On some stochastic properties in Devaney's chaos

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Abstract

The author complements recent results obtained by Wu, Xu, Lin and Ruan (2004, 2005) and Abraham, Biau and Cadre (2002) on the links between several topological and stochastic properties of dynamical systems.

KEYWORDS : Chaos, mixing, topological mixing, sensitivity to initial conditions, great deviations, decay of correlations.

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1 Introduction

There has been recently a surge of scientific activity on the links between the topological and the stochastic properties of deterministic dynamical systems, with the aim of bridging the gap between the topological and the probabilistic approaches to chaos. The topological approach lays at the origin of the most popular definition of chaos, the one given by Devaney [6], according to which a continuous map f of a metric space (M, d) into itself is chaotic if

i) f is *topologically transitive*, that is, for every nonempty open sets $U, V \subset M$, there is an integer $n > 0$ such that $U \cap f^n V \neq \emptyset$;

ii) the set of periodic points of f is dense in M; and

iii) f is *sensitive to initial conditions*, that is, there exists a constant $\sigma > 0$ (sensitivity constant) such that for every point $x \in M$ and every open neighborhood N_x of x, there is an integer $n \geq 0$ such that

$$\sup_{y \in N_x} d(T^n x, T^n y) > \sigma.$$

Since its introduction by Devaney, this definition has been extensively investigated, with research focusing mainly on finding possible redundancies between conditions *i), ii), iii)* and proposing alternative ones. Thus, it has been proved by Banks & al. [4] that *i)* and *ii)* imply *iii)*, and by Assaf IV and Gadbois [2] that, for *general* maps and spaces, this is the only redundancy in the definition, although it has been proved by Vellekoop and Berglund [11] that *i)* implies *ii)* and *iii)* if M is an interval of $\mathbb{R}$. Other definitions of chaos have been proposed by Li and Yorke, Wiggins and Martelli, among others [9].

Another approach, which aims at characterizing the dynamics of a dynamical system through its stochastic properties lays at the heart of an increasingly important topic in ergodic theory, namely the investigation of the *decay of correlations property* using tools from functional analysis and spectral theory - this approach can be exemplified by the works of Liverani [7, 8], Baladi [3] and Viana [12], among others.

The links between the two approaches are being gradually investigated in the literature. In a recent article, Abraham, Biau and Cadre [1] have investigated the links between several topological properties and sensitivity to initial conditions for *measure-preserving* dynamical systems on nontrivial metric spaces endowed with a borelian probability measure.

More recently, Xu, Lin and Ruan [14] have investigated the links between the decay of correlations property and sensitivity to initial conditions for smooth mappings on compact connex smooth riemannian manifolds. In another paper, they have investigated the links between a great deviations inequality and sensitivity to initial conditions in the same context, under a topological mixing assumption [13].

In this paper, we shall extend the results of Abraham, Biau and Cadre, and Wu, Xu, Lin and Ruan by emphasizing the *topological mixing* property. Thus, we shall be able to relax the invariance assumption made by Abraham, Biau and Cadre, and the differentiability and great deviations assumptions made by Wu, Xu, Lin and Ruan.

2 Definitions and review of recent results

Let (M, d) be a metric space which is not reduced to a single point, let $\mathcal{B}_M$ denote the Borel sigma-field on (M, d), and let f be a measurable map from M to M. Let $\mathrm{supp}(\mu)$ denote the *support* of any probability measure μ on $(M, \mathcal{B}_M)$, that is to say, the complement with respect to M of the largest μ-negligible open set.

We shall say that f is *topologically mixing* if for any nonempty open sets U, $V \subset M$, there is an integer $n \geq 0$ such that $U \cap f^k V \neq \emptyset$ for every $k \geq n$.

We shall say that f is *measure-preserving* or that the measure μ is *invariant* if $\mu(f^{-1}B) = \mu(B)$ for every measurable set B.

In this, setting, Abraham, Biau and Cadre [1] have established the following result.

