LONG TIME BEHAVIOR OF THE TWO-DIMENSIONAL VLASOV EQUATION WITH A STRONG EXTERNAL MAGNETIC FIELD

E. FRÉNOD and E. SONNENDRÜCKER
Laboratoire de Mathématiques et Applications des Mathématiques
Université de Bretagne Sud, 1 rue de la loi, F-56000 Vannes
Emanuel.Frenod@univ-ubs.fr

Institut Élie Cartan, Université Henri Poincaré
Nancy 1, F-54506 Vandœuvre-lès-Nancy cedex
Eric.Sonnendrucker@antares.iecn.u-nancy.fr

When charged particles are submitted to a large external magnetic field, their movement in first approximation occurs along the magnetic field lines and obeys a one dimensional Vlasov equation along these field lines. However, when observing the particles on a sufficiently long time scale, a drift phenomenon perpendicular to the magnetic field lines superposes to this first movement. In this paper, we present a rigorous asymptotic analysis of the two-dimensional Vlasov equation when the magnetic field tends to infinity, the observation time scale growing accordingly. Techniques based on the two-scale convergence and the introduction of a second problem enable us to find an equation verified by the weak limit of the distribution function.

1. Introduction

Following our first paper \(^5\), we continue the investigation of the Vlasov equation with a strong external magnetic field. This problem is of great importance in many plasma devices like electron guns, diodes and most of all tokamaks. In the first paper, we proved that in the limit when the magnetic field goes to infinity, the three-dimensional Vlasov equation reduces to a one dimensional equation along the magnetic field lines. In a device like the tokamak the geometry of which is toroidal, the particles then circle around the device. But when watching the particles on a long enough time scale, if those are submitted to a self consistent or external electric field, a drift phenomenon perpendicular to the magnetic field occurs and this is what we want to investigate in this paper.

In order to describe this behavior, we consider a strong magnetic field \(\frac{M}{\varepsilon}\), where \(\varepsilon\) is a small parameter and with \(M = e_1\) where \(e_1\) is the first vector of the canonical basis of \(\mathbb{R}^3\) \((e_1,e_2,e_3)\). We also assume that the charged particles are submitted to
an external electric field deriving from a potential the derivative of which vanishes in the direction of the magnetic field. This assumption which is somewhat restrictive enables us to decouple the two transverse directions from the parallel direction and thus consider only what happens in the plane orthogonal to the magnetic field, which is a simplified description of the physical problem. Moreover, the observation time scale we consider is $\frac{1}{\varepsilon}$.

In the context we chose, no force acts on the particles in the $x_1$-direction. Hence the problem may be modeled by a two-dimensional Vlasov-equation. The natural position variable is then $\mathbf{x} = (x_2, x_3) \in \mathbb{R}^2_2$, the velocity variable is $\mathbf{v} = (v_2, v_3) \in \mathbb{R}^2_v$ and the time space is $[0, T)$. We introduce the notation $\Omega = \mathbb{R}^2_2 \times \mathbb{R}^2_v$, $\mathcal{O} = [0, T) \times \mathbb{R}^2_v$ and $\mathcal{Q} = [0, T) \times \Omega$.

Now, the space-velocity distribution $f_\varepsilon(t, \mathbf{x}, \mathbf{v})$, describing the above phenomenon for a particle species $s$ is the solution of:

$$
\left\{
\begin{array}{ll}
\frac{\partial f_\varepsilon}{\partial t} + \frac{\mathbf{v}}{\varepsilon} \cdot \nabla_\mathbf{x} f_\varepsilon + \lambda_s \left( -\nabla_\mathbf{v} p + \frac{\mathbf{v} \times \mathcal{M}}{\varepsilon} \right) \cdot \nabla_\mathbf{v} f_\varepsilon = 0, \\
f_\varepsilon\big|_{t=0} = f_{0_\varepsilon},
\end{array}
\right.
$$

(1.1)

where $\lambda_s$ is a dimensionless constant linked to the charge over mass ratio of the species $s$ and thus characterizing this species of particles, where

$$
\mathbf{v} \times \mathcal{M} = \begin{pmatrix} v_3 \\ -v_2 \end{pmatrix},
$$

and where the electric potential $p(t, \mathbf{x}) \in C^1(\Omega, C^2(\mathbb{R}^2_v))$, $C^2_v$ being the space of bounded and twice continuously differentiable functions. The initial data satisfies

$$
f_{0_\varepsilon} \geq 0, \quad \int_\Omega f_{0_\varepsilon} \, d\mathbf{x} \, d\mathbf{v} < \infty, \quad \int_\Omega f_{0_\varepsilon}^2 \, d\mathbf{x} \, d\mathbf{v} < \infty.
$$

(1.2)

**Remark 1.1** We assume a $L^2$-bound on $f_{0_\varepsilon}$ in order to get a $L^2$-bound on $f_\varepsilon$. This enables to get two-scale and weak-* convergence of $f_\varepsilon$. Of course we could assume any $L^q$-bound with $q > 1$ without changing the results and methods presented here. Here, as we consider a linear Vlasov equation, we do not need any assumption on the moments of $f_{0_\varepsilon}$.

With conditions (1.2), equation (1.1) has for any positive $\varepsilon$ a solution which is positive, which belongs to $L^\infty(0, T; L^1 \cap L^2(\Omega))$ and which satisfies

$$
\|f_\varepsilon\|_{L^\infty(0, T; L^1(\Omega))} \leq C,
$$

(1.3)

for some constant $C$.

