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TWO-SCALE EXPANSION OF A SINGULARLY PERTURBED CONVECTION EQUATION

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ABSTRACT. – In magnetic fusion, a plasma is constrained by a very large magnetic field, which introduces a new time scale, namely the period of rotation of the particles around the magnetic field lines. This new time scale is very restrictive for numerical simulation, which makes it important to find approximate models of the Vlasov–Poisson equation where it is removed. The gyrokinetic models aim at exactly this. Such models have been derived in the physics literature for several decades now, but only in the last few years there have been rigorous mathematical derivations. Those have only addressed the limit when the magnetic field becomes infinite. We consider here the Vlasov equation in different physical regimes for which small parameters are identified, and cast the obtained dimensionless equations into the abstract framework of a singularly perturbed convection equation. In this framework we derive an asymptotic expansion with respect to the small parameter of its solution, and characterize the terms of the expansion. The proofs make use of Allaire's two-scale convergence. © 2001 Éditions scientifiques et médicales Elsevier SAS

Keywords: Vlasov equation, Singular perturbation, Two scale convergence, Asymptotic expansion

RÉSUMÉ. – Lors de la fusion par confinement magnétique, un plasma est confiné par un champ magnétique très fort, qui introduit une nouvelle échelle de temps qui est la période de rotation des particules autour des lignes de champ magnétique. Cette nouvelle échelle de temps est très pénalisante pour la simulation numérique ce qui rend essentiel l'utilisation de modèles approchés de l'équation de Vlasov–Poisson dans lesquels cette échelle n'apparaît pas. Les modèles gyrocinétiques ont été introduits dans ce but. De tels modèles existent dans la littérature de physique depuis plusieurs décennies, mais des dérivations mathématiquement rigoureuses ont seulement été réalisées récemment. Celles-ci ont considéré la limite de l'équation de Vlasov–Poisson quand le champ magnétique tend vers l'infini. Nous considérons ici l'équation de Vlasov dans différents régimes physiques pour lesquels nous identifions des paramètres petits. Nous replaçons les équations adimensionnées obtenues dans le cadre abstrait d'une équation de convection singulièrement perturbée. Dans ce cadre nous dérivons un développement asymptotique par rapport au petit paramètre de la solution, et caractérisons les différents termes du développement. Les preuves sont basées sur la convergence à deux échelles introduite par Allaire.© 2001 Éditions scientifiques et médicales Elsevier SAS

Mots Clés: Equation de Vlasov, Perturbation singulière, Convergence à deux échelles, Développement asymptotique

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1. Introduction

One of the great challenges in plasma physics is still to obtain energy through the thermonuclear fusion process. There are essentially two ways which are currently explored to achieve this feat: inertial confinement fusion (ICF) and magnetic confinement fusion (MCF). The magnetic confinement is performed in large toroidal devices called tokamaks, and, as the name tells us, the plasma is confined by using an external magnetic field which needs to be very large. One tool which is used for the understanding of the behavior of plasmas in tokamaks is numerical simulation. The MCF plasma dynamics can be described by the Vlasov equation coupled to Poisson's equation. However this model is very difficult for numerical simulation of tokamaks in particular because the large magnetic field introduces a very restrictive time step for numerical stability. In the past other models like the guiding center model or the gyro-kinetic model have been introduced to simulate such plasmas.

In previous papers [2–5] limits of the Vlasov or the Vlasov–Poisson equations were investigated in different regimes, corresponding to different small parameters but having in common a large magnetic field which is the situation occurring in tokamaks. Related work has also been presented by other authors [6–9]. The next step which we wish to address here is to characterize all the terms of a two scale asymptotic expansion in the large magnetic field, or equivalently the large cyclotron frequency, regime of the linear Vlasov equation with a given electric field.

Let us make this discussion more precise by introducing the Vlasov equation, for a given electric field $\mathbf{E}(\mathbf{x},t)$ and magnetic field $\mathbf{B}(\mathbf{x},t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_x f + \frac{q}{m}\big(\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)\big)\cdot\nabla_v f = 0. \tag{1.1}$$

In order to introduce the small and large quantities, we need to define some characteristic scales: $\overline{t}$ stands for a characteristic time, $\overline{L}$ for a characteristic length, $\overline{v}$ for a characteristic velocity. We now define new variables t', $\mathbf{x}'$ and $\mathbf{v}'$, by $t = \overline{t}t'$, $\mathbf{x} = \overline{L}\mathbf{x}'$, and $\mathbf{v} = \overline{v}\mathbf{v}'$, making the characteristic scales the unities. In the same way, we define the scaling factors for the fields: $\overline{E}$ for the electric field and $\overline{B}$ for the magnetic field and the new fields $\mathcal{E}$ and $\mathcal{B}$ are given by: $\overline{E}\mathcal{E}(\mathbf{x}',t') = \mathbf{E}(\overline{L}\mathbf{x}',\overline{t}t')$ and $\overline{B}\mathcal{B}(\mathbf{x}',t') = \mathbf{b}(\overline{L}\mathbf{x}',\overline{t}t')$. Last, defining a scaling factor $\overline{f}$ for the distribution function, noticing that f is a distribution function on the phase-space it is natural to define the new distribution function by:

$$\overline{f} f'(\mathbf{x}',\mathbf{v}',t') = \overline{L}^3\overline{v}^3 f(\overline{L}\mathbf{x}',\overline{v}\mathbf{v}',\overline{t}t'). \tag{1.2}$$

With those new variables and fields we obtain that f' is solution of:

$$\frac{\partial f'}{\partial t'} + \frac{\overline{v}\overline{t}}{\overline{L}}\mathbf{v}'\cdot\nabla_{x'} f' + \left(\frac{q\overline{E}\overline{t}}{m\overline{v}}\mathcal{E}(\mathbf{x}',t') + \frac{q\overline{B}\overline{t}}{m}\mathbf{v}'\times\mathcal{B}(\mathbf{x}',t')\right)\cdot\nabla_{v'} f' = 0. \tag{1.3}$$

Now, we introduce the characteristic cyclotron frequency: $\overline{\omega}_c = q\overline{B}/m$ and the characteristic Larmor radius: $\overline{a}_L = \overline{v}/\overline{\omega}_c$. Using those physical quantities, (1.3) becomes:

$$\frac{\partial f'}{\partial t'} + \overline{t}\overline{\omega}_c\frac{\overline{a}_L}{\overline{L}}\mathbf{v}'\cdot\nabla_{x'} f' + \left(\overline{t}\overline{\omega}_c\frac{\overline{E}}{\overline{v}\overline{B}}\mathcal{E}(\mathbf{x}',t') + \overline{t}\overline{\omega}_c\mathbf{v}'\times\mathcal{B}(\mathbf{x}',t')\right)\cdot\nabla_{v'} f' = 0. \tag{1.4}$$

Assuming the magnetic field is strong consists essentially in setting

$$\overline{t}\,\overline{\omega_c} = \frac{1}{\varepsilon} \quad \text{and} \quad \frac{\overline{E}}{\overline{v}\,\overline{B}} = \varepsilon. \tag{1.5}$$

On the other hand we may assume that the length scale L is large compared to the Larmor radius, i.e. $\overline{a}_L/L = \varepsilon$. Then the rescaled Vlasov equation writes:

$$\frac{\partial f'}{\partial t'} + \mathbf{v}' \cdot \nabla_{x'} f' + \left(\mathcal{E}(\mathbf{x}', t') + \frac{1}{\varepsilon}\mathbf{v} \times \mathcal{B}(\mathbf{x}', t')\right) \cdot \nabla_{v'} f' = 0. \tag{1.6}$$

We may also assume that the length scale L is comparable to the Larmor radius in the direction orthogonal to the magnetic field, remaining large in the magnetic field direction. In this regime, called "Finite Larmor Radius Regime", the Vlasov equation reads

$$\frac{\partial f'}{\partial t'} + \mathbf{v}'_{\parallel} \cdot \nabla_{x'} f' + \frac{1}{\varepsilon}\mathbf{v}'_{\perp} \cdot \nabla_{x'} f' + \left(\mathcal{E}(\mathbf{x}', t') + \frac{1}{\varepsilon}\mathbf{v} \times \mathcal{B}(\mathbf{x}', t')\right) \cdot \nabla_{v'} f' = 0, \tag{1.7}$$

where we denote by $\parallel$ and $\perp$ the directions parallel and perpendicular to the magnetic field.

