Mathematical Models and Methods in Applied Sciences

LONG TIME BEHAVIOR OF THE TWO-DIMENSIONAL VLASOV EQUATION WITH A STRONG EXTERNAL MAGNETIC FIELD

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When charged particles are submitted to a large external magnetic field, their movement in first approximation occurs along the magnetic field lines and obeys a one dimensional Vlasov equation along these field lines. However, when observing the particles on a sufficiently long time scale, a drift phenomenon perpendicular to the magnetic field lines superposes to this first movement. In this paper, we present a rigorous asymptotic analysis of the two-dimensional Vlasov equation when the magnetic field tends to infinity, the observation time scale growing accordingly. Techniques based on the two-scale convergence and the introduction of a second problem enable us to find an equation verified by the weak limit of the distribution function.

1. Introduction

Following our first paper [5], we continue the investigation of the Vlasov equation with a strong external magnetic field. This problem is of great importance in many plasma devices like electron guns, diodes and most of all tokamaks. In the first paper, we proved that in the limit when the magnetic field goes to infinity, the three-dimensional Vlasov equation reduces to a one dimensional equation along the magnetic field lines. In a device like the tokamak the geometry of which is toroidal, the particles then circle around the device. But when watching the particles on a long enough time scale, if those are submitted to a self consistent or external electric field, a drift phenomenon perpendicular to the magnetic field occurs and this is what we want to investigate in this paper.

In order to describe this behavior, we consider a strong magnetic field $\frac{\mathcal{M}}{\varepsilon}$, where ε is a small parameter and with $\mathcal{M} = \mathbf{e}_1$ where $\mathbf{e}_1$ is the first vector of the canonical basis of $I\!R^3$ $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. We also assume that the charged particles are submitted to an external electric field deriving from a potential the derivative of which vanishes in the direction of the magnetic field. This assumption which is somewhat restrictive enables us to decouple the two transverse directions from the parallel direction and thus consider only what happens in the plane orthogonal to the magnetic field, which is a simplified description of the physical problem. Moreover, the observation time scale we consider is $\frac{1}{\varepsilon}$.

In the context we chose, no force acts on the particles in the x_1-direction. Hence the problem may be modeled by a two-dimensional Vlasov-equation. The natural position variable is then $\mathbf{x} = (x_2, x_3) \in \mathbb{R}^2_x$, the velocity variable is $\mathbf{v} = (v_2, v_3) \in \mathbb{R}^2_v$ and the time space is $[0, T)$. We introduce the notation $\Omega = \mathbb{R}^2_x \times \mathbb{R}^2_v$, $\mathcal{O} = [0, T) \times \mathbb{R}^2_x$ and $\mathcal{Q} = [0, T) \times \Omega$.

Now, the space-velocity distribution $f_s^\varepsilon(t, \mathbf{x}, \mathbf{v})$, describing the above phenomenon for a particle species s is the solution of:

$$\begin{cases} \dfrac{\partial f_s^\varepsilon}{\partial t} + \dfrac{\mathbf{v}}{\varepsilon} \cdot \nabla_x f_s^\varepsilon + \lambda_s \left(\dfrac{-\nabla_x p}{\varepsilon} + \mathbf{v} \times \dfrac{\mathcal{M}}{\varepsilon^2}\right) \cdot \nabla_v f_s^\varepsilon = 0, \\ f_s^\varepsilon{}_{|t=0} = f_{s0}, \end{cases} \tag{1.1}$$

where λ_s is a dimensionless constant linked to the charge over mass ratio of the species s and thus characterizing this species of particles, where

$$\mathbf{v} \times \mathcal{M} = \begin{pmatrix} v_3 \\ -v_2 \end{pmatrix},$$

and where the electric potential $p(t, \mathbf{x}) \in C^1(\mathbb{R}_t, C_b^2(\mathbb{R}^2_x))$, C_b^2 being the space of bounded and twice continuously differentiable functions. The initial data satisfies

$$f_{s0} \geq 0, \qquad \int_\Omega f_{s0}\, d\mathbf{x} d\mathbf{v} < \infty, \qquad \int_\Omega f_{s0}^2\, d\mathbf{x} d\mathbf{v} < \infty. \tag{1.2}$$

Remark 1.1 We assume a L^2-bound on f_{s0} in order to get a L^2-bound on f_s^ε. This enables to get two-scale and weak$-*$ convergence of f_s^ε. Of course we could assume any L^q-bound with $q > 1$ without changing the results and methods presented here. Here, as we consider a linear Vlasov equation, we do not need any assumption on the moments of f_{s0}.

With conditions (1.2), equation (1.1) has for any positive ε a solution which is positive, which belongs to $L^\infty(0, T; L^1 \cap L^2(\Omega))$ and which satisfies

$$\|f_s^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{1.3}$$

for some constant C.

From bound (1.3), we may deduce that we can extract a subsequence such that

$$f_s^\varepsilon \rightharpoonup f_s \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak} - *, \tag{1.4}$$

but also, see [1,9], for any $\psi(t,\tau,\mathbf{x},\mathbf{v})$, regular, with compact support in $t,\mathbf{x},\mathbf{v}$ and $2\pi/\lambda_s-$periodic in τ, there exists a profile $F_s(t,\tau,\mathbf{x},\mathbf{v}) \in L^\infty(0,T;L^\infty_{2\pi/\lambda_s}(I\!R_\tau;L^2(\Omega)))$, also called the two scale limit of f_s^ε, such that

$$\int_Q f_s^\varepsilon (\psi)^\varepsilon \, dtd\mathbf{x}d\mathbf{v} \to \int_Q \int_0^{2\pi/\lambda_s} F_s \psi \, d\tau dt d\mathbf{x} d\mathbf{v}, \tag{1.5}$$

where $(\psi)^\varepsilon = \psi(t, \frac{t}{\varepsilon^2}, \mathbf{x}, \mathbf{v})$. Moreover the weak$-*$ limit f_s of $(f_s^\varepsilon)_\varepsilon$ is expressed in term of the two scale limit F_s

$$f_s = \int_0^{2\pi/\lambda_s} F_s(.,\tau,.,.) \, d\tau. \tag{1.6}$$

Notice that we adopt the following convention: if $\lambda_s < 0$, then $F_s \leq 0$.
Using this tool we shall derive the equation verified by F_s and then formula (1.6) will give us the equation for f_s for each species s of particles. Moreover, those equations will no more be genuine kinetic equations, but merely involve $\mathbf{v}$ as a parameter. Hence we shall be able, integrating over the velocity variable $\mathbf{v}$, to obtain one unique equation satisfied by the charge density

$$\rho = \sum_s q_s \int_{I\!R^2_v} f_s \, d\mathbf{v}, \tag{1.7}$$

of the whole bunch of particles, where q_s is the dimensionless charge of species s.