From bound (1.3), we may deduce that we can extract a subsequence such that

$$
f_\varepsilon \rightharpoonup f_\varepsilon \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-*},
$$

(1.4)
but also, see \(^1\), for any \(\psi(t, \tau, x, v)\), regular, with compact support in \(t, x, v\) and \(2\pi/\lambda_s\)-periodic in \(\tau\), there exists a profile \(F_\psi(t, \tau, x, v) \in L^\infty(0, T; L^\infty_{2\pi/\lambda_s}(\mathbb{R}^3; L^2(\Omega)))\), also called the two scale limit of \(f^\varepsilon\), such that

\[
\int_\Omega f^\varepsilon(\psi)^\varepsilon dt dx dv \to \int_\Omega \int_0^{2\pi/\lambda_s} F_\psi dt dx dv,
\]

(1.5)

where \((\psi)^\varepsilon = \psi(t, \frac{2\pi}{\lambda_s}, x, v)\). Moreover the weak-* limit \(f_s\) of \((f^\varepsilon)^s\) is expressed in term of the two scale limit \(F_s\)

\[
f_s = \int_0^{2\pi/\lambda_s} F_s(., \tau, ., .) d\tau.
\]

(1.6)

Notice that we adopt the following convention: if \(\lambda_s < 0\), then \(F_s < 0\).

Using this tool we shall derive the equation verified by \(F_s\) and then formula (1.6) will give us the equation for \(f_s\) for each species \(s\) of particles. Moreover, those equations will no more be genuine kinetic equations, but merely involve \(v\) as a parameter. Hence we shall be able, integrating over the velocity variable \(v\), to obtain one unique equation satisfied by the charge density

\[
\rho = \sum_s q_s \int_{\mathbb{R}^3} f_s dv,
\]

(1.7)

of the whole bunch of particles, where \(q_s\) is the dimensionless charge of species \(s\).

Let us now state the main results of the paper:

**Theorem 1.1** Under assumptions (1.2), the sequence \((f^\varepsilon(\psi(t, x, v))^s)\) of solutions of (1.1) weak-* converges to \(f(\psi(t, x, v))\) unique solution of

\[
\begin{aligned}
\frac{\partial f_s}{\partial t} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x f_s + \frac{1}{2} \left(\Delta_p (v \times \mathcal{M})\right) \cdot \nabla_v f_s = 0, \\
f_s|_{t=0} = \frac{1}{2\pi} \int_0^{2\pi} f_0(x, r(v, \tau)) dr,
\end{aligned}
\]

(1.8)

where \(r(v, \tau)\) is the rotation of angle \(\tau\) applied to \(v\).

Moreover

**Theorem 1.2** Assuming (1.2) satisfied for a set of particles, the charge density \(\rho\) defined by (1.7) of the set of particles is the unique solution of

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x \rho = 0, \\
\rho|_{t=0} = \sum_s q_s \int_{\mathbb{R}^3} f_0(x, v) dv dr
\end{aligned}
\]

(1.9)
This model corresponds to the guiding-center model which has been used by physicists for some time.\textsuperscript{4,6,8,10} This guiding center model tells that charged particles submitted to an Electric field $E$ and to a Magnetic field $B$ undergo a drift with velocity $||B||^{-2}(E \times B)$. The drift velocity being independent on charge and mass of the considered particles.

In our result the drift is obtained as a long time behavior, with drift velocity $(-\nabla_x p \times \mathcal{M})$, the independence on charge and mass standing in the fact that the $\lambda_k$-coefficients are not present in the limit equations (1.8) and (1.9).

\textbf{Remark 1.2} Notice that looking for the guiding center model as a long time behavior does not enable to treat the case when the electric and the large magnetic fields are not orthogonal. Indeed, in this case, the usual time scale motion makes that every particle runs to infinity before the long time scale is reached.

As a consequence of this, if we do not make the orthogonality assumption, we simply cannot provide our analysis.

We are now proceeding with the demonstration of these results, which is outlined in the following way: First, using oscillating test functions, we determine a constraint equation for the profile $F_\varepsilon$. As a consequence of this constraint equation, we have that $F_\varepsilon$ may be expressed in terms of a function $G$ depending on an oscillating velocity variable. Then removing the main oscillation in $f^\varepsilon$ and building ad-hoc oscillating test functions, we derive the equation satisfied by $G$. Finally we deduce the equation for the weak limit of $f^\varepsilon$ and prove the theorems.

\textbf{Remark 1.3} We mention that every two-scale and weak-* convergence has to be understood up to subsequences. Nevertheless, since every limit equation we obtain has a unique solution, we may finally deduce that the whole considered sequences converge.