We notice that the Vlasov equation with a large magnetic field (1.6) or (1.7) can be cast into the framework of a singularly perturbed convection equation.

So, let us now consider the Initial-Value problem for a convection equation with a small parameter ε:

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{a} \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u_\varepsilon = 0, \quad \mathbf{x} \in \mathbb{R}^d,\ t > 0, \tag{1.8}$$

$$u_\varepsilon(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{1.9}$$

In (1.8), we assume that $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ verifies

$$\nabla \cdot \mathbf{b} = 0. \tag{1.10}$$

In order to get existence and uniqueness of the solution of (1.8)–(1.9), we need to make some regularity assumptions. In particular, we shall assume in the sequel that

$$\big(\mathbf{a}(\cdot, t), \mathbf{b}(\cdot, t)\big) \in W^{1,\infty}\big(\mathbb{R}^d\big). \tag{1.11}$$

As for $\mathbf{a}$, the only requirement is enough smoothness in order to be able to carry out the asymptotic expansion. We shall not try here to find the minimal required smoothness.

The aim of this paper is to prove that, under suitable hypotheses on $\mathbf{b}$, u_ε admits an asymptotic expansion of the form:

$$u_\varepsilon(\mathbf{x}, t) = \sum_{k \geqslant 0} \varepsilon^k U^k\left(\mathbf{x}, t, \frac{t}{\varepsilon}\right), \tag{1.12}$$

where the functions $U^k(x, t, \theta)$ are periodic in θ and will be characterized.

Before stating our results, let us introduce some notations: We shall denote by $\theta \mapsto X(\theta; \mathbf{x}, t)$ the solution of

$$\begin{aligned} &\frac{\mathrm{d}X}{\mathrm{d}\theta} = \mathbf{b}(X, t), \\ &X(0) = \mathbf{x}, \end{aligned} \tag{1.13}$$

and assume that the solutions of this system are periodic of period 2π in θ. Moreover, we shall denote by:

$$\mathbf{a}^0(\mathbf{x},t) = \frac{1}{2\pi}\int_0^{2\pi} \nabla X(\theta;\mathbf{x},t)^{-1}\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)\,\mathrm{d}\theta, \tag{1.14}$$

and

$$\tilde{\mathbf{a}}^0(\mathbf{x},t) = \frac{1}{2\pi}\int_0^{2\pi} \alpha(\mathbf{x},t,\theta)\,\mathrm{d}\theta, \tag{1.15}$$

with

$$\alpha(\mathbf{x},t,\theta) = \nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big) - \frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right), \tag{1.16}$$

where $\nabla X(\theta;\mathbf{x},t)$ stands for the Jacobian matrix of $\mathbf{x}\mapsto X(\theta;\mathbf{x},t)$.

Notice that $\tilde{\mathbf{a}}^0 = \mathbf{a}^0$ when $\mathbf{b} = \mathbf{b}(\mathbf{x})$ does not depend on t. Indeed $X(\theta;\mathbf{x},t) = X(\theta;\mathbf{x})$ does not depend on t either.

The main results of the paper are the following: First we have theorems characterizing the terms of the expansion:

THEOREM 1.1. – *The first term of the expansion* (1.12) *is given by*

$$U^0(\mathbf{x},t,\theta) = V^0\big(X(-\theta;\mathbf{x},t),t\big), \tag{1.17}$$

where the function V^0 is the solution of the initial value problem:

$$\frac{\partial V^0}{\partial t} + \tilde{\mathbf{a}}^0\cdot\nabla V^0 = 0, \quad \mathbf{x}\in\mathbb{R}^d,\ t>0, \tag{1.18}$$

$$V^0(\mathbf{x},0) = u_0(\mathbf{x}). \tag{1.19}$$

THEOREM 1.2. – *The function U^0 verifies the equation*:

$$\frac{\partial U^0}{\partial t}(\mathbf{x},t,\theta) + \mathbf{a}^0(\mathbf{x},t)\cdot\nabla U^0(\mathbf{x},t,\theta) = 0. \tag{1.20}$$

THEOREM 1.3. – *For $k\geqslant 1$ the function U^k is given by*:

$$U^k(\mathbf{x},t,\theta) = V^k\big(X(-\theta;\mathbf{x},t),t\big) + W^k\big(X(-\theta;\mathbf{x},t),t,\theta\big), \tag{1.21}$$

with

$$W^k(\mathbf{x},t,\theta) = -\int_0^{\theta}\left(\frac{\partial U^{k-1}}{\partial t} + \mathbf{a}\cdot\nabla U^{k-1}\right)\big(X(\sigma;\mathbf{x},t),t,\sigma\big)\,\mathrm{d}\sigma, \tag{1.22}$$

and where the function V^k is the solution of the initial value problem

$$\begin{aligned}\frac{\partial V^k}{\partial t} + \tilde{\mathbf{a}}^0\cdot\nabla V^k = -\frac{1}{2\pi}\int_0^{2\pi}\bigg[&\frac{\partial W^k}{\partial t}(\mathbf{x},t,\theta)\\ &+ \alpha(\mathbf{x},t,\theta)\cdot\nabla W^k(\mathbf{x},t,\theta)\bigg]\,\mathrm{d}\theta, \quad \mathbf{x}\in\mathbb{R}^d,\ t>0,\end{aligned} \tag{1.23}$$

$$V^k(\mathbf{x},0) = 0. \tag{1.24}$$

Our first approximation result, justifying the first term of the two-scale expansion is the following:

THEOREM 1.4. – *We assume that* $u_0 \in L^p(\mathbb{R}^d)$, $1 < p \leqslant \infty$. *Then, for any* $T > 0$, *the solution* u_ε *of* (1.8), (1.9) *stays bounded in* $L^\infty(0,T;L^p(\mathbb{R}^d))$ *and two-scale converges to the function* $U^0 \in L^\infty(0,T;L^\infty_\sharp(\mathbb{R};L^p(\mathbb{R}^d))$ *characterized by Theorem* 1.1.

Moreover, if $u_0 \in L^2(\mathbb{R}^d) \cap L^{2q}(\mathbb{R}^d)$, *for some* $q > 1$, *we have*:

$$\lim_{\varepsilon\to 0} \int_{Q_T} \left| u_\varepsilon(\mathbf{x},t) - U^0\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = 0. \tag{1.25}$$

In this theorem, $L^\infty_\sharp(\mathbb{R})$ stands for the space of 2π-periodic functions being in $L^\infty(\mathbb{R})$ and Q_T for $[0,T) \times \mathbb{R}^d$.