Let us now state the main results of the paper:

Theorem 1.1 Under assumptions (1.2), the sequence $(f_s^\varepsilon(t,\mathbf{x},\mathbf{v}))_\varepsilon$ of solutions of (1.1) weak$-*$ converges to $f_s(t,\mathbf{x},\mathbf{v})$ unique solution of

$$\begin{cases} \dfrac{\partial f_s}{\partial t} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x f_s + \dfrac{1}{2}(\Delta p \, (\mathbf{v} \times \mathcal{M})) \cdot \nabla_v f_s = 0, \\ f_{s|t=0} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_{s0}(\mathbf{x}, \mathbf{r}(\mathbf{v},\tau)) \, d\tau, \end{cases} \tag{1.8}$$

where $\mathbf{r}(\mathbf{v},\tau)$ is the rotation of angle τ applied to $\mathbf{v}$.

Moreover

Theorem 1.2 Assuming (1.2) satisfied for a set of particles, the charge density ρ defined by (1.7) of the set of particles is the unique solution of

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x \rho = 0, \\ \rho_{|t=0} = \displaystyle\sum_s q_s \int_{I\!R^2_v} f_{s0}(\mathbf{x},\mathbf{v}) \, d\mathbf{v} d\tau. \end{cases} \tag{1.9}$$

This model corresponds to the guiding-center model which has been used by physicists for some time [4,6,8,10]. This guiding center model tells that charged particles submitted to an Electric field $\mathbf{E}$ and to a Magnetic field $\mathbf{B}$ undergo a drift with velocity $||\mathbf{B}||^{-2}(\mathbf{E}\times\mathbf{B})$. The drift velocity being independent on charge and mass of the considered particles.
In our result the drift is obtained as a long time behavior, with drift velocity $(-\nabla_x p \times \mathcal{M})$, the independence on charge and mass standing in the fact that the λ_s−coefficients are not present in the limit equations (1.8) and (1.9).

Remark 1.2 Notice that looking for the guiding center model as a long time behavior does not enable to treat the case when the electric and the large magnetic fields are not orthogonal. Indeed, in this case, the usual time scale motion makes that every particle runs to infinity before the long time scale is reached.
As a consequence of this, if we do not make the orthogonality assumption, we simply cannot provide our analysis.

We are now proceeding with the demonstration of these results, which is outlined in the following way: First, using oscillating test functions, we determine a constraint equation for the profile F_s. As a consequence of this constraint equation, we have that F_s may be expressed in terms of a function G depending on an oscillating velocity variable. Then removing the main oscillation in f_s^ε and building ad-hoc oscillating test functions, we derive the equation satisfied by G. Finally we deduce the equation for the weak limit of f_s^ε and prove the theorems.

Remark 1.3 We mention that every two-scale and weak−∗ convergence has to be understood up to subsequences. Nevertheless, since every limit equation we obtain has a unique solution, we may finally deduce that the whole considered sequences converge.

2. The constraint equation and its consequences

We multiply equation (1.1), where we do not write down explicitely for the moment the index s, by $(\psi)^\varepsilon = \psi(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{v})$ where $\psi(t,\tau,\mathbf{x},\mathbf{v})$ is regular, with compact support in $(t,\mathbf{x},\mathbf{v})$ and $2\pi/\lambda-$ periodic in τ. We then integrate by parts and get the following weak formulation with oscillating test functions:

$$\int_Q f^\varepsilon[(\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon^2}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{\mathbf{v}}{\varepsilon}\cdot(\nabla_x\psi)^\varepsilon + \lambda(\frac{-\nabla_x p}{\varepsilon} + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon^2})\cdot(\nabla_v\psi)^\varepsilon]\,dt\,d\mathbf{x}\,d\mathbf{v}$$
$$= -\int_\Omega f_0\psi(0,0,.,.)\,d\mathbf{x}\,d\mathbf{v}. \quad (2.1)$$

Multiplying (2.1) by ε^2 and passing to the limit gives

$$\int_Q\int_0^{2\pi/\lambda} F(\frac{\partial\psi}{\partial\tau} + \lambda(\mathbf{v}\times\mathcal{M})\cdot\nabla_v\psi)\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v} = 0, \quad (2.2)$$

which is, as we saw in Frénod and Sonnendrücker [5], equivalent to the following constraint equation:

$$\frac{\partial F}{\partial \tau} + \lambda(\mathbf{v} \times \mathcal{M}) \cdot \nabla_v F = 0 \text{ in } \mathcal{D}'(I\!\!R_\tau \times I\!\!R^2_v), \tag{2.3}$$

for almost every $t \in [0, T)$ and almost every $\mathbf{x} \in I\!\!R^2_x$.

We now introduce the transformation $\mathbf{r}(., \tau) : I\!\!R^2 \to I\!\!R^2$ defined by

$$\mathbf{r}(\mathbf{v}, \tau) = [v_2 \cos\tau - v_3 \sin\tau]\mathbf{e}_2 + [v_2 \sin\tau + v_3 \cos\tau]\mathbf{e}_3. \tag{2.4}$$

From (2.3) we may deduce (see [5]) that there exists a function

$$G \equiv G(t, \mathbf{x}, \mathbf{u}) \in L^\infty(0, T; L^2(I\!\!R^2_x \times I\!\!R^2_u)),$$

such that

$$F(t, \tau, \mathbf{x}, \mathbf{v}) = G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)), \tag{2.5}$$

where we denote, similarly to $\mathbf{x}$ and $\mathbf{v}$, $\mathbf{u} = (u_2, u_3)$.

3. Obtention of an equation for G

In order to deduce the equation satisfied by G, we shall use first, in the weak formulation, test functions satisfying the constraint (2.3). This will yield a weak formulation in which one term is multiplied by $\frac{1}{\varepsilon}$. Choosing then appropriate test functions and using an ad hoc change of variables, we shall obtain the limit of this term. The deduction of the equation for G will then be straightforward.