\section{The constraint equation and its consequences}

We multiply equation (1.1), where we do not write down explicitly for the moment the index $s$, by $(\varepsilon)^2 = \psi(t, \tau, x, v)$ where $\psi(t, \tau, x, v)$ is regular, with compact support in $(t, x, v)$ and $2\pi/\lambda-$periodic in $\tau$. We then integrate by parts and get the following weak formulation with oscillating test functions:

$$
\int_Q f^\varepsilon \left[ \frac{\partial \psi}{\partial t} \right]^\varepsilon + \frac{1}{\varepsilon^2} \left( \frac{\partial \psi}{\partial \tau} \right)^\varepsilon + \nabla_x \psi (\nabla_x \psi)^\varepsilon + (\nabla_x \psi)^\varepsilon = -\int_Q \int_\Omega f_0 \psi(0,0,\ldots) \, dx \, dv.
$$

Multiplying (2.1) by $\varepsilon^2$ and passing to the limit gives

$$
\int_Q \int_\Omega 2\pi/\lambda \, F(\frac{\partial \psi}{\partial \tau} + \lambda (v \times \mathcal{M}) \cdot \nabla_v \psi) \, d\tau \, dx \, dv = 0,
$$

(2.2)
which is, as we saw in Frénod and Sonnendrücker \(^5\), equivalent to the following constraint equation:

\[
\frac{\partial F}{\partial \tau} + \lambda (v \times \mathcal{M}) \cdot \nabla_v F = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}_+^3),
\]  

(2.3)

for almost every \( t \in [0, T) \) and almost every \( x \in \mathbb{R}_+^3 \).

We now introduce the transformation \( r(\cdot, \tau) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by

\[
r(v, \tau) = [v_2 \cos \tau - v_3 \sin \tau] e_2 + [v_2 \sin \tau + v_3 \cos \tau] e_3.
\]  

(2.4)

From (2.3) we may deduce (see \(^5\)) that there exists a function

\[
G \equiv G(t, x, u) \in L^\infty(0, T; L^2(\mathbb{R}_+^2 \times \mathbb{R}_+^3)),
\]

such that

\[
F(t, \tau, x, v) = G(t, x, r(v, \lambda \tau)),
\]

(2.5)

where we denote, similarly to \( x \) and \( v \), \( u = (u_2, u_3) \).

3. Obtention of an equation for \( G \)

In order to deduce the equation satisfied by \( G \), we shall use first, in the weak formulation, test functions satisfying the constraint (2.3). This will yield a weak formulation in which one term is multiplied by \( \frac{1}{\varepsilon} \). Choosing then appropriate test functions and using an ad hoc change of variables, we shall obtain the limit of this term. The deduction of the equation for \( G \) will then be straightforward.

For any function \( \varphi(t, x, u) \) regular and with compact support, we define \( \psi(t, \tau, x, v) = \varphi(t, x, r(v, \lambda \tau)) \). Then the function \( \psi \) is \( 2\pi/\lambda \)-periodic, satisfies the constraint, and using \((\psi)^\varepsilon\) as a test function in (2.1) leads to:

\[
\int_{\Omega} f^\varepsilon \left[ \frac{\partial \psi}{\partial t} + \frac{v}{\varepsilon} \cdot (\nabla_v \psi)^\varepsilon + \lambda \frac{-\nabla_z P}{\varepsilon} \cdot (\nabla_v \psi)^\varepsilon \right] dt \, dx \, dv = - \int_{\Omega} f_0 \psi(0, 0, \cdot, \cdot) \, dx \, dv.
\]  

(3.1)

We now introduce the change of variables

\[
v \rightarrow u = r(v, \lambda \frac{t}{\varepsilon^2}),
\]  

(3.2)

whose reverse transformation is

\[
u \rightarrow v = r(u, -\lambda \frac{t}{\varepsilon^2}).
\]  

(3.3)

Using the definition of \( \psi \), and defining \( g^\varepsilon(t, x, u) \) by \( g^\varepsilon(t, x, u) = f^\varepsilon(t, x, r(u, -\lambda \frac{t}{\varepsilon^2})) \) which is equivalent to \( f^\varepsilon(t, x, v) = g^\varepsilon(t, x, r(v, \lambda \frac{t}{\varepsilon^2})) \), since

\[
\frac{\partial \psi}{\partial v_2}(t, \tau, x, v) = \cos \lambda \tau \frac{\partial \varphi}{\partial u_2}(t, x, r(v, \lambda \tau)) + \sin \lambda \tau \frac{\partial \varphi}{\partial u_3}(t, x, r(v, \lambda \tau)),
\]


$$\frac{\partial \psi}{\partial u_3}(t, \tau, x, v) = -\sin \lambda \tau \frac{\partial \varphi}{\partial u_2}(t, x, r(v, \lambda \tau)) + \cos \lambda \tau \frac{\partial \varphi}{\partial u_3}(t, x, r(v, \lambda \tau)),$$

the change of variables (3.2) transforms the equation (3.1) into

$$\int_{Q'} g^\varepsilon(t, x, u) \left[ \frac{\partial \varphi}{\partial t}(t, x, u) ight] dt dx du$$

$$+ \frac{1}{\varepsilon} \{ (u_2 \cos \lambda t \varepsilon + u_3 \sin \lambda t \varepsilon^2) \frac{\partial \varphi}{\partial x_2}(t, x, u)$$

$$+ (-u_2 \sin \lambda t \varepsilon + u_3 \cos \lambda t \varepsilon^2) \frac{\partial \varphi}{\partial x_3}(t, x, u)$$

$$+ \lambda \lambda t \cos \lambda t \varepsilon^2 + (\partial \varphi t \sin \lambda t \varepsilon^2 \frac{\partial \varphi}{\partial u_3}(t, x, u)) \} dt dx dv$$

$$= -\int_\Omega f_0 \varphi(0, x, u) dx du - \int_\Omega f_0 \varphi(0, x, u) dx du. \quad (3.4)$$

Now we focus, for a few pages, on passing to the limit in (3.4). For this purpose, we need to characterize the weak--* limit and the two scale limit of $g^\varepsilon$, and to compute the limit of the term containing the $\frac{t}{\varepsilon}$ in factor.