For the justification of the following terms of the expansion, we study the kth order approximation $U^0(\mathbf{x},t,\frac{t}{\varepsilon}) + \varepsilon U^1(\mathbf{x},t,\frac{t}{\varepsilon}) + \cdots + \varepsilon^k U^k(\mathbf{x},t,\frac{t}{\varepsilon})$ of u_ε. For this purpose, setting $u^0_\varepsilon(\mathbf{x},t) = u_\varepsilon(\mathbf{x},t)$, and $v^0_\varepsilon(\mathbf{x},t) = v_\varepsilon(\mathbf{x},t) = u_\varepsilon(X(\frac{t}{\varepsilon};\mathbf{x},t),t)$, we introduce the functions u^k_ε and v^k_ε defined respectively by:

$$\begin{aligned} u^k_\varepsilon(\mathbf{x},t) &= \frac{1}{\varepsilon}\left(u^{k-1}_\varepsilon(\mathbf{x},t) - U^{k-1}\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\right) \\ &= \frac{1}{\varepsilon^k}\left(u_\varepsilon(\mathbf{x},t) - U^0\left(\mathbf{x},t,\frac{t}{\varepsilon}\right) - \varepsilon U^1\left(\mathbf{x},t,\frac{t}{\varepsilon}\right) - \cdots - \varepsilon^{k-1}U^{k-1}\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\right), \end{aligned} \tag{1.26}$$

and

$$v^k_\varepsilon = \frac{1}{\varepsilon}\left(v^{k-1}_\varepsilon - V^{k-1}\right) - W^k_\varepsilon, \qquad W^k_\varepsilon(\mathbf{x},t) = W^k\left(\mathbf{x},t,\frac{t}{\varepsilon}\right), \tag{1.27}$$

where the function W^k is defined by (1.22). We have:

THEOREM 1.5. – *We assume that* $u_0 \in L^p(\mathbb{R}^d)$, $1 < p \leqslant \infty$. *Then, for* $k \geqslant 1$, *if the function* W^k *is "smooth enough", for example* $\partial W^k/\partial t, \nabla W^k \in L^\infty((0,T)\times(0,2\pi);L^p(\mathbb{R}^d))$, u^k_ε *two-scale converges to the function* $U^k \in L^\infty(0,T;L^\infty_\sharp(\mathbb{R};L^p(\mathbb{R}^d))$ *characterized by Theorem* 1.3, *whereas* v^k_ε *converges two-scale and in* $L^\infty(0,T;L^p(\mathbb{R}^d))$ *weak-$*$ to the associated function* V^k. *If moreover,* $u_0 \in L^2(\mathbb{R}^d)$, *we have*:

$$\lim_{\varepsilon\to 0} \int_{Q_T} \left| u^k_\varepsilon(\mathbf{x},t) - U^k\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \tag{1.28}$$

and v^k_ε *converges to* V^k *in* $L^2(Q_T)$ *strong.*

The paper is organized as follows: in the first part we formally characterize the different terms of the expansion (1.12) and prove Theorems 1.1 to 1.3 and in the second part we prove approximation results given by the different terms in Theorems 1.4 and 1.5. In the third part, we briefly present two simple applications of the previously built framework to the Vlasov equation.

2. Characterization of the terms of the expansion

Plugging expansion (1.12) into equation (1.8), we obtain:

$$\frac{1}{\varepsilon}\left(\frac{\partial U^0}{\partial\theta}+\mathbf{b}\cdot\nabla U^0\right)+\sum_{k\geqslant 1}\varepsilon^{k-1}\left(\frac{\partial U^k}{\partial\theta}+\mathbf{b}\cdot\nabla U^k+\frac{\partial U^{k-1}}{\partial t}+\mathbf{a}\cdot\nabla U^{k-1}\right)=0.$$

Identifying the terms of the same order in ε, this yields at the order -1

$$\frac{\partial U^0}{\partial\theta}+\mathbf{b}\cdot\nabla U^0=0, \tag{2.1}$$

and at the order $k-1$, $k\geqslant 1$,

$$\frac{\partial U^k}{\partial\theta}+\mathbf{b}\cdot\nabla U^k=-\left(\frac{\partial U^{k-1}}{\partial t}+\mathbf{a}\cdot\nabla U^{k-1}\right). \tag{2.2}$$

On the other hand, the initial condition (1.9), yields

$$U^0(\mathbf{x},0,0)=u_0(\mathbf{x}), \tag{2.3}$$

$$U^k(\mathbf{x},0,0)=0, \quad k\geqslant 1. \tag{2.4}$$

Equations (2.1) and (2.2) are of the form

$$\frac{\partial U}{\partial\theta}+\mathbf{b}\cdot\nabla U=S, \tag{2.5}$$

where the variable t only appears as a parameter and where

$$\theta\mapsto S(\theta;\mathbf{x}) \quad \text{is periodic of period } 2\pi. \tag{2.6}$$

Thanks to the regularity assumption (1.11), equation (2.5) can be solved by the method of characteristics. As $\mathbf{b}$ does not depend on θ, the solution of

$$\frac{\mathrm{d}X}{\mathrm{d}\theta}=\mathbf{b}(X),$$
$$X(\sigma)=\mathbf{x},$$

is given by $\theta\mapsto X(\theta-\sigma;\mathbf{x})$. Hence any solution of (2.5) is of the form

$$U(\mathbf{x},\theta)=V\big(X(-\theta;\mathbf{x})\big)+\int_0^\theta S\big(X(\sigma-\theta;\mathbf{x}),\sigma\big)\,\mathrm{d}\sigma, \tag{2.7}$$

where

$$V(\mathbf{x})=U(\mathbf{x},\theta=0).$$

Remember that we assume that the vector field $\mathbf{b}$ is such that

$$\theta\mapsto X(\mathbf{x},\theta) \quad \text{is periodic of period } 2\pi. \tag{2.8}$$

In the case of the Vlasov equation this is the magnetic field, and the periodic orbits are the trajectories in the magnetic field. We now express a condition under which U defined by (2.7) is also periodic in θ of period 2π:

LEMMA 2.1. – *Under assumptions* (2.6) *and* (2.8), *equation* (2.5) *is solvable in the class of* 2π*-periodic functions in* θ *if and only if*

$$\int_0^{2\pi} S\big(X(\theta;\mathbf{x}),\theta\big)\,\mathrm{d}\theta = 0 \quad \forall \mathbf{x}\in\mathbb{R}^d. \tag{2.9}$$

Proof. – Due to assumption (2.6), the function U defined by (2.7) is 2π-periodic in θ if and only if

$$\int_0^{\theta+2\pi} S\big(X(\sigma-\theta-2\pi;\mathbf{x}),\sigma\big)\,\mathrm{d}\sigma = \int_0^{\theta} S\big(X(\sigma-\theta;\mathbf{x}),\sigma\big)\,\mathrm{d}\sigma,$$

or equivalently

$$\int_\theta^{\theta+2\pi} S\big(X(\sigma-\theta;\mathbf{x}),\sigma\big)\,\mathrm{d}\sigma = 0.$$

Using assumption (2.8), this condition becomes

$$\int_0^{2\pi} S\big(X(\sigma-\theta;\mathbf{x}),\sigma\big)\,\mathrm{d}\sigma = 0.$$

Noticing that:

$$X(\sigma-\theta;\mathbf{x}) = X\big(\sigma; X(-\theta;\mathbf{x})\big),$$

and setting $\mathbf{y} = X(-\theta;\mathbf{x})$, we get

$$\int_0^{2\pi} S\big(X(\sigma;\mathbf{y}),\sigma\big)\,\mathrm{d}\sigma = 0 \quad \forall \mathbf{y}\in\mathbb{R}^d,$$

which proves our lemma. □

Let us now recall the dependence on t and write $X(\theta;\mathbf{x},t)$ instead of $X(\theta;\mathbf{x})$. Then the solution (2.7) of (2.5) reads

$$U(\mathbf{x},t,\theta) = V\big(X(-\theta;\mathbf{x},t),t\big) + \int_0^{\theta} S\big(X(\sigma-\theta;\mathbf{x},t),t,\sigma\big)\,\mathrm{d}\sigma, \tag{2.10}$$

and the solvability condition (2.9) becomes:

$$\int_0^{2\pi} S\big(X(\theta;\mathbf{x},t),t,\theta\big)\,\mathrm{d}\theta = 0 \quad \forall \mathbf{x}\in\mathbb{R}^d. \tag{2.11}$$

Let us now characterize the functions U^k, $k \geqslant 0$. Using (2.10), we deduce on the one hand from (2.1) that U^0 is of the form:

$$U^0(\mathbf{x}, t, \theta) = V^0\big(X(-\theta; \mathbf{x}, t), t\big), \tag{2.12}$$

i.e. (1.17) and, on the other hand from (2.2) that U^k, $k \geqslant 1$, is of the form (1.21) where W^k is defined by (1.22). At this stage, we have proved the first part of Theorems 1.1 and 1.3.

