{10}-s^8+2\,s^6-s^2+1}{s^6\alpha^2}-\frac{\left(s^4-1\right)\left(s^6-s^2+1\right)}{s^3\alpha^3}\right) \\
\left.\frac{K_{\mathrm{ad}}(K_3)-1}{\alpha-s}\right|_{\alpha=s} &= \frac{2}{s}\left(\phi(\widetilde{Q}(K_3))+\frac{s^4+4s^2+1}{s\,(s^4-1)}\right)
\end{aligned}$$

And to compare with section 5.2

$$\begin{aligned}
H_{\mathrm{ad}}(U_0) &= \frac{\left(v^2+zv-1\right)\left(v^2-zv-1\right)}{z^2v^2} = -1+\frac{\left(v-\frac{1}{v}\right)^2}{z^2} \\
\frac{H_{\mathrm{ad}}(K_3)}{H_{\mathrm{ad}}(U_0)} &= 1-3\left(v-\frac{1}{v}\right)+\left(v-\frac{1}{v}\right)^2\left(\frac{v+4}{v+1}+\left(v^2+4\right)z^2+z^4\right) \\
f\left(\frac{H_{\mathrm{ad}}(K_3)}{H_{\mathrm{ad}}(U_0)}\right) &= 1+3\delta+\left(\frac{5}{2}+5z^2+z^4\right)\delta^2+O(\delta^3) \\
&= 1+3\delta+\delta^2\left(\frac{9}{2}-2\right)+\delta^2z^2\left(2+(z^2+3)\right)+O(\delta^3) \\
&= 1+\delta^2\left(\frac{W(K_3)}{2}-V_2(K_3)\right)+\delta\psi(\widetilde{Q}(K_3))+O(\delta^3)
\end{aligned}$$

References

[1] **D Bar-Natan**, *On the Vassiliev knot invariants*, Topology 34 no. 2 (1995), 423–472.

[2] **D Bar-Natan, S Garoufalidis, L Rozansky, D P Thurston**, *Wheels, wheeling, and the Kontsevich integral of the unknot*, Israel Journal of Mathematics 119 (2000) 217–237.

[3] **A B Berger, I Stassen**, *The skein relation for the* $(\mathfrak{g}_2, V)$ *-link invariant*, Comment. Math. Helv. 75 (2000), no. 1, 134–155.

[4] **A M Cohen, M A A van Leeuwen, B Lisser**, *LiE, a package for lie group computations*, CAN. Amsterdam (1992).

[5] **T Deguchi**, *Multivariable invariants of colored links generalizing the Alexander polynomial.* Proceedings of the Conference on Quantum Topology (Manhattan, KS, 1993) (River Edge, NJ), World Sci. Publishing, 1994, 67–85.

[6] **V G Drinfel'd**, *Quasi-Hopf algebras*, Leningrad Math. J. 1 (1990) 1419–1457

[7] **N Geer**, *Etingof-Kazhdan quantization of Lie superbialgebras*, preprint math.QA/0409563 (2004). To appear in Advances in Mathematics.

[8] **N Geer**, *The Kontsevich integral and quantized Lie superalgebras*, Algebraic and Geometric Topology 5 (2005), paper no. 45, pages 1111–1139.

[9] **P Freyd, J Hoste, W B R Lickorish, K Millet, A Ocneanu and D Yetter**, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. 12 (1985), 239–246.

[10] **V F R Jones**, *A polynomial invariant of knots via Von Neumann algebras*, Bull. Amer. Math. Soc. 12 (1985), 103–111.

[11] **V G Kac**, *Lie superalgebras*, Advances in math. 26 (1977), 8–96.

[12] **R M Kashaev**, *A link invariant from quantum dilogarithm.* Modern Phys. Lett. A 10 (1995), no. 19, 1409–1418.

[13] **L H Kauffman**, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. 312 (1990), 417–471.

[14] **T T Q Le, J Murakami**, *Representation of the category of tangles by Kontsevich's iterated integral*, Comm. Math. Phys. 168 (1995), no. 3, 535–562.

[15] **T T Q Le, J Murakami**, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Compositio Math. 102 (1996), no. 1, 41–64.

[16] **T T Q Le, J Murakami**, *Parallel version of the universal Vassiliev-Kontsevich invariant*, J. Pure Appl. Algebra 121 (1997), no. 3, 271–291.

[17] **J Murakami**, *A state model for the multivariable Alexander polynomial.* Pacific J. Math. 157 (1993), no. 1, 109–135.

[18] **J Murakami, T Ohtsuki**, *Topological quantum field theory for the universal quantum invariant.*, Comm. Math. Phys. 188 (1997), no. 3, 501–520.

[19] **B Patureau-Mirand**, *Caractères sur l'algèbre de diagrammes trivalents* Λ, Geom. Topol. 6 (2002), 563–607.

[20] **J Przytycki, P Traczyk**, *Conway Algebras and Skein Equivalence of Links*, Proc. Amer. Math. Soc. 100 (1987) 744–748.

[21] **M Scheunert**, *The presentation and q deformation of special linear Lie superalgebras*, J. Math. Phys. 34 (1993), no. 8, 3780–3808.

[22] **A Sergeev**, *The invariant polynomials on simple Lie superalgebras*, Represent. Theory 3, no.10, 250–280 (1999).

[23] **T Stanford**, *Finite-type invariants of knots, links, and graphs*, Topology 35 (1996), no. 4, 1027–1050.

[24] **H Thys**, *R-matrice universelle pour $U_h(D(2,1,x))$ et invariant d'entrelacs associé* Bull. Soc. Math. Fr. 130, no.2, 309–336 (2002).

[25] **V Vassiliev**, *Cohomology of knot spaces, Theory of singularities and its applications* (V.I. Arnold, ed), Amer. Math. Soc., Providence (1990).

[26] **P Vogel**, *Algebraic structures on modules of diagrams*, preprint (1995) to appear in Inventiones Mathematicae.

[27] **Y M Zou**, *Deformation of the universal enveloping algebra of* $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, Canadian Math. Bull., Vol 39(4)(1996), 499–506.

[28] **Y M Zou**, *Finite Dimensional Representations of* $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, J. Algebra 169(3) (1994), 827–846.

L.M.A.M. Université de Bretagne-Sud
Centre de Recherche, Campus de Tohannic, BP 573
F-56017 Vannes, France

Email: `bertrand.patureau@univ-ubs.fr`

URL: `http://www.univ-ubs.fr/lmam/patureau/`