First, we have

**Lemma 3.1** The two scale limit of $g^\varepsilon$ is the function $G$ defined by (2.5). And, since $G$ does not depend on $\tau$, the weak--* limit $g$ of $g^\varepsilon$ is $(2\pi/\lambda)G$.

**Proof.** Since the sequence $(f^\varepsilon)$ is bounded, we have that $(g^\varepsilon)$ is also bounded. Hence $(g^\varepsilon)$ two-scale and weak--* converges.

We take any function $\varphi(t, \tau, x, u)$, regular, with compact support in $(t, x, u)$ and $2\pi/\lambda$-- periodic in $\tau$. On the one hand, denoting by $\tilde{G}$ the two scale limit of $g^\varepsilon$, we have

$$\int_{Q'} g^\varepsilon(t, x, u) \varphi(t, \frac{t}{\varepsilon^2}, x, u) dt dx du \rightarrow \int_{Q'} \int_0^{2\pi/\lambda} \tilde{G}(t, \tau, x, u) \varphi(t, \tau, x, u) dt dt dx dv. \quad (3.5)$$

On the other hand,

$$\int_{Q'} g^\varepsilon(t, x, u) \varphi(t, \frac{t}{\varepsilon^2}, x, u) dt dx du =$$

$$\int_Q g^\varepsilon(t, x, r(v, \frac{t}{\varepsilon^2})) \varphi(t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) dt dx dv =$$

$$\int_Q f^\varepsilon(t, x, v) \varphi(t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) dt dx dv,$$

which converges to

$$\int_Q \int_0^{2\pi/\lambda} F(t, \tau, x, v) \varphi(t, \tau, x, r(v, \lambda \tau)) dt dt dx dv =$$
Long time behavior of the two-dimensional Vlasov equation with a strong external magnetic field

\[ \int_{Q} \int_{0}^{2\pi/\lambda} G(t, x, r(\mathbf{v}, \lambda \tau)) \varphi(t, \tau, x, r(\mathbf{v}, \lambda \tau)) \, d\tau \, dt \, dx \, dv = \]
\[ \int_{Q} \int_{0}^{2\pi/\lambda} G(t, x, u) \varphi(t, \tau, x, u) \, d\tau \, dt \, dx \, du. \]

Thus, we may conclude
\[ \tilde{G}(t, \tau, x, u) = G(t, x, u), \]
which implies that \( G \) is the two scale limit of \( g^\varepsilon \). The first part of the Lemma is proved. Moreover, using that the link between the two-scale and the weak-* limits and the fact that \( G \) does not depend on \( \tau \), we have:

\[ g(t, \tau, u) = \int_{0}^{2\pi/\lambda} \tilde{G}(t, \tau, x, u) \, d\tau, \]
\[ = \int_{0}^{2\pi/\lambda} G(t, x, u) \, d\tau, \]
\[ = \frac{2\pi}{\lambda} G(t, x, u), \]

proving the second part of Lemma 3.1 \( \Box \).

Concerning the computation of the term containing the \( \frac{1}{\varepsilon} \) factor in (3.4) we have:

**Lemma 3.2** Denoting by \( G(t, x, u) \) the two scale limit of \( g^\varepsilon \), which is also the function linked to the two scale limit \( F \) of \( f^\varepsilon \) by (2.5), the following formula holds true for all test functions \( \varphi(t, x, u) \):

\[ \lim_{\varepsilon \to 0} \left\{ \int_{Q} g^\varepsilon(t, x, u) \frac{1}{\varepsilon} [(u_2 \cos \frac{t}{\varepsilon^2} + u_3 \sin \frac{t}{\varepsilon^2}) \frac{\partial \varphi}{\partial x_2}(t, x, u) \right. \]
\[ + (-u_2 \sin \frac{t}{\varepsilon^2} + u_3 \cos \frac{t}{\varepsilon^2}) \frac{\partial \varphi}{\partial x_3}(t, x, u) \]
\[ + \lambda ((-\partial_{x_2} p) \cos \frac{t}{\varepsilon^2} + (\partial_{x_3} p) \sin \frac{t}{\varepsilon^2}) \frac{\partial \varphi}{\partial u_2}(t, x, u) \]
\[ + \lambda ((-\partial_{x_3} p) \sin \frac{t}{\varepsilon^2} - (\partial_{x_2} p) \cos \frac{t}{\varepsilon^2}) \frac{\partial \varphi}{\partial u_3}(t, x, u) \, dt \, dx \, du \}
\[ = \frac{2\pi}{\lambda} \int_{Q} G(t, x, u) \left[ \partial_{x_2} p \frac{\partial \varphi}{\partial x_2} - \partial_{x_3} p \frac{\partial \varphi}{\partial x_3} \right] 
\[ + \frac{1}{2} (\partial_{x_2}^2 p + \partial_{x_3}^2 p) (-u_3 \frac{\partial \varphi}{\partial u_2} + u_2 \frac{\partial \varphi}{\partial u_3}) \, dt \, dx \, du. \quad (3.6) \]

**Proof.** As in the singular perturbation theory, see \( ^7 \), we are looking for the limit of the inner product of \( g^\varepsilon \) by a function which is not bounded but which converges to 0 in the sense of distributions. Indeed since

\[ \frac{1}{\varepsilon} \cos \frac{t}{\varepsilon^2} = \frac{d}{dt} \left( \frac{\varepsilon \sin \frac{t}{\varepsilon^2}}{\lambda} \right), \quad \frac{1}{\varepsilon} \sin \frac{t}{\varepsilon^2} = \frac{d}{dt} \left( -\frac{\varepsilon \cos \frac{t}{\varepsilon^2}}{\lambda} \right). \quad (3.7) \]
those functions converges to 0 in the sense of distributions. Using this Remark, we develop a method which is in the same spirit as the one used by Cioranescu and Murat [2,3].

