MODIFIED $6j$-SYMBOLS AND 3-MANIFOLD INVARIANTS

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Abstract. We show that the renormalized quantum invariants of links and graphs in the 3-sphere, derived from tensor categories in [14], lead to modified $6j$-symbols and to new state sum 3-manifold invariants. We give examples of categories such that the associated standard Turaev-Viro 3-manifold invariants vanish but the secondary invariants may be non-zero. The categories in these examples are pivotal categories which are neither ribbon nor semi-simple and have an infinite number of simple objects.

Dedicated to Jose Maria Montesinos on the occasion of his 65th birthday

Introduction

The numerical $6j$-symbols associated with the Lie algebra $sl_2(\mathbb{C})$ were first introduced in theoretical physics by Eugene Wigner in 1940 and Giulio (Yoel) Racah in 1942. They were extensively studied in the quantum theory of angular momentum, see for instance [10]. In mathematics, $6j$-symbols naturally arise in the study of semisimple monoidal tensor categories. In this context, the $6j$-symbols are not numbers but rather tensors on 4 variables running over certain multiplicity spaces. The $6j$-symbols have interesting topological applications: one can use them to write down state sums on knot diagrams and on triangulations of 3-manifolds yielding topological invariants of knots and of 3-manifolds, see [17], [19], [3].

The aim of this paper is to introduce and to study "modified" $6j$-symbols. This line of research extends our previous paper [14] where we introduced so-called modified quantum dimensions of objects of a monoidal tensor category. These new dimensions are particularly interesting when the usual quantum dimensions are zero since the modified dimensions may be non-zero. The standard $6j$-symbols can be viewed as a far-reaching generalization of quantum dimensions of objects. Therefore it is natural to attempt to "modify" their definition following the ideas of [14]. We discuss here a natural setting for such a modification based on a notion of an ambidextrous pair. In this setting, our constructions produce an interesting (and new) system of tensors. These tensors share some of the properties of the standard $6j$-symbols including the fundamental Biedenharn-Elliott identity. We

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suggest a general scheme allowing to derive state-sum topological invariants of links in 3-manifolds from the modified $6j$-symbols.

As an example, we study the modified $6j$-symbols associated with a version of the category of representations of $sl_2(\mathbb{C})$. More precisely, we introduce a Hopf algebra $\bar{U}_q(\mathfrak{sl}(2))$ and study the monoidal category of weight $\bar{U}_q(\mathfrak{sl}(2))$-modules at a complex root of unity q of odd order r. This category is neither ribbon nor semi-simple and has an infinite number of isomorphism classes of simple objects parametrized by elements of $\mathbb{C}/2r\mathbb{Z} \approx \mathbb{C}^*$. The standard quantum dimensions of these objects and the standard $6j$-symbols are generically zero. We show that the modified quantum dimensions are generically non-zero.

As an application, for each pivotal category with additional data, including a G-grading (where G is an abelian group), we construct a topological state-sum invariant of triples (a closed oriented 3-manifold M, a link in M, an element of $H^1(M;G)$). This invariant generalizes the standard Turaev-Viro invariant arising from a quantized simple Lie algebra at a root of unity (for $G = \{1\}$). We also show that the category of weight $\bar{U}_q(\mathfrak{sl}(2))$-modules allows the required additional data with $G = \mathbb{C}/2r\mathbb{Z}$ and thus leads to state-sum 3-manifold invariants. The latter invariants are closely related to Kashaev's invariants introduced in his foundational paper where he first stated the volume conjecture. Indeed, the generalized Kashaev invariants [11] associated to quantized simple Lie algebras are special cases of the invariants defined here (see Theorems 20 and 49 of [13]). Theorems 15 and 49 of [13] imply that the constructions of this paper apply to the categories of finite dimensional weight modules over non-restricted quantum groups studied by C. De Concini, V. Kac, C. Procesi, N. Reshetikhin, and M. Rosso in [6, 7, 8, 9].

Note finally that Francesco Costantino and Jun Murakami [5] has recently showed that the asymptotic behavior of the $6j$-symbols arising from the category of weight $\bar{U}_q(\mathfrak{sl}(2))$-modules is related to the hyperbolic volume of truncated tetrahedra. This gives new evidence that the 6j-symbols and 3-manifold invariants defined in this paper are related to the hyperbolic volume and the volume conjecture.

1. Pivotal tensor categories

We recall the definition of a pivotal tensor category, see for instance, [2]. A *tensor category* $\mathcal{C}$ is a category equipped with a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product, an associativity constraint, a unit object $\mathbb{I}$, and left and right unit constraints such that the Triangle and Pentagon Axioms hold. When the associativity constraint and the left and right unit constraints are all identities we say that $\mathcal{C}$ is a *strict* tensor category. By MacLane's coherence theorem, any tensor category is equivalent to a strict tensor category. To simplify the exposition, we formulate further definitions only for strict tensor categories; the reader will easily extend them to arbitrary tensor categories.

A strict tensor category $\mathcal{C}$ has a *left duality* if for each object V of $\mathcal{C}$ there are an object V^* of $\mathcal{C}$ and morphisms

$$b_V : \ \mathbb{I} \to V \otimes V^* \quad \text{and} \quad d_V : \ V^* \otimes V \to \mathbb{I} \tag{1}$$

such that

$$(\mathrm{Id}_V \otimes d_V)(b_V \otimes \mathrm{Id}_V) = \mathrm{Id}_V \qquad \text{and} \qquad (d_V \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes b_V) = \mathrm{Id}_{V^*}\,.$$

A left duality determines for every morphism $f : V \to W$ in $\mathcal{C}$ the dual (or transposed) morphism $f^* : W^* \to V^*$ by

$$f^* = (d_W \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{W^*} \otimes f \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{W^*} \otimes b_V),$$

and determines for any objects V, W of $\mathcal{C}$, an isomorphism $\gamma_{V,W} : W^* \otimes V^* \to (V \otimes W)^*$ by

$$\gamma_{V,W} = (d_W \otimes \mathrm{Id}_{(V\otimes W)^*})(\mathrm{Id}_{W^*} \otimes d_V \otimes \mathrm{Id}_W \otimes \mathrm{Id}_{(V\otimes W)^*})(\mathrm{Id}_{W^*} \otimes \mathrm{Id}_{V^*} \otimes b_{V\otimes W}).$$

Similarly, $\mathcal{C}$ has a *right duality* if for each object V of $\mathcal{C}$ there are an object $V^\bullet$ of $\mathcal{C}$ and morphisms

$$b'_V : \ \mathbb{I} \to V^\bullet \otimes V \quad \text{and} \quad d'_V : \ V \otimes V^\bullet \to \mathbb{I} \tag{2}$$

such that

$$(\mathrm{Id}_{V^\bullet} \otimes d'_V)(b'_V \otimes \mathrm{Id}_{V^\bullet}) = \mathrm{Id}_{V^\bullet} \qquad \text{and} \qquad (d'_V \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes b'_V) = \mathrm{Id}_V\,.$$

The right duality determines for every morphism $f : V \to W$ in $\mathcal{C}$ the dual morphism $f^\bullet : W^\bullet \to V^\bullet$ by

$$f^\bullet = (\mathrm{Id}_{V^\bullet} \otimes d'_W)(\mathrm{Id}_{V^\bullet} \otimes f \otimes \mathrm{Id}_{W^\bullet})(b'_V \otimes \mathrm{Id}_{W^\bullet}),$$

and determines for any objects V, W, an isomorphism $\gamma'_{V,W} : W^\bullet \otimes V^\bullet \to (V \otimes W)^\bullet$ by

$$\gamma'_{V,W} = (\mathrm{Id}_{(V\otimes W)^\bullet} \otimes d'_V)(\mathrm{Id}_{(V\otimes W)^\bullet} \otimes \mathrm{Id}_V \otimes d'_W \otimes \mathrm{Id}_{V^\bullet})(b'_{V\otimes W} \otimes \mathrm{Id}_{W^\bullet} \otimes \mathrm{Id}_{V^\bullet}).$$

A *pivotal category* is a tensor category with left duality $\{b_V, d_V\}_V$ and right duality $\{b'_V, d'_V\}_V$ which are compatible in the sense that $V^* = V^\bullet$, $f^* = f^\bullet$, and $\gamma_{V,W} = \gamma'_{V,W}$ for all V, W, f as above.

A tensor category $\mathcal{C}$ is said to be an *Ab-category* if for any objects V, W of $\mathcal{C}$, the set of morphism $\mathrm{Hom}(V, W)$ is an additive abelian group and both composition and tensor multiplication of morphisms are bilinear. Composition of morphisms induces a commutative ring structure on the abelian group $K = \mathrm{End}(\mathbb{I})$. The resulting ring is called the *ground ring* of $\mathcal{C}$. For any objects V, W of $\mathcal{C}$ the abelian group $\mathrm{Hom}(V, W)$ becomes a left K-module via $kf = k \otimes f$ for $k \in K$ and $f \in \mathrm{Hom}(V, W)$. An object V of $\mathcal{C}$ is *simple* if $\mathrm{End}(V) = K\,\mathrm{Id}_V$.

2. Invariants of graphs in S^2

From now on and up to the end of the paper the symbol $\mathcal{C}$ denotes a pivotal tensor Ab-category with ground ring K, left duality (1), and right duality (2).

A morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m$ in $\mathcal{C}$ can be represented by a box and arrows as in Figure 1. Here the plane of the picture is oriented counterclockwise, and this orientation determines the numeration of the arrows attached to the bottom and top sides of the box. More generally, we allow such boxes with some arrows directed up and some arrows directed down. For example, if all the bottom arrows in the above picture and redirected upward, then the box represents a morphism $V_1^* \otimes \cdots \otimes V_n^* \to W_1 \otimes \cdots \otimes W_m$. The boxes as above are called *coupons*. Each coupon has distinguished bottom and top sides and all incoming and outgoing arrows can be attached only to these sides.

By a graph we always mean a finite graph with oriented edges (we allow loops and multiple edges with the same vertices). By a $\mathcal{C}$-colored ribbon graph in an oriented surface Σ, we mean a graph embedded in Σ whose edges are colored by objects of $\mathcal{C}$ and whose vertices lying in $\operatorname{Int}\Sigma = \Sigma - \partial\Sigma$ are thickened to coupons colored by morphisms of $\mathcal{C}$. The edges of a ribbon graph do not meet each other and may meet the coupons only at the bottom and top sides. The intersection of a $\mathcal{C}$-colored ribbon graph in Σ with $\partial\Sigma$ is required to be empty or to consist only of vertices of valency 1.

We define a category of $\mathcal{C}$-colored ribbon graphs $\mathrm{Gr}_{\mathcal{C}}$. The objects of $\mathrm{Gr}_{\mathcal{C}}$ are finite sequences of pairs (V, ε), where V is an object of $\mathcal{C}$ and $\varepsilon = \pm$. The morphisms of $\mathrm{Gr}_{\mathcal{C}}$ are isotopy classes of $\mathcal{C}$-colored ribbon graphs Γ embedded in $\mathbb{R} \times [0, 1]$. The (1-valent) vertices of such Γ lying on $\mathbb{R} \times 0$ are called the inputs. The colors and orientations of the edges of Γ incident to the inputs (enumerated from the left to the right) determine an object of $\mathrm{Gr}_{\mathcal{C}}$ called the source of Γ. Similarly, the (1-valent) vertices of Γ lying on $\mathbb{R} \times 1$ are called the outputs; the colors and orientations of the edges of Γ incident to the outputs determine an object of $\mathrm{Gr}_{\mathcal{C}}$ called the target of Γ. We view Γ as a morphism in $\mathrm{Gr}_{\mathcal{C}}$ from the source object to the target object. More generally, formal linear combinations over K of $\mathcal{C}$-colored ribbon graphs in $\mathbb{R} \times [0, 1]$ with the same input and source are also viewed as morphisms in $\mathrm{Gr}_{\mathcal{C}}$. Composition, tensor multiplication, and

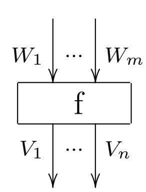


Figure 1

left and right duality in $\mathrm{Gr}_{\mathcal{C}}$ are defined in the standard way, cf. [18]. This makes $\mathrm{Gr}_{\mathcal{C}}$ into a pivotal Ab-category.

The well-known Reshetikhin-Turaev construction defines a K-linear functor $G : \mathrm{Gr}_{\mathcal{C}} \to \mathcal{C}$ preserving both left and right dualities. This functor is compatible with tensor multiplication and transforms an object (V, ε) of $\mathrm{Gr}_{\mathcal{C}}$ to V if $\varepsilon = +$ and to V^* if $\varepsilon = -$. The definition of G goes by splitting the ribbon graphs into "elementary" pieces, cf. [18]. The left duality in $\mathcal{C}$ is used to assign morphisms in $\mathcal{C}$ to the small right-oriented cup-like and cap-like arcs. The right duality is used to assign morphisms in $\mathcal{C}$ to the small left-oriented cup-like and cap-like arcs. Invariance of G under plane isotopy is deduced from the conditions $f^* = f^\bullet$ and $\gamma_{V,W} = \gamma'_{V,W}$ in the definition of a pivotal tensor category.

Under certain assumptions, one can thicken the usual (non-ribbon) graphs in S^2 into ribbon graphs. The difficulty is that thickening of a vertex to a coupon is not unique. We recall a version of thickening for trivalent graphs following [18].

By *basic data* in $\mathcal{C}$ we mean a family $\{V_i\}_{i\in I}$ of simple objects of $\mathcal{C}$ numerated by elements of a set I with involution $I \to I$, $i \mapsto i^*$ and a family of isomorphisms $\{w_i : V_i \to V^*_{i^*}\}_{i\in I}$ such that $\mathrm{Hom}_{\mathcal{C}}(V_i, V_j) = 0$ for distinct $i, j \in I$ and

$$d_{V_i}(w_{i^*} \otimes \mathrm{Id}_{V_i}) = d'_{V_{i^*}}(\mathrm{Id}_{V_{i^*}} \otimes w_i) \colon V_{i^*} \otimes V_i \to \mathbb{I} \qquad (3)$$

for all $i \in I$. In particular, V_i is not isomorphic to V_j for $i \neq j$.

For any $i, j, k \in I$, consider the multiplicity module

$$H^{ijk} = \mathrm{Hom}(\mathbb{I}, V_i \otimes V_j \otimes V_k).$$

The K-modules $H^{ijk}, H^{jki}, H^{kij}$ are canonically isomorphic. Indeed, let $\sigma(i, j, k)$ be the isomorphism

$$H^{ijk} \to H^{jki}, \;\; x \mapsto d_{V_i} \circ (\mathrm{Id}_{V_i^*} \otimes x \otimes \mathrm{Id}_{V_i}) \circ b'_{V_i}.$$

Using the functor $G : \mathrm{Gr}_{\mathcal{C}} \to \mathcal{C}$, one easily proves that

$$\sigma(k, i, j)\, \sigma(j, k, i)\, \sigma(i, j, k) = \mathrm{Id}_{H^{ijk}}\,.$$

Identifying the modules $H^{ijk}, H^{jki}, H^{kij}$ along these isomorphisms we obtain a *symmetrized multiplicity module* $H(i, j, k)$ depending only on the cyclically ordered triple (i, j, k). This construction may be somewhat disturbing, especially if some (or all) of the indices i, j, k are equal. We give therefore a more formal version of the same construction. Consider a set X consisting of three (distinct) elements $\{a, b, c\}$ with cyclic order $a < b < c < a$. For any function $f : X \to I$, the construction above yields canonical K-isomorphisms

$$H^{f(a),f(b),f(c)} \cong H^{f(b),f(c),f(a)} \cong H^{f(c),f(a),f(b)}$$

of the modules determined by the linear orders on X compatible with the cyclic order. Identifying these modules along these isomorphisms we obtain a module $H(f)$ independent of the choice of a linear order on X compatible with the cyclic order. If $i = f(a), j = f(b), k = f(c)$, then we write $H(i, j, k)$ for $H(f)$.

By a labeling of a graph we mean a function assigning to every edge of the graph an element of I. By a trivalent graph we mean a (finite oriented) graph whose vertices all have valency 3. Let Γ be a labeled trivalent graph in S^2. Using the standard orientation of S^2 (induced by the right-handed orientation of the unit ball in $\mathbb{R}^3$), we cyclically order the set X_v of 3 half-edges adjacent to any given vertex v of Γ. The labels of the edges determine a function $f_v : X_v \to I$ as follows: if a half-edge e adjacent to v is oriented towards v, then $f_v(e) = i$ is the label of the edge of Γ containing e; if a half-edge e adjacent to v is oriented away from v, then $f_v(e) = i^*$. Set $H_v(\Gamma) = H(f_v)$ and $H(\Gamma) = \otimes_v H_v(\Gamma)$ where v runs over all vertices of Γ.

Consider now a labeled trivalent graph $\Gamma \subset S^2$ as above, endowed with a family of vectors $h = \{h_v \in H_v(\Gamma)\}_v$, where v runs over all vertices of Γ. We thicken Γ into a $\mathcal{C}$-colored ribbon graph on S^2 as follows. First, we insert inside each edge e of Γ a coupon with one edge outgoing from the bottom along e and with one edge outgoing from the top along e in the direction opposite to the one on e. If e is labeled with $i \in I$, then these two new (smaller) edges are labeled with V_i and V_{i^*}, respectively, and the coupon is labeled with $w_i : V_i \to V_{i^*}^*$ as in Figure 2.

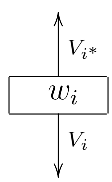


Figure 2

Next, we thicken each vertex v of Γ to a coupon so that the three half-edges adjacent to v yield three arrows adjacent to the top side of the coupon and oriented towards it. If $i, j, k \in I$ are the labels of these arrows (enumerated from the left to the right), then we color this coupon with the image of h_v under the natural isomorphism $H_v(\Gamma) \to H^{ijk}$. Denote the resulting $\mathcal{C}$-colored ribbon graph by $\Omega_{\Gamma,h}$. Then $\mathbb{G}(\Gamma, h) = G(\Omega_{\Gamma,h})$ is an isotopy invariant of the pair (Γ, h) independent of the way in which the vertices of Γ are thickened to coupons.

In the next section we describe a related but somewhat different approach to invariants of colored ribbon graphs and labeled trivalent graphs in S^2.