Theorem 2.1 *If supp$(\mu) = M$ and the mapping f preserves the probability measure μ and is topologically mixing, then f is sensitive to initial conditions.*

Proof. See Abraham, Biau and Cadre [1]. An essential element of the proof is Poincaré's recurrence theorem, a proof of which is given, for example, by Petersen [10]. □

Now, assume that (M, d) is a compact connex riemannian manifold endowed with the riemannian norm and the corresponding distance. Assume moreover that $f : M \to M$ is C^1 and that μ is a probability measure which is absolutely continuous with respect to the riemannian volume on M. Let us call any continuous function $g : M \to \mathbb{R}$ an *observable*.

Let ϕ, ψ be two μ-integrable observables; one defines the *correlation function* of φ and ψ on nonnegative integers - a more standard statistical terminology would be *covariance function* - by

$$C_{\varphi,\psi}(n) = \int_M (\varphi \circ f^n)\, \psi \, d\mu - \int_M \varphi \, d\mu \int_M \psi \, d\mu.$$

The mapping f is said to be *mixing* if $C_{\varphi,\psi}(n) = o(1)$ for every observables φ, ψ. It is said to be *exponentially mixing* if there is $\tau \in [0, 1)$ such that $\tau^{-n} C_{\varphi,\psi}(n) = O(1)$ for every observables φ, ψ.

Xu, Lin and Ruan [14] have established the following result.

Theorem 2.2 *If f is mixing, then f is sensitive to initial conditions.*

Proof. See Xu, Lin and Ruan [14]. □

Now, following Wu, Xu, Lin and Ruan [13], we shall say that (f, μ) satisfies the great deviations principle if for every observable φ and every $\varepsilon > 0$, there exists $h(\varepsilon, \varphi) > 0$ such that

$$\mu(\{x \in M : |\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) - \int \varphi d\mu| > \varepsilon\}) \leq e^{-nh(\varepsilon,\varphi)},$$

for all $n \geq 1$ sufficiently large. Then, Wu, Xu, Lin and Ruan prove the following result.

Theorem 2.3 *If (f, μ) satisfies the great deviations principle and f is topologically mixing, then f is sensitive to initial conditions.*

Proof. See Wu, Xu, Lin and Ruan [13]. □

In the sequel, we shall generalize theorems 2.2 and 2.3 and complement theorem 2.1.

3 Topological mixing and sensitivity

Let us state our main result. Assume f is a continuous map on a compact metric space (M, d) which is not reduced to a single point.

Theorem 3.1 *If f is topologically mixing, then f is sensitive to initial conditions.*

In order to prove this theorem, we shall use the following lemmas.

Lemma 3.1 *If f is topologically transitive, then for every nonempty open set $U \subset M$, there is an increasing sequence of nonnegative integers $(n_k)_{k\geq 0}$ such that for every $k \geq 0$, one has*

$$f^{n_k}U \cap U \neq \emptyset.$$

Proof of lemma 3.1. It is a straightforward consequence of proposition 39 in Block and Coppel [5] - see also lemma 1 in Xu, Lin and Ruan [14]. □

Lemma 3.2 *If f is topologically transitive and is not sensitive to initial conditions, then f is not topologically mixing.*

Proof of lemma 3.2. Assume that f is not sensitive to initial conditions. This means that for any real $\sigma > 0$, there is $x_\sigma \in M$ and an open neighborhood $N(x_\sigma)$ of x_σ such that

$$\sup_{n\geq 0}\sup_{y\in N(x_\sigma)} d(f^n x_\sigma, f^n y) \leq \sigma.$$

Let us define

$$V_\sigma = \{x \in M : d(x, N(x_\sigma)) \equiv \inf_{y \in N(x_\sigma)} d(x, y) \leq 4\,\sigma\}.$$

Since M is not reduced to one point, there is σ_0 such that $\overline{V_{\sigma_0}} \equiv M \backslash V_{\sigma_0}$ is a nonempty open set.