We now need to determine the equations satisfied by the functions $V^k = V^k(\mathbf{x}, t)$, $k \geqslant 0$. To this purpose, we shall write the solvability condition for equation (2.2) corresponding to $k+1$ in the class of functions that are 2π-periodic in θ:

$$\int_0^{2\pi} \left(\frac{\partial U^k}{\partial t} + \mathbf{a} \cdot \nabla U^k\right)\big(X(\theta; \mathbf{x}, t), t, \theta\big)\, \mathrm{d}\theta = 0. \tag{2.13}$$

The next step is to express condition (2.13) for U^k of the form (1.21). More generally, let us first evaluate the expression

$$\int_0^{2\pi} \left(\frac{\partial U}{\partial t} + \mathbf{a} \cdot \nabla U\right)\big(X(\theta; \mathbf{x}, t), t, \theta\big)\, \mathrm{d}\theta$$

when

$$U(\mathbf{x}, t, \theta) = W\big(X(-\theta; \mathbf{x}, t), t, \theta\big). \tag{2.14}$$

For this purpose, we need to verify some useful identities:

LEMMA 2.2. – *For $\theta, \sigma \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, $t > 0$, we have*:

$$\nabla X\big(\sigma; X(-\theta; \mathbf{x}, t), t\big) = \nabla X(\sigma - \theta; \mathbf{x}, t) \nabla X(-\theta; \mathbf{x}, t)^{-1}, \tag{2.15}$$

and

$$\begin{aligned} &\frac{\partial X}{\partial t}\big(\sigma; X(-\theta; \mathbf{x}, t), t\big) \\ &\quad = \frac{\partial X}{\partial t}(\sigma - \theta; \mathbf{x}, t) - \nabla X(\sigma - \theta; \mathbf{x}, t) \nabla X(-\theta; \mathbf{x}, t)^{-1} \frac{\partial X}{\partial t}(-\theta; \mathbf{x}, t). \end{aligned} \tag{2.16}$$

Proof. – We start from the identity

$$X(\sigma - \theta; \mathbf{x}, t) = X\big(\sigma, X(-\theta; \mathbf{x}, t), t\big).$$

Taking the derivative with respect to $\mathbf{x}$, we find

$$\nabla X(\sigma - \theta; \mathbf{x}, t) = \nabla X\big(\sigma; X(-\theta; \mathbf{x}, t), t\big) \nabla X(-\theta; \mathbf{x}, t),$$

i.e. (2.15). Taking the derivative with respect to t, we get:

$$\frac{\partial X}{\partial t}(\sigma - \theta; \mathbf{x}, t) = \frac{\partial X}{\partial t}\big(\sigma; X(-\theta; \mathbf{x}, t), t\big) + \nabla X\big(\sigma; X(-\theta; \mathbf{x}, t), t\big) \frac{\partial X}{\partial t}(-\theta; \mathbf{x}, t)$$

which together with (2.15) yields (2.16). □

In the case when $\sigma=\theta$, (2.15) and (2.16) become respectively:

$$\nabla X\big(\theta;X(-\theta;\mathbf{x},t),t\big)=\nabla X(-\theta;\mathbf{x},t)^{-1} \tag{2.17}$$

and

$$\frac{\partial X}{\partial t}\big(\theta;X(-\theta;\mathbf{x},t),t\big)=-\nabla X(-\theta;\mathbf{x},t)^{-1}\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t). \tag{2.18}$$

We can then prove

LEMMA 2.3. – *If the function U is of the form* (2.14)*, we have*:

$$\begin{aligned}&\int_0^{2\pi}\left[\frac{\partial U}{\partial t}+\mathbf{a}\cdot\nabla U\right]\big(X(\theta;\mathbf{x},t),t,\theta\big)\,\mathrm{d}\theta\\
&\quad=\int_0^{2\pi}\left[\frac{\partial W}{\partial t}(\mathbf{x},t,\theta)+\nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\cdot\nabla W(\mathbf{x},t,\theta)\right]\mathrm{d}\theta.\end{aligned} \tag{2.19}$$

Proof. – Taking the derivative of (2.14) with respect to t, we first obtain:

$$\frac{\partial U}{\partial t}(\mathbf{x},t,\theta)=\frac{\partial W}{\partial t}\big(X(-\theta;\mathbf{x},t),t,\theta\big)+\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\cdot\nabla W\big(X(-\theta;\mathbf{x},t),t,\theta\big)$$

whence

$$\frac{\partial U}{\partial t}\big(X(\theta;\mathbf{x},t),t,\theta\big)=\frac{\partial W}{\partial t}(\mathbf{x},t,\theta)+\frac{\partial X}{\partial t}\big(-\theta;X(\theta;\mathbf{x},t),t\big)\cdot\nabla W(\mathbf{x},t,\theta)$$

and due to (2.18)

$$\frac{\partial U}{\partial t}\big(X(\theta;\mathbf{x},t),t,\theta\big)=\frac{\partial W}{\partial t}(\mathbf{x},t,\theta)-\left(\nabla X(\theta;\mathbf{x},t)^{-1}\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\cdot\nabla W(\mathbf{x},t,\theta).$$

Then, taking the derivative of (2.14) with respect to $\mathbf{x}$, we find:

$$\nabla U(\mathbf{x},t,\theta)=\nabla X(-\theta;\mathbf{x},t)^T\,\nabla W\big(X(-\theta;\mathbf{x},t),t,\theta\big)$$

hence, using (2.17)

$$\begin{aligned}(\mathbf{a}\cdot\nabla U)\big(X(\theta;\mathbf{x},t),t,\theta\big)&=\big(\nabla X\big(-\theta;X(\theta;\mathbf{x},t),t\big)\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)\big)\cdot\nabla W(\mathbf{x},t,\theta)\\
&=\big(\nabla X(\theta;\mathbf{x},t)^{-1}\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)\big)\cdot\nabla W(\mathbf{x},t,\theta).\end{aligned}$$

So we get

$$\begin{aligned}&\left[\frac{\partial U}{\partial t}+\mathbf{a}\cdot\nabla U\right](X(\theta;\mathbf{x},t),t,\theta)\\
&\quad=\frac{\partial W}{\partial t}(\mathbf{x},t,\theta)+\left(\nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\right)\cdot\nabla W(\mathbf{x},t,\theta)\end{aligned}$$

and in particular (2.19). Thus the lemma is proved. □

We also have:

COROLLARY 2.4. – *If the function U is of the form*

$$U(\mathbf{x},t,\theta)=V\big(X(-\theta;\mathbf{x},t),t\big) \tag{2.20}$$

for a function $V=V(\mathbf{x},t)$, we have:

$$\int_0^{2\pi}\left[\frac{\partial U}{\partial t}+\mathbf{a}\cdot\nabla U\right]\big(X(\theta;\mathbf{x},t),t,\theta\big)\,\mathrm{d}\theta=2\pi\left[\frac{\partial V}{\partial t}+\tilde{\mathbf{a}}^0\cdot\nabla V\right](\mathbf{x},t). \tag{2.21}$$

We then verify

LEMMA 2.5. – *We have*:

$$\nabla\cdot\mathbf{a}^0=\nabla\cdot\tilde{\mathbf{a}}^0=\frac{1}{2\pi}\int_0^{2\pi}(\nabla\cdot\mathbf{a})\big(X(\theta;\mathbf{x},t),t\big)\,\mathrm{d}\theta. \tag{2.22}$$