3. Cutting of graphs and ambidextrous pairs

Let $T \subset S^2$ be a $\mathcal{C}$-colored ribbon graph and let e be an edge of T colored with a simple object V of $\mathcal{C}$. Cutting T at a point of e, we obtain a $\mathcal{C}$-colored ribbon graph in $\mathbb{R} \times [0, 1]$ with one input and one output such that the edges incident to the input and output are oriented downward and colored with V. We denote this $\mathcal{C}$-colored ribbon graph by T_V^e or shorter by T_V. Clearly, $T_V \in \mathrm{End}_{\mathrm{Gr}_{\mathcal{C}}}((V, +))$.

Note that the closure of T_V (obtained by connecting the endpoints of T_V by an arc in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ disjoint from the rest of T_V) is isotopic to T in S^2. We call T_V a *cutting presentation* of T and let $\langle T_V \rangle \in K$ denote the isotopy invariant of T_V defined from the equality $G(T_V) = \langle T_V \rangle \, \mathrm{Id}_V$.

In the following definition and in the sequel, a ribbon graph is *trivalent* if all its coupons are adjacent to 3 half-edges.

Definition 1. *Let $\{V_i, w_i\}_{i\in I}$ be basic data in $\mathcal{C}$. Let I_0 be a subset of I invariant under the involution $i \mapsto i^*$ and $\mathsf{d} : I_0 \to K$ be a mapping such that $\mathsf{d}(i) = \mathsf{d}(i^*)$ for all $i \in I_0$. Let $\mathcal{T}_{I_0}$ be the class of $\mathcal{C}$-colored connected trivalent ribbon graphs in S^2 such that the colors of all edges belong to the set $\{V_i\}_{i\in I}$ and the color of at least one edge belongs to the set $\{V_i\}_{i\in I_0}$. The pair (I_0, d) is* trivalent-ambidextrous *if for any $T \in \mathcal{T}_{I_0}$ and for any two cutting presentations T_{V_i}, T_{V_j} of T with $i, j \in I_0$,*

$$\mathsf{d}(i)\langle T_{V_i}\rangle = \mathsf{d}(j)\langle T_{V_j}\rangle. \tag{4}$$

To simplify notation, we will say that the pair (I_0, d) in Definition 1 is *t-ambi*. For a t-ambi pair (I_0, d) we define a function $G' : \mathcal{T}_{I_0} \to K$ by

$$G'(T) = \mathsf{d}(i)\langle T_{V_i}\rangle\,, \tag{5}$$

where T_{V_i} is any cutting presentation of T with $i \in I_0$. The definition of a t-ambi pair implies that G' is well defined.

The invariant G' can be extended to a bigger class of $\mathcal{C}$-colored ribbon graphs in S^2. We say that a coupon of a ribbon graph is straight if both its bottom and top sides are incident to exactly one arrow. We can remove a straight coupon and unite the incident arrows into a (longer) edge, see Figure 3. We call this operation *straightening*. A *quasi-trivalent ribbon graph* is a ribbon graph in S^2 such that straightening it at all straight coupons we obtain a trivalent ribbon graph.

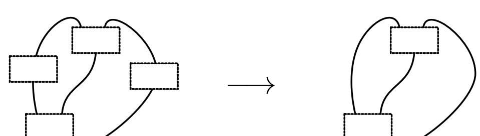


FIGURE 3. Straightening a quasi-trivalent ribbon graph

Lemma 2. *Let (I_0, d) be a t-ambi pair in $\mathcal{C}$ and $\overline{\mathcal{T}}_{I_0}$ be the class of connected quasi-trivalent ribbon graphs in S^2 such that the colors of all edges belong to the set $\{V_i\}_{i\in I}$, the color of at least one edge belongs to the set $\{V_i\}_{i\in I_0}$, and all straight coupons are colored with isomorphisms in $\mathcal{C}$. Then Formula* (5) *determines a well-defined function $G' : \overline{\mathcal{T}}_{I_0} \to K$.*

Proof. Given an endomorphism f of a simple object V of $\mathcal{C}$, we write $\langle f \rangle$ for the unique $a \in K$ such that $f = a\,\mathrm{Id}_V$. Observe that if $f : V \to W$ and $g : W \to V$

are homomorphisms of simple objects and f is invertible, then

$$\langle fg\rangle = \langle gf\rangle \,. \tag{6}$$

To see this, set $a = \langle fg\rangle \in K$ and $b = \langle gf\rangle \in K$. Then

$$a\,\mathrm{Id}_W = fg(ff^{-1}) = f(gf)f^{-1} = bff^{-1} = b\,\mathrm{Id}_W \ .$$

Thus, $a = b$.

Consider now a $\mathcal{C}$-colored quasi-trivalent ribbon graph T in S^2 and a straight coupon of T such that the arrow adjacent to the bottom side is outgoing and colored with V_i, $i \in I_0$ while the arrow adjacent to the top side is incoming and colored with V_j, $j \in I_0$. By assumption, the coupon is colored with an isomorphism $V_i \to V_j$. Formula (6) implies that $\langle T_{V_i}\rangle = \langle T_{V_j}\rangle$, where T_{V_i} is obtained by cutting T at a point of the bottom arrow and T_{V_j} is obtained by cutting T at a point of the top arrow. By the properties of a t-ambi pair, $\mathsf{d}(i) = \mathsf{d}(j)$. Hence Formula (5) yields the same element of K for these two cuttings. The cases where some arrows adjacent to the straight coupon are oriented up (rather than down) are considered similarly, using the identity $\mathsf{d}(i) = \mathsf{d}(i^*)$. Therefore cutting T at two different edges that unite under straightening, we obtain on the right-hand side of (5) the same element of K. When we cut T at two edges which do not unite under straightening, a similar claim follows from the definitions. □

We can combine the invariant G' with the thickening of trivalent graphs to obtain invariants of trivalent graphs in S^2. Suppose that $\Gamma \subset S^2$ is a labeled connected trivalent graph such that the label of at least one edge of Γ belongs to I_0. We define

$$\mathbb{G}'(\Gamma) \in H(\Gamma)^\star = \mathrm{Hom}_K(H(\Gamma), K)$$

as follows. Pick any family of vectors $h = \{h_v \in H_v(\Gamma)\}_v$, where v runs over all vertices of Γ. The $\mathcal{C}$-colored ribbon graph $\Omega_{\Gamma,h}$ constructed at the end of Section 2 belongs to the class $\overline{\mathcal{T}}_{I_0}$ defined in Lemma 2. Set

$$\mathbb{G}'(\Gamma)(\otimes_v h_v) = G'(\Omega_{\Gamma,h}) \in K.$$

By the properties of G', the vector $\mathbb{G}'(\Gamma) \in H(\Gamma)^\star$ is an isotopy invariant of Γ. Both $H(\Gamma)$ and $\mathbb{G}'(\Gamma)$ are preserved under the reversion transformation inverting the orientation of an edge of Γ and replacing the label of this edge, i, with i^*. This can be easily deduced from Formula (3).

4. Modified $6j$-symbols

Let $\mathcal{C}$ be a pivotal tensor Ab-category with ground ring K, basic data $\{V_i, w_i : V_i \to V_{i^*}^*\}_{i\in I}$, and t-ambi pair (I_0, d). To simplify the exposition, we assume that there is a well-defined direct summation of objects in $\mathcal{C}$. We define a system of tensors called modified $6j$-symbols.

Let i, j, k, l, m, n be six elements of I such that at least one of them is in I_0. Consider the labeled trivalent graph $\Gamma = \Gamma(i, j, k, l, m, n) \subset \mathbb{R}^2 \subset S^2$ given in Figure 4. By definition,

$$H(\Gamma) = H(i, j, k^*) \otimes_K H(k, l, m^*) \otimes_K H(n, l^*, j^*) \otimes_K H(m, n^*, i^*).$$

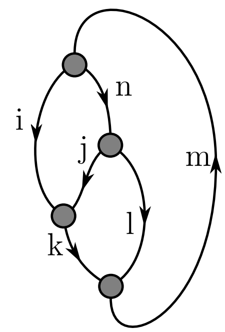


FIGURE 4. $\Gamma(i, j, k, l, m, n)$

We define the modified 6j-symbol of the tuple (i, j, k, l, m, n) to be

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \mathbb{G}'(\Gamma) \in H(\Gamma)^\star = \mathrm{Hom}_K(H(\Gamma), K). \tag{7}$$

It follows from the definitions that the modified 6j-symbols have the symmetries of an oriented tetrahedron. In particular,

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & k^* & i^* \\ m & n & l \end{vmatrix} = \begin{vmatrix} k & l & m \\ n^* & i & j^* \end{vmatrix}.$$

These equalities hold because the labeled trivalent graphs in S^2 defining these 6j-symbols are related by isotopies and reversion transformations described above.

To describe $H(\Gamma)^\star$, we compute the duals of the symmetrized multiplicity modules. For any indices $i, j, k \in I$ such that at least one of them lies in I_0, we define a pairing

$$(\,,)_{ijk} : H(i, j, k) \otimes_K H(k^*, j^*, i^*) \to K \tag{8}$$

by

$$(x, y)_{ijk} = \mathbb{G}'(\Theta)(x \otimes y)\,,$$

where $x \in H(i, j, k) = H^{ijk}$ and $y \in H(k^*, j^*, i^*) = H^{k^*j^*i^*}$ and $\Theta = \Theta_{i,j,k}$ is the theta graph with vertices u, v and three edges oriented from v to u and labeled with i, j, k, see Figure 5. Clearly,

$$H_u(\Theta) = H^{ijk} = H(i, j, k) \quad \text{and} \quad H_v(\Theta) = H^{k^*j^*i^*} = H(k^*, j^*, i^*)$$

so that we can use x, y as the colors of u, v, respectively. It follows from the definitions that the pairing $(,)_{ijk}$ is invariant under cyclic permutations of i,j,k and $(x,y)_{ijk} = (y,x)_{k^*j^*i^*}$ for all $x \in H(i,j,k)$ and $y \in H(k^*,j^*,i^*)$.

We now give sufficient conditions for the pairing $(,)_{ijk}$ to be non-degenerate. We say that a pair $(i,j) \in I^2$ is *good* if $V_i \otimes V_j$ splits as a finite direct sum of some V_k's (possibly with multiplicities) such that $k \in I_0$ and $\mathsf{d}(k)$ is an invertible element of K. By duality, a pair $(i,j) \in I^2$ is good if and only if the pair (j^*, i^*) is good. We say that a triple $(i,j,k) \in I^3$ is *good* if at least one of the pairs $(i,j), (j,k), (k,i)$ is good. Clearly, the goodness of a triple (i,j,k) is preserved under cyclic permutations and implies the goodness of the triple (k^*, j^*, i^*). Note also that if a triple (i,j,k) is good, then at least one of the indices i,j,k belongs to I_0 so that we can consider the pairing (8).

Lemma 3. *If the triple $(i,j,k) \in I^3$ is good, then the pairing (8) is non-degenerate.*

Proof. For any $i,j,k \in I$, consider the K-modules

$$H^{ij}_k = \operatorname{Hom}(V_k, V_i \otimes V_j) \quad \text{and} \quad H^k_{ij} = \operatorname{Hom}(V_i \otimes V_j, V_k).$$

Consider the homomorphism

$$H^{ij}_k \to H^{ijk^*}, \; y \mapsto (y \otimes w_{k^*}^{-1})\, b_{V_k}. \tag{9}$$

It is easy to see that this is an isomorphism. Composing with the canonical isomorphism $H^{ijk^*} \to H(i,j,k^*)$ we obtain an isomorphism $a^{ij}_k : H^{ij}_k \to H(i,j,k^*)$. Similarly, we have an isomorphism

$$a^k_{ij} : H^k_{ij} \to H^{kj^*i^*} = H(k,j^*,i^*), \; x \mapsto (x \otimes w_{j^*}^{-1} \otimes w_{i^*}^{-1})(\operatorname{Id}_{V_i} \otimes b_{V_j} \otimes \operatorname{Id}_{V_i^*}) b_{V_i}.$$

Let $(,)^{ij}_k : H^{ij}_k \otimes_K H^k_{ij} \to K$ be the bilinear pairing whose value on any pair $(y \in H^{ij}_k, x \in H^k_{ij})$ is computed from the equality

$$(y,x)^{ij}_k \operatorname{Id}_{V_k} = \mathsf{d}(k) xy : V_k \to V_k.$$

It follows from the definitions that under the isomorphism

$$a^{ij}_k \otimes a^k_{ij} : H^{ij}_k \otimes_K H^k_{ij} \to H(i,j,k^*) \otimes_K H(k,j^*,i^*)$$

the pairing $(,)^{ij}_k$ is transformed into $(,)_{ijk^*}$.

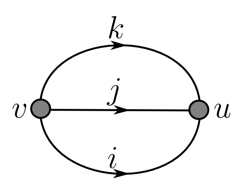


FIGURE 5. $\Theta_{i,j,k}$

We can now prove the claim of the lemma. By cyclic symmetry, it is enough to consider the case where the pair (i,j) is good. Then H_k^{ij} and H_{ij}^k are free K-modules of the same finite rank and the pairing $(,)_k^{ij}$ is non-degenerate. Therefore the pairing $(,)_{ijk^*}$ is non-degenerate. □

We say that a 6-tuple $(i,j,k,l,m,n) \in I^6$ is *good* if all the triples

$$(i,j,k^*),(k,l,m^*),(n,l^*,j^*),(m,n^*,i^*) \tag{10}$$

are good. This property of a 6-tuple is invariant under symmetries of an oriented tetrahedron acting on the labels of the edges. The previous lemma implies that if (i,j,k,l,m,n) is good, then the ambient module of the modified 6j-symbol (7) is

$$H(k,j^*,i^*) \otimes_K H(m,l^*,k^*) \otimes_K H(j,l,n^*) \otimes_K H(i,n,m^*). \tag{11}$$

Remark 4. *In the more general setting of pivotal tensor Ab-categories (possibly not allowing direct summation), one can define a good pair $(i,j) \in I^2$ by requiring that the identity automorphism of $V_i \otimes V_j$ expands as a finite sum of compositions $V_i \otimes V_j \to V_k \to V_i \otimes V_j$ with $k \in I$, cf. the notion of domination in* [18].

5. Comparison with the standard 6j-symbols

We compare the modified 6j-symbols defined above with standard 6j-symbols derived from decompositions of tensor products of simple objects into direct sums.

We keep notation of Section 4 and begin with two simple lemmas.

Lemma 5. *For any $i,j,k,l,m \in I$, the formula $(f,g) \mapsto (f \otimes \mathrm{Id}_{V_l})g$ defines a K-homomorphism*

$$H_k^{ij} \otimes_K H_m^{kl} \to \mathrm{Hom}(V_m, V_i \otimes V_j \otimes V_l). \tag{12}$$

If (i,j) is good, then the direct sum of these homomorphisms is an isomorphism

$$\bigoplus_{k \in I} H_k^{ij} \otimes_K H_m^{kl} \to \mathrm{Hom}(V_m, V_i \otimes V_j \otimes V_l). \tag{13}$$

Proof. The first claim is obvious; we prove the second claim. If (i,j) is good, then for every $k \in I$, the K-modules H_k^{ij} and H_{ij}^k are free of the same finite rank. Fix a basis $\{\alpha_{k,r}\}_{r \in R_k}$ of H_k^{ij} where R_k is a finite indexing set. Then there is a basis $\{\alpha^{k,r}\}_{r \in R_k}$ of H_{ij}^k such that $\alpha^{k,r}\alpha_{k,s} = \delta_{r,s}\,\mathrm{Id}_{V_k}$ for all $r,s \in R_k$. Clearly,

$$\mathrm{Id}_{V_i \otimes V_j} = \sum_{k \in I, r \in R_k} \alpha_{k,r}\alpha^{k,r}.$$

We define a K-homomorphism $\mathrm{Hom}(V_m, V_i \otimes V_j \otimes V_l) \to \bigoplus_{k \in I} H_k^{ij} \otimes H_m^{kl}$ by

$$f \mapsto \sum_{k \in I,\, r \in R_k} \alpha_{k,r} \otimes_K (\alpha^{k,r} \otimes \mathrm{Id}_{V_l})f.$$

This is the inverse of the homomorphism (13). □

Similarly, we have the following lemma.

Lemma 6. *For any $i, j, l, m, n \in I$, the formula $(f, g) \mapsto (\mathrm{Id}_{V_i} \otimes f)g$ defines a K-homomorphism*

$$H_n^{jl} \otimes_K H_m^{in} \to \mathrm{Hom}(V_m, V_i \otimes V_j \otimes V_l). \tag{14}$$

If the pair (j, l) is good, then the direct sum of these homomorphisms is an isomorphism

$$\bigoplus_{n\in I} H_n^{jl} \otimes_K H_m^{in} \to \mathrm{Hom}(V_m, V_i \otimes V_j \otimes V_l). \tag{15}$$

Suppose that both pairs (i, j) and (j, l) are good. Composing the isomorphism (13) with the inverse of the isomorphism (15) we obtain an isomorphism

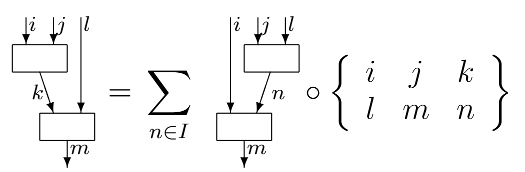
$$\bigoplus_{k\in I} H_k^{ij} \otimes_K H_m^{kl} \to \bigoplus_{n\in I} H_n^{jl} \otimes_K H_m^{in}. \tag{16}$$

Restricting to the summand on the left-hand side corresponding to k and projecting to the summand on the right-hand side corresponding to n we obtain a homomorphism

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} : H_k^{ij} \otimes_K H_m^{kl} \to H_n^{jl} \otimes_K H_m^{in}. \tag{17}$$

This is the (standard) $6j$-symbol determined by i, j, k, l, m, n. We emphasize that it is defined only when the pairs (i, j) and (j, l) are good. Note that this $6j$-symbol is equal to zero unless both k and n belong to I_0.

The following equality is an equivalent graphical form of the same definition. It indicates that the composition of the homomorphisms (17) and (14) summed up over all $n \in I$ is equal to the homomorphism (12):



Our next aim is to relate the $6j$-symbol (17) to the modified $6j$-symbol (7). Recall the isomorphism $a_k^{ij} : H_k^{ij} \to H(i, j, k^*)$ introduced in the proof of Lemma 3. Using these isomorphisms, we can rewrite the $6j$-symbol (17) as a homomorphism

$$H(i, j, k^*) \otimes_K H(k, l, m^*) \to H(j, l, n^*) \otimes_K H(i, n, m^*). \tag{18}$$

Since (j, l) is good, $H(j, l, n^*)^\star = H(n, l^*, j^*)$. Assuming that (i, n) is good, we write $H(i, n, m^*)^\star = H(m, n^*, i^*)$ and consider the homomorphism

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}^\sigma : H(i, j, k^*) \otimes_K H(k, l, m^*) \otimes_K H(n, l^*, j^*) \otimes_K H(m, n^*, i^*) \to K$$

adjoint to (18). This homomorphism has the same source module as the modified 6j-symbol (7).