For any $\varphi$, we build an oscillating test function $\alpha$ whose $\tau-$derivative when applied in $\tau = \frac{1}{\varepsilon}$ gives $\frac{1}{\varepsilon} \left( \left( u_2 \cos \lambda \frac{\partial}{\partial x_2} + u_3 \sin \lambda \frac{\partial}{\partial x_3} \right) \frac{\partial \varphi}{\partial x_2}(t, x, u) \right.$
\[ + \frac{1}{\varepsilon} \left( (-\partial_{x_2}p) \sin \lambda \frac{\partial}{\partial x_2} + (\partial_{x_3}p) \cos \lambda \frac{\partial}{\partial x_3} \right) \frac{\partial \varphi}{\partial u_2}(t, x, u) \]
\[ + \left( (-\partial_{x_3}p) \sin \lambda \frac{\partial}{\partial x_3} - (\partial_{x_2}p) \cos \lambda \frac{\partial}{\partial x_2} \right) \frac{\partial \varphi}{\partial u_3}(t, x, u), \tag{3.8} \]
and we consider
\[ \beta(t, \tau, x, v) = \alpha(t, \tau, x, r(v, \lambda \tau)). \tag{3.9} \]

Applying the constraint operator to $\beta$, we get
\[ \frac{\partial \beta}{\partial \tau} + \lambda(v \times \mathcal{M}) \cdot \nabla_v \beta = \frac{\partial \alpha}{\partial \tau} + \lambda \frac{\partial r(v, \lambda \tau)}{\partial \tau} \cdot \nabla_v \alpha + \lambda(v \times \mathcal{M}) \cdot \nabla_v \alpha = \left( \frac{\partial \alpha}{\partial \tau} \right)(t, \tau, x, r(v, \lambda \tau)), \tag{3.10} \]
and the computation of $\frac{\partial \alpha}{\partial \tau}$ gives
\[ \left( \frac{\partial \alpha}{\partial \tau} \right)(t, \tau, x, u) = \left( u_2 \cos \lambda \frac{\partial}{\partial x_2} + u_3 \sin \lambda \frac{\partial}{\partial x_3} \right) \frac{\partial \varphi}{\partial x_2}(t, x, u) \]
\[ + \left( (-\partial_{x_2}p) \sin \lambda \frac{\partial}{\partial x_2} + (\partial_{x_3}p) \cos \lambda \frac{\partial}{\partial x_3} \right) \frac{\partial \varphi}{\partial u_2}(t, x, u) \]
\[ + \left( (-\partial_{x_3}p) \sin \lambda \frac{\partial}{\partial x_3} - (\partial_{x_2}p) \cos \lambda \frac{\partial}{\partial x_2} \right) \frac{\partial \varphi}{\partial u_3}(t, x, u). \tag{3.11} \]

Beside this, easy computations give
\[ \left( \frac{\partial \beta}{\partial t} \right)^\varepsilon = \left( \frac{\partial \alpha}{\partial t} \right)^\varepsilon \left( \frac{t}{\varepsilon^2}, x, r(v, \lambda \frac{t}{\varepsilon^2}) \right), \tag{3.12} \]
Long time behavior of the two-dimensional Vlasov equation with a strong external magnetic field

\[
\frac{\partial \beta}{\partial t} = \left( \frac{\partial \alpha}{\partial t} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})),
\]

(3.13)

\[
\frac{\partial \beta}{\partial v_2} = \cos \frac{t}{\varepsilon^2} \left( \frac{\partial \alpha}{\partial v_2} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) + \sin \frac{t}{\varepsilon^2} \left( \frac{\partial \alpha}{\partial v_3} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})),
\]

(3.14)

\[
\frac{\partial \beta}{\partial v_3} = -\sin \frac{t}{\varepsilon^2} \left( \frac{\partial \alpha}{\partial v_2} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) + \cos \frac{t}{\varepsilon^2} \left( \frac{\partial \alpha}{\partial v_3} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})).
\]

(3.15)

Having those formulae at hand, we use test functions

\[
\varepsilon (\beta)^* = \varepsilon \beta(t, \frac{t}{\varepsilon^2}, x, v),
\]

(3.16)
in the weak formulation (2.1). We get

\[
\int_Q f^* \varepsilon \left( \frac{\partial \beta}{\partial t} \right)^* + \frac{1}{\varepsilon} \left( \frac{\partial \beta}{\partial T} \right)^* + \nabla \cdot (\nabla \beta)^* + \lambda (-\nabla_p + v \times \frac{M}{\varepsilon} \cdot (\nabla \beta)^*) \ dt \ dx \ dv
\]

(3.17)