Proof. – Let $\varphi\in C_c^\infty(\mathbb{R}^d)$. We can write

$$\int_{\mathbb{R}^d}\mathbf{a}^0(\mathbf{x},t)\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}=\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^d}\nabla X(\theta;\mathbf{x},t)^{-1}\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\theta.$$

Making the change of variable

$$\mathbf{y}=X(\theta;\mathbf{x},t)\Leftrightarrow\mathbf{x}=X(-\theta;\mathbf{y},t),$$

we have:

$$\begin{aligned}&\int_{\mathbb{R}^d}\big(\nabla X(\theta;\mathbf{x},t)^{-1}\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)\big)\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&\quad=\int_{\mathbb{R}^d}\mathbf{a}(\mathbf{y},t)\cdot\big(\nabla X\big(\theta;X(-\theta;\mathbf{y},t),t\big)^{-T}\nabla\varphi\big(X(-\theta;\mathbf{y},t)\big)\big)\,\mathrm{d}\mathbf{y}.\end{aligned}$$

Thanks to (2.17), this last integral is equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathbf{a}(\mathbf{y},t)\cdot\big(\nabla X\big(\theta;X(-\theta;\mathbf{y},t),t\big)^{-T}\nabla\varphi\big(X(-\theta;\mathbf{y},t)\big)\big)\,\mathrm{d}\mathbf{y}\\&\quad=\int_{\mathbb{R}^d}\mathbf{a}(\mathbf{y},t)\cdot\nabla_y\big(\varphi\big(X(-\theta;\mathbf{y},t)\big)\big)\,\mathrm{d}\mathbf{y}\\&\quad=-\int_{\mathbb{R}^d}\nabla\cdot\mathbf{a}\,\varphi\big(X(-\theta;\mathbf{y},t)\big)\,\mathrm{d}\mathbf{y}=-\int_{\mathbb{R}^d}(\nabla\cdot\mathbf{a})\big(X(\theta;\mathbf{x},t)\big)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Hence we get:

$$\int_{\mathbb{R}^d}\mathbf{a}^0(\mathbf{x},t)\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}=-\frac{1}{2\pi}\int_{\mathbb{R}^d}\int_0^{2\pi}(\nabla\cdot\mathbf{a})\big(X(\theta;\mathbf{x},t)\big)\,\mathrm{d}\theta\,\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

i.e.

$$\nabla\cdot\mathbf{a}^0=\frac{1}{2\pi}\int_0^{2\pi}(\nabla\cdot\mathbf{a})\big(X(\theta;\mathbf{x},t)\big)\,\mathrm{d}\theta.$$

In the same way, we can write

$$\int_{\mathbb{R}^d}\left(\nabla X(\theta;\mathbf{x},t)^{-1}\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\cdot\nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}$$
$$=\int_{\mathbb{R}^d}\left(\nabla X\big(\theta;X(-\theta;\mathbf{y},t),t\big)^{-1}\frac{\partial X}{\partial t}\big(\theta;X(-\theta;\mathbf{y},t),t\big)\right)\cdot\nabla\varphi\big(X(-\theta;\mathbf{y},t)\big)\,\mathrm{d}\mathbf{y}=0.$$

Thanks to (2.18) and (2.17) this last integral is equal to:

$$-\int_{\mathbb{R}^d}\left(\nabla X\big(\theta;X(-\theta;\mathbf{y},t),t\big)^{-1}\nabla X(-\theta;\mathbf{y},t)^{-1}\frac{\partial X}{\partial t}(-\theta;\mathbf{y},t)\right)\cdot\nabla\varphi\big(X(-\theta;\mathbf{y},t)\big)\,\mathrm{d}\mathbf{y}$$
$$=-\int_{\mathbb{R}^d}\frac{\partial X}{\partial t}(-\theta;\mathbf{y},t)\cdot\nabla\varphi\big(X(-\theta;\mathbf{y},t)\big)\,\mathrm{d}\mathbf{y}=-\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\varphi\big(X(-\theta;\mathbf{y},t)\big)\,\mathrm{d}\mathbf{y}$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}=0.$$

It follows that $\nabla\cdot\mathbf{a}^0=\nabla\cdot\tilde{\mathbf{a}}^0$. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. – As already noticed (1.17) is a consequence of formula (2.10), and the solvability condition for equation (2.2) corresponding to $k=1$ reads, due to Corollary 2.4,

$$\frac{\partial V^0}{\partial t}+\tilde{\mathbf{a}}^0\cdot V^0=0.$$

On the other hand, we deduce from (2.3) and (2.12) that $V^0(\mathbf{x},0)=U^0(\mathbf{x},0,0)=u_0(\mathbf{x})$. □

As a consequence of Theorem 1.1, we can now deduce Theorem 1.2 as follows:

Proof of Theorem 1.2. – We deduce from (2.12) that

$$\frac{\partial U^0}{\partial t}(\mathbf{x},t,\theta)=\frac{\partial V^0}{\partial t}\big(X(-\theta;\mathbf{x},t),t\big)+\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\cdot\nabla V^0\big(X(-\theta;\mathbf{x},t),t\big),$$

and

$$\nabla U^0\big(X(-\theta,\mathbf{x},t)\big)=\nabla X(-\theta;\mathbf{x},t)^T\nabla V^0\big(X(-\theta;\mathbf{x},t),t\big)$$

hence thanks to equation (1.18) satisfied by U^0, we have:

$$\frac{\partial U^0}{\partial t}(\mathbf{x},t,\theta)+\nabla X(-\theta;\mathbf{x},t)^{-1}\left(\tilde{\mathbf{a}}^0\big(X(-\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)\cdot\nabla U^0(\mathbf{x},t,\theta)=0.$$

We then need to evaluate:

$$\nabla X(-\theta;\mathbf{x},t)^{-1}\left(\tilde{\mathbf{a}}^0\big(X(-\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\Big[\nabla X(-\theta;\mathbf{x},t)^{-1}\nabla X\big(\sigma,X(-\theta;\mathbf{x},t),t\big)^{-1}\mathbf{a}\big(\sigma,X(-\theta;\mathbf{x},t),t\big)$$
$$-\frac{\partial X}{\partial t}\big(\sigma,X(-\theta;\mathbf{x},t),t\big)\Big]\,\mathrm{d}\sigma-\nabla X(-\theta;\mathbf{x},t)^{-1}\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t).$$

Using (2.15) and (2.16), we find:

$$\nabla X(-\theta;\mathbf{x},t)^{-1}\left(\tilde{\mathbf{a}}^0\big(X(-\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\nabla X(\sigma-\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\sigma-\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(\sigma-\theta;\mathbf{x},t)\right.$$
$$\left.+\nabla X(\sigma-\theta;\mathbf{x},t)\nabla X(-\theta;\mathbf{x},t)^{-1}\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)\mathrm{d}\sigma-\nabla X(-\theta;\mathbf{x},t)\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\nabla X(\sigma-\theta;\mathbf{x},t)^{-1}\mathbf{a}(\sigma-\theta,\mathbf{x},t)\,\mathrm{d}\sigma.$$

We then deduce from the 2π-periodicity of $\theta\mapsto X(-\theta;\mathbf{x},t)$ that

$$\nabla X(-\theta;\mathbf{x},t)^{-1}\left(\tilde{\mathbf{a}}^0\big(X(-\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\nabla X(\sigma;\mathbf{x},t)^{-1}\mathbf{a}\big(X(\sigma;\mathbf{x},t),t\big)\,\mathrm{d}\sigma=\tilde{\mathbf{a}}^0(\mathbf{x},t)$$

which proves the result. □

Let us now come to the determination of the functions V^k and U^k for $k\geqslant 1$. Assuming U^{k-1} and consequently W^k are known, we obtain V^k and U^k thanks to Theorem 1.3 which we are now able to prove:

Proof of Theorem 1.3. – Formula (1.21) has already been obtained from (2.10) In order to determine the function V^k, we write the solvability condition (2.13) of equation (1.23) corresponding to $k+1$. We deduce from Lemma 2.3 and from Corollary 2.4 that

$$\int_0^{2\pi}\left(\frac{\partial U^k}{\partial t}+\mathbf{a}\cdot\nabla U^k\right)\big(X(-\theta;\mathbf{x},t),t,\theta\big)\,\mathrm{d}\theta$$
$$=2\pi\left(\frac{\partial V^k}{\partial t}+\tilde{\mathbf{a}}_0\cdot\nabla V^k\right)(\mathbf{x},t)$$
$$-\int_0^{2\pi}\left(\frac{\partial W^k}{\partial t}(\mathbf{x},t,\theta)+\left(\nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\right)\cdot\nabla W^k(\mathbf{x},t,\theta)\right)\mathrm{d}\theta$$

whence the first equation (1.23). On the other hand, we have:

$$V^k(\mathbf{x},0)=U^k(\mathbf{x},0,0)-W^k(\mathbf{x},0,0)=0$$

due to (2.4) and the definition (1.22) of W^k. □

Notice that a result analogous to Theorem 1.2 can also be proved.

Let us now explicit the case $k=1$, where it is easy to find a more explicit equation. We shall compute the source term of the initial-value problem explicitely in function of V^0. We start by computing

$$W^1(\mathbf{x},t,\theta)=-\int_0^\theta\left(\frac{\partial U^0}{\partial t}+\mathbf{a}\cdot\nabla U^0\right)\big(X(\sigma;\mathbf{x},t),t,\sigma\big)\,\mathrm{d}\sigma.$$

Setting

$$\tilde{\mathbf{A}}(\mathbf{x},t,\theta)=\frac{1}{\theta}\int_0^\theta\alpha(\mathbf{x},t,\theta)\,\mathrm{d}\sigma, \tag{2.23}$$

we obtain the:

THEOREM 2.6. – *We have*

$$W^1(\mathbf{x},t,\theta)=\theta\big(\tilde{\mathbf{a}}^0(\mathbf{x},t)-\tilde{\mathbf{A}}(\mathbf{x},t,\theta)\big)\cdot\nabla V^0(\mathbf{x},t), \tag{2.24}$$

and

$$\begin{aligned}&-\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial W^1}{\partial t}(\mathbf{x},t,\theta)+\alpha(\mathbf{x},t,\theta)\cdot\nabla W^1(\mathbf{x},t,\theta)\right]\mathrm{d}\theta\\&\quad=\frac{1}{2\pi}\left[\int_0^{2\pi}\theta\left(\frac{\partial\tilde{\mathbf{A}}}{\partial t}(\mathbf{x},t,\theta)-\frac{\partial\tilde{\mathbf{a}}^0}{\partial t}(\mathbf{x},t)\right)\right.\\&\qquad+\theta\big(\nabla\tilde{\mathbf{A}}(\mathbf{x},t,\theta)-\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\alpha(\mathbf{x},t,\theta)-\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big(\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta)\\&\qquad\left.-\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\big)\,\mathrm{d}\theta\right]\cdot\nabla V^0(\mathbf{x},t).\end{aligned} \tag{2.25}$$

Proof. – Starting from (2.12), we obtain as in the proof of Theorem 1.2

$$\begin{aligned}&\left(\frac{\partial U^0}{\partial t}+\mathbf{a}\cdot\nabla U^0\right)(\mathbf{x},t,\theta)\\&\quad=\frac{\partial V^0}{\partial t}\big(X(-\theta;\mathbf{x},t),t\big)+\left(\nabla X(-\theta;\mathbf{x},t)\mathbf{a}(\mathbf{x},t)+\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t)\right)\cdot\nabla V^0\big(X(-\theta;\mathbf{x},t),t\big),\end{aligned}$$

that is

$$\begin{aligned}\left(\frac{\partial U^0}{\partial t}+\mathbf{a}\cdot\nabla U^0\right)\big(X(\sigma;\mathbf{x},t),t,\sigma\big)&=\frac{\partial V^0}{\partial t}X\big(-\sigma;X(\sigma;\mathbf{x},t),t\big)\\&\quad+\Big(\nabla X\big(-\sigma;X(\sigma;\mathbf{x},t),t\big)\mathbf{a}\big(X(\sigma;\mathbf{x},t),t\big)\\&\quad+\frac{\partial X}{\partial t}\big(-\sigma;X(\sigma;\mathbf{x},t),t\big)\Big)\cdot\nabla V^0\big(-\sigma;X(\sigma;\mathbf{x},t),t\big),\end{aligned}$$

and due to (2.17)–(2.18)

$$\left(\frac{\partial U^0}{\partial t} + \mathbf{a}\cdot\nabla U^0\right)\big(X(\sigma;\mathbf{x},t),t,\sigma\big)$$
$$= \frac{\partial V^0}{\partial t}(\mathbf{x},t) + \nabla X(\sigma;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\sigma;\mathbf{x},t),t\big) - \frac{\partial X}{\partial t}(\sigma;\mathbf{x},t)\right)\cdot\nabla V^0(\mathbf{x},t).$$

It follows, using (2.23)

$$W^1(\mathbf{x},t,\theta) = -\theta\left(\frac{\partial V^0}{\partial t}(\mathbf{x},t) + \tilde{\mathbf{A}}(\mathbf{x},t,\theta)\cdot\nabla V^0(\mathbf{x},t)\right),$$

that is (2.24) using (1.18).

Concerning (2.25), we first compute

$$-\int_0^{2\pi}\frac{\partial W^1}{\partial t}(\mathbf{x},t,\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\theta\left(\frac{\partial\tilde{\mathbf{A}}}{\partial t}(\mathbf{x},t,\theta) - \frac{\partial\tilde{\mathbf{a}}^0}{\partial t}(\mathbf{x},t)\right)\cdot\nabla V^0(\mathbf{x},t)\,\mathrm{d}\theta$$
$$+\int_0^{2\pi}\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\cdot\nabla\frac{\partial V^0}{\partial t}(\mathbf{x},t)\,\mathrm{d}\theta.$$

Using (1.18), we have:

$$\nabla\frac{\partial V^0}{\partial t}(\mathbf{x},t) = -\nabla\big(\tilde{\mathbf{a}}^0(\mathbf{x},t)\cdot\nabla V^0(\mathbf{x},t)\big) = -\big(\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)^T\nabla V^0(\mathbf{x},t) - \big(\nabla\nabla V^0(\mathbf{x},t)\big)\tilde{\mathbf{a}}^0(\mathbf{x},t),$$

and then

$$-\int_0^{2\pi}\frac{\partial W^1}{\partial t}(\mathbf{x},t,\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\theta\left(\frac{\partial\tilde{\mathbf{A}}}{\partial t}(\mathbf{x},t,\theta) - \frac{\partial\tilde{\mathbf{a}}^0}{\partial t}(\mathbf{x},t)\right)\cdot\nabla V^0(\mathbf{x},t)\,\mathrm{d}\theta$$
$$-\int_0^{2\pi}\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\cdot\Big(\big(\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)^T\nabla V^0(\mathbf{x},t)\Big)\,\mathrm{d}\theta \tag{2.26}$$
$$-\int_0^{2\pi}\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\cdot\Big(\big(\nabla\nabla V^0(\mathbf{x},t)\big)\tilde{\mathbf{a}}^0(\mathbf{x},t)\Big)\,\mathrm{d}\theta.$$