For any function $\varphi(t, \mathbf{x}, \mathbf{u})$ regular and with compact support, we define $\psi(t, \tau, \mathbf{x}, \mathbf{v}) = \varphi(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau))$. Then the function ψ is $2\pi/\lambda-$periodic, satisfies the constraint, and using $(\psi)^\varepsilon$ as a test function in (2.1) leads to:

$$\int_Q f^\varepsilon[(\frac{\partial \psi}{\partial t})^\varepsilon + \frac{\mathbf{v}}{\varepsilon} \cdot (\nabla_x \psi)^\varepsilon + \lambda(\frac{-\nabla_x p}{\varepsilon}) \cdot (\nabla_v \psi)^\varepsilon]\, dt\, d\mathbf{x}\, d\mathbf{v} = -\int_\Omega f_0 \psi(0, 0, ., .)\, d\mathbf{x}\, d\mathbf{v}. \tag{3.1}$$

We now introduce the change of variables

$$\mathbf{v} \to \mathbf{u} = \mathbf{r}(\mathbf{v}, \lambda\frac{t}{\varepsilon^2}), \tag{3.2}$$

whose reverse transformation is

$$\mathbf{u} \to \mathbf{v} = \mathbf{r}(\mathbf{u}, -\lambda\frac{t}{\varepsilon^2}). \tag{3.3}$$

Using the definition of ψ, and defining $g^\varepsilon(t, \mathbf{x}, \mathbf{u})$ by $g^\varepsilon(t, \mathbf{x}, \mathbf{u}) = f^\varepsilon(t, \mathbf{x}, \mathbf{r}(\mathbf{u}, -\lambda\frac{t}{\varepsilon^2}))$ which is equivalent to $f^\varepsilon(t, \mathbf{x}, \mathbf{v}) = g^\varepsilon(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\frac{t}{\varepsilon^2}))$, since

$$\frac{\partial \psi}{\partial v_2}(t, \tau, \mathbf{x}, \mathbf{v}) = \cos\lambda\tau \frac{\partial \varphi}{\partial u_2}(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)) + \sin\lambda\tau \frac{\partial \varphi}{\partial u_3}(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)),$$

$$\frac{\partial \psi}{\partial v_3}(t,\tau,\mathbf{x},\mathbf{v}) = -\sin\lambda\tau\frac{\partial \varphi}{\partial u_2}(t,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau)) + \cos\lambda\tau\frac{\partial \varphi}{\partial u_3}(t,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau)),$$

the change of variables (3.2) transforms the equation (3.1) into

$$\begin{aligned}
&\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})[\frac{\partial \varphi}{\partial t}(t,\mathbf{x},\mathbf{u}) \\
&+ \frac{1}{\varepsilon}\{(u_2\cos\lambda\frac{t}{\varepsilon^2} + u_3\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial \varphi}{\partial x_2}(t,\mathbf{x},\mathbf{u}) \\
&+ (-u_2\sin\lambda\frac{t}{\varepsilon^2} + u_3\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial \varphi}{\partial x_3}(t,\mathbf{x},\mathbf{u}) \\
&+ \lambda[((-\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial \varphi}{\partial u_2}(t,\mathbf{x},\mathbf{u}) \\
&+ ((-\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} + (-\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial \varphi}{\partial u_3}(t,\mathbf{x},\mathbf{u})]\}]\,dt\,d\mathbf{x}\,d\mathbf{v} \\
&\qquad = -\int_\Omega f_0\psi(0,0,\mathbf{x},\mathbf{v})\,d\mathbf{x}\,d\mathbf{v} = -\int_\Omega f_0\varphi(0,\mathbf{x},\mathbf{u})\,d\mathbf{x}\,d\mathbf{u}. \qquad (3.4)
\end{aligned}$$

Now we focus, for a few pages, on passing to the limit in (3.4). For this purpose, we need to characterize the weak$-*$ limit and the two scale limit of g^ε, and to compute the limit of the term containing the $\frac{1}{\varepsilon}$ in factor.

First, we have

Lemma 3.1 The two scale limit of g^ε is the function G defined by (2.5). And, since G does not depend on τ, the weak$-*$ limit g of g^ε is $(2\pi/\lambda)G$.

Proof. Since the sequence (f^ε) is bounded, we have that (g^ε) is also bounded. Hence (g^ε) two-scale and weak$-*$ converges.

We take any function $\varphi(t,\tau,\mathbf{x},\mathbf{u})$, regular, with compact support in $(t,\mathbf{x},\mathbf{u})$ and $2\pi/\lambda-$ periodic in τ. On the one hand, denoting by $\bar{G}$ the two scale limit of g^ε, we have

$$\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})\varphi(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})\,dt\,d\mathbf{x}\,d\mathbf{u} \to \int_{Q'}\int_0^{2\pi/\lambda} \bar{G}(t,\tau,\mathbf{x},\mathbf{u})\varphi(t,\tau,\mathbf{x},\mathbf{u})\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v}. \qquad (3.5)$$

On the other hand,

$$\begin{aligned}
&\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})\varphi(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})\,dt\,d\mathbf{x}\,d\mathbf{u} = \\
&\qquad \int_Q g^\varepsilon(t,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))\varphi(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))\,dt\,d\mathbf{x}\,d\mathbf{v} = \\
&\qquad\qquad \int_Q f^\varepsilon(t,\mathbf{x},\mathbf{v})\varphi(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))\,dt\,d\mathbf{x}\,d\mathbf{v},
\end{aligned}$$

which converges to

$$\int_Q\int_0^{2\pi/\lambda} F(t,\tau,\mathbf{x},\mathbf{v})\varphi(t,\tau,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau))\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v} =$$

$$\int_Q \int_0^{2\pi/\lambda} G(t,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau))\varphi(t,\tau,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau))\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{v} =$$
$$\int_{Q'} \int_0^{2\pi/\lambda} G(t,\mathbf{x},\mathbf{u})\varphi(t,\tau,\mathbf{x},\mathbf{u})\,d\tau\,dt\,d\mathbf{x}\,d\mathbf{u}.$$

Thus, we may conclude

$$\bar{G}(t,\tau,\mathbf{x},\mathbf{u}) = G(t,\mathbf{x},\mathbf{u}),$$

which implies that G is the two scale limit of g^ε. The first part of the Lemma is proved. Moreover, using that the link between the two-scale and the weak$-*$ limits and the fact that G does not depend on τ, we have:

$$\begin{aligned} g(t,\tau,\mathbf{u}) &= \int_0^{2\pi/\lambda} \bar{G}(t,\tau,\mathbf{x},\mathbf{u}))\,d\tau, \\ &= \int_0^{2\pi/\lambda} G(t,\mathbf{x},\mathbf{u})\,d\tau, \\ &= \frac{2\pi}{\lambda} G(t,\mathbf{x},\mathbf{u}), \end{aligned}$$

prooving the second part of Lemma 3.1 □ .