Now, using formula (3.10), the expression of \( f^* \) in term of \( g^* \) and the expressions of the derivatives of \( \beta \) (3.11)-(3.15), equation (3.17) becomes

\[
\int_Q g^* (t, x, r(v, \frac{t}{\varepsilon^2})) \left[ \varepsilon \left( \frac{\partial \alpha}{\partial t} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) \right]
\]

\[
+ \frac{1}{\varepsilon} \left( \frac{\partial \alpha}{\partial r} \right) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) + v \cdot (\nabla \alpha) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2}))
\]

\[- \lambda \left( \frac{(\partial \alpha)}{(\partial x)} \cos \frac{t}{\varepsilon^2} - \frac{(\partial \alpha)}{(\partial y)} \sin \frac{t}{\varepsilon^2} \right) \cdot (\nabla \alpha) (t, \frac{t}{\varepsilon^2}, x, r(v, \frac{t}{\varepsilon^2})) \] \ dx \ dv

(3.18)

In equation (3.18) we make the change of variable (3.2), it gives

\[
\int_{Q'} g^* (t, x, u) \varepsilon \left( \frac{\partial \alpha}{\partial t} \right) (t, \frac{t}{\varepsilon^2}, x, u)
\]

\[
+ \frac{1}{\varepsilon} \left( \frac{\partial \alpha}{\partial r} \right) (t, \frac{t}{\varepsilon^2}, x, u) + r(u, -\lambda \frac{t}{\varepsilon^2} \cdot (\nabla \alpha) (t, \frac{t}{\varepsilon^2}, x, u)
\]

\[- \lambda \left( \frac{(\partial \alpha)}{(\partial x)} \cos \frac{t}{\varepsilon^2} - \frac{(\partial \alpha)}{(\partial y)} \sin \frac{t}{\varepsilon^2} \right) \cdot (\nabla \alpha) (t, \frac{t}{\varepsilon^2}, x, u) \] \ dt \ dx \ du

(3.19)
which, passing to the limit in $\varepsilon$, using Lemma 3.1 gives

$$
\lim_{\varepsilon \to 0} \left\{ \int_{Q'} g^\varepsilon(t, x, u) \frac{1}{\varepsilon} \left( \frac{\partial \alpha}{\partial \tau} \right) (t, \frac{t}{\varepsilon^2}, x, u) dt dx du \right\} =
- \int_{Q'} G \int_0^{2\pi/\lambda} \left[ \begin{array}{c}
u_2 \cos \lambda \tau + u_3 \sin \lambda \tau \\ -u_2 \sin \lambda \tau + u_3 \cos \lambda \tau \end{array} \right] \cdot (\nabla_x \alpha)(t, \tau, x, u)
- \lambda \left( \frac{\partial_x \psi}{\partial \tau} \cos \lambda \tau - \frac{\partial_x \psi}{\partial \psi} \sin \lambda \tau \right) \cdot (\nabla_u \psi)(t, \tau, x, u) \right] d\tau dx du,
$$

(3.20)

and

$$
\int_{Q'} g^\varepsilon(t, x, u) \frac{1}{\varepsilon} \left( \frac{\partial \alpha}{\partial \tau} \right) (t, \frac{t}{\varepsilon^2}, x, u) dt dx du =
\int_{Q'} g^\varepsilon(t, x, u) \frac{1}{\varepsilon} \left[ \begin{array}{c}
u_2 \cos \lambda \tau + u_3 \sin \lambda \tau \\ -u_2 \sin \lambda \tau + u_3 \cos \lambda \tau \end{array} \right] \cdot (\nabla_x \alpha)(t, \tau, x, u)
+ \left( \frac{\partial_x \psi}{\partial \tau} \cos \lambda \tau + \frac{\partial_x \psi}{\partial \psi} \sin \lambda \tau \right) \frac{\partial \phi}{\partial u_2}(t, x, u)
+ \lambda \left( \frac{\partial_x \psi}{\partial \tau} \cos \lambda \tau - \frac{\partial_x \psi}{\partial \psi} \sin \lambda \tau \right) \frac{\partial \phi}{\partial u_3}(t, x, u) \right] dt dx du,
$$

(3.21)

is exactly the term whose limit we want to compute in (3.6).

At this level, formula (3.20) gives the sought limit in terms of $\alpha$. Hence we now have to replace, in the right hand side of (3.20) $\alpha$ by its expression in term of $\varphi$ and to integrate in $\tau$. This step is straightforward but fastidious. The detailed computation yielding the following formula is given in the appendix.

We have

$$
\int_0^{2\pi/\lambda} \left[ \begin{array}{c}
u_2 \cos \lambda \tau + u_3 \sin \lambda \tau \\ -u_2 \sin \lambda \tau + u_3 \cos \lambda \tau \end{array} \right] \cdot (\nabla_x \alpha)(t, \tau, x, u)
- \lambda \left( \frac{\partial_x \psi}{\partial \tau} \cos \lambda \tau - \frac{\partial_x \psi}{\partial \psi} \sin \lambda \tau \right) \cdot (\nabla_u \psi)(t, \tau, x, u) \right] d\tau =
\frac{2\pi}{\lambda} \left( \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial u_2} + \frac{\partial \phi}{\partial u_3} \right) + \frac{\pi}{\lambda} \left( \frac{\partial_x \psi}{\partial u_2} + \frac{\partial_x \psi}{\partial u_3} \right) \left( -u_3 \frac{\partial \phi}{\partial u_2} + u_2 \frac{\partial \phi}{\partial u_3} \right).
$$

(3.22)

Using then (3.22) in (3.20) gives the Lemma $\Box$.