Lemma 7. *For any $i,j,k,l,m,n \in I$ such that the pairs $(i,j),(j,l)$, (i,n) are good and $m,n \in I_0$,*

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}^{\sigma} = \mathsf{d}(n) \left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| . \tag{19}$$

Proof. Pick any $x_1 \in H^{ij}_k$, $x_2 \in H^{kl}_m$. By Lemma 6,

$$(x_1 \otimes \mathrm{Id}_{V_l})x_2 = \sum_{n' \in I} \sum_{r \in R_{n'}} (\mathrm{Id}_{V_i} \otimes y_1^{n',r}) y_2^{n',r},$$

where $y_1^{n',r} \in H^{jl}_{n'}$, $y_2^{n',r} \in H^{in'}_m$, and $R_{n'}$ is a finite set of indices. By definition,

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} (x_1 \otimes x_2) = \sum_{r \in R_n} y_1^{n,r} \otimes y_2^{n,r}.$$

Under our assumptions on i,j,k,l,m,n, the pairings $(\,,)^{jl}_n$ and $(\,,)^{in}_m$ introduced in the proof of Lemma 3 are non-degenerate, and we use them to identify $H^{jl}_n = (H^n_{jl})^\star$ and $H^{in}_m = (H^m_{in})^\star$. Consider the homomorphism

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_+ : H^{ij}_k \otimes_K H^{kl}_m \otimes_K H^n_{jl} \otimes_K H^m_{in} \to K \tag{20}$$

adjoint to $\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}$. The computations above show that

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_+ (x_1 \otimes x_2 \otimes x_3 \otimes x_4) = \sum_{r \in R_n} (y_1^{n,r}, x_3)^{jl}_n \; (y_2^{n,r}, x_4)^{in}_m \tag{21}$$

for any $x_3 \in H^n_{jl}$ and $x_4 \in H^m_{in}$.

Consider the $\mathcal{C}$-colored ribbon graphs Γ_1, $\Gamma_2^{n',r}$, $\Gamma_3^{n,r}$ in Figure 6 (it is understood that an edge with label $s \in I$ is colored with V_s). It is clear that

$$G'(\Gamma_1) = \sum_{n' \in I, r \in R_{n'}} G'\left(\Gamma_2^{n',r}\right) = \sum_{n' \in I, r \in R_{n'}} \delta_{n,n'} \, \mathsf{d}(n)^{-1} (y_1^{n,r}, x_3)^{jl}_n \, G'(\Gamma_3^{n,r}) \tag{22}$$

where the second equality follows from the fact that $xy = \mathsf{d}(n)^{-1}(y,x)\,\mathrm{Id}_{V_n}$ for $x \in H^n_{jl}$ and $y \in H^{jl}_n$. Similarly, $G'(\Gamma_3^{n,r}) = (y_2^{n,r}, x_4)^{in}_m$ (here we use the inclusion $m \in I_0$). Therefore

$$\mathsf{d}(n)\, G'(\Gamma_1) = \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_+ (x_1 \otimes x_2 \otimes x_3 \otimes x_4).$$

Rewriting this equality in terms of the symmetrized multiplicity modules, we obtain the claim of the lemma. □

We say that a 6-tuple $(i,j,k,l,m,n) \in I^6$ is *strongly good* if $m,n \in I_0$ and the pairs (i,j), (j,l), (i,n), and (k,l) are good. A strongly good 6-tuple (i,j,k,l,m,n) is good in the sense of Section 4, so that both associated $6j$-symbols $\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}^{\sigma}$ and $\left|\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right|$ lie in the K-module (11). Lemma 7 yields equality (19) understood as an equality in this K-module.

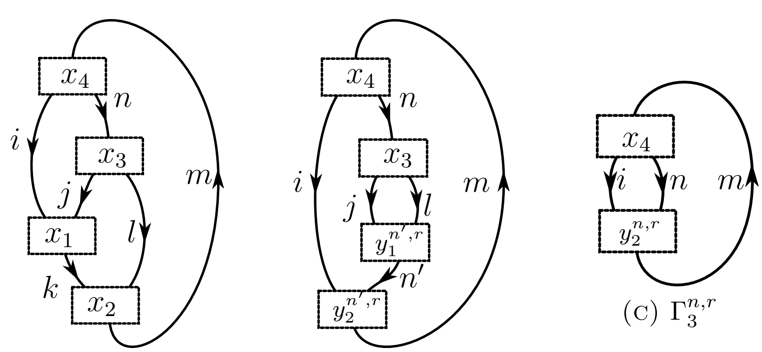
Unfortunately, the notion of a strongly good 6-tuple is not invariant under symmetries of an oriented tetrahedron. To make it symmetric, one has to add more conditions on the labels. We say that a tuple $(i,j,k,l,m,n) \in I^6$ is *admissible* if all the indices i,j,k,l,m,n belong to I_0 and the pairs (i,j), (j,l), (i,n), (k,l), (j,k^*), (k^*,i), (l,m^*), (m^*,k), (n,l^*), (j^*,n), (m,n^*), (i^*,m) are good. Admissible 6-tuples are strongly good, and the notion of an admissible 6-tuple is preserved under the symmetries of an oriented tetrahedron.

6. Properties of the modified $6j$-symbols

Given a good triple $(i,j,k) \in I^3$ and a tensor product of several K-modules such that among the factors there is a matched pair $H(i,j,k)$, $H(k^*,j^*,i^*)$, we may contract any element of this tensor product using the pairing (8). This operation is called the contraction along $H(i,j,k)$ and denoted by $*_{ijk}$. For example, an element $x \otimes y \otimes z \in H(i,j,k) \otimes_K H(k^*,j^*,i^*) \otimes_K H$, where H is a K-module, contracts into $(x,y)z \in H$.

Theorem 8 (The Biedenharn-Elliott identity). *Let $j_0, j_1, \ldots, j_8$ be elements of I such that the tuples $(j_1,j_2,j_5,j_8,j_0,j_7)$ and $(j_5,j_3,j_6,j_4,j_0,j_8)$ are strongly good, $j_7, j_8 \in I_0$, and the pair (j_2,j_3) is good. Set*

$$J = \{j \in I \mid H_j^{j_2 j_3} \neq 0\} \subset I_0.$$



(A) Γ_1 (B) $\Gamma_2^{n',r}$ (C) $\Gamma_3^{n,r}$

Figure 6

If the pairs (j_1, j) and (j, j_4) are good for all $j \in J$, then all 6-tuples defining the $6j$-symbols in the following formula are strongly good and

$$\sum_{j\in J} \mathsf{d}(j)\ *_{j_2j_3j^*} *_{jj_4j_7^*} *_{j_1jj_6^*} \left(\begin{vmatrix} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{vmatrix} \otimes \begin{vmatrix} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{vmatrix} \otimes \begin{vmatrix} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{vmatrix} \right)$$

$$= *_{j_5j_8j_0^*} \left(\begin{vmatrix} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{vmatrix} \otimes \begin{vmatrix} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{vmatrix} \right). \qquad (23)$$

Here both sides lie in the tensor product of six K-modules

$$H(j_6, j_3^*, j_5^*), H(j_5, j_2^*, j_1^*), H(j_0, j_4^*, j_6^*), H(j_1, j_7, j_0^*), H(j_2, j_8, j_7^*), H(j_3, j_4, j_8^*).$$

Proof. The claim concerning the strong goodness follows directly from the definitions. We verify (23). Recall that if (i, j) is a good pair, and V_k is a summand of $V_i \otimes V_j$, then $\mathsf{d}(k)$ is invertible. Therefore if $\mathsf{d}(j_7)$ is not invertible, then V_{j_7} can not be a summand of $V_{j_2} \otimes V_{j_8}$, and so both sides of (23) are equal to 0. Similarly if $\mathsf{d}(j_8)$ is not invertible, then V_{j_8} can not be a summand of $V_{j_3} \otimes V_{j_4}$, and so both sides of (23) are equal to 0. We assume from now on that $\mathsf{d}(j_7)$ and $\mathsf{d}(j_8)$ are invertible in K.

The Pentagon Axiom for the tensor multiplication in $\mathcal{C}$ implies that

$$\sum_{j\in J} \left(I_{j_0}^{j_1j_7} \otimes \begin{Bmatrix} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{Bmatrix} \right) \left(\begin{Bmatrix} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{Bmatrix} \otimes I_{j_2j_3}^{j} \right) \left(I_{j_6j_4}^{j_0} \otimes \begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{Bmatrix} \right)$$
$$= \left(\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{Bmatrix} \otimes I_{j_8}^{j_3j_4} \right) P_{23} \left(\begin{Bmatrix} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{Bmatrix} \otimes I_{j_5}^{j_1j_2} \right) \qquad (24)$$

where I_i^{jk} is the identity automorphism of H_i^{jk}, I_{jk}^i is the identity automorphism of H_{jk}^i, and P_{23} is the permutation of the second and third factors in the tensor product (cf. Theorem VI.1.5.1 of [18]). Both sides of (24) are homomorphisms

$$H_{j_0}^{j_6j_4} \otimes H_{j_6}^{j_5j_3} \otimes H_{j_5}^{j_1j_2} \to H_{j_0}^{j_1j_7} \otimes H_{j_7}^{j_2j_8} \otimes H_{j_8}^{j_3j_4}.$$

We can rewrite all $6j$-symbols in (24) in the form (18). Note that the homomorphism (18) carries any $x \in H(i, j, k^*) \otimes_K H(k, l, m^*)$ to

$$*_{ijk^*} *_{klm^*} \left(\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\sigma} \otimes x \right) = \mathsf{d}(n) *_{ijk^*} *_{klm^*} \left(\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \otimes x \right),$$

where we suppose that the tuple (i, j, k, l, m, n) is strongly good and consider both $6j$-symbols as vectors in the module (11).

Denote by C_j the tensor product of the three $6j$-symbols in the j-th term of the sum on the left hand side of (23). Denote by D the tensor products of the two $6j$-symbols on the right hand side of (23). We can rewrite (24) as

$$\sum_{j\in J} \mathsf{d}(j_8)\,\mathsf{d}(j_7)\,\mathsf{d}(j)\, *_{j_2 j_3 j^*} *_{j j_4 j_7^*} *_{j_1 j j_6^*} *_{j_6 j_4 j_0^*} *_{j_1 j_2 j_5^*} *_{j_5 j_3 j_6^*}(C_j\otimes x)$$
$$= \mathsf{d}(j_8)\,\mathsf{d}(j_7) *_{j_1 j_2 j_5^*} *_{j_5 j_8 j_0^*} *_{j_5 j_3 j_6^*} *_{j_6 j_4 j_0^*}(D\otimes x)$$

for all $x \in H(j_6, j_4, j_0^*) \otimes H(j_5, j_3, j_6^*) \otimes H(j_1, j_2, j_5^*)$. Since $\mathsf{d}(j_7)$ and $\mathsf{d}(j_8)$ are invertible elements of K, the previous equality implies that

$$\sum_{j\in J} \mathsf{d}(j) *_{j_2 j_3 j^*} *_{j j_4 j_7^*} *_{j_1 j j_6^*} (C_j) = *_{j_5 j_8 j_0^*}(D)\,.$$

This proves the theorem. □

Theorem 9 (The orthonormality relation)**.** *Let i, j, k, l, m, p be elements of I such that $k, m \in I_0$ and the pairs (i, j), (j, l), (p, l), (k, l) are good. Set*

$$N = \{n \in I \mid H^{jl}_n \neq 0 \text{ and } H^{in}_m \neq 0\} \subset I_0\,.$$

If the pair (i, n) is good for all $n \in N$, then both 6-tuples defining the $6j$-symbols in the following formula are good (in fact, the first one is strongly good) and

$$\mathsf{d}(k)\sum_{n\in N}\mathsf{d}(n) *_{inm^*} *_{jln^*}\left(\begin{vmatrix} i & j & p\\ l & m & n\end{vmatrix}\otimes\begin{vmatrix} k & j^* & i\\ n & m & l\end{vmatrix}\right)$$
$$= \delta_{k,p}\,\mathrm{Id}(i, j, k^*)\otimes \mathrm{Id}(k, l, m^*),$$

where $\delta_{k,p}$ is the Kronecker symbol and $\mathrm{Id}(a, b, c)$ is the canonical element of $H(a, b, c) \otimes_K H(c^*, b^*, a^*)$ determined by the duality pairing.

Proof. Consider any $i, j, l, m \in I$ such that the pairs (i, j) and (j, l) are good and consider the associated isomorphism (16). Restricting the inverse isomorphism to the summand in the source corresponding to $n \in I$ and projecting into the summand in the target corresponding to $k \in I$ we obtain a homomorphism

$$\begin{Bmatrix} i & j & k\\ l & m & n\end{Bmatrix}_{\mathrm{inv}} : H^{jl}_n \otimes_K H^{in}_m \to H^{ij}_k \otimes_K H^{kl}_m. \tag{25}$$

These homomorphisms corresponding to fixed i, j, l, m and various $k, n \in I$ form a block-matrix of the isomorphism inverse to (16). Therefore,

$$\sum_{n\in I}\begin{Bmatrix} i & j & k\\ l & m & n\end{Bmatrix}_{\mathrm{inv}}\circ\begin{Bmatrix} i & j & p\\ l & m & n\end{Bmatrix} = \delta^k_p(i, j, l, m) \tag{26}$$

where $\delta^k_p(i, j, l, m)$ is zero if $k \neq p$ and is the identity automorphism of $H^{ij}_k \otimes_K H^{kl}_m$ if $k = p$.

Switching to the symmetrized multiplicity modules as in Section 5, we can rewrite (25) as a homomorphism

$$H(j, l, n^*) \otimes_K H(i, n, m^*) \to H(i, j, k^*) \otimes_K H(k, l, m^*). \tag{27}$$

Since (i,j) is good, $H(i,j,k^*)^\star = H(k,j^*,i^*)$. Assuming that (k,l) is good, we can write $H(k,l,m^*)^\star = H(m,l^*,k^*)$. Consider the homomorphism

$$\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}^\sigma_{\mathrm{inv}} : H(j,l,n^*) \otimes_K H(i,n,m^*) \otimes_K H(k,j^*,i^*) \otimes_K H(m,l^*,k^*) \to K$$

adjoint to (27). This homomorphism has the same source module as the $6j$-symbol $\left|\begin{matrix} k & j^* & i \\ n & m & l \end{matrix}\right|$, where we suppose that $k \in I_0$.

The argument similar to that in Lemma 7 (using the graphs Γ_4 and Γ_5 in Figure 7) shows that if the pairs $(i,j),(j,l),(k,l)$ are good and $k,m \in I_0$, then

$$\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}^\sigma_{\mathrm{inv}} = \mathsf{d}(k) \left|\begin{matrix} k & j^* & i \\ n & m & l \end{matrix}\right| . \tag{28}$$

Assuming additionally that the pair (i,n) is good, we can view both $6j$-symbols in (28) as vectors in

$$H(n,l^*,j^*) \otimes_K H(m,n^*,i^*) \otimes_K H(i,j,k^*) \otimes_K H(k,l,m^*) .$$

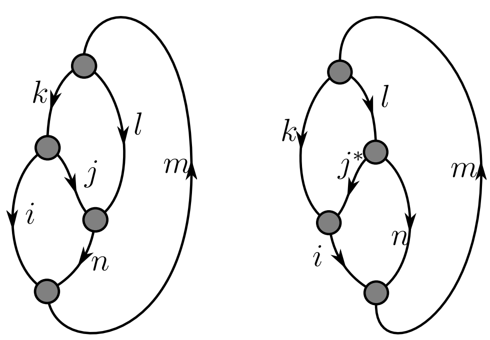
Then the homomorphism (27) carries any $y \in H(j,l,n^*) \otimes_K H(i,n,m^*)$ to

$$*_{jln^*} *_{inm^*} \left(\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}^\sigma_{\mathrm{inv}} \otimes y \right) = \mathsf{d}(k) *_{jln^*} *_{inm^*} \left(\left|\begin{matrix} k & j^* & i \\ n & m & l \end{matrix}\right| \otimes y \right) .$$

Now we can rewrite (26) as

$$\sum_{n \in N} \mathsf{d}(n)\, \mathsf{d}(k) *_{jln^*} *_{inm^*} *_{ijk^*} *_{klm^*} \left(\left|\begin{matrix} i & j & p \\ l & m & n \end{matrix}\right| \otimes \left|\begin{matrix} k & j^* & i \\ n & m & l \end{matrix}\right| \otimes x \right) = \delta_{k,p}\, x$$

for all $x \in H(i,j,k^*) \otimes_K H(k,l,m^*)$. This implies the claim of the theorem. □



(A) Γ_4 (B) Γ_5

Figure 7

7. Ambidextrous objects and standard $6j$-symbols

To apply the results of the previous sections, we must construct a pivotal Ab-category $\mathcal{C}$ with basic data $\{V_i, w_i\}_{i\in I}$ and a t-ambi pair (I_0, d). In this and the next sections, we give examples of such objects using the technique of ambidextrous objects introduced in [14]. We first briefly recall this technique.

7.1. **Ambidextrous objects.** Let $\mathcal{C}$ be a (strict) ribbon Ab-category, i.e., a (strict) pivotal tensor Ab-category with braiding and twist. We denote the braiding morphisms in $\mathcal{C}$ by $c_{V,W} : V \otimes W \to W \otimes V$ and the duality morphisms b_V, d_V, b'_V, d'_V as in Section 1. We assume that the ground ring K of $\mathcal{C}$ is a field.

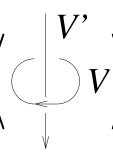
For an object J of $\mathcal{C}$ and an endomorphism f of $J \otimes J$, set

$$\mathrm{tr}_L(f) = (d_J \otimes \mathrm{Id}_J) \circ (\mathrm{Id}_{J^*} \otimes f) \circ (b'_J \otimes \mathrm{Id}_J) \in \mathrm{End}(J),$$

$$\mathrm{tr}_R(f) = (\mathrm{Id}_J \otimes d'_J) \circ (f \otimes \mathrm{Id}_{J^*}) \circ (\mathrm{Id}_J \otimes b_J) \in \mathrm{End}(J).$$

An object J of $\mathcal{C}$ is *ambidextrous* if $\mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for all $f \in \mathrm{End}(J \otimes J)$.