Concerning the computation of the term containing the $\frac{1}{\varepsilon}$ factor in (3.4) we have:

Lemma 3.2 Denoting by $G(t,\mathbf{x},\mathbf{u})$ the two scale limit of g^ε, which is also the function linked to the two scale limit F of f^ε by (2.5), the following formula holds true for all test functions $\varphi(t,\mathbf{x},\mathbf{u})$:

$$\begin{aligned} \lim_{\varepsilon\to 0}\{ \int_{Q'} & g^\varepsilon(t,\mathbf{x},\mathbf{u})\frac{1}{\varepsilon}[(u_2\cos\lambda\frac{t}{\varepsilon^2} + u_3\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial x_2}(t,\mathbf{x},\mathbf{u}) \\ &+(-u_2\sin\lambda\frac{t}{\varepsilon^2} + u_3\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial x_3}(t,\mathbf{x},\mathbf{u}) \\ &+\lambda((-\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_2}(t,\mathbf{x},\mathbf{u}) \\ &+\lambda((-\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} - (\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_3}(t,\mathbf{x},\mathbf{u})\,dt\,d\mathbf{x}\,d\mathbf{u}\} = \\ & -\frac{2\pi}{\lambda}\int_{Q'} G(t,\mathbf{x},\mathbf{u})[\partial_{x_3}p\frac{\partial\varphi}{\partial x_2} - \partial_{x_2}p\frac{\partial\varphi}{\partial x_3} \\ &\qquad +\frac{1}{2}(\partial^2_{x_2}p + \partial^2_{x_3}p)(-u_3\frac{\partial\varphi}{\partial u_2} + u_2\frac{\partial\varphi}{\partial u_3})]\,dt\,d\mathbf{x}\,d\mathbf{u}. \end{aligned} \quad (3.6)$$

Proof. As in the singular perturbation theory, see [7], we are looking for the limit of the inner product of g^ε by a function which is not bounded but which converges to 0 in the sense of distributions. Indeed since

$$\frac{1}{\varepsilon}\cos\lambda\frac{t}{\varepsilon^2} = \frac{d}{dt}\left(\frac{\varepsilon}{\lambda}\sin\lambda\frac{t}{\varepsilon^2}\right),\quad \frac{1}{\varepsilon}\sin\lambda\frac{t}{\varepsilon^2} = \frac{d}{dt}\left(-\frac{\varepsilon}{\lambda}\cos\lambda\frac{t}{\varepsilon^2}\right). \quad (3.7)$$

those functions converges to 0 in the sense of distributions. Using this Remark, we develop a method which is in the same spirit as the one used by Cioranescu and Murat [2,3].

For any φ, we build an oscillating test function α whose $\tau-$derivative when applied in $\tau = \frac{t}{\varepsilon^2}$ gives $\frac{1}{\varepsilon}\,[(u_2 \cos\lambda\frac{t}{\varepsilon^2} + u_3 \sin\lambda\frac{t}{\varepsilon^2})\,\frac{\partial\varphi}{\partial x_2} + (-u_2 \sin\lambda\frac{t}{\varepsilon^2} + u_3\cos\lambda\frac{t}{\varepsilon^2})\,\frac{\partial\varphi}{\partial x_3} + \lambda((-\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_2} + \lambda((-\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} - (\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_3}$. Choosing the function α as a test function in the weak formulation (2.1) gives first an expression of the limit we want to compute in term of α. Then using the expression of the function α, an easy but fastidious integration leads to the result.

For any regular $\varphi(t, \mathbf{x}, \mathbf{u})$, we define

$$\begin{aligned}
\alpha(t,\tau,\mathbf{x},\mathbf{u}) &= \frac{1}{\lambda}(u_2\sin\lambda\tau - u_3\cos\lambda\tau)\frac{\partial\varphi}{\partial x_2}(t,\mathbf{x},\mathbf{u})\\
&+\frac{1}{\lambda}(u_2\cos\lambda\tau + u_3\sin\lambda\tau)\frac{\partial\varphi}{\partial x_3}(t,\mathbf{x},\mathbf{u})\\
&+((-\partial_{x_2}p)\sin\lambda\tau - (\partial_{x_3}p)\cos\lambda\tau)\frac{\partial\varphi}{\partial u_2}(t,\mathbf{x},\mathbf{u})\\
&\qquad+((\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau)\frac{\partial\varphi}{\partial u_3}(t,\mathbf{x},\mathbf{u}),
\end{aligned}\tag{3.8}$$

and we consider

$$\beta(t,\tau,\mathbf{x},\mathbf{v}) = \alpha(t,\tau,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau)).\tag{3.9}$$

Applying the constraint operator to β, we get

$$\begin{aligned}
&\frac{\partial\beta}{\partial\tau} + \lambda(\mathbf{v}\times\mathcal{M})\cdot\nabla_v\beta = \\
&\quad\frac{\partial\alpha}{\partial\tau} + \lambda\frac{\partial\mathbf{r}}{\partial\tau}(\mathbf{v},\lambda\tau)\cdot\nabla_u\alpha + \lambda(\mathbf{v}\times\mathcal{M})\cdot\nabla_u\alpha = \left(\frac{\partial\alpha}{\partial\tau}\right)(t,\tau,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\tau)),
\end{aligned}\tag{3.10}$$

and the computation of $\frac{\partial\alpha}{\partial\tau}$ gives

$$\begin{aligned}
\left(\frac{\partial\alpha}{\partial\tau}\right)(t,\tau,\mathbf{x},\mathbf{u}) &= (u_2\cos\lambda\tau + u_3\sin\lambda\tau)\frac{\partial\varphi}{\partial x_2}(t,\mathbf{x},\mathbf{u})\\
&+(-u_2\sin\lambda\tau + u_3\cos\lambda\tau)\frac{\partial\varphi}{\partial x_3}(t,\mathbf{x},\mathbf{u})\\
&+\lambda((-\partial_{x_2}p)\cos\lambda\tau + (\partial_{x_3}p)\sin\lambda\tau)\frac{\partial\varphi}{\partial u_2}(t,\mathbf{x},\mathbf{u})\\
&\qquad+\lambda((-\partial_{x_2}p)\sin\lambda\tau - (\partial_{x_3}p)\cos\lambda\tau)\frac{\partial\varphi}{\partial u_3}(t,\mathbf{x},\mathbf{u}).
\end{aligned}\tag{3.11}$$