Since Lemma 3.1 and Lemma 3.2 give the two scale- and weak- * -limits of $g^\varepsilon$ and the limit of the term containing the $\frac{1}{\varepsilon}$ in (3.4), we can pass to the limit and obtain

$$
\frac{2\pi}{\lambda} \int_{Q'} G \left( \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial u_2} + \frac{\partial \phi}{\partial u_3} \right) + \frac{\pi}{\lambda} \left( \frac{\partial_x \psi}{\partial u_2} + \frac{\partial_x \psi}{\partial u_3} \right) \left( -u_3 \frac{\partial \phi}{\partial u_2} + u_2 \frac{\partial \phi}{\partial u_3} \right) dt dx du
$$

(3.23)
for all \( \varphi(t, x, v) \) regular and with compact support, which is the weak formulation of the equation for \( G \) we are seeking. Thus noticing
\[
-\partial_{x_2} p \frac{\partial \varphi}{\partial x_2} + \partial_{x_3} p \frac{\partial \varphi}{\partial x_3} = (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x \varphi,
\]
and
\[
u_3 \frac{\partial \varphi}{\partial u_2} - u_2 \frac{\partial \varphi}{\partial u_3} = (u \times \mathcal{M}) \cdot \nabla_u \varphi,
\]
and denoting
\[
\Delta p = \partial_{x_2}^2 p + \partial_{x_3}^2 p,
\]
we have proved the

**Theorem 3.3** Under assumptions (1.2), the sequence \((f^\epsilon(t, x, v))_\epsilon\) of solutions of (1.1) two-scale converges to a profile \(F(t, \tau, x, v) \in L^\infty(0, T; L^2_{2\pi/\lambda}(\mathbb{R}_\tau; L^2(\Omega)))\) satisfying
\[
F(t, \tau, x, v) = G(t, x, r(v, \lambda \tau)),
\]
where \(r(v, \tau)\) is given by (2.4) and where \(G \equiv G(t, x, u) \in L^\infty(0, T; L^2(\mathbb{R}_2 \times \mathbb{R}_u))\) is the unique solution of
\[
\begin{align*}
\frac{\partial G}{\partial \tau} + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x G + \frac{1}{2}(\Delta p (u \times \mathcal{M})) \cdot \nabla_u G &= 0, \\
G|_{\tau=0} &= \frac{\lambda}{2\pi} f_0(x, u).
\end{align*}
\]

**Remark 3.1** As a consequence of the uniqueness of the solution of (3.28), we have that the whole sequences \((f^\epsilon)\) and \((g^\epsilon)\) two-scale and weak-* converge.

4. **Equation for \( f \), proof of the Theorems**

Using the relation (1.6) linking \( f \) and \( F \), we have
\[
f = \int_0^{2\pi/\lambda} F(\tau, \tau, \cdot) d\tau = \int_0^{2\pi/\lambda} G(\tau, \cdot, r(\tau, \lambda \tau)) d\tau,
\]
and noticing
\[
(r(v, \lambda \tau) \times \mathcal{M}) \cdot \nabla_v G(t, x, r(v, \lambda \tau)) = (v \times \mathcal{M}) \cdot \nabla_v [G(t, x, r(v, \lambda \tau))],
\]
replacing \( u \) by \( r(v, \lambda \tau) \) in (3.28) gives
\[
\begin{align*}
\frac{\partial}{\partial \tau} [G(t, x, r(v, \lambda \tau))] + (-\nabla_x p \times \mathcal{M}) \cdot \nabla_x [G(t, x, r(v, \lambda \tau))] \\
+ \frac{1}{2\lambda} (\Delta p (v \times \mathcal{M}) \cdot \nabla_u [G(t, x, r(v, \lambda \tau))]) &= 0, \\
G(0, x, r(v, \lambda \tau)) &= \frac{\lambda}{2\pi} f_0(x, r(v, \lambda \tau)).
\end{align*}
\]
Integrating (4.3) in $\tau$ from 0 to $2\pi/\lambda$ finally gives equation (1.8) and achieve the proof of Theorem 1.1 $\square$.

Noticing
$$\nabla_v \cdot (\Delta p (v \times \mathcal{M})) = 0,$$
and
$$\frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} f_0(x, r(v, \tau)) \, d\nu d\tau = \int_{\mathbb{R}^2} f_0(x, v) \, dv,$$
the proof of Theorem 1.2 is straightforward integrating (1.8) in $v$ and summing in $s$ $\square$.

Appendix: Computation yielding (3.22)
In this appendix, we give the fastidious computation yielding formula (3.22). We have
$$\int_0^{2\pi/\lambda} \left( \begin{array}{c} \frac{u_2 \cos \lambda \tau + u_3 \sin \lambda \tau}{} \\ -u_2 \sin \lambda \tau + u_3 \cos \lambda \tau \end{array} \right) \left( \begin{array}{c} \nabla_x \alpha \\ \nabla_y \alpha \end{array} \right) - \lambda \left( \begin{array}{c} (\partial_{x_v} P) \cos \lambda \tau - (\partial_{x_v} P) \sin \lambda \tau \\ (\partial_{x_v} P) \sin \lambda \tau + (\partial_{x_v} P) \cos \lambda \tau \end{array} \right) \left( \begin{array}{c} \nabla_y \alpha \\ \nabla_x \alpha \end{array} \right) \right) \, d\tau =$$
$$\int_0^{2\pi/\lambda} \left[ r(u, -\lambda \tau) \cdot (\nabla_x \alpha) - \lambda r(\nabla_x P, \lambda \tau) \cdot (\nabla_y \alpha) \right] \, d\tau. \quad (A.1)$$