Let $Rib_{\mathcal{C}}$ be the category of $\mathcal{C}$-colored ribbon graphs and let $F : Rib_{\mathcal{C}} \to \mathcal{C}$ be the usual ribbon functor (see [18]). Let T_V be a $\mathcal{C}$-colored (1,1)-ribbon graph whose open string is oriented downward and colored with a simple object V of $\mathcal{C}$. Then $F(T_V) \in \mathrm{End}_{\mathcal{C}}(V) = K\,\mathrm{Id}_V$. Let $< T_V > \in K$ be such that $F(T_V) = < T_V >\ \mathrm{Id}_V$. For any objects V, V' of $\mathcal{C}$ such that V' is simple, set

$S'(V, V') = \Big\langle$ 

$\Big\rangle \in K\,.$

Fix basic data $\{V_i, w_i\}_{i\in I}$ in $\mathcal{C}$ and a simple ambidextrous object J of $\mathcal{C}$. Set

$$I_0 = I_0(J) = \{i \in I : S'(J, V_i) \neq 0\}\,. \tag{29}$$

For $i \in I_0$, set

$$\mathsf{d}_J(i) = \frac{S'(V_i, J)}{S'(J, V_i)} \in K\,. \tag{30}$$

We view $\mathsf{d}_J(i)$ as the modified quantum dimension of V_i determined by J.

Theorem 10 ([14]). *Let L be a $\mathcal{C}$-colored ribbon $(0,0)$-graph having an edge e colored with V_i where $i \in I_0$. Cutting L at e, we obtain a colored ribbon (1,1)-graph $T^e_{V_i} = T_{V_i}$ whose closure is L. Then the product $\mathsf{d}_J(i) < T_{V_i} > \in K$ is independent of the choice of e and yields an isotopy invariant of L.*

Corollary 11. *The pair $(I_0 = I_0(J), \mathsf{d}_J)$ is a t-ambi pair in $\mathcal{C}$.*

7.2. **Example: The standard quantum 6j-symbols.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let q be a primitive complex root of unity of order $2r$, where r is a positive integer. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo $\mathbb{C}$-algebra associated to $\mathfrak{g}$. This algebra is presented by the generators E_k, F_k, K_k, K_k^{-1}, where $k = 1, \ldots, m$ and the usual relations. Let $\widehat{U}_q(\mathfrak{g})$ be the Hopf algebra obtained as the quotient of $U_q(\mathfrak{g})$ by the two-sided ideal generated by $E_k^r, F_k^r, K_k^r - 1$ with $k = 1, \ldots, m$. It is known that $\widehat{U}_q(\mathfrak{g})$ is a finite dimensional ribbon Hopf algebra. The category of $\widehat{U}_q(\mathfrak{g})$-modules of finite complex dimension is a ribbon Ab-category. It gives rise to a ribbon Ab-category $\mathcal{C}$ obtained by annihilating all negligible morphisms (see Section XI.4 of [18]).

Let I be the set of weights belonging to the Weyl alcove determined by $\mathfrak{g}$ and r. For $i \in I$, denote by V_i the simple weight module with highest weight i. By [20], there is an involution $I \to I, i \to i^*$ and morphisms $\{w_i : V_i \to V_{i^*}^*\}_{i \in I}$ satisfying Equation (3). Then $\mathcal{C}$ is a modular category with basic data $\{V_i\}_{i \in I}$.

Let $J = \mathbb{C}$ be the unit object of $\mathcal{C}$. By Lemma 1 of [14], the object J is ambidextrous and by Corollary 11 the pair $(I_0 = I_0(J), \mathsf{d}_J)$ is a t-ambi pair. The general theory of [14] implies that $I_0 = I$ and d_J is the usual quantum dimension.

We can apply the techniques of Sections 4 and 5 to the basic data $\{V_i, w_i\}_{i \in I}$, and the t-ambi pair $(I = I_0(J), \mathsf{d}_J)$. It is easy to check that the corresponding modified 6j-symbols are the usual quantum 6j-symbols associated to $\mathfrak{g}$. Note that here all pairs $(i, j) \in I^2$ are good.

8. Example: Quantum 6j-symbols from $U_q^H(\mathfrak{sl}(2))$ at roots of unity

In this section we consider a category of modules over the quantization $\bar{U}_q^H(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ introduced in [14]. The usual quantum dimensions and the standard 6j-symbols associated to this category are generically zero. We equip this category with basic data and a t-ambi pair leading to non-trivial 6j-symbols.

Set $q = \mathrm{e}^{i\pi/r} \in \mathbb{C}$, where r is a positive integer. We use the notation q^x for $\mathrm{e}^{xki\pi/r}$, where $x \in \mathbb{C}$ or x is an endomorphism of a finite dimensional vector space.

Let $U_q(\mathfrak{sl}(2))$ be the standard quantization of $\mathfrak{sl}(2)$, i.e. the $\mathbb{C}$-algebra with generators E, F, K, K^{-1} and the following defining relations:

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{31}$$

This algebra is a Hopf algebra with coproduct Δ, counit ε, and antipode S defined by the formulas

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K, & \Delta(F) = K^{-1} \otimes F + F \otimes 1, & \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \\ \varepsilon(E) = \varepsilon(F) = 0, & \varepsilon(K) = \varepsilon(K^{-1}) = 1, & \\ S(E) = -EK^{-1}, & S(F) = -KF, & S(K) = K^{-1}. \end{array}$$

Let $U_q^H(\mathfrak{sl}(2))$ be the $\mathbb{C}$-algebra given by the generators E, F, K, K^{-1}, H, relations (31), and the following additional relations:

$$HK = KH, \qquad HK^{-1} = K^{-1}H, \qquad [H,E] = 2E, \qquad [H,F] = -2F.$$

The algebra $U_q^H(\mathfrak{sl}(2))$ is a Hopf algebra with coproduct Δ, counit ε, and antipode S defined as above on E, F, K, K^{-1} and defined on H by the formulas

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \varepsilon(H) = 0, \qquad S(H) = -H.$$

Following [14], we define $\bar{U}_q^H(\mathfrak{sl}(2))$ to be the quotient of $U_q^H(\mathfrak{sl}(2))$ by the relations $E^r = F^r = 0$. It is easy to check that the operations above turn $\bar{U}_q^H(\mathfrak{sl}(2))$ into a Hopf algebra.

Let V be a $\bar{U}_q^H(\mathfrak{sl}(2))$-module. An eigenvalue $\lambda \in \mathbb{C}$ of the operator $H : V \to V$ is called a *weight* of V and the associated eigenspace $E_\lambda(V)$ is called a *weight space*. We call V a *weight module* if V is finite-dimensional, splits as a direct sum of weight spaces, and $q^H = K$ as operators on V.

Given two weight modules V and W, the operator H acts as $H \otimes 1 + 1 \otimes H$ on $V \otimes W$. So, $E_\lambda(V) \otimes E_\mu(W) \subset E_{\lambda+\mu}(V \otimes W)$ for all $\lambda, \mu \in \mathbb{C}$. Moreover, $q^{\Delta(H)} = \Delta(K)$ as operators on $V \otimes W$. Thus, $V \otimes W$ is a weight module.

Let $\mathcal{C}^H$ be the tensor Ab-category of weight $\bar{U}_q^H(\mathfrak{sl}(2))$-modules. By Section 6.2 of [14], $\mathcal{C}^H$ is a ribbon Ab-category with ground ring $\mathbb{C}$. In particular, for any object V in $\mathcal{C}^H$, the dual object and the duality morphisms are defined as follows: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$\begin{aligned}
&b_V : \mathbb{C} \to V \otimes V^* \text{ is given by } 1 \mapsto \sum v_j \otimes v_j^*,\\
&d_V : V^* \otimes V \to \mathbb{C} \text{ is given by } f \otimes w \mapsto f(w),\\
&d'_V : V \otimes V^* \to \mathbb{C} \text{ is given by } v \otimes f \mapsto f(K^{1-r}v),\\
&b'_V : \mathbb{C} \to V^* \otimes V \text{ is given by } 1 \mapsto \sum v_j^* \otimes K^{r-1}v_j,
\end{aligned} \tag{32}$$

where $\{v_j\}$ is a basis of V and $\{v_j^*\}$ is the dual basis of V^*.

For an isomorphism classification of simple weight $U_q(\mathfrak{sl}(2))$-modules (i.e., modules on which K acts diagonally), see for example [16], Chapter VI. This classification implies that simple weight $\bar{U}_q^H(\mathfrak{sl}(2))$-modules are classified up to isomorphism by highest weights. For $i \in \mathbb{C}$, we denote by V_i the simple weight $\bar{U}_q^H(\mathfrak{sl}(2))$-module of highest weight $i + r - 1$. This notation differs from the standard labeling of highest weight modules. Note that $V_{-r+1} = \mathbb{C}$ is the trivial module and V_0 is the so called Kashaev module.

We now define basic data in $\mathcal{C}^H$. Set $I = \mathbb{C}$ and define an involution $i \mapsto i^*$ on I by $i^* = i$ if $i \in \mathbb{Z}$ and $i^* = -i$ if $i \in \mathbb{C} \setminus \mathbb{Z}$.

Lemma 12. *There are isomorphisms $\{w_i : V_i \to (V_{i^*})^*\}_{i \in I}$ satisfying* (3).

Proof. If $i \in \mathbb{C} \setminus \mathbb{Z}$, then $V_i^* \cong V_{-i}$, see [14] . We take an arbitrary isomorphism $V_i \to (V_{i^*})^* = V_{-i}^*$ for w_i and choose w_{i^*} so that it satisfies (3). If $i \in \mathbb{Z}$, then $i = i^*$ and there is a unique integer d such that $1 \le d \le r$ and $d \equiv i \pmod r$. Let v_0 be a highest weight vector of V_i. Set $v_j = F^j v_0$ for $j = 1, \dots, d-1$. Then $\{v_0, \dots, v_{d-1}\}$ is a basis of V_i and in particular, $\dim V_i = d$. Let $\{v_j^*\}$ be the basis of V_i^* dual to $\{v_j\}$ so that $v_j^*(v_k) = \delta_{j,k}$. Define an isomorphism $w_i : V_i \to (V_i)^*$ by $v_0 \mapsto v_{d-1}^*$. To verify (3), it suffices to check that the morphism $(w_i^{-1})^* w_i : V_i \to V_i^{**} = V_i$ is multiplication by K^{1-r}. To do this it is enough to calculate the image of v_0. A direct calculation shows that

$$S(F)^{d-1} v_0 = (-1)^{d-1} q^{-(d-1)} v_{d-1} = q^{r(d-1)-(d-1)} v_{d-1}.$$

It follows that $F^{d-1} v_{d-1}^* = q^{(r-1)(d-1)} v_0^*$. Thus for $k = 0, \dots, d-1$,

$$(w_i^{-1})^*(v_{d-1}^*)(v_k^*) = v_{d-1}^*(w_i^{-1}(v_k^*)) = \delta_{k,0}\, q^{-(r-1)(d-1)}.$$

In other words, $(w_i^{-1})^*(v_{d-1}^*) = q^{-(r-1)(d-1)} v_0^{**}$. Under the standard identification $V_i^{**} = V_i$, we obtain

$$(w_i^{-1})^* w_i(v_0) = (w_i^{-1})^*(v_{d-1}^*) = q^{-(r-1)(d-1)} v_0.$$

The proof is completed by noticing that $K^{1-r} v_0 = q^{-(r-1)(d-1)} v_0$. □

Set $I_0 = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z} \subset I = \mathbb{C}$. We say that a simple weight $\bar{U}_q^H(\mathfrak{sl}(2))$-module V_i is *typical* if $i \in I_0$. By [14], every typical module $J = V_i$ is ambidextrous and the set $I_0(J)$ defined by (29) is equal to I_0. Formula (30) defines a function $\mathsf{d}_J : I_0 \to \mathbb{C}$. As shown in [14], up to multiplication by a non-zero complex number, d_J is equal to the function $\mathsf{d} : I_0 \to \mathbb{C}$ defined by

$$\mathsf{d}(k) = \frac{1}{\prod_{j=0}^{r-2} \{k-j-1\}} \quad \text{for} \quad k \in I_0 \tag{33}$$

(here $\{a\} = q^a - q^{-a}$ for all $a \in \mathbb{C}$). Note in particular that

$$\mathsf{d}(k) = \mathsf{d}(k+2r) \tag{34}$$

for all $k \in I_0$. Theorem 10 implies the following lemma.

Lemma 13. (I_0, d) *is a t-ambi pair in* $\mathcal{C}^H$.

We can apply the techniques of Sections 4 and 5 to the category $\mathcal{C}^H$ with basic data $\{V_i, w_i\}_{i \in I}$ and the t-ambi pair (I_0, d). This yields a 6j-symbol $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ for all indices $i, j, k, l, m, n \in I = \mathbb{C}$ with at least one of them in I_0. In the case $i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z}$ this 6j-symbol is identified with a complex number: for any numbers $i, j, k \in \mathbb{C} \setminus \mathbb{Z}$ we have that $\dim H(i,j,k) = 0$ or $\dim H(i,j,k) = 1$. In the last case, there is a natural choice of an isomorphism $H(i,j,k) \simeq \mathbb{C}$. Hence the 6$j$-symbols can be viewed as taking value in $\mathbb{C}$ and can be effectively computed via certain recurrence relations forming a "skein calculus" for $\mathcal{C}^H$-colored ribbon trivalent graphs. These relations and the resulting explicit formulas for the $6j$-symbols are discussed in [12]. We formulate here one computation.

Example 14. *For $r = 3$ and $q = e^{i\pi/3}$ we can compute $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ for all $i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z}$ as follows.*

(1) *If $k = i + j + 2$, $l = n - j - 2$, and $m = n + i + 2$, then*
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \{n+1\}\{n+2\}.$$

(2) *If $k = i + j$, $l = n - j - 2$, and $m = n + i$, then*
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \{j+2\}\{n+1\}.$$

(3) *If $k = i + j$, $l = n - j$, and $m = n + i$, then*
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = -\left(q^{i+j-n} + q^{i-j+n} + q^{-i+j+n} + q^{i-j-n} + q^{-i+j-n} + q^{-i-j+n}\right).$$

(4) *For tuples $i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z}$ obtained from the tuples in (1), (2), (3) by tetrahedral permutations, the value of the $6j$-symbol is computed by (1), (2), (3). In all other cases*
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = 0.$$

Note finally that in the present setting the admissibility condition of Section 5 is generically satisfied.

9. Example: Quantum $6j$-symbols from $U_q(\mathfrak{sl}(2))$ at roots of unity

In this section we construct a category of $U_q(\mathfrak{sl}(2))$-modules which is pivotal but *a priori* not ribbon. We use the results of Section 8 to provide this category with basic data and a t-ambi pair.

Let r, q, and $U_q(\mathfrak{sl}(2))$ be as in Section 8. Assume additionally that r is odd and set $r' = \frac{r-1}{2}$. Let $\bar{U}_q(\mathfrak{sl}(2))$ be the Hopf algebra obtained as the quotient of $U_q(\mathfrak{sl}(2))$ by the relations $E^r = F^r = 0$.

A $\bar{U}_q(\mathfrak{sl}(2))$-module V is a *weight module* if V is finite dimensional and K act diagonally on V. We say that $v \in V$ is a *weight vector* with *weight* $\tilde{\lambda} \in \mathbb{C}/2r\mathbb{Z}$ if $Kv = q^{\tilde{\lambda}}v$. Let $\mathcal{C}$ be the tensor Ab-category of weight $\bar{U}_q(\mathfrak{sl}(2))$-modules (the ground ring is $\mathbb{C}$).

Lemma 15. *The category $\mathcal{C}$ is a pivotal Ab-category with duality morphisms* (32).

Proof. It is a straightforward calculation to show that the left and right duality are compatible and define a pivotal structure on $\mathcal{C}$. □

If V is a weight $\bar{U}_q(\mathfrak{sl}(2))$-module containing a weight vector v of weight $\tilde{\lambda}$ such that $Ev = 0$ and $V = U_q(\mathfrak{sl}(2))v$, then V is a *highest weight module* with *highest weight* $\tilde{\lambda}$. The classification of simple weight $U_q(\mathfrak{sl}(2))$-modules (see Chapter VI of [16]) implies that highest weight $\bar{U}_q(\mathfrak{sl}(2))$-modules are classified up to isomorphism by their highest weights. For any $\tilde{i} \in \mathbb{C}/2r\mathbb{Z}$, we denote by $V_{\tilde{i}}$ the highest weight module with highest weight $\tilde{i} + \widetilde{r-1}$, where $\widetilde{r-1} \in \mathbb{C}/2r\mathbb{Z}$ is the projection of $r-1$ to $\mathbb{C}/2r\mathbb{Z}$. The following lemma yields basic data in $\mathcal{C}$.

Lemma 16. *Set $I^{\mathcal{C}} = (\mathbb{C} \setminus \mathbb{Z})/2r\mathbb{Z}$ and define an involution $\tilde{i} \mapsto \tilde{i}^*$ on $I^{\mathcal{C}}$ by $\tilde{i}^* = -\tilde{i}$. Then there are isomorphisms $\{w_{\tilde{i}} : V_{\tilde{i}} \to (V_{\tilde{i}^*})^*\}_{\tilde{i} \in I^{\mathcal{C}}}$ satisfying Equation* (3)*.*

Proof. The lemma follows as in the proof of Lemma 12. □

Theorem 17. *Let $\mathsf{d} : I^{\mathcal{C}} \to \mathbb{C}$ be the function*

$$\mathsf{d}(\tilde{k}) = \frac{1}{\prod_{j=0}^{r-2} \left\{ \tilde{k} - \tilde{1} - \tilde{j} \right\}} \tag{35}$$

for any $\tilde{k} \in I^{\mathcal{C}}$. Then the pair $(I_0 = I^{\mathcal{C}}, \mathsf{d})$ is t-ambi in $\mathcal{C}$.