Beside this, easy computations give

$$(\frac{\partial\beta}{\partial t})^\varepsilon = \left(\frac{\partial\alpha}{\partial t}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})),\tag{3.12}$$

$$(\frac{\partial\beta}{\partial x_i})^\varepsilon = \left(\frac{\partial\alpha}{\partial x_i}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})), \tag{3.13}$$

$$(\frac{\partial\beta}{\partial v_2})^\varepsilon = \cos\lambda\frac{t}{\varepsilon^2}\left(\frac{\partial\alpha}{\partial u_2}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})) + \sin\lambda\frac{t}{\varepsilon^2}\left(\frac{\partial\alpha}{\partial u_3}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})), \tag{3.14}$$

$$(\frac{\partial\beta}{\partial v_3})^\varepsilon = -\sin\lambda\frac{t}{\varepsilon^2}\left(\frac{\partial\alpha}{\partial u_2}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})) + \cos\lambda\frac{t}{\varepsilon^2}\left(\frac{\partial\alpha}{\partial u_3}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})). \tag{3.15}$$

Having those formulae at hand, we use test functions

$$\varepsilon(\beta)^\varepsilon = \varepsilon\beta(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{v}), \tag{3.16}$$

in the weak formulation (2.1). We get

$$\int_Q f^\varepsilon[\varepsilon(\frac{\partial\beta}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\beta}{\partial\tau})^\varepsilon + \mathbf{v}\cdot(\nabla_x\beta)^\varepsilon + \lambda(-\nabla_x p + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon})\cdot(\nabla_v\beta)^\varepsilon]\,dt\,d\mathbf{x}\,d\mathbf{v} = -\varepsilon\int_\Omega f_0\beta(0,0,.,.)\,d\mathbf{x}\,d\mathbf{v}. \tag{3.17}$$

Now, using formula (3.10), the expression of f^ε in term of g^ε and the expressions of the derivatives of β (3.11)-(3.15), equation (3.17) becomes

$$\begin{aligned}\int_Q & g^\varepsilon(t,\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))[\varepsilon\left(\frac{\partial\alpha}{\partial t}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))\\ &+\frac{1}{\varepsilon}\left(\frac{\partial\alpha}{\partial\tau}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2})) + \mathbf{v}\cdot(\nabla_x\alpha)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))\\ &-\lambda\begin{pmatrix}(\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} - (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2}\\ (\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2}\end{pmatrix}\cdot(\nabla_u\alpha)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{r}(\mathbf{v},\lambda\frac{t}{\varepsilon^2}))]dtd\mathbf{x}d\mathbf{v}\\ &= -\varepsilon\int_\Omega f_0\alpha(0,0,.,.)\,d\mathbf{x}\,d\mathbf{v}.\end{aligned} \tag{3.18}$$

In equation (3.18) we make the change of variable (3.2), it gives

$$\begin{aligned}\int_{Q'} & g^\varepsilon(t,\mathbf{x},\mathbf{u})[\varepsilon\left(\frac{\partial\alpha}{\partial t}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})\\ &+\frac{1}{\varepsilon}\left(\frac{\partial\alpha}{\partial\tau}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u}) + \mathbf{r}(\mathbf{u},-\lambda\frac{t}{\varepsilon^2})\cdot(\nabla_x\alpha)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})\\ &-\lambda\begin{pmatrix}(\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} - (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2}\\ (\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2}\end{pmatrix}\cdot(\nabla_u\alpha)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})]dtd\mathbf{x}d\mathbf{u}\\ &= -\varepsilon\int_\Omega f_0\alpha(0,0,.,.)\,d\mathbf{x}\,d\mathbf{u},\end{aligned} \tag{3.19}$$

which, passing to the limit in ε, using Lemma 3.1 gives

$$\begin{aligned}
\lim_{\varepsilon\to 0}\{\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})\frac{1}{\varepsilon}\left(\frac{\partial\alpha}{\partial\tau}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})dtd\mathbf{x}d\mathbf{u}\} = \\
-\int_{Q'} G\int_0^{2\pi/\lambda}[\begin{pmatrix} u_2\cos\lambda\tau + u_3\sin\lambda\tau \\ -u_2\sin\lambda\tau + u_3\cos\lambda\tau\end{pmatrix}\cdot(\nabla_x\alpha)(t,\tau,\mathbf{x},\mathbf{u}) \\
-\lambda\begin{pmatrix}(\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau \\ (\partial_{x_2}p)\sin\lambda\tau + (\partial_{x_3}p)\cos\lambda\tau\end{pmatrix}\cdot(\nabla_u\alpha)(t,\tau,\mathbf{x},\mathbf{u})]\,d\tau\;dtd\mathbf{x}d\mathbf{u},
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})\frac{1}{\varepsilon}\left(\frac{\partial\alpha}{\partial\tau}\right)(t,\frac{t}{\varepsilon^2},\mathbf{x},\mathbf{u})dtd\mathbf{x}d\mathbf{u} = \\
\int_{Q'} g^\varepsilon(t,\mathbf{x},\mathbf{u})\frac{1}{\varepsilon}[(u_2\cos\lambda\frac{t}{\varepsilon^2} + u_3\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial x_2}(t,\mathbf{x},\mathbf{u}) \\
+(-u_2\sin\lambda\frac{t}{\varepsilon^2} + u_3\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial x_3}(t,\mathbf{x},\mathbf{u}) \\
+\lambda((-\partial_{x_2}p)\cos\lambda\frac{t}{\varepsilon^2} + (\partial_{x_3}p)\sin\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_2}(t,\mathbf{x},\mathbf{u}) \\
+\lambda((-\partial_{x_2}p)\sin\lambda\frac{t}{\varepsilon^2} - (\partial_{x_3}p)\cos\lambda\frac{t}{\varepsilon^2})\frac{\partial\varphi}{\partial u_3}(t,\mathbf{x},\mathbf{u})]\,dt\,d\mathbf{x}\,d\mathbf{u},
\end{aligned} \tag{3.21}$$

is exactly the term whose limit we want to compute in (3.6).