Secondly,

$$-\int_0^{2\pi}\alpha(\mathbf{x},t,\theta)\cdot\nabla W^1(\mathbf{x},t,\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\alpha(\mathbf{x},t,\theta)\cdot\Big[\theta\big(\nabla\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)^T\nabla V^0(\mathbf{x},t)\Big]\,\mathrm{d}\theta$$
$$+\int_0^{2\pi}\alpha(\mathbf{x},t,\theta)\cdot\Big[\nabla\nabla V^0(\mathbf{x},t)\Big(\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\Big)\Big]\,\mathrm{d}\theta$$
$$= \int_0^{2\pi}\theta\big(\nabla\tilde{\mathbf{A}}(\mathbf{x},t,\theta) - \nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\alpha(\mathbf{x},t,\theta)\cdot\nabla V^0(\mathbf{x},t)\,\mathrm{d}\theta$$

$$+\int_0^{2\pi}\big(\nabla\nabla V^0(\mathbf{x},t)\big)\alpha(\mathbf{x},t,\theta)\cdot\big(\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta)-\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\big)\,\mathrm{d}\theta. \tag{2.27}$$

Summing (2.26) and (2.27), we get:

$$\begin{aligned}
&-\int_0^{2\pi}\left[\frac{\partial W^1}{\partial t}(\mathbf{x},t,\theta)+\alpha(\mathbf{x},t,\theta)\cdot\nabla W^1(\mathbf{x},t,\theta)\right]\mathrm{d}\theta\\
&\quad=\left[\int_0^{2\pi}\theta\left(\frac{\partial\tilde{\mathbf{A}}}{\partial t}(\mathbf{x},t,\theta)-\frac{\partial\tilde{\mathbf{a}}^0}{\partial t}(\mathbf{x},t)\right)\right.\\
&\qquad+\theta\big(\nabla\tilde{\mathbf{A}}(\mathbf{x},t,\theta)-\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\alpha(\mathbf{x},t,\theta)-\nabla\tilde{\mathbf{a}}^0(\mathbf{x},t)\big(\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta)\\
&\qquad\left.-\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\big)\,\mathrm{d}\theta\right]\cdot\nabla V^0(\mathbf{x},t)\\
&\qquad-\int_0^{2\pi}\big(\nabla\nabla V^0(\mathbf{x},t)\big)\big(\alpha(\mathbf{x},t,\theta)-\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\cdot\big(\theta\big(\tilde{\mathbf{A}}(\mathbf{x},t,\theta)-\tilde{\mathbf{a}}^0(\mathbf{x},t)\big)\big)\,\mathrm{d}\theta.
\end{aligned}$$

In this equality, the last integral is of the form $\int_0^{2\pi}(S\frac{\mathrm{d}}{\mathrm{d}\theta}\xi)\cdot\xi\,\mathrm{d}\theta$ where S is symmetric and independent on θ and ξ is 2π-periodic, hence it gives 0. Then we obtain (2.25). □

3. Approximation results

After having formally derived a two-scale expansion of the initial value problem (1.8)–(1.9), we now want to to justify it rigorously, i.e. study the approximation properties of the partial sums:

$$U^0\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)+\varepsilon U^1\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)+\cdots+\varepsilon^k U^k\left(\mathbf{x},t,\frac{t}{\varepsilon}\right).$$

Let us start by presenting a straightforward generalization of the results of N'Guetseng [10] and Allaire [1]. Let X be a Banach space and let $q\in[1,\infty)$; we denote by X' the dual space of X, $\langle\cdot,\cdot\rangle$ the duality bracket between X' and X and q' the conjugate exponent of q, such that $1/q+1/q'=1$. We denote by $C_\sharp(\mathbb{R};X)$ the space of continuous 2π-periodic functions on $\mathbb{R}$, with values in X. Then given a sequence (u_ε) of functions of $L^{q'}(0,T;X')$ and a function $U^0=U^0(t,\theta)$ in $L^{q'}((0,T)\times(0,2\pi);X')=L^{q'}((0,T);L^{q'}(0,2\pi;X'))$, we shall say that

$$u_\varepsilon\to U^0\quad\text{two-scale when }\varepsilon\to 0,$$

if, for any function $\psi\in L^q(0,T;C_\sharp(\mathbb{R};X))$, we have

$$\lim_{\varepsilon\to 0}\int_0^T\left\langle u_\varepsilon(t),\psi\left(t,\frac{t}{\varepsilon}\right)\right\rangle\mathrm{d}t=\frac{1}{2\pi}\int_0^T\int_0^{2\pi}\left\langle U^0(t,\theta),\psi(t,\theta)\right\rangle\mathrm{d}\theta\,\mathrm{d}t.$$

In particular, it follows that

$$u_\varepsilon \to u^0 = \frac{1}{2\pi}\int_0^{2\pi} U^0(\cdot,\theta)\,\mathrm{d}\theta \quad \text{in } L^{q'}(0,T;X') \text{ weak-*}.$$

Then, can prove:

THEOREM 3.1. – *Given a sequence* (u_ε) *bounded in* $L^{q'}(0,T;X')$, *there exists an extracted subsequence* (*denoted in the same way*) *and a function* $U^0 \in L^{q'}((0,T)\times(0,2\pi);X')$ *such that*

$$u_\varepsilon \to U^0 \quad \textit{two-scale}.$$

Moreover, if X *is a Hilbert space,* $q=2$ *and* $U^0 \in L^2(0,T;C_\sharp(\mathbb{R};X'))$ *and if in addition*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^2(0,T;X)} = \frac{1}{\sqrt{2\pi}}\|U^0\|_{L^2((0,T)\times(0,2\pi);X)},$$

we have:

$$\lim_{\varepsilon\to 0}\int_0^T \left\|u_\varepsilon(t) - U^0\left(t,\frac{t}{\varepsilon}\right)\right\|_X^2 \mathrm{d}t = 0.$$

The two-scale convergence notions being precised, we can now prove our first approximation result,

Proof of Theorem 1.4. – We first prove that u_ε remains bounded. Multiplying (1.8) by pu_ε^{p-1}, we obtain, as $\nabla\cdot\mathbf{b}=0$ the energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u_\varepsilon^p\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d}(\nabla\cdot\mathbf{a})u_\varepsilon^p\,\mathrm{d}\mathbf{x} \leqslant \|\nabla\cdot\mathbf{a}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d} u_\varepsilon^p\,\mathrm{d}\mathbf{x}. \tag{3.1}$$

Then using the Gronwall lemma we get:

$$\int_{\mathbb{R}^d} u_\varepsilon^p\,\mathrm{d}\mathbf{x} \leqslant \int_{\mathbb{R}^d} u_0^p\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{T\|\nabla\cdot\mathbf{a}\|},$$

so that u_ε remains bounded in $L^\infty(0,T;L^p(\mathbb{R}^d))$. As $p>1$, we can apply Theorem 3.1: there exists an extracted subsequence of the sequence (u_ε), still denoted by (u_ε), and a function $U^0 \in L^\infty(0,T;L^\infty_\sharp(\mathbb{R};L^\infty(\mathbb{R}^d))$ such that

$$u_\varepsilon \to U^0 \quad \text{two-scale}.$$

Let then $\psi=\psi(\mathbf{x},t,\theta)$ be a function of class C^1, 2π-periodic in θ such that $\psi(\cdot,\cdot,\theta)\in C_c^1(\mathbb{R}^d\times[0,T))$. We set $\psi_\varepsilon(\mathbf{x},t)=\psi(\mathbf{x},t,\frac{t}{\varepsilon})$, so that $\psi_\varepsilon \in C_c^1(\mathbb{R}^d\times[0,T))$. We then write that u_ε is a weak solution of (1.8), (1.9). Setting $Q_T=\mathbb{R}^d\times[0,T)$, we obtain in particular that

$$\int_{Q_T} u_\varepsilon\left(\frac{\partial\psi_\varepsilon}{\partial t} + \nabla\cdot(\psi_\varepsilon\mathbf{a}) + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla\psi_\varepsilon\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} u_0\psi_\varepsilon(\cdot,0)\,\mathrm{d}\mathbf{x} = 0. \tag{3.2}$$