A proof of Theorem 17 will be given at the end of the section using the following preliminaries. Recall the category $\mathcal{C}^H = \bar{U}_q^H(\mathfrak{sl}(2))$-mod from Section 8. Let $\varphi : \mathcal{C}^H \to \mathcal{C}$ be the functor which forgets the action of H. Consider the functors $G_{\mathcal{C}}$ and $G_{\mathcal{C}^H}$ associated to $\mathcal{C}$ and $\mathcal{C}^H$, respectively (see Section 2). The functor $\varphi : \mathcal{C}^H \to \mathcal{C}$ induces (in the obvious way) a functor $\varphi_{\mathrm{Gr}} : \mathrm{Gr}_{\mathcal{C}^H} \to \mathrm{Gr}_{\mathcal{C}}$ such that the following square diagram commutes:

$$\begin{array}{ccc} \mathrm{Gr}_{\mathcal{C}^H} & \xrightarrow{\varphi_{\mathrm{Gr}}} & \mathrm{Gr}_{\mathcal{C}} \\ {\scriptstyle G_{\mathcal{C}^H}} \downarrow & & \downarrow {\scriptstyle G_{\mathcal{C}}} \\ \mathcal{C}^H & \xrightarrow[\varphi]{} & \mathcal{C} \end{array} \tag{36}$$

We say that the highest weight $\bar{U}_q(\mathfrak{sl}(2))$-module $V_{\tilde{i}}$ is *typical* if $\tilde{i} \in I^{\mathcal{C}} \cup \{\tilde{0}, \tilde{r}\}$. In other words, $V_{\tilde{i}}$ is typical if its highest weight is in $I^{\mathcal{C}} \cup \{\widetilde{-1}, \widetilde{2r'}\} \subset \mathbb{C}/2r\mathbb{Z}$. A typical module $V_{\tilde{i}}$ has dimension r and its weight vectors have the weights $\tilde{i} + 2\tilde{k}$ where $k = -r', -r'+1, \ldots, r'$. It can be shown that if a typical module V is a submodule of a finite dimensional $\bar{U}_q(\mathfrak{sl}(2))$-module W, then V is a direct summand of W. Therefore, if $V_{\tilde{i}}$ and $V_{\tilde{j}}$ are typical modules such that $\tilde{i} + \tilde{j} \notin \mathbb{Z}/2r\mathbb{Z}$, then

$$V_{\tilde{i}} \otimes V_{\tilde{j}} \cong \bigoplus_{k=-r'}^{r'} V_{\tilde{i}+\tilde{j}+2\tilde{k}}. \tag{37}$$

As in Section 8, for any $i \in \mathbb{C}$, we denote by V_i a weight module in $\mathcal{C}^H$ of highest weight $i + r - 1$. Clearly, $\varphi(V_i) \simeq V_{\tilde{i}}$ for all $i \in \mathbb{C}$, where $\tilde{i} = i \,(\mathrm{mod}\, 2r\mathbb{Z})$ and $\simeq$

denotes isomorphism in $\mathcal{C}$. To specify an isomorphism $\varphi(V_i) \simeq V_{\tilde{i}}$, we fix for each $i \in \mathbb{C}$ a highest weight vector in $V_i \in \mathcal{C}^H$ and we fix for each $a \in \mathbb{C}/2r\mathbb{Z}$ a highest weight vector in $V_a \in \mathcal{C}$. Then for every $i \in \mathbb{C}$, there is a canonical isomorphism $\varphi(V_i) \to V_{\tilde{i}}$ in $\mathcal{C}$ carrying the highest weight vector of V_i into the highest weight vector of $V_{\tilde{i}}$. To simplify notation, we write in the sequel $\varphi(V_i) = V_{\tilde{i}}$ for all $i \in \mathbb{C}$.

We say that a triple $(i, j, k) \in \mathbb{C}^3$ has *height* $i + j + k \in \mathbb{C}$. For $i, j, k \in \mathbb{C}$, set

$$H^{ijk} = \mathrm{Hom}_{\mathcal{C}^H}(\mathbb{I}, V_i \otimes V_j \otimes V_k) \quad \text{and} \quad H_k^{ij} = \mathrm{Hom}_{\mathcal{C}^H}(V_k, V_i \otimes V_j).$$

Lemma 18. *Let $i, j, k \in \mathbb{C} \setminus \mathbb{Z}$. If the height of (i, j, k) does not belong to the set $\{-2r', -2r' + 2, \ldots, 2r'\}$, then $H^{ijk} = 0$. If the height of (i, j, k) belongs to the set $\{-2r', -2r' + 2, \ldots, 2r'\}$, then the composition of the homomorphisms*

$$\begin{aligned} H^{ijk} = \mathrm{Hom}_{\mathcal{C}^H}(\mathbb{I}, V_i \otimes V_j \otimes V_k) &\xrightarrow{\varphi} \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, \varphi(V_i) \otimes \varphi(V_j) \otimes \varphi(V_k)) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_{\tilde{i}} \otimes V_{\tilde{j}} \otimes V_{\tilde{k}}) \end{aligned}$$

is an isomorphism.

Proof. We begin with a simple observation. Since $i \in \mathbb{C} \setminus \mathbb{Z}$, the character formula for V_i is $\sum_{l=-r'}^{r'} u^{i+2l}$ where the coefficient of u^a is the dimension of the a-weight space. Therefore, the character formula for $V_i \otimes V_j$ is

$$\left(\sum_{l=-r'}^{r'} u^{i+2l} \right) \left(\sum_{m=-r'}^{r'} u^{j+2m} \right) = \sum_{l,m=-r'}^{r'} u^{i+j+2l+2m}. \tag{38}$$

As we know, Formula (9) defines an isomorphism $H^{ijk} \simeq H^{ij}_{-k}$. We claim that $\dim(H^{ij}_{-k}) = 1$ if the height of (i, j, k) belongs to $\{-2r', -2r' + 2, \ldots, 2r'\}$ and $H^{ij}_{-k} = 0$, otherwise. To see this, we consider two cases.

Case 1: $i + j \in \mathbb{Z}$. Since $k \notin \mathbb{Z}$, the height of (i, j, k) is not an integer and does not belong to the set $\{-2r', -2r' + 2, \ldots, 2r'\}$. Equation (38) implies that all the weights of $V_i \otimes V_j$ are integers. Since $k \notin \mathbb{Z}$, we have $H^{ij}_{-k} = 0$.

Case 2: $i + j \notin \mathbb{Z}$. It can be shown that if $V \in \mathcal{C}^H$ is a typical module which is a sub-module of a module $W \in \mathcal{C}^H$, then V is a direct summand of W. Combining with Equation (38) we obtain that

$$V_i \otimes V_j \simeq \bigoplus_{l=-r'}^{r'} V_{i+j+2l}. \tag{39}$$

Therefore $H^{ij}_{-k} \neq 0$ if and only if $i + j + k \in \{-2r', -2r' + 2, \ldots, 2r'\}$. In addition, Formula (39) implies that if $H^{ij}_{-k} \neq 0$ then $\dim(H^{ij}_{-k}) = 1$. This proves the claim above and the first statement of the lemma.

To prove the second part of the lemma, assume that the height of (i, j, k) is in $\{-2r', -2r' + 2, \ldots, 2r'\}$. It is enough to show that the homomorphism

$$\mathrm{Hom}_{\mathcal{C}^H}(\mathbb{I}, V_i \otimes V_j \otimes V_k) \xrightarrow{\varphi} \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, \varphi(V_i) \otimes \varphi(V_j) \otimes \varphi(V_k))$$

is an isomorphism. This homomorphism is injective by the very definition of the forgetful functor. So it suffices to show that the domain and the range have the same dimension. From the claim above, $\dim(H^{ij}_{-k}) = 1$ and thus the domain is one-dimensional. The assumption on the height of (i, j, k) combined with Equation (37) implies that $\dim \operatorname{Hom}_{\mathcal{C}}(\varphi(V_{-k}), \varphi(V_i) \otimes \varphi(V_j)) = 1$. Thus, the range is also one-dimensional. This completes the proof of the lemma. □

We say that a triple $\tilde{i}, \tilde{j}, \tilde{k} \in \mathbb{C}/2r\mathbb{Z}$ has *integral height* if $\tilde{i} + \tilde{j} + \tilde{k} \in \mathbb{Z}/2r\mathbb{Z}$ and $\tilde{i} + \tilde{j} + \tilde{k}$ is even. In this case the *height* of the triple $(\tilde{i}, \tilde{j}, \tilde{k})$ is the unique (even) $n \in \{-2r', -2r' + 2, \ldots, 2r'\}$ such that $\tilde{i} + \tilde{j} + \tilde{k} = n \,(\operatorname{mod} 2r)$.

By an $I^{\mathcal{C}}$-colored ribbon graph we mean a $\mathcal{C}$-colored ribbon graphs such that all edges are colored with modules $V_{\tilde{i}}$ with $\tilde{i} \in I^{\mathcal{C}} = (\mathbb{C} \setminus \mathbb{Z})/2r\mathbb{Z}$. To each $I^{\mathcal{C}}$-colored trivalent coupon in S^2 we assign a triple of elements of $I^{\mathcal{C}}$: each arrow attached to the coupon contributes a term $\varepsilon\tilde{i} \in I^{\mathcal{C}}$ where $V_{\tilde{i}}$ is the color of the arrow and $\varepsilon = +1$ if the arrow is oriented towards the coupon, and $\varepsilon = -1$. Such a coupon has *integral height* if the associated triple has integral height. We say that a $I^{\mathcal{C}}$-colored trivalent ribbon graph in S^2 has *integral heights* if all its coupons have integral height. The *total height* of an $I^{\mathcal{C}}$-colored trivalent ribbon graph with integral heights is defined to be the sum of the heights of all its coupons. This *total height* is an even integer.

Lemma 19. *Let T be a $I^{\mathcal{C}}$-colored trivalent ribbon graph in S^2. If T has a coupon which does not have integral height, then $G_{\mathcal{C}}(T_V) = 0$ for any cutting presentation T_V of T.*

Proof. Let $(\tilde{i}, \tilde{j}, \tilde{k})$ be the triple associated with a coupon of T which does not have integral height. An argument similar to the one in the proof of Lemma 18 shows that $\operatorname{Hom}_{\mathcal{C}}(\mathbb{I}, V_{\tilde{i}} \otimes V_{\tilde{j}} \otimes V_{\tilde{k}}) = 0$. This forces $G_{\mathcal{C}}(T_V) = 0$ for any cutting presentation T_V of T. □

Lemma 20. *Let T be a connected $I^{\mathcal{C}}$-colored trivalent ribbon graph in S^2 with no inputs and outputs and with integral heights. There exists a $\mathcal{C}^H$-colored trivalent ribbon graph $\tilde{T}$ such that $\varphi_{\mathrm{Gr}}(\tilde{T}) = T$ if and only if the total height of T is zero. If the total height of T is non-zero, then $G_{\mathcal{C}}(T_V) = 0$ for any cutting presentation T_V of T.*

Proof. Given a CW-complex space X and an abelian group G, we denote by $C_n(X; G)$ the abelian group of cellular n-chains of X with coefficients in G.

Consider a connected $I^{\mathcal{C}}$-colored trivalent ribbon graph S with integral heights and possibly with inputs and outputs. Let $|S|$ be the underlying 1-dimensional CW-complex of S and $|\partial S| \subset |S|$ be the set of univalent vertices (corresponding to the inputs and outputs of S). The coloring of S determines a 1-chain $\tilde{c} \in$

$C_1(|S|;\mathbb{C}/2r\mathbb{Z})$. Consider also the 0-chain

$$c_\partial = \sum_{x\in|\partial S|} C_x x \in C_0(|\partial S|;\mathbb{C}/2r\mathbb{Z}),$$

where $C_x \in \mathbb{C}/2r\mathbb{Z}$ is the label of the only edge of S adjacent to x if this edge is oriented towards x and minus this label otherwise. The heights of the coupons of S form a 0-chain $w \in C_0(|S| \setminus |\partial S|; 2\mathbb{Z})$. Clearly,

$$\partial\tilde{c} = c_\partial + w \,(\mathrm{mod}\, 2r\mathbb{Z}). \tag{40}$$

Suppose that the 0-chain c_∂ lifts to a certain 0-chain $b \in C_0(|\partial S|;\mathbb{C})$ such that $[b+w] = 0 \in H_0(|S|;\mathbb{C}) = \mathbb{C}$. We claim that then the 1-chain $\tilde{c}$ lifts to a 1-chain $c \in C_1(|S|,\mathbb{C})$ such that $\partial c = b + w$. Indeed, pick any $c' \in C_1(|S|;\mathbb{C})$ such that $\tilde{c} = c' \,(\mathrm{mod}\, 2r\mathbb{Z})$. Set $x = \partial c' - b - w \in C_0(|S|;\mathbb{C})$. By (40), $x \in C_0(|S|;2r\mathbb{Z})$. Clearly, $[x] = -[b+w] = 0 \in H_0(|S|;2r\mathbb{Z})$. Then $x = \partial\delta$ for some $\delta \in C_1(|S|;2r\mathbb{Z})$ and $c = c' - \delta$ satisfies the required conditions.

Let us now prove the first statement of the lemma. By assumption, $|\partial T| = \emptyset$. As above, the coloring of T determines a 1-chain $\tilde{c} \in C_1(|T|;\mathbb{C}/2r\mathbb{Z})$. Let $w \in C_0(|T|;2\mathbb{Z})$ be the 0-chain formed by the heights of the coupons of T. The total height of T is $[w] \in H_0(|T|;2\mathbb{Z}) = 2\mathbb{Z}$. If $[w] = 0$, then the preceding argument (with $b = 0$) shows that $\tilde{c}$ lifts to a 1-chain $c \in C_1(|T|;\mathbb{C})$ such that $\partial c = w$. The chain c determines a coloring of all edges of T by the modules $\{V_i \mid i \in \mathbb{C}\}$. Clearly, the height of any coupon is equal to the corresponding value of w. In particular, all these heights belong to the set $\{-2r', -2r'+2, \ldots, 2r'\}$. By Lemma 18, this coloring of edges can be uniquely extended to a $\mathcal{C}^H$-coloring $\check{T}$ of our graph such that $\varphi_{\mathrm{Gr}}(\check{T}) = T$. Conversely, if $\varphi_{\mathrm{Gr}}(\check{T}) = T$ then with the same notation $w = \partial c$ and thus the total height of T is zero.

Suppose now that $[w] \neq 0$. Consider the cutting presentation T_V where $V = V_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \mathbb{C} \setminus \mathbb{Z} \,(\mathrm{mod}\, 2r\mathbb{Z})$ is the color of an edge of T. Here $|\partial T_V|$ is the two-point set formed by the extremities of T_V and c_∂ is the 0-chain $\tilde{\alpha} \times$ (the input vertex minus the output vertex). As

$$[w] = [\partial\tilde{c}] - [c_\partial] = 0 \in H_0(|T_V|;\mathbb{C}/2r\mathbb{Z}),$$

we have $[w] \in 2r\mathbb{Z}$. Pick $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ such that $\tilde{\alpha} = \alpha \,(\mathrm{mod}\, 2r\mathbb{Z})$. Consider the 0-chain

$$b = \alpha \times (\text{the input vertex minus the output vertex}) - [w] \times (\text{the output vertex}).$$

Clearly, $[w+b] = 0 \in H_0(|T_V|;\mathbb{C})$. The argument at the beginning of the proof and Lemma 18 imply that there is $\check{T} \in \mathrm{Hom}_{\mathrm{Gr}_{\mathcal{C}^H}}(V_\alpha, V_{\alpha+[w]})$ such that $\varphi_{\mathrm{Gr}}(\check{T}) = T_V$. But then $G_{\mathcal{C}^H}(\check{T}) = 0$ because it is a morphism between non-isomorphic simple modules. Therefore $G_{\mathcal{C}}(T_V) = G_{\mathcal{C}}(\varphi_{\mathrm{Gr}}(\check{T})) = \varphi(G_{\mathcal{C}^H}(\check{T})) = 0$. □

9.1. **Proof of Theorem 17.** Let $T_{V_{\tilde{i}}}$ and $T_{V_{\tilde{j}}}$ be cutting presentations of a connected $I^{\mathcal{C}}$-colored trivalent ribbon graph T in S^2 with no inputs and outputs. We claim that

$$\mathsf{d}(\tilde{i}) < G_{\mathcal{C}}(T_{V_{\tilde{i}}}) >= \mathsf{d}(\tilde{j}) < G_{\mathcal{C}}(T_{V_{\tilde{j}}}) > . \tag{41}$$

By the previous two lemmas it is enough to consider the case where all coupons of T have integral height and the total height of T is zero. By Lemma 20, there is a $\mathcal{C}^H$-colored trivalent ribbon graph $\check{T}$ such that $\varphi_{\mathrm{Gr}}(\check{T}) = T$. Cutting $\check{T}$ at the same edges, we obtain $\mathcal{C}^H$-colored graphs $\check{T}_{V_i}$ and $\check{T}_{V_j}$ carried by φ_{Gr} to $T_{V_{\tilde{i}}}$ and $T_{V_{\tilde{j}}}$, respectively. From Diagram (36) we have $< G_{\mathcal{C}^H}(\check{T}_{V_i}) >=< G_{\mathcal{C}}(T_{V_{\tilde{i}}}) >$ and $< G_{\mathcal{C}^H}(\check{T}_{V_j}) >=< G_{\mathcal{C}}(T_{V_{\tilde{j}}}) >$. Lemma 13 implies that $\mathsf{d}(i) < G_{\mathcal{C}^H}(\check{T}_{V_i}) >$ $= \mathsf{d}(j) < G_{\mathcal{C}^H}(\check{T}_{V_j}) >$. Since $\mathsf{d}(i) = \mathsf{d}(\tilde{i})$ and $\mathsf{d}(j) = \mathsf{d}(\tilde{j})$, these formulas imply Formula (41).

Remark 21. *The category $\mathcal{C}$ with basic data $\{V_{\tilde{i}}, w_{\tilde{i}}\}_{\tilde{i}\in I}$ and t-ambi pair $(I^{\mathcal{C}} = (\mathbb{C}\setminus\mathbb{Z})/2r\mathbb{Z}, \mathsf{d})$ determines modified 6j-symbols $\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix}$ for $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n} \in I^{\mathcal{C}}$. Using Lemma 18, one observes that as in Section 8, the 6j-symbols of this section can be viewed as taking values in $\mathbb{C}$. Moreover, these complex valued symbols preserve the tetrahedral symmetry of 6j-symbols (see* [12] *for explicit formulas). The values of these 6j-symbols are essentially the same as the values of the 6j-symbols derived from the category $\mathcal{C}^H$. More precisely, let $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n} \in I^{\mathcal{C}}$. If the total height of $\Gamma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ is non-zero, then $\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix} = 0$. If the total height of $\Gamma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ is zero, then for some lifts $i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z}$ of $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}$,*

$$\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix} = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_{\mathcal{C}^H}.$$

Example 14 computes these 6j-symboles for $r = 3$ and $q = \mathrm{e}^{i\pi/3}$.