At this level, formula (3.20) gives the seeked limit in terms of α. Hence we now have to replace, in the right hand side of (3.20) α by its expression in term of φ and to integrate in τ. This step is straightforward but fastidious. The detailed computation yielding the following formula is given in the appendix.
We have

$$\begin{aligned}
\int_0^{2\pi/\lambda}[\begin{pmatrix} u_2\cos\lambda\tau + u_3\sin\lambda\tau \\ -u_2\sin\lambda\tau + u_3\cos\lambda\tau\end{pmatrix}\cdot(\nabla_x\alpha) \\
-\lambda\begin{pmatrix}(\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau \\ (\partial_{x_2}p)\sin\lambda\tau + (\partial_{x_3}p)\cos\lambda\tau\end{pmatrix}\cdot(\nabla_u\alpha)]\,d\tau = \\
\frac{2\pi}{\lambda}(\partial_{x_3}p\frac{\partial\varphi}{\partial x_2} - \partial_{x_2}p\frac{\partial\varphi}{\partial x_3}) + \frac{\pi}{\lambda}(\partial^2_{x_2}p + \partial^2_{x_3}p)(-u_3\frac{\partial\varphi}{\partial u_2} + u_2\frac{\partial\varphi}{\partial u_3})].
\end{aligned} \tag{3.22}$$

Using then (3.22) in (3.20) gives the Lemma $\square$.

Since Lemma 3.1 and Lemma 3.2 give the two scale$-$ and weak$-*-$limits of g^ε and the limit of the term containing the $\frac{1}{\varepsilon}$ in (3.4), we can pass to the limit and obtain

$$\frac{2\pi}{\lambda}\int_{Q'} G[\frac{\partial\varphi}{\partial t} - \partial_{x_3}p\frac{\partial\varphi}{\partial x_2} + \partial_{x_2}p\frac{\partial\varphi}{\partial x_3} + \frac{1}{2}(\partial^2_{x_2}p + \partial^2_{x_3}p)(u_3\frac{\partial\varphi}{\partial u_2} - u_2\frac{\partial\varphi}{\partial u_3})]\;dtd\mathbf{x}d\mathbf{u}$$

$$= -\int_\Omega f_0 \varphi(0, \mathbf{x}, \mathbf{u})\, d\mathbf{x} d\mathbf{u}, \tag{3.23}$$

for all $\varphi(t, \mathbf{x}, \mathbf{v})$ regular and with compact support, which is the weak formulation of the equation for G we are seeking. Thus noticing

$$-\partial_{x_3} p \frac{\partial \varphi}{\partial x_2} + \partial_{x_2} p \frac{\partial \varphi}{\partial x_3} = (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x \varphi, \tag{3.24}$$

and

$$u_3 \frac{\partial \varphi}{\partial u_2} - u_2 \frac{\partial \varphi}{\partial u_3} = (\mathbf{u} \times \mathcal{M}) \cdot \nabla_u \varphi, \tag{3.25}$$

and denoting

$$\Delta p = \partial^2_{x_2} p + \partial^2_{x_3} p, \tag{3.26}$$

we have proved the

Theorem 3.3 Under assumptions (1.2), the sequence $(f^\varepsilon(t, \mathbf{x}, \mathbf{v}))_\varepsilon$ of solutions of (1.1) two-scale converges to a profile $F(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^\infty_{2\pi/\lambda}(\mathbb{R}_\tau; L^2(\Omega)))$ satisfying

$$F(t, \tau, \mathbf{x}, \mathbf{v}) = G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)), \tag{3.27}$$

where $\mathbf{r}(\mathbf{v}, \tau)$ is given by (2.4) and where $G \equiv G(t, \mathbf{x}, \mathbf{u}) \in L^\infty(0, T; L^2(\mathbb{R}^2_x \times \mathbb{R}^2_u))$ is the unique solution of

$$\begin{cases} \dfrac{\partial G}{\partial t} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x G + \dfrac{1}{2}(\Delta p\, (\mathbf{u} \times \mathcal{M})) \cdot \nabla_u G = 0, \\ G_{|t=0} = \dfrac{\lambda}{2\pi} f_0(\mathbf{x}, \mathbf{u}). \end{cases} \tag{3.28}$$

Remark 3.1 As a consequence of the uniqueness of the solution of (3.28), we have that the whole sequences (f^ε) and (g^ε) two-scale and weak$-*$ converge.

4. Equation for f, proof of the Theorems

Using the relation (1.6) linking f and F, we have

$$f = \int_0^{2\pi/\lambda} F(., \tau, ., .)\, d\tau = \int_0^{2\pi/\lambda} G(., ., \mathbf{r}(., \lambda\tau))\, d\tau, \tag{4.1}$$

and noticing

$$(\mathbf{r}(\mathbf{v}, \lambda\tau) \times \mathcal{M})) \cdot \nabla_u G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)) = (\mathbf{v} \times \mathcal{M}) \cdot \nabla_v [G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau))], \tag{4.2}$$

replacing $\mathbf{u}$ by $\mathbf{r}(\mathbf{v}, \lambda\tau)$ in (3.28) gives

$$\begin{cases} \dfrac{\partial}{\partial t}[G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau))] + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x [G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau))] \\ \qquad + \dfrac{1}{2}(\Delta p\, (\mathbf{v} \times \mathcal{M})) \cdot \nabla_v [G(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau))] = 0, \\ G(0, \mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)) = \dfrac{\lambda}{2\pi} f_0(\mathbf{x}, \mathbf{r}(\mathbf{v}, \lambda\tau)). \end{cases} \tag{4.3}$$

Integrating (4.3) in τ from 0 to $2\pi/\lambda$ finally gives equation (1.8) and achieve the proof of Theorem 1.1 $\square$.

Noticing

$$\nabla_v \cdot (\Delta p\,(\mathbf{v} \times \mathcal{M})) = 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{I\!R_v^2} f_0(\mathbf{x}, \mathbf{r}(\mathbf{v}, \tau))\, d\mathbf{v} d\tau = \int_{I\!R_v^2} f_0(\mathbf{x}, \mathbf{v})\, d\mathbf{v},$$

the proof of Theorem 1.2 is straightforward integrating (1.8) in $\mathbf{v}$ and summing in s $\square$.