Now, according to lemma 3.1, there is an increasing sequence of nonnegative integers $(n_k)_{k \geq 0}$ such that for every $k \geq 0$, one has

$$f^{n_k} N(x_{\sigma_0}) \cap N(x_{\sigma_0}) \neq \emptyset.$$

Let $z_k \in f^{n_k} N(x_{\sigma_0})$. On one side, if $z_k \in f^{n_k} N(x_{\sigma_0}) \cap N(x_{\sigma_0})$, then $z_k \in V_{\sigma_0}$ since $d(z_k, N(x_{\sigma_0})) = 0$.
On the other side, if

$$z_k \notin f^{n_k} N(x_{\sigma_0}) \cap N(x_{\sigma_0}),$$

that is, if

$$z_k \in f^{n_k} N(x_{\sigma_0}) \cap \overline{N(x_{\sigma_0})},$$

then, for any $y \in N(x_{\sigma_0})$ and any point $x \in f^{n_k} N(x_{\sigma_0}) \cap N(x_{\sigma_0})$, one has

$$\begin{aligned} d(z_k, y) &\leq d(z_k, x) + d(x, y) \leq d(z_k, f^{n_k} x_{\sigma_0}) + d(f^{n_k} x_{\sigma_0}, x) + d(x, x_{\sigma_0}) + d(x_{\sigma_0}, y) \\ &\leq 4\,\sigma_0. \end{aligned}$$

Hence, $z_k \in V_{\sigma_0}$. Accordingly, for every $k \geq 0$,

$$f^{n_k} N(x_{\sigma_0}) \subset V_{\sigma_0}$$

and so, $f^{n_k} N(x_{\sigma_0}) \cap \overline{V_{\sigma_0}} = \emptyset$, which contradicts topological mixing. □

Proof of theorem 3.1. Topological mixing clearly implies topological transitivity, hence the result. □

Theorem 3.1 is a generalization of the result of Wu, Xu, Lin and Ruan given above (theorem 2.3) to general nontrivial compact metric spaces for mappings which are not necessarily differentiable and do not necessarily satisfy the great deviations principle. It complements the result of Abraham, Biau and Cadre given above (theorem 2.1) in the situation where f is not measure-preserving.

4 Decay of correlations and sensitivity

As an application of theorem 3.1, we are now going to establish a generalization of the result obtained by Xu, Lin and Ruan (theorem 2.2) on the links between mixing and sensitivity to initial conditions. As above, assume (M, d) is a compact metric space endowed with its borelian σ-algebra $\mathcal{B}_M$ and a probability measure μ. Assume f is a continuous mapping from M to M, which is not necessarily μ-preserving. Then, we have the following result.

Theorem 4.1 *If supp(μ) $= M$ and f is mixing, then f is topologically mixing.*

A straightforward consequence of this theorem is the following result.

Proposition 4.1 *If supp(μ) $= M$ and f is mixing, then f is sensitive to initial conditions.*

Proof of theorem 4.1. Assume that there exists two nonempty open sets $U, V \subset M$ and an increasing sequence $(n_k)_{k\geq 1}$ of nonnegative integers such that, for any $k \geq 1$,

$$U \cap f^{n_k} V = \emptyset.$$

It is then possible to construct two continuous functions $h : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ such that

$$\int h \cdot d\mu > 0, \int g \cdot d\mu > 0$$

and satisfying

$$\begin{cases} h(x) & \geq & 0 & \text{if } x \in V \\ h(x) & = & 0 & \text{otherwise} \end{cases}$$

and

$$\begin{cases} g(x) & \geq & 0 & \text{if } x \in U \\ g(x) & = & 0 & \text{otherwise.} \end{cases}$$

To see this, let x_0 be any point in U and let $B(x_0, r_0)$ be a nonempty open ball such that $B(x_0, r_0) \subset U$. Then $\overline{B}(x_0, r_0/2) \subset U$, where $\overline{B}(x_0, r_0/2)$ denotes the closed ball with center x_0 and radius $r_0/2$, and the map h given, for any $x \in M$, by

$$h(x) = 1 - \frac{d(x, \overline{B}(x_0, r_0/2))}{d(x, \overline{B}(x_0, r_0/2)) + d(x, \overline{U})}$$

satisfies the required properties. The map g can be defined in a perfectly analogous manner.

Now, by construction, for any $k \geq 1$,

$$\int h \cdot (g \circ f^{n_k}) \cdot d\mu = 0.$$

Thus, for every nonnegative integer k, $C_{g,h}(n_k) = -\int h \cdot d\mu \cdot \int g \cdot d\mu < 0$ is a negative constant, which contradicts the mixing assumption, hence the result. □

Proof of proposition 4.1. Under the assumptions, f is topologically mixing according to theorem 4.1, and hence sensitive to initial conditions according to theorem 3.1. □

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