10. Three-manifold invariants

In this section we derive a topological invariant of links in closed orientable 3-manifolds from a suitable pivotal tensor Ab-category. We also show that Sections 7.2 and 9 yield examples of such categories.

10.1. **Topological preliminaries.** Let M be a closed orientable 3-manifold and L a link in M. Following [4], we use the term *quasi-regular triangulation* of M for a decomposition of M as a union of embedded tetrahedra such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices, edges, and (2-dimensional) faces. Quasi-regular triangulations differ from usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Nevertheless, the edges of a quasi-regular triangulation have distinct ends. A *Hamiltonian link* in a quasi-regular triangulation $\mathcal{T}$ is a set $\mathcal{L}$ of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$. Then the union of the edges of $\mathcal{T}$ belonging to $\mathcal{L}$ is a link L in M. We call the pair $(\mathcal{T},\mathcal{L})$ an *H-triangulation* of (M,L).

Proposition 22 ([4], Proposition 4.20)**.** *Any pair (a closed connected orientable 3-manifold M, a non-empty link $L \subset M$) admits an H-triangulation.*

10.2. **Algebraic preliminaries.** Let $\mathcal{C}$ be a pivotal tensor Ab-category with ground ring K, basic data $\{V_i, w_i : V_i \to V_{i^*}^*\}_{i\in I}$, and t-ambi pair (I_0, d). As in Section 7.1, we assume that K is a field. To define the associated 3-manifold invariant we need the following requirements on $\mathcal{C}$. Fix an abelian group G. Suppose that $\mathcal{C}$ is *G-graded* in the sense that for all $g \in G$, we have a class $\mathcal{C}_g$ of object of $\mathcal{C}$ such that

(1) $\mathbb{I} \in \mathcal{C}_0$,
(2) if $V \in \mathcal{C}_g$, then $V^* \in \mathcal{C}_{-g}$,
(3) if $V \in \mathcal{C}_g$, $V' \in \mathcal{C}_{g'}$, then $V \otimes V' \in \mathcal{C}_{g+g'}$,
(4) if $V \in \mathcal{C}_g$, $V' \in \mathcal{C}_{g'}$, and $g \neq g'$, then $\mathrm{Hom}_{\mathcal{C}}(V, V') = \{0\}$.

We shall assume that G contains a set X with the following properties:

(1) X is symmetric: $-X = X$,
(2) G can not be covered by a finite number of translated copies of X, in other words, for any $g_1, \dots, g_n \in G$, we have $\bigcup_{i=1}^n (g_i + X) \neq G$,
(3) if $g \in G \setminus X$, then the set $I^g = \{i \in I \,|\, V_i \in \mathcal{C}_g\}$ is finite and every object of $\mathcal{C}_g$ is isomorphic to a direct sum of a finite family of objects $\{V_i \,|\, i \in I^g\}$,
(4) $I_0 = \cup_{g\in G\setminus X} I^g$ and $\mathsf{d}(I_0) \subset K^*$.

The third condition and the definition of a basic data imply that for $g \in G \setminus X$, every simple object of $\mathcal{C}_g$ is isomorphic to V_i for a unique $i \in I^g$. Note also that given $g_1, g_2 \in G \setminus X$ with $g_1 + g_2 \in G \setminus X$, any pair $(i \in I^{g_1}, j \in I^{g_2})$ is good in the sense of Section 4.

Finally, we need a map $\mathsf{b} : I_0 \to K$ such that

(1) $\mathsf{b}(i) = \mathsf{b}(i^*)$, for all $i \in I_0$,
(2) for any $g, g_1, g_2 \in G \setminus X$ with $g + g_1 + g_2 = 0$ and for all $j \in I^g$,

$$\mathsf{b}(j) = \sum_{j_1 \in I^{g_1},\, j_2 \in I^{g_2}} \mathsf{b}(j_1)\, \mathsf{b}(j_2) \dim(H(j, j_1, j_2)).$$

Denoting by b_g the formal sum $\mathsf{b}_g = \sum_{j\in I^g} \mathsf{b}(j) V_j$, one obtains $\mathsf{b}_{g_1} \otimes \mathsf{b}_{g_2} = \mathsf{b}_{g_1+g_2}$ whenever $g_1, g_2, g_1+g_2 \in G\setminus X$. The map $g \mapsto \mathsf{b}_g$ can be seen as a "representation" of $G \setminus X$ in the Grothendieck ring of $\mathcal{C}$.

We briefly describe a way to derive a map b as above from characters. A *character* is a map $\chi : I \to K$ satisfying

(1) $\chi(j^*) = \chi(j)$ for all $j \in I_0$,

(2) if $g_1, g_2, g = g_1 + g_2 \in G \setminus X$ and $j_1 \in I^{g_1}$, $j_2 \in I^{g_2}$ then

$$\chi(j_1)\chi(j_2) = \sum_{j \in I^g} \dim(H_j^{j_1 j_2})\chi(j),$$

(3) for any $g \in G \setminus X$, the element $\mathcal{D}_g = \sum_{j \in I^g} \chi(j)^2$ of K is non-zero.

Lemma 23. *If $\chi : I \to K$ is a character then the map $G \setminus X \to K$, $g \mapsto \mathcal{D}_g$ is a constant function with value $\mathcal{D}$. Moreover, the map $\mathsf{b} = \frac{1}{\mathcal{D}}\chi$ restricted to I_0 satisfies two properties listed above.*

Proof. If $g_1, g_2, g = g_1 + g_2 \in G \setminus X$ and $j_1 \in I^{g_1}$, then

$$\chi(j_1)\mathcal{D}_{g_2} = \sum_{j_2 \in I^{g_2}, j \in I^g} \dim(H_j^{j_1 j_2})\chi(j)\chi(j_2).$$

It follows that if $g_1, g_2, g_3 \in G \setminus X$ and $g_1 + g_2 + g_3 = 0$, then for all $j_1 \in I^{g_1}$,

$$\chi(j_1)\mathcal{D}_{g_2} = \sum_{j_2 \in I^{g_2}, j_3 \in I^{g_3}} \dim(H(j_1, j_2, j_3))\chi(j_2)\chi(j_3) = \chi(j_1)\mathcal{D}_{g_3}$$

Since $g_2 + g_3 = -g_1 \in G \setminus X$ property (3) of χ implies that $\chi(j_1) \neq 0$ for some $j_1 \in I^{g_1}$. Therefore, $\mathcal{D}_{g_2} = \mathcal{D}_{g_3}$. Finally, for any elements g_1, g_2 of $G \setminus X$, there exists $g \in G \setminus (X \cup (X - g_1) \cup (X - g_2))$. By the previous argument, $\mathcal{D}_{g_1} = \mathcal{D}_g = \mathcal{D}_{g_2}$. Now it is easy to see $\mathsf{b} = \frac{1}{\mathcal{D}}\chi$ has all the desired properties. □

The usual dimension $\dim_K$ in a category whose objects are finite dimensional vector spaces over K is a character. Then Lemma 23 yields a map b satisfying the requirements above: for $i \in I^g$,

$$\mathsf{b}(i) = \dim_K V_i / \sum_{j \in I^g} (\dim_K V_j)^2 \tag{42}$$

where g is an arbitrary element of $G \setminus X$.

10.3. **A state sum invariant.** We start from the algebraic data described in the previous subsection and produce a topological invariant of a triple (M, L, h), where M is a closed connected oriented 3-manifold, $L \subset M$ is a non-empty link, and $h \in H^1(M, G)$.

Let $(\mathcal{T}, \mathcal{L})$ be an H-triangulation of (M, L). By a *G-coloring* of $\mathcal{T}$, we mean a G-valued 1-cocycle Φ on $\mathcal{T}$, that is a map from the set of oriented edges of $\mathcal{T}$ to G such that

(1) the sum of the values of Φ on the oriented edges forming the boundary of any face of $\mathcal{T}$ is zero and
(2) $\Phi(-e) = -\Phi(e)$ for any oriented edge e of $\mathcal{T}$, where $-e$ is e with opposite orientation.

Each G-coloring Φ of $\mathcal{T}$ represents a cohomology class $[\Phi] \in H^1(M,G)$.

A *state* of a G-coloring Φ is a map φ assigning to every oriented edge e of $\mathcal{T}$ an element $\varphi(e)$ of $I^{\Phi(e)}$ such that $\varphi(-e) = \varphi(e)^*$ for all e. The set of all states of Φ is denoted $\mathrm{St}(\Phi)$. The identities $\mathsf{d}(\varphi(e)) = \mathsf{d}(\varphi(-e))$ and $\mathsf{b}(\varphi(e)) = \mathsf{b}(\varphi(-e))$ allow us to use the notation $\mathsf{d}(\varphi(e))$ and $\mathsf{b}(\varphi(e))$ for non-oriented edges.

We call a G-coloring of $(\mathcal{T},\mathcal{L})$ *admissible* if it takes values in $G \setminus X$. Given an admissible G-coloring Φ of $(\mathcal{T},\mathcal{L})$, we define a certain partition function (state sum) as follows. For each tetrahedron T of $\mathcal{T}$, we choose its vertices v_1, v_2, v_3, v_4 so that the (ordered) triple of oriented edges $(\overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4})$ is positively oriented with respect to the orientation of M. Here by $\overrightarrow{v_1v_2}$ we mean the edge oriented from v_1 to v_2, etc. For each $\varphi \in \mathrm{St}(\Phi)$, set

$$|T|_\varphi = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \text{ where } \begin{cases} i = \varphi(\overrightarrow{v_2v_1}), & j = \varphi(\overrightarrow{v_3v_2}), \quad k = \varphi(\overrightarrow{v_3v_1}), \\ l = \varphi(\overrightarrow{v_4v_3}), & m = \varphi(\overrightarrow{v_4v_1}), \quad n = \varphi(\overrightarrow{v_4v_2}). \end{cases}$$

This $6j$-symbol belongs to the tensor product of 4 multiplicity modules associated to the faces of T and does not depend on the choice of the numeration of the vertices of T compatible with the orientation of M. This follows from the tetrahedral symmetry of the (modified) $6j$-symbol discussed in Section 4. Note that any face of $\mathcal{T}$ belongs to exactly two tetrahedra of $\mathcal{T}$, and the associated multiplicity modules are dual to each other. The tensor product of the $6j$-symbols $|T|_\varphi$ associated to all tetrahedra T of $\mathcal{T}$ can be contracted using this duality. We denote by cntr the tensor product of all these contractions. Let $\mathcal{T}_1$ be the set of unoriented edges $\mathcal{T}$ and let $\mathcal{T}_3$ the set of tetrahedra of $\mathcal{T}$. Set

$$TV(\mathcal{T},\mathcal{L},\Phi) = \sum_{\varphi\in\mathrm{St}(\Phi)} \prod_{e\in\mathcal{T}_1\setminus\mathcal{L}} \mathsf{d}(\varphi(e)) \prod_{e\in\mathcal{L}} \mathsf{b}(\varphi(e)) \ \mathrm{cntr}\left(\bigotimes_{T\in\mathcal{T}_3} |T|_\varphi\right) \in K. \quad (43)$$

Theorem 24. *$TV(\mathcal{T},\mathcal{L},\Phi)$ depends only on the isotopy class of L in M and the cohomology class $[\Phi] \in H^1(M,G)$. It does not depend on the choice of the H-triangulation of (M,L) and on the choice of Φ in its cohomology class.*

A proof of this theorem will be given in the next section.

Lemma 25. *Any $h \in H^1(M,G)$ can be represented by an admissible G-coloring on an arbitrary quasi-regular triangulation $\mathcal{T}$ of M.*

Proof. Take any G-coloring Φ of $\mathcal{T}$ representing h. We say that a vertex v of $\mathcal{T}$ is *bad* for Φ if there is an oriented edge e in $\mathcal{T}$ outgoing from v such that $\Phi(e) \in X$. It is clear that Φ is admissible if and only if Φ has no bad vertices. We show how to modify Φ in its cohomology class to reduce the number of bad vertices. Let v be a bad vertex for Φ and let E_v be the set of all oriented edges of $\mathcal{T}$ outgoing from v. Pick any

$$g \in G \setminus \left(\bigcup_{e\in E_v} (\Phi(e) + X)\right).$$

Let c be the G-valued 0-cochain on $\mathcal{T}$ assigning g to v and 0 to all other vertices. The 1-cocycle $\Phi+\delta c$ takes values in $G\setminus X$ on all edges of $\mathcal{T}$ incident to v and takes the same values as Φ on all edges of $\mathcal{T}$ not incident to v. Here we use the fact that the edges of $\mathcal{T}$ are not loops which follows from the quasi-regularity of $\mathcal{T}$. The transformation $\Phi \mapsto \Phi + \delta c$ decreases the number of bad vertices. Repeating this argument, we find a 1-cocycle without bad vertices. □

We represent any $h \in H^1(M,G)$ by an admissible G-coloring Φ of $\mathcal{T}$ and set

$$TV_{\mathcal{C},\mathsf{b}}(M,L,h) = TV(\mathcal{T},\mathcal{L},\Phi) \in K.$$

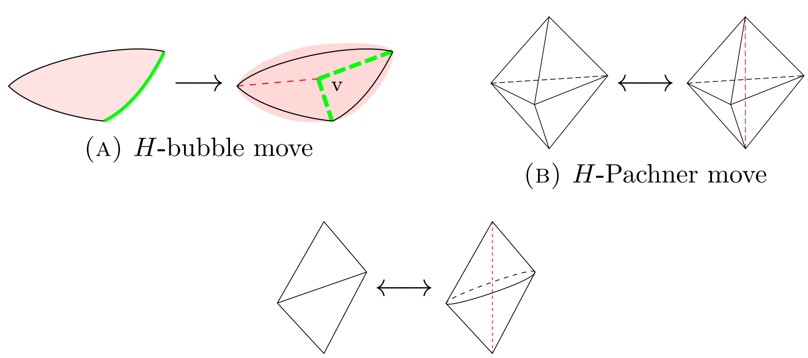
By Theorem 24, $TV_{\mathcal{C},\mathsf{b}}(M,L,h)$ is a topological invariant of the triple (M,L,h).

11. Proof of Theorem 24

Throughout this section, we keep notation of Theorem 24. We begin by explaining that any two H-triangulations of (M,L) can be related by elementary moves adding or removing vertices, edges, etc. We call an elementary move positive if it adds edges and negative if it removes edges.

The first type of elementary moves are the so-called H-bubble moves. The *positive H-bubble move* starts with a choice of a face $F = v_1v_2v_3$ of $\mathcal{T}$ such that at least one of its edges, say v_2v_3, is in $\mathcal{L}$. Consider two tetrahedra of $\mathcal{T}$ meeting along F. We unglue these tetrahedra along F and insert a 3-ball between the resulting two copies of F. We triangulate this 3-ball by adding a vertex v at its center and three edges vv_1, vv_2, vv_3. The edge v_2v_3 is removed from $\mathcal{L}$ and replaced by the edges $\{vv_2, vv_3\}$. This move can be visualized as in the transformation of Figure 8a (where the bold (green) edges belong to $\mathcal{L}$). The inverse move is the *negative H-bubble move*.

The second type of elementary moves is the *H-Pachner* $2 \leftrightarrow 3$ *moves* shown in Figure 8b. It is understood that the newly added edge on the right is not



(A) H-bubble move

(B) H-Pachner move

(C) H-lune move

Figure 8. Elementary moves

an element of $\mathcal{L}$. The negative H-Pachner move is allowed only when the edge common to the three tetrahedra on the right is not in $\mathcal{L}$.

Proposition 26 ([4], Proposition 4.23)**.** *Let L be a non-empty link in a closed connected orientable 3-manifold M. Any two H-triangulations of (M, L) can be related by a finite sequence of H-bubble moves and H-Pachner moves in the class of H-triangulations of (M, L).*

We will need one more type of moves on H-triangulations of (M, L) called the *H-lune moves*. It is represented in Figure 8c where for the negative H-lune move we require that the disappearing edge is not in $\mathcal{L}$. The H-lune move may be expanded as a composition of H-bubble moves and H-Pachner moves (see [4], Section 2.1), but it will be convenient for us to use the H-lune moves directly.

The following lemma is an algebraic analog of the H-bubble move.

Lemma 27. *Let $g_1, g_2, g_3, g_4, g_5, g_6 \in G \setminus X$ with $g_3 = g_1 + g_2$, $g_6 = g_2 + g_4$ and $g_5 = g_1 + g_6$. If $i \in I^{g_1}$, $j \in I^{g_2}$, $k \in I^{g_3}$, then*

$$\mathsf{d}(k) \sum_{l \in I^{g_4}, m \in I^{g_5}, n \in I^{g_6}} \mathsf{d}(n)\, \mathsf{b}(l)\, \mathsf{b}(m) *_{klm^*} *_{inm^*} *_{jln^*} \left(\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \otimes \begin{vmatrix} k & j^* & i \\ n & m & l \end{vmatrix} \right)$$
$$= \mathsf{b}(k)\, \mathrm{Id}(i, j, k^*) \tag{44}$$

Proof. We can apply the orthonormality relation (Theorem 9) to the tuple consisting of i, j, k, arbitrary $l \in I^{g_4}$, $m \in I^{g_5}$, and $p = k$. Note that $k, m \in I_0$ and the pair (i, j) is good because $V_i \otimes V_j \in \mathcal{C}_{g_1+g_2} = \mathcal{C}_{g_3}$ and $g_3 \notin X$. Similarly, the pairs (j, l) and (k, l) are good. Analyzing the grading in $\mathcal{C}$, we observe that the set N appearing in the orthonormality relation is a subset of I^{g_6}. Moreover, if $n \in I^{g_6} \setminus N$ then

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = 0$$

as $H(j, l, n^*) = 0$ or $H(i, n, m^*) = 0$ for such n. If we multiply the orthogonality relation by $\mathsf{b}(l)\, \mathsf{b}(m)$, apply $*_{klm^*}$, and sum over all pairs $(l, m) \in I^{g_4} \times I^{g_5}$ we obtain that the left hand side of (44) is equal to

$$\sum_{(l,m) \in I^{g_4} \times I^{g_5}} \mathsf{b}(l)\, \mathsf{b}(m) *_{klm^*} (\mathrm{Id}(i, j, k^*) \otimes \mathrm{Id}(k, l, m^*)) \tag{45}$$

Finally, using the equality $*_{klm^*}(\mathrm{Id}(k, l, m^*)) = \dim(H(k, l, m^*))$ and the relations satisfied by b, we obtain that the expression (45) is equal to $\mathsf{b}(k)\, \mathrm{Id}(i, j, k^*)$. $\square$

Lemma 28. *Let Φ be an admissible G-coloring of $\mathcal{T}$. Suppose that $(\mathcal{T}', \mathcal{L}')$ is an H-triangulation obtained from $(\mathcal{T}, \mathcal{L})$ by a negative H-Pachner, H-lune or H-bubble move. Then Φ restricts to an admissible G-coloring Φ' of $\mathcal{T}'$ and*

$$TV(\mathcal{T}, \mathcal{L}, \Phi) = TV(\mathcal{T}', \mathcal{L}', \Phi'). \tag{46}$$

Proof. The values of Φ' form a subset of the set of values of Φ, and therefore the admissibility of Φ implies the the admissibility of Φ'.