Appendix: Computation yielding (3.22)

In this appendix, we gives the fastidious computation yielding formula (3.22). We have

$$\int_0^{2\pi/\lambda} [\begin{pmatrix} u_2\cos\lambda\tau + u_3\sin\lambda\tau \\ -u_2\sin\lambda\tau + u_3\cos\lambda\tau \end{pmatrix} \cdot (\nabla_x\alpha) - \lambda\begin{pmatrix} (\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau \\ (\partial_{x_2}p)\sin\lambda\tau + (\partial_{x_3}p)\cos\lambda\tau \end{pmatrix} \cdot (\nabla_u\alpha)\,]\, d\tau =$$

$$\int_0^{2\pi/\lambda} [\mathbf{r}(\mathbf{u}, -\lambda\tau) \cdot (\nabla_x\alpha) - \lambda\mathbf{r}(\nabla_x p, \lambda\tau) \cdot (\nabla_u\alpha)]\, d\tau. \qquad \text{(A.1)}$$

Using the definition of α in terms of φ, we have

$$\alpha(t, \tau, \mathbf{x}, \mathbf{u}) = \frac{1}{\lambda}\mathbf{r}(\mathbf{u}, \frac{\pi}{2} - \lambda\tau) \cdot \nabla_x\varphi(t, \mathbf{x}, \mathbf{u}) + \mathbf{r}(\nabla_x p, \frac{\pi}{2} + \lambda\tau) \cdot \nabla_u\varphi(t, \mathbf{x}, \mathbf{u}). \quad \text{(A.2)}$$

Hence the integral to compute becomes

$$\begin{aligned}\int_0^{2\pi/\lambda} [\frac{1}{\lambda}\mathbf{r}(\mathbf{u}, -\lambda\tau) \cdot (\nabla_x(\nabla_x\varphi))\mathbf{r}(\mathbf{u}, \frac{\pi}{2} - \lambda\tau) \\ -\lambda\mathbf{r}(\nabla_x p, \lambda\tau) \cdot (\nabla_u(\nabla_u\varphi))\mathbf{r}(\nabla_x p, \frac{\pi}{2} + \lambda\tau) \\ + \mathbf{r}(\mathbf{u}, -\lambda\tau) \cdot \nabla_x\{\mathbf{r}(\nabla_x p, \frac{\pi}{2} + \lambda\tau) \cdot \nabla_u\varphi\} \\ -\mathbf{r}(\nabla_x p, \lambda\tau) \cdot \nabla_u\{\frac{1}{\lambda}\mathbf{r}(\mathbf{u}, \frac{\pi}{2} - \lambda\tau) \cdot \nabla_x\varphi\}]\, d\tau.\end{aligned} \qquad \text{(A.3)}$$

Now, since $(\nabla_x(\nabla_x\varphi))$ and $(\nabla_u(\nabla_u\varphi))$ are symmetric matrices, we easyly deduce

$$\int_0^{2\pi/\lambda} \mathbf{r}(\mathbf{u}, -\lambda\tau) \cdot (\nabla_x(\nabla_x\varphi))\mathbf{r}(\mathbf{u}, \frac{\pi}{2} - \lambda\tau)\, d\tau =$$

$$\int_0^{2\pi/\lambda} \mathbf{r}(\nabla_x p, \lambda\tau) \cdot (\nabla_u(\nabla_u\varphi))\mathbf{r}(\nabla_x p, \frac{\pi}{2} + \lambda\tau)\, d\tau = 0, \qquad \text{(A.4)}$$

and computing

$$\nabla_x\{\mathbf{r}(\nabla_x p, \frac{\pi}{2} + \lambda\tau) \cdot \nabla_u\varphi\} \text{ and } \nabla_u\{\frac{1}{\lambda}\mathbf{r}(\mathbf{u}, \frac{\pi}{2} - \lambda\tau) \cdot \nabla_x\varphi\}, \qquad \text{(A.5)}$$

(A.3) gives:

$$\int_0^{2\pi/\lambda} [\begin{pmatrix} u_2\cos\lambda\tau + u_3\sin\lambda\tau \\ -u_2\sin\lambda\tau + u_3\cos\lambda\tau \end{pmatrix} \cdot (\nabla_x\alpha) - \lambda \begin{pmatrix} (\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau \\ (\partial_{x_2}p)\sin\lambda\tau + (\partial_{x_3}p)\cos\lambda\tau \end{pmatrix} \cdot (\nabla_u\alpha)]\, d\tau =$$

$$\begin{aligned}
&\int_0^{2\pi/\lambda} [(u_2\cos\lambda\tau + u_3\sin\lambda\tau)\{ \\
&+((-\partial^2_{x_2x_2}p)\sin\lambda\tau - (\partial^2_{x_2x_3}p)\cos\lambda\tau)(\frac{\partial\varphi}{\partial u_2}) + ((-\partial_{x_2}p)\sin\lambda\tau - (\partial_{x_3}p)\cos\lambda\tau)(\frac{\partial^2\varphi}{\partial x_2\partial u_2}) \\
&+((\partial^2_{x_2x_2}p)\cos\lambda\tau - (\partial^2_{x_2x_3}p)\sin\lambda\tau)(\frac{\partial\varphi}{\partial u_3}) + ((\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau)(\frac{\partial^2\varphi}{\partial x_2\partial u_3})\} \\
&+(-u_2\sin\lambda\tau + u_3\cos\lambda\tau)\{ \\
&+((-\partial^2_{x_2x_3}p)\sin\lambda\tau - (\partial^2_{x_3x_3}p)\cos\lambda\tau)(\frac{\partial\varphi}{\partial u_2}) + ((-\partial_{x_2}p)\sin\lambda\tau - (\partial_{x_3}p)\cos\lambda\tau)(\frac{\partial^2\varphi}{\partial x_3\partial u_2}) \\
&+((\partial^2_{x_2x_3}p)\cos\lambda\tau - (\partial^2_{x_3x_3}p)\sin\lambda\tau)(\frac{\partial\varphi}{\partial u_3}) + ((\partial_{x_2}p)\cos\lambda\tau - (\partial_{x_3}p)\sin\lambda\tau)(\frac{\partial^2\varphi}{\partial x_3\partial u_3})\} \\
&+((-\partial_{x_2}p)\cos\lambda\tau + (\partial_{x_3}p)\sin\lambda\tau)\{\sin\lambda\tau(\frac{\partial\varphi}{\partial x_2}) + \cos\lambda\tau(\frac{\partial\varphi}{\partial x_3}) \\
&+(u_2\sin\lambda\tau - u_3\cos\lambda\tau)(\frac{\partial^2\varphi}{\partial x_2\partial u_2}) + (u_2\cos\lambda\tau + u_3\sin\lambda\tau)(\frac{\partial^2\varphi}{\partial x_3\partial u_2})\} \\
&+((-\partial_{x_2}p)\sin\lambda\tau - (\partial_{x_3}p)\cos\lambda\tau)\{-\cos\lambda\tau(\frac{\partial\varphi}{\partial x_2}) + \sin\lambda\tau(\frac{\partial\varphi}{\partial x_3}) \\
&(u_2\sin\lambda\tau - u_3\cos\lambda\tau)(\frac{\partial^2\varphi}{\partial x_2\partial u_3}) + (u_2\cos\lambda\tau + u_3\sin\lambda\tau)(\frac{\partial^2\varphi}{\partial x_3\partial u_3})\}]d\tau.
\end{aligned} \tag{A.6}$$