Using the definition of $\alpha$ in terms of $\varphi$, we have
$$\alpha(t, \tau, x, u) = \frac{1}{\lambda} r(u, \frac{\pi}{2} - \lambda \tau) \cdot \nabla_x \varphi(t, x, u) + r(\nabla_x P, \frac{\pi}{2} + \lambda \tau) \cdot \nabla_y \varphi(t, x, u). \quad (A.2)$$
Hence the integral to compute becomes
$$\int_0^{2\pi/\lambda} \left[ \frac{1}{\lambda} r(u, -\lambda \tau) \cdot (\nabla_x \varphi) \right] r(u, \frac{\pi}{2} - \lambda \tau)$$
$$- \lambda r(\nabla_x P, \lambda \tau) \cdot r(\nabla_x \varphi) r(\nabla_y \varphi)$$
$$+ r(u, -\lambda \tau) \cdot \nabla_x \left\{ r(\nabla_x P, \frac{\pi}{2} + \lambda \tau) \cdot \nabla_y \varphi \right\}$$
$$- r(\nabla_x P, \lambda \tau) \cdot \nabla_y \left\{ \frac{1}{\lambda} r(u, \frac{\pi}{2} - \lambda \tau) \cdot \nabla_x \varphi \right\} \right] \, d\tau. \quad (A.3)$$

Now, since $(\nabla_x \varphi)$ and $(\nabla_y \varphi)$ are symmetric matrices, we easily deduce
$$\int_0^{2\pi/\lambda} r(u, -\lambda \tau) \cdot (\nabla_x (\nabla_y \varphi)) r(u, \frac{\pi}{2} - \lambda \tau) \, d\tau =$$
$$\int_0^{2\pi/\lambda} r(\nabla_x P, \lambda \tau) \cdot (\nabla_y (\nabla_x \varphi)) r(\nabla_x P, \frac{\pi}{2} + \lambda \tau) \, d\tau = 0, \quad (A.4)$$
and computing
$$\nabla_x \left\{ r(\nabla_x P, \frac{\pi}{2} + \lambda \tau) \cdot \nabla_y \varphi \right\} \text{ and } \nabla_y \left\{ \frac{1}{\lambda} r(u, \frac{\pi}{2} - \lambda \tau) \cdot \nabla_x \varphi \right\}, \quad (A.5)$$
\[ \int_0^{2\pi/\lambda} \left[ \left( \frac{u_x \cos \sigma \tau + u_y \sin \sigma \tau}{-u_x \sin \sigma \tau + u_y \cos \sigma \tau} \right) \cdot (\nabla, \alpha) - \lambda \left( \frac{\partial}{\partial u_2} \cos \sigma \tau - \frac{\partial}{\partial u_3} \sin \sigma \tau \right) \cdot (\nabla, \alpha) \right] \, d\tau = \]

\[ \int_0^{2\pi/\lambda} \left[ \left( \frac{u_x \cos \sigma \tau + u_y \sin \sigma \tau}{-u_x \sin \sigma \tau + u_y \cos \sigma \tau} \right) \cdot (\nabla, \alpha) - \lambda \left( \frac{\partial}{\partial u_2} \cos \sigma \tau - \frac{\partial}{\partial u_3} \sin \sigma \tau \right) \cdot (\nabla, \alpha) \right] \, d\tau. \]

In order to provide the computations in (A.6) we use (to reduce the number of terms) \[ \int_0^{2\pi/\lambda} \sin \sigma \tau \cos \sigma \tau \, d\tau = 0. \] We then get

\[ \int_0^{2\pi/\lambda} \left[ \frac{\partial \phi}{\partial u_3} \cos^2 \sigma \tau + \sin^2 \sigma \tau \right] (u_x \, \partial_{u_x}) (u_y \, \partial_{u_y}) \right] \, d\tau \]

\[ + \left( \frac{\partial \phi}{\partial u_3} \right) \left( -u_x \cos \sigma \tau + u_y \sin \sigma \tau \right) \left( \frac{\partial^2 \phi}{\partial u^2_{u_2}} \right) + u_x \sin \sigma \tau \left( \frac{\partial^2 \phi}{\partial u^2_{u_3}} \right) + u_x \cos \sigma \tau \left( \frac{\partial^2 \phi}{\partial u^2_{u_3}} \right) \]

\[ \left( -u_x \cos \sigma \tau + u_y \sin \sigma \tau \right) \left( \frac{\partial^2 \phi}{\partial u^2_{u_3}} \right) + u_x \sin \sigma \tau \left( \frac{\partial^2 \phi}{\partial u^2_{u_3}} \right) + u_x \cos \sigma \tau \left( \frac{\partial^2 \phi}{\partial u^2_{u_3}} \right) \]

Finally, using that \[ \int_0^{2\pi/\lambda} \cos^2 \sigma \tau \sin \sigma \tau \, d\tau = \pi / \lambda \] and \[ \int_0^{2\pi/\lambda} \sin^2 \sigma \tau \sin \sigma \tau \, d\tau = 0, \] and \[ \int_0^{2\pi/\lambda} \sin \sigma \tau \sin \sigma \tau \, d\tau = \pi / \lambda, \] (A.7) gives

\[ \frac{2\pi}{\lambda} \left( \frac{\partial \phi}{\partial u_2} \right) - \frac{\partial \phi}{\partial u_3} + \frac{u_x}{\partial u_2} \frac{\partial \phi}{\partial u_3} + \frac{u_y}{\partial u_3} \frac{\partial \phi}{\partial u_2}, \]

which is the left hand side of (3.22).