The rest of the proof is similar to the one in [18, Section VII.2.3]. In particular, we can translate the H-Pachner, H-lune, and H-bubble move into algebraic identities: the Biedenharn-Elliott identity, the orthonormality relation and Equation (44), respectively. The first two identities require certain pairs of indices to be good. To see that all relevant pairs in this proof are good we make the following observation. If e_1, e_2, e_3 are consecutive oriented edges of a 2-face of $\mathcal{T}$, then $\Phi(e_1) + \Phi(e_2) = \Phi(e_3) \in G \setminus X$, and the semi-simplicity of $\mathcal{C}_{\Phi(e_3)}$ implies that the pair $(\varphi(e_1), \varphi(e_2))$ is good for any $\varphi \in \mathrm{St}(\Phi)$. Translating into the language of 6j-symbols, we obtain that all the 6j-symbols involved in our state sums are admissible.

We will now give a detailed proof of (46) for a negative H-bubble move. Let $\varphi' \in \mathrm{St}(\Phi')$ and $S \subset \mathrm{St}(\Phi)$ be the set of all states φ of Φ extending φ'. It is enough to show that the term $TV_{\varphi'}$ of $TV(\mathcal{T}', \mathcal{L}', \Phi')$ associated to φ' is equal to the sum TV_S of the terms of $TV(\mathcal{T}, \mathcal{L}, \Phi)$ associated to the states in the set S. Here it is equivalent to work with the positive H-bubble move, which we do for convenience.

Recall the description of the positive bubble given at the beginning of this section. Let v, v_1, v_2, v_3 (resp. $F = v_1v_2v_3$) be the vertices (resp. face) given in this description (see Figure 9). Set $i = \varphi'(v_3v_1)$, $j = \varphi'(v_1v_2)$ and $k = \varphi'(v_3v_2)$. Let T_1 and T_2 be the two new tetrahedra of $\mathcal{T}$ and f_1, f_2, f_3 be their faces $vv_2v_3, vv_3v_1, vv_1v_2$. A state $\varphi \in S$ is determined by the values l, m, n of φ on the edges v_2v, v_3v, v_1v, respectively (see Figure 9).

As explained above, in the bubble move two tetrahedra meeting along F are unglued and a 3-ball is inserted between the resulting two copies F_1 and F_2 of F. Note that F_1 and F_2 are faces of $\mathcal{T}$. Also this 3-ball is triangulated with the two tetrahedra T_1 and T_2.

For a fixed state on a triangulation, denote by $*_f$ the contraction along a face f. One can write

$$TV_{\varphi'} = *_F(\mathsf{b}(\varphi'(v_3v_2))X)$$

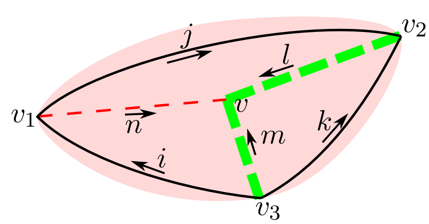


FIGURE 9. $T_1 \cup T_2$ colored by $\varphi \in S$

where X is all the factors of the state sum for φ' except for $*_F$ and $\mathsf{b}(\varphi'(v_3v_2))$. Since all $\varphi \in S$ restrict to φ', we have that TV_S is equal to

$$*_{F_1}*_{F_2}\left(X \otimes \sum_{\varphi\in S} \mathsf{d}(\varphi(v_3v_2))\,\mathsf{d}(\varphi(v_1v))\,\mathsf{b}(\varphi(v_2v))\,\mathsf{b}(\varphi(v_3v))\,*_{f_1}*_{f_2}*_{f_3}\left(|T_1|_\varphi \otimes |T_2|_\varphi\right)\right)$$

where the additional factors come from the triangulation of the 3-ball and the fact that the link goes through v_3v, vv_2 instead of v_3v_2. Here, the contraction $*_{F_1}*_{F_2}$ is equal to $*_{ijk^*}*_{ijk^*}$. On the other hand, one has $*_F(X) = *_{ijk^*}(X) = *_{ijk^*}*_{ijk^*}(X \otimes \mathrm{Id}(i,j,k^*))$. Hence the equality $TV_{\varphi'} = TV_S$ follows from

$$\sum_{\varphi\in S} \mathsf{d}(\varphi(v_3v_2))\,\mathsf{d}(\varphi(v_1v))\,\mathsf{b}(\varphi(v_2v))\,\mathsf{b}(\varphi(v_3v))\,*_{f_1}*_{f_2}*_{f_3}\left(|T_1|_\varphi \otimes |T_2|_\varphi\right)$$
$$= \mathrm{Id}(i,j,k^*)\,\mathsf{b}(\varphi'(v_3v_2))$$

which is exactly the identity established in Lemma 27. □

Let $\mathcal{T}_0$ be the set of vertices of $\mathcal{T}$. Let δ be the coboundary operator from the G-valued 0-cochains on $\mathcal{T}$ to the G-valued 1-cochains on $\mathcal{T}$.

Lemma 29. *Let $v_0 \in \mathcal{T}_0$ and $c : \mathcal{T}_0 \to G$ be a map such that $c(v) = 0$ for all $v \neq v_0$ and $c(v_0) \notin X$. If Φ and $\Phi + \delta c$ are admissible G-colorings of $\mathcal{T}$, then $TV(\mathcal{T},\mathcal{L},\Phi) = TV(\mathcal{T},\mathcal{L},\Phi+\delta c)$*

Proof. In the proof, we shall use the language of skeletons of 3-manifolds dual to the language of triangulations (see, for instance [18, 4]). A skeleton of M is a 2-dimensional polyhedron P in M such that $M\setminus P$ is a disjoint union of open 3-balls and locally P looks like a plane, or a union of 3 half-planes with common boundary line in $\mathbb{R}^3$, or a cone over the 1-skeleton of a tetrahedron. A typical skeleton of M is constructed from a triangulation T of M by taking the union P_T of the 2-cells dual to its edges. This construction establishes a bijective correspondence $T \leftrightarrow P_T$ between the quasi-regular triangulations T of M and the skeletons P of M such that every 2-face of P is a disk adjacent to two distinct components of $M - P$. To specify a Hamiltonian link L in a triangulation T, we provide some faces of P_T with dots such that each component of $M - P_T$ is adjacent to precisely two (distinct) dotted faces. These dots correspond to the intersections of L with the 2-faces. The notion of a G-coloring on an H-triangulation T of (M,L) can be rephrased in terms of P_T as a 2-cycle on P_T with coefficients in G, that is a function assigning an element of G to every oriented 2-face of P_T such that opposite orientations of a face give rise to opposite elements of G and the sum of the values of the function on three faces of P_T sharing a common edge and coherently oriented is always equal to zero. The notion of a state on T can be also rephrased in terms of P_T. The state sum $TV(T,L,\Phi)$ can be rewritten in terms of $P = P_T$ in the obvious way, and will be denoted $TV(P,\Phi)$ in the rest of the proof. The moves on the H-triangulations may also be translated to this dual language and give the well-known Matveev-Piergallini moves on skeletons adjusted to the setting of Hamiltonian links, see [4]. We shall use the Matveev-Piergallini moves on dotted skeletons dual to the H-Pachner move and to the H-lune moves. Instead of the dual H-bubble move we use the so-called b-move $P \to P'$. The b-move adds to a dotted skeleton $P \subset M$ a dotted 2-disk $D \subset M$ such that the circle ∂D lies on a dotted face f of P, bounds a small 2-disk $D' \subset f$ containing the dot of f, and the 2-sphere $D \cup D'$ bounds an embedded 3-ball in M meeting P solely along D'. Note that a dual H-bubble move on dotted skeletons is a composition of a b-move with a dual H-lune move.

We apply the b-move to the polyhedron $P = P_T$ as follows. Consider the open 3-ball of $M - P_T$ surrounding the vertex v_0. Assume first that the closure B of this open ball is an embedded closed 3-ball in M. The 2-sphere ∂B meets L at two dots arising as the intersection of P with the two edges of L adjacent to v_0. We call these dots the south pole and the north pole of B. We apply the b-move $P \to P' = P \cup D$ at the south pole of B. Here $D \subset B$ is a 2-disk such that $D \cap P = \partial D = \partial D'$, where D' is a small disk in P centered at the south pole and contained in a face f of P. The given G-coloring Φ of P induces a G-coloring Φ' of P' which coincides with Φ on the faces of P distinct from f and assigns to the faces $f - D'$, D, D' of P' the elements $\varphi(f)$, $g = c(v_0) \notin X$, $\varphi(f) - g$ of G, respectively. Here the orientation D' is induced by the one of M restricted to B and $f - D'$, D are oriented so that $\partial D' = \partial D = -\partial(\overline{f - D'})$ in the category of oriented manifolds. Next, we push the equatorial circle ∂D towards the north pole of ∂B. This transformation changes P' by isotopy in M and a sequence of Matveev-Piergallini moves dual to the H-Pachner move and to the H-lune moves. This is accompanied by the transformation of the G-coloring Φ' of P' which keeps the colors of the faces not lying on ∂B or lying on ∂B to the north of the (moving) equatorial circle ∂D and deduces g from the Φ-colors for the faces lying on ∂B to the south of ∂D. The color of the face $\operatorname{Int} D$ remains g throughout the transformations. These G-colorings are admissible because all colors of the faces of the southern hemisphere are given by $\Phi + \delta c$ whereas the colors of the faces of the northern hemisphere are given by Φ. Hence, by Lemma 28, $TV(P', \Phi')$ is preserved through this isotopy of ∂D on ∂B. Finally, at the end of the isotopy, ∂D becomes a small circle surrounding the northern pole of ∂B. Applying the inverse b-move, we now remove D and obtain the skeleton P with the G-coloring $\Phi + \delta(c)$. Hence $TV(P, \Phi) = TV(P, \Phi + \delta(c))$ which is equivalent to the claim of the lemma. In the case where the 3-ball B is not embedded in M, essentially the same argument applies. The key observation is that since $\mathcal{T}$ is a quasi-triangulation, the edges of $\mathcal{T}$ adjacent to v_0 are not loops, and therefore the ball B does not meet itself along faces of P (though it may meet itself along vertices and/or edges of P). □

Lemma 30. *If Φ and Φ' are two admissible G-colorings of $\mathcal{T}$ representing the same class in $H^1(M;G)$, then $TV(\mathcal{T},\mathcal{L},\Phi) = TV(\mathcal{T},\mathcal{L},\Phi')$.*

Proof. As Φ and Φ' represent the same cohomology class, $\Phi' = \Phi + \delta c_1 + \cdots + \delta c_n$ where $c_i : \mathcal{T}_0 \to G$ is a 0-cochain taking non-zero value at a single vertex v_i for all $i = 1, ..., n$. We prove the desired equality by induction on n. If $n = 0$ then $\Phi' = \Phi$ and the equality is clear. Otherwise, let E_1 be the set of (oriented) edges of $\mathcal{T}$ beginning at v_1. Pick any

$$g \in G \setminus \Big[X \cup \bigcup_{e\in E_1} \big(\Phi(e) + X\big) \cup \bigcup_{e\in E_1} \big(c_1(v_1) + \Phi'(e) + X\big)\Big].$$

Let $c : \mathcal{T}_0 \to G$ be the map given by $c(v_1) = g$ and $c(v) = 0$ for all $v \neq v_1$. Then $\Phi + \delta c$ and $\Phi + \delta c + \delta c_2 + \cdots + \delta c_n = \Phi' + \delta(c - c_1)$ are admissible colorings. Lemma 29 and the induction assumption imply that

$$\begin{aligned} TV(\mathcal{T},\mathcal{L},\Phi) &= TV(\mathcal{T},\mathcal{L},\Phi + \delta c) \\ &= TV(\mathcal{T},\mathcal{L},\Phi + \delta c + \delta c_2 + \cdots + \delta c_n) \\ &= TV(\mathcal{T},\mathcal{L},\Phi'). \end{aligned}$$

□

Theorem 31. *Let $(\mathcal{T},\mathcal{L})$ and $(\mathcal{T}',\mathcal{L}')$ be two H-triangulations of (M,L) such that $(\mathcal{T}',\mathcal{L}')$ is obtained from $(\mathcal{T},\mathcal{L})$ by a single H-Pachner move, H-bubble move, or H-lune move. Then for any admissible G-colorings Φ and Φ' on $\mathcal{T}$ and $\mathcal{T}'$ respectively, representing the same class in $H^1(M;G)$,*

$$TV(\mathcal{T},\mathcal{L},\Phi) = TV(\mathcal{T}',\mathcal{L}',\Phi').$$

Proof. For concreteness, assume that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a negative move. The admissible G-coloring Φ of $\mathcal{T}$ restricts to an admissible G-coloring Φ'' of $\mathcal{T}'$ which represent the same class in $H^1(M;G)$. Now Lemma 28 implies that $TV(\mathcal{T},\mathcal{L},\Phi) = TV(\mathcal{T}',\mathcal{L}',\Phi'')$ and Lemma 30 implies that $TV(\mathcal{T}',\mathcal{L}',\Phi'') = TV(\mathcal{T}',\mathcal{L}',\Phi')$. □

Proof of Theorem 24. From Proposition 26 we know that any two H-triangulation of (M,L) are related by a finite sequence of elementary moves. Then the result follows from the Theorem 31 by induction on the number of moves. □

Remark 32. *The invariant $TV(\mathcal{T},\mathcal{L},\Phi)$ may be computed in terms of more general singular triangulations of 3-manifolds. Such triangulation are obtained by gluing pairs of 2-faces of a finite set of tetrahedra where the pairs in question are allowed to include two 2-faces of the same tetrahedron. Singular triangulations may be more economical and more suitable for computations, cf. Section 12.2 below. We briefly explain why state sums on singular triangulations give the same invariant. Let $\mathcal{T}$ be a singular triangulation of M and $\mathcal{L}$ be a Hamiltonial link in $\mathcal{T}$, i.e., a set of edges of $\mathcal{T}$ whose union L contains all vertices of*

$\mathcal{T}$ and is a link in M. The notions of admissible G-colorings and their states directly generalize to this setting. If Φ is an admissible G-coloring of $\mathcal{T}$ representing $h \in H^1(M,G)$ then we define $TV(\mathcal{T},\mathcal{L},\Phi)$ by (43). We claim that $TV(\mathcal{T},\mathcal{L},\Phi) = TV(M,L,h)$. The proof is quite involved and we give only an outline (see [4] for a detailed argument in a similar setting). First, one finds a sequence of triples $(\mathcal{T},\mathcal{L},\Phi) = (\mathcal{T}_0,\mathcal{L}_0,\Phi_0)$,..., $(\mathcal{T}_n,\mathcal{L}_n,\Phi_n)$ such that

1. *for all i, $\mathcal{T}_i$ is a singular triangulation of M, $\mathcal{L}_i$ is a Hamiltonian link in $\mathcal{T}_i$ representing L, and Φ_i is an admissible G-coloring of $\mathcal{T}_i$,*
2. *for all i, $(\mathcal{T}_{i+1},\mathcal{L}_{i+1})$ is obtained from $(\mathcal{T}_i,\mathcal{L}_i)$ be an H-bubble move or an H-Pachner $2 \leftrightarrow 3$ move where Φ_{i+1} restricts to Φ_i if the move is positive or Φ_i restricts to Φ_{i+1} if the move is negative,*
3. *$(\mathcal{T}_n,\mathcal{L}_n)$ is an H-triangulation of (M,L).*

The difficulty in the construction of such a sequence lies in the fact that a positive H-Pachner $2 \leftrightarrow 3$ move can yield a triangulation without admissible G-colorings. This can be overcome by only doing positive H-Pachner $2 \leftrightarrow 3$ moves near a vertex created by a bubble move. The proof of Lemma 28 shows that $TV(\mathcal{T}_i,\mathcal{L}_i,\Phi_i) = TV(\mathcal{T}_{i+1},\mathcal{L}_{i+1},\Phi_{i+1})$ for all i and therefore $TV(\mathcal{T},\mathcal{L},\Phi) = TV(M,L,h)$.

12. Examples of 3-manifold invariants

12.1. **The Turaev-Viro invariants.** The invariants defined in Theorem 24 include the standard Turaev-Viro invariants arising from quantized simple Lie algebras at roots of unity. Indeed, let $\widetilde{U}_q(\mathfrak{g})$ be the restricted quantum group associated to a simple Lie algebra $\mathfrak{g}$ of type A, B, C, D at a primitive root of unity q of even order (see [18, XI.6.3]). Let $\mathcal{C}$ be the modular category formed from finite dimensional representations of $\widetilde{U}_q(\mathfrak{g})$ modulo negligible morphisms (see [18, XI]).

Theorem 33. *The category $\mathcal{C}$ is a pivotal tensor Ab-category with basic data and t-ambi pair (I, qdim) where I is the set of isomorphism classes of simple objects of $\mathcal{C}$ and* qdim *is the usual quantum dimension in $\mathcal{C}$. In addition, $\mathcal{C}$ is G-graded where $G = \{1\}$ is the trivial group and $X = \emptyset$. Then $\mathcal{C}$ admits a map $\mathsf{b}\colon I_0 = I \to \mathbb{C}$ satisfying all requirements of Section 10.2 and such that $TV_{\mathcal{C},\mathsf{b}}(M,L,h)$ is independent of L and equal to the usual Turaev-Viro invariant of M for any link $L \subset M$ and $h = 0 \in H^1(M,G) = \{0\}$.*

Proof. First, $\mathcal{C}$ is a $\mathbb{C}$-linear pivotal category with basic data indexed by I, see [18, XI], [20]. Equation (4) holds by definition of the functor $G : \mathrm{Gr}_{\mathcal{C}} \to \mathcal{C}$ of Section 2 and so (I, qdim) is a t-ambi pair. By definition, $\mathcal{C} = \mathcal{C}_1$ is G-graded. The category $\mathcal{C}$ is semi-simple with finitely many isomorphism classes of simple objects and qdim is non-zero on all simple objects. Thus, $G = \{1\}$ and $X = \emptyset$ satisfy all the requirements above. By Lemma 23 applied to the character $\mathrm{qdim} : I \to \mathbb{C}$, the map $\mathsf{b} = \frac{1}{\mathcal{D}}\mathrm{qdim}$ has the desired properties. Here $\mathcal{D} = \sum_{j\in I} \mathrm{qdim}(j)^2$.