In order to provide the computations in (A.6) we use (to reduce the number of terms) $\int_0^{2\pi/\lambda}\sin\lambda\tau\cos\lambda\tau\, d\tau = 0$. We then get

$$\begin{aligned}
&\int_0^{2\pi/\lambda} [(\frac{\partial\varphi}{\partial x_2})(\cos^2\lambda\tau + \sin^2\lambda\tau)(\partial_{x_3}p) - (\frac{\partial\varphi}{\partial x_3})(\cos^2\lambda\tau + \sin^2\lambda\tau)(\partial_{x_2}p) \\
&+(\frac{\partial\varphi}{\partial u_2})(-u_2(\cos^2\lambda\tau - \sin^2\lambda\tau)(\partial^2_{x_2x_3}p) - u_3\sin^2\lambda\tau(\partial^2_{x_2x_2}p) - u_3\cos^2\lambda\tau(\partial^2_{x_3x_3}p)) \\
&+(\frac{\partial\varphi}{\partial u_3})(u_3(\cos^2\lambda\tau - \sin^2\lambda\tau)(\partial^2_{x_2x_3}p) + u_2\cos^2\lambda\tau(\partial^2_{x_2x_2}p) + u_2\sin^2\lambda\tau(\partial^2_{x_3x_3}p)) \\
&+(\frac{\partial^2\varphi}{\partial x_2\partial u_2})(-u_2(\partial_{x_3}p)(\cos^2\lambda\tau - \sin^2\lambda\tau) + u_3(\partial_{x_2}p)(\cos^2\lambda\tau - \sin^2\lambda\tau)) \\
&+(\frac{\partial^2\varphi}{\partial x_2\partial u_3})(u_2(\partial_{x_2}p)(\cos^2\lambda\tau - \sin^2\lambda\tau) + u_3(\partial_{x_3}p)(\cos^2\lambda\tau - \sin^2\lambda\tau)) \\
&+(\frac{\partial^2\varphi}{\partial x_3\partial u_2})(-u_2(\partial_{x_2}p)(\cos^2\lambda\tau - \sin^2\lambda\tau) - u_3(\partial_{x_3}p)(\cos^2\lambda\tau - \sin^2\lambda\tau)) \\
&+(\frac{\partial^2\varphi}{\partial x_3\partial u_3})(-u_2(\partial_{x_3}p)(\cos^2\lambda\tau - \sin^2\lambda\tau) + u_3(\partial_{x_2}p)(\cos^2\lambda\tau - \sin^2\lambda\tau))]d\tau.
\end{aligned} \tag{A.7}$$

Finally, using that $\int_0^{2\pi/\lambda}(\cos^2\lambda\tau + \sin^2\lambda\tau)d\tau = 2\pi/\lambda$, $\int_0^{2\pi/\lambda}(\cos^2\lambda\tau - \sin^2\lambda\tau)d\tau = 0$, $\int_0^{2\pi/\lambda}\cos^2\lambda\tau d\tau = \pi/\lambda$, and $\int_0^{2\pi/\lambda}\sin^2\lambda\tau d\tau = \pi/\lambda$, (A.7) gives

$$\frac{2\pi}{\lambda}(\partial_{x_3}p\frac{\partial\varphi}{\partial x_2} - \partial_{x_2}p\frac{\partial\varphi}{\partial x_3}) + \frac{\pi}{\lambda}(\partial^2_{x_2}p + \partial^2_{x_3}p)(-u_3\frac{\partial\varphi}{\partial u_2} + u_2\frac{\partial\varphi}{\partial u_3}), \tag{A.8}$$

which is the left hand side of (3.22) □ .

1. G. Allaire. Homogeneization and Two-scale Convergence. *SIAM J. Math. Anal.*, XXIII(6):1482–1518, 1992.
2. D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs. In Research Note in Mathematics H. Brezis, J. L. Lions, editor, *Nonlinear partial differential equations and their application, Collège de France Seminar*, volume 2, pages 98–138. Pitman, London, 1982.
3. D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs ii. In Research Note in Mathematics H. Brezis, J. L. Lions, editor, *Nonlinear partial differential equations and their application, Collège de France Seminar*, volume 3, pages 154–178. Pitman, London, 1982.
4. D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee. Nonlinear gyrokinetic equations. *Phys. Fluids*, XXVI(12):3524–3535, 1983.
5. E. Frénod and E. Sonnendrücker. Homogenization of the Vlasov equation and of the Vlasov-Poisson system with a strong external magnetic field. *to appear.*
6. W. W. Lee. Gyrokinetic approach in particle simulation. *Phys. Fluids*, 26(2):556–562, 1983.
7. J. L. Lions. *Pertubations singulières dans les problemes aux limites et en controle optimal.* Number 323 in Lecture notes in Mathematics. Springer-Verlag, 1973.
8. Robert G. Littlejohn. Hamiltonian formulation of guiding center motion. *Phys. Fluids*, 24, 1981.
9. G. N'Guetseng. A general convergence result for a functional related to the theory of homogeneization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.
10. Theodore G. Northrop. The guiding center approximation to charged particle motion. *Ann. Phys.*, 15:79–101, 1961.