We have

$$\frac{\partial \psi_\varepsilon}{\partial t}(\mathbf{x}, t) = \frac{\partial \psi}{\partial t}\left(\mathbf{x}, t, \frac{t}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial \psi}{\partial \theta}\left(\mathbf{x}, t, \frac{t}{\varepsilon}\right),$$

that is, shorthanded,

$$\frac{\partial \psi_\varepsilon}{\partial t} = \left(\frac{\partial \psi}{\partial t}\right)_\varepsilon + \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial \theta}\right)_\varepsilon,$$

so that, multiplying (3.2) by ε, we get:

$$\int_{Q_T} u_\varepsilon\left(\left(\frac{\partial \psi}{\partial \theta}\right)_\varepsilon + \mathbf{b}\cdot(\nabla\psi)_\varepsilon\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+\,\varepsilon\left[\int_{Q_T} u_\varepsilon\left(\left(\frac{\partial \psi}{\partial t}\right)_\varepsilon + \nabla\cdot(\mathbf{a}\psi_\varepsilon)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} u_0\psi_\varepsilon(\cdot, 0)\,\mathrm{d}\mathbf{x}\right] = 0.$$

Using that u_ε two-scale converges to U^0 and the fact that the last two integrals remain bounded, we obtain when passing to the limit in ε

$$\int_0^{2\pi}\int_{Q_T} U^0\left(\frac{\partial \psi}{\partial \theta} + \mathbf{b}\cdot\nabla\psi\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}\theta = 0.$$

We thus find that $(\mathbf{x}, \theta) \mapsto U^0(\mathbf{x}, t, \theta)$ is the solution of (2.1) so that U^0 is of the form (2.12) and is 2π-periodic in θ due to hypothesis (2.6).

We now need to verify that the function V^0 so introduced is indeed well characterized by (1.18). For this we choose in (3.2)

$$\psi(\mathbf{x}, t, \theta) = \varphi\big(X(-\theta; \mathbf{x}, t), t\big), \quad \varphi \in C_c^1\big(\mathbb{R}^d \times [0, T)\big),$$

so that

$$\frac{\partial \psi}{\partial \theta} + \mathbf{b}\cdot\nabla\psi = 0.$$

This time we obtain:

$$\int_{Q_T} u_\varepsilon\left(\left(\frac{\partial \psi}{\partial t}\right)_\varepsilon + \nabla\cdot(\mathbf{a}\psi_\varepsilon)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} u_0\psi(\cdot, 0, 0)\,\mathrm{d}\mathbf{x} = 0,$$

and using the two scale convergence of u_ε to U^0 once more

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T} U^0\left(\frac{\partial \psi}{\partial t} + \nabla\cdot(\mathbf{a}\psi)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}\theta + \int_{\mathbb{R}^d} u_0\psi(\cdot, 0, 0)\,\mathrm{d}\mathbf{x} = 0.$$

Noticing, as previously, that

$$\frac{\partial \psi}{\partial t}(\mathbf{x}, t, \theta) = \frac{\partial \varphi}{\partial t}\big(X(-\theta; \mathbf{x}, t), t\big) + \frac{\partial X}{\partial t}(-\theta; \mathbf{x}, t), t)\cdot\nabla\varphi\big(X(-\theta; \mathbf{x}, t), t\big),$$

$$\nabla\psi(\mathbf{x},t,\theta)=\nabla X(-\theta;\mathbf{x},t)^T\nabla\varphi\big(X(-\theta;\mathbf{x},t),t\big),$$

and replacing U^0 by its expression (2.12), we find:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T}V^0\big(X(-\theta;\mathbf{x},t),t\big)\bigg[\frac{\partial\varphi}{\partial t}\big(X(-\theta;\mathbf{x},t),t\big)+\bigg(\nabla X(-\theta;\mathbf{x},t)\mathbf{a}(\mathbf{x},t)$$
$$+\frac{\partial X}{\partial t}(-\theta;\mathbf{x},t),t\bigg)\cdot\nabla\varphi\big(X(-\theta;\mathbf{x},t),t\big)+\nabla\cdot\Big(\mathbf{a}(\mathbf{x},t)\varphi\big(X(-\theta;\mathbf{x},t),t\big)\Big)\bigg]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}\theta$$
$$+\int_{\mathbb{R}^d}u_0\varphi(\cdot,0)\,\mathrm{d}\mathbf{x}=0.$$

Setting $\mathbf{y}=X(-\theta;\mathbf{x},t)$ and using (2.17) and (2.18) with $\mathbf{x}$ replaced by $\mathbf{y}$ and θ by $-\theta$, and using Lemma 2.5, we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T}V^0(\mathbf{y},t)\bigg(\frac{\partial\varphi}{\partial t}(\mathbf{y},t)+\nabla X(\theta,\mathbf{y},t)^{-1}\bigg(\mathbf{a}\big(X(\theta;\mathbf{y},t),t\big)-\frac{\partial X}{\partial t}(\theta;\mathbf{y},t)\bigg)\cdot\nabla\varphi(\mathbf{y},t)$$
$$+\nabla\cdot\Big(\nabla X(\theta,\mathbf{y},t)^{-1}\mathbf{a}\big(X(\theta;\mathbf{y},t),t\big)\varphi(\mathbf{y},t)\Big)\bigg)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\,\mathrm{d}\theta+\int_{\mathbb{R}^d}u_0\varphi(\cdot,0)\,\mathrm{d}\mathbf{x}=0,$$

which is also, using the definition of $\tilde{\mathbf{a}}_0$

$$\int_{Q_T}V^0(\mathbf{y},t)\bigg(\frac{\partial\varphi}{\partial t}(\mathbf{y},t)+\nabla\cdot(\tilde{\mathbf{a}}_0\varphi)\bigg)\mathrm{d}\mathbf{y}\,\mathrm{d}t+\int_{\mathbb{R}^d}u_0\varphi(\cdot,0)\,\mathrm{d}\mathbf{x}=0.$$

In other words, V^0 is a weak solution of (1.18). This proves the first part of the theorem.

Assume now that $u_0\in L^2(\mathbb{R}^d)\cap L^{2q}(\mathbb{R}^d)$. We notice that $U^0\in C^0([0,T];C_\sharp(\mathbb{R};L^2(\mathbb{R}^d)))$. On the other hand, equation (1.8) being linear in u_ε, we get, multiplying it by u_ε that u_ε^2 also verifies (1.8) with the initial condition $u_\varepsilon^2(\mathbf{x},0)=u_0^2(\mathbf{x})$. Then as $u_0^2(\mathbf{x})\in L^q(\mathbb{R}^d)$ with $q>1$, we can apply the same technique as previously to show that u_ε^2 two scale converges to a limit which is necessarily U_0^2. It follows that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^2(Q_T)}=\frac{1}{\sqrt{2\pi}}\big\|U^0\big\|_{L^2(Q_T\times(0,2\pi))}.$$

We can then apply the second part of Theorem 3.1 which gives here

$$\int_{Q_T}\bigg|u_\varepsilon(\mathbf{x},t)-U^0\bigg(\mathbf{x},t,\frac{t}{\varepsilon}\bigg)\bigg|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\to 0,$$

which completes the proof of the theorem. □

We can reformulate the previous results with the function v_ε defined by:

$$v_\varepsilon(\mathbf{x},t)=u_\varepsilon\bigg(X\bigg(\frac{t}{\varepsilon};\mathbf{x},t\bigg),t\bigg)\Leftrightarrow u_\varepsilon(\mathbf{x},t)=v_\varepsilon\bigg(X\bigg(-\frac{t}{\varepsilon};\mathbf{x},