Let L be a link in M. Let $(\mathcal{T},\mathcal{L})$ be an H-triangulation of (M,L) and Φ be the unique G-coloring of $\mathcal{T}$ assigning 1 to all edges. The states of Φ are the maps from the set of oriented edges of $\mathcal{T}$ to I carrying any edge with opposite orientations to dual elements of I. Then

$$TV_{\mathcal{C},\mathsf{b}}(\mathcal{T},\mathcal{L},\Phi) = \sum_{\varphi\in \mathrm{St}(\Phi)} \prod_{e\in\mathcal{T}_1\setminus\mathcal{L}} \mathrm{qdim}(\varphi(e)) \prod_{e\in\mathcal{L}} \mathsf{b}(\varphi(e))\ \mathrm{cntr}\left(\bigotimes_{T\in\mathcal{T}_3} |T|_\varphi\right) \tag{47}$$

$$= \mathcal{D}^{-v} \sum_{\varphi\in \mathrm{St}(\Phi)} \prod_{e\in\mathcal{T}_1} \mathrm{qdim}(\varphi(e))\ \mathrm{cntr}\left(\bigotimes_{T\in\mathcal{T}_3} |T|_\varphi\right) \tag{48}$$

where v is the number of edges of $\mathcal{L}$ (i.e., the number of vertices of $\mathcal{T}$) and the second equality follows from the definition of b. The resulting formula is the standard expression for the Turaev-Viro invariant of M. □

12.2. **Quantum $\mathfrak{sl}(2)$.** Set $G = \mathbb{C}/2\mathbb{Z}$ and $X = \mathbb{Z}/2\mathbb{Z} \subset G$. Let $\mathcal{C}$ be the pivotal tensor Ab-category with basic data and t-ambi pair defined in Section 9. This category is G-graded: for $\overline{\alpha} \in G$, the class $\mathcal{C}_{\overline{\alpha}}$ consists of the modules on which the central element $K^r \in U_q(\mathfrak{sl}(2))$ act as multiplication by the scalar $q^{r\overline{\alpha}} = e^{i\pi\overline{\alpha}}$. For example, for any $\tilde{i} \in \mathbb{C}/2r\mathbb{Z}$, the simple object $V_{\tilde{i}} \in \mathcal{C}$ is graded with $\tilde{i} + \widetilde{r-1} (\mathrm{mod}\, 2\mathbb{Z}) = \tilde{i} (\mathrm{mod}\, 2\mathbb{Z}) \in G$ (since r is odd). It is easy to check that G, X, and the constant function $\mathsf{b} = r^{-2}$ satisfy the requirements of Section 10.2 (cf. (37)). The constructions above derive from this data a state-sum topological invariant $TV(M,L,h) = TV_{\mathcal{C},\mathsf{b}}(M,L,h)$ of links in 3-manifolds. To put this invariant into perspective note that the standard Turaev-Viro-type 3-manifold invariants can not be derived from $\mathcal{C}$ because $\mathcal{C}$ contains infinitely many isomorphism classes of simple objects and is not semi-simple. Moreover, the usual quantum dimensions of simple objects in $\mathcal{C}$ are generically equal to zero and so are the usual $6j$-symbols associated with $\mathcal{C}$. Note also that by Remark 21 the modified $6j$-symbols associated with $\mathcal{C}$ can be treated as complex numbers.

We now illustrate the definition of $TV(M,L,h)$ with a few computations. First, consider the triple (an oriented 3-sphere S^3, an unknot $L_0 \subset S^3, h = 0$). The 3-sphere has a quasi-regular triangulation $\mathcal{T} = T \cup_{\partial T} \overline{T}$ where T is an oriented tetrahedron and $\overline{T}$ is a copy of T with opposite orientation. Let v_1, v_2, v_3, v_4 be the vertices of T numbered so that the triple of oriented edges $(\overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4})$ is positively oriented. Let $\mathcal{L}$ be the set of edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1$. Then $(\mathcal{T},\mathcal{L})$ is an H-triangulation of (S^3, L_0). Pick an admissible G-coloring Φ of $\mathcal{T}$. For any state φ of Φ,

$$|T|_\varphi = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \quad \text{and} \quad |\overline{T}|_\varphi = \begin{vmatrix} k & j* & i \\ n & m & l \end{vmatrix}$$

where i, j, k, l, m, n are given by the same formulas as in Section 10.3. Then

$$TV(S^3, L_0, 0) = \sum_{i,j,k,l,m,n} \mathsf{d}(k)\,\mathsf{d}(n)\,\mathsf{b}(i)\,\mathsf{b}(j)\,\mathsf{b}(l)\,\mathsf{b}(m) \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} k & j* & i \\ n & m & l \end{vmatrix}$$

$$= \sum_{i,j,k,l,m} \mathsf{b}(i)\,\mathsf{b}(j)\,\mathsf{b}(l)\,\mathsf{b}(m) = r^5 (r^{-2})^4 = r^{-3}$$

where the second equality follows from the orthonormality relation for the modified 6j-symbols and the third equality holds because $\mathsf{b} = r^{-2}$.

From the census by Ben Burton [1], there is a singular triangulation $\mathcal{T}$ of the lens space $L_{5,2}$ with only one tetrahedron. Let v_1, v_2, v_3, v_4 be the vertices of a standard tetrahedron T. The singular triangulation $\mathcal{T}$ of $L_{5,2}$ is obtained by glueing the face $v_2v_3v_4$ with $v_3v_4v_1$ and glueing the face $v_1v_2v_4$ with $v_3v_1v_2$. This triangulation has one vertex, two faces, and two edges (see Figure 10). We orient $L_{5,2}$ so that the triple of oriented edges $(\overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4})$ is positively oriented. Denote by $\overrightarrow{a}$ (resp. $\overrightarrow{b}$) the oriented edges $\overrightarrow{v_2v_3} = \overrightarrow{v_3v_4} = \overrightarrow{v_4v_1}$ (resp. $\overrightarrow{v_2v_4} = \overrightarrow{v_3v_1} = \overrightarrow{v_1v_2}$) of $\mathcal{T}$. The 1-homology of $L_{5,2}$ is computed by

$$H_1(L_{5,2}, \mathbb{Z}) = (\mathbb{Z}[\overrightarrow{a}] \oplus \mathbb{Z}[\overrightarrow{b}])/(2[\overrightarrow{a}] - [\overrightarrow{b}], 5[\overrightarrow{a}]) = (\mathbb{Z}/5\mathbb{Z})[\overrightarrow{a}].$$

A cohomology class $h \in H^1(L_{5,2}, \mathbb{C}/2\mathbb{Z})$ is determined by the value $h([\overrightarrow{a}]) = \frac{2k}{5} \in \mathbb{C}/2\mathbb{Z}$ where $k = k_h \in \mathbb{Z}$ is determined by h up to addition of an element of $5\mathbb{Z}$. If $h \neq 0$, then $k \notin 5\mathbb{Z}$ and the $\mathbb{C}/2\mathbb{Z}$-cocycle Φ : {oriented edges of $\mathcal{T}$} $\to \mathbb{C}/2\mathbb{Z}$ carrying $[\overrightarrow{a}]$ to $\frac{2k}{5} \pmod{2\mathbb{Z}}$ and $[\overrightarrow{b}]$ to $\frac{4k}{5} \pmod{2\mathbb{Z}}$ is an admissible $\mathbb{C}/2\mathbb{Z}$-coloring of $\mathcal{T}$ representing h. Set $\alpha = \frac{2k}{5} + 2k = \frac{12k}{5} \in \mathbb{C}/5\mathbb{Z}$. Then Φ has nine states described as follows:

$$\mathrm{St}(\Phi) = \left\{ \varphi \mid \varphi(\overrightarrow{a}) = \alpha + m_a, \varphi(\overrightarrow{b}) = 2\alpha + m_b \text{ where } m_a, m_b \in \{-2, 0, 2\} \right\}.$$

For any state φ of Φ,

$$|T|_\varphi = \begin{vmatrix} -\varphi(\overrightarrow{b}) & -\varphi(\overrightarrow{a}) & \varphi(\overrightarrow{b}) \\ -\varphi(\overrightarrow{a}) & \varphi(\overrightarrow{a}) & -\varphi(\overrightarrow{b}) \end{vmatrix}. \tag{49}$$

Now we restrict ourselves to the case $r = 3$. A state φ of Φ is determined by a pair (m_a, m_b) as above. One can explicitly compute $|T|_\varphi$ using (49) and the

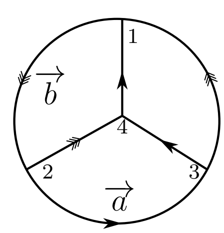


FIGURE 10. A singular triangulation of $L(5, 2)$ with one tetrahedron.

symmetries of the $6j$-symbols allowing to write $|T|_\varphi$ as in Example 14. Doing this for all nine φ, we obtain that

$$(m_a, m_b) : \begin{vmatrix} -\varphi(\overrightarrow{b}) & -\varphi(\overrightarrow{a}) & \varphi(\overrightarrow{b}) \\ -\varphi(\overrightarrow{a}) & \varphi(\overrightarrow{a}) & -\varphi(\overrightarrow{b}) \end{vmatrix} = \text{a symmetric } 6j\text{-symbol} = \text{formula from Example 14}$$

$$(-2,-2) : \begin{vmatrix} 2\alpha-2 & \alpha-2 & 2-2\alpha \\ \alpha-2 & 2-\alpha & 2\alpha-2 \end{vmatrix} = \begin{vmatrix} 2-2\alpha & 2-\alpha & 2\alpha-2 \\ 2-\alpha & \alpha-2 & 2-2\alpha \end{vmatrix} = \{5-3\alpha\}\{6-4\alpha\}$$

$$(-2,0) : \begin{vmatrix} 2\alpha & 2\alpha & 2-\alpha \\ 2\alpha & \alpha-2 & 2-\alpha \end{vmatrix} = \begin{vmatrix} -2\alpha & 2-\alpha & 2\alpha \\ 2-\alpha & \alpha-2 & -2\alpha \end{vmatrix} = \{3-\alpha\}\{4-\alpha\}$$

$$(-2,2) : \begin{vmatrix} 2-\alpha & 2-\alpha & -2\alpha-2 \\ -2\alpha-2 & \alpha-2 & 2\alpha+2 \end{vmatrix} = \begin{vmatrix} -2\alpha-2 & 2-\alpha & 2\alpha+2 \\ 2-\alpha & \alpha-2 & -2\alpha-2 \end{vmatrix} = \{2\alpha+3\}\{4\alpha+4\}$$

$$(0,-2) : \begin{vmatrix} 2-2\alpha & -\alpha & 2\alpha-2 \\ -\alpha & \alpha & 2-2\alpha \end{vmatrix} = \begin{vmatrix} 2-2\alpha & -\alpha & 2\alpha-2 \\ -\alpha & \alpha & 2-2\alpha \end{vmatrix} = \{3-2\alpha\}\{4-2\alpha\}$$

$$(0,0) : \begin{vmatrix} -2\alpha & -\alpha & 2\alpha \\ -\alpha & \alpha & -2\alpha \end{vmatrix} = \begin{vmatrix} -2\alpha & -\alpha & 2\alpha \\ -\alpha & \alpha & -2\alpha \end{vmatrix} = -(q^{-5\alpha}+q^{-3\alpha}+q^{-\alpha}+q^{\alpha}+q^{3\alpha}+q^{5\alpha})$$

$$(0,2) : \begin{vmatrix} 2\alpha+2 & \alpha & -2\alpha-2 \\ \alpha & -\alpha & 2\alpha+2 \end{vmatrix} = \begin{vmatrix} -2\alpha-2 & -\alpha & 2\alpha+2 \\ -\alpha & \alpha & -2\alpha-2 \end{vmatrix} = \{-3-3\alpha\}\{-2-3\alpha\}$$

$$(2,-2) : \begin{vmatrix} \alpha+2 & \alpha+2 & 2\alpha-2 \\ 2\alpha-2 & -\alpha-2 & 2-2\alpha \end{vmatrix} = \begin{vmatrix} 2-2\alpha & -\alpha-2 & 2\alpha-2 \\ -\alpha-2 & \alpha+2 & 2-2\alpha \end{vmatrix} = \{3+3\alpha\}\{4+\alpha\}$$

$$(2,0) : \begin{vmatrix} -2\alpha & -2\alpha & \alpha+2 \\ -2\alpha & -\alpha-2 & \alpha+2 \end{vmatrix} = \begin{vmatrix} -2\alpha & -\alpha-2 & 2\alpha \\ -\alpha-2 & \alpha+2 & -2\alpha \end{vmatrix} = \{3-4\alpha\}\{4-4\alpha\}$$

$$(2,2) : \begin{vmatrix} -2\alpha-2 & -\alpha-2 & 2\alpha+2 \\ -\alpha-2 & \alpha+2 & -2\alpha-2 \end{vmatrix} = \begin{vmatrix} -2\alpha-2 & -\alpha-2 & 2\alpha+2 \\ -\alpha-2 & \alpha+2 & -2\alpha-2 \end{vmatrix} = \{-1-2\alpha\}\{-\alpha\}$$

For all $x \in \mathbb{C}/6\mathbb{Z} \setminus \mathbb{Z}/6\mathbb{Z}$, $\mathsf{b}(x) = \dfrac{1}{9}$ and $\mathsf{d}(x) = \dfrac{1}{\{x-1\}\{x-2\}} = -\dfrac{\{x\}}{\{3x\}}$. Thus

$$\mathsf{d}(\varphi(a)) = \mathsf{d}(\alpha+m_a) = -\frac{\{\alpha+m_a\}}{\{3\alpha\}} \text{ and } \mathsf{d}(\varphi(b)) = \mathsf{d}(2\alpha+m_b) = -\frac{\{2\alpha+m_b\}}{\{6\alpha\}}.$$

Taking for $\mathcal{L}$ a single edge, we obtain:

$$TV(L_{5,2}, L = \vec{a}, h) = \sum_\varphi \mathsf{b}(\varphi(a))\,\mathsf{d}(\varphi(b))|T|_\varphi = \frac{1}{3}(1 - q^{2\alpha} - q^{-2\alpha})$$

and

$$TV(L_{5,2}, L = \vec{b}, h) = \sum_\varphi \mathsf{b}(\varphi(b))\,\mathsf{d}(\varphi(a))|T|_\varphi = \frac{1}{3}(2 + q^{2\alpha} + q^{-2\alpha}).$$

Note that $q^\alpha \in \mathbb{C}$ is a primitive fifth root of unity depending on h. Our computations show in particular that the knots $\vec{a}$ and $\vec{b}$ in $L_{5,2}$ are not isotopic and that $TV(L_{5,2}, L, h)$ does depend on the choice of h for $L = \vec{a}$ and for $L = \vec{b}$.

13. Totally symmetric 6j-symbols

Let $\mathcal{C}$ be a pivotal tensor Ab-category with ground ring K, basic data $\{V_i, w_i : V_i \to V_{i^*}^*\}_{i\in I}$, and t-ambi pair (I_0, d). Recall that the associated modified $6j$-symbols have the symmetries of an oriented tetrahedron. The modified $6j$-symbols are totally symmetric if they are invariant under the full group of symmetries of a tetrahedron. More precisely, suppose that for every good triple $i, j, k \in I$, we have an isomorphism $\eta(i, j, k) : H(i, j, k) \to H(k, j, i)$ satisfying the following conditions:

$$\eta(i,j,k) = \eta(j,k,i) = \eta(k,i,j),$$
$$\eta(k,j,i) \circ \eta(i,j,k) = \mathrm{id} : H(i,j,k) \to H(i,j,k)$$

and

$$(,)_{kji}\,(\eta(i,j,k) \otimes \eta(k^*,j^*,i^*)) = (,)_{ijk} : H(i,j,k) \otimes_K H(k^*,j^*,i^*) \to K$$

where $(,)_{ijk}$ is the pairing (8). We say that the modified $6j$-symbols are *totally symmetric* if for any good tuple $(i, j, k, l, m, n) \in I^6$

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \circ [\eta(k^*,j,i) \otimes \eta(m^*,l,k) \otimes \eta(j^*,l^*,n) \otimes \eta(i^*,n^*,m)] = \begin{vmatrix} j & i & k \\ m^* & l^* & n^* \end{vmatrix}.$$

For ribbon $\mathcal{C}$, the associated modified $6j$-symbols are totally symmetric (see [18], Chapter VI). The isomorphisms $\eta(i, j, k)$ in this case are determined by so-called half-twists. A half-twist in $\mathcal{C}$ is a family $\{\theta'_i \in K\}_{i\in I}$ such that for all $i \in I$, we have $\theta'_{i^*} = \theta'_i$ and the twist $V_i \to V_i$ is equal to $(\theta'_i)^2\, \mathrm{id}_{V_i}$.

Lemma 34. *The modified $6j$-symbols defined in Section 9 are totally symmetric.*

Proof. The category $\mathcal{C}^H$ is ribbon with half-twist $(\theta'_i)_{i\in I} = \left(q^{(i/2)^2-(r')^2}\right)_{i\in I}$, see [14]. This gives rise to a family of isomorphisms

$$\eta(i,j,k) : H_{\mathcal{C}^H}(i,j,k) \to H_{\mathcal{C}^H}(k,j,i)$$

making the modified $6j$-symbols associated with $\mathcal{C}^H$ totally symmetric. Using the isomorphisms provided by Lemma 18, one can check that this family induces a well-defined family of isomorphisms

$$\eta(i,j,k) : H_{\mathcal{C}}(i,j,k) \to H_{\mathcal{C}}(k,j,i).$$

The latter family makes the modified $6j$-symbols associated with $\mathcal{C}$ totally symmetric. □

Remark 35. *For a category with totally symmetric $6j$-symbols, the construction of Section 10 may be applied to links in non-oriented closed 3-manifolds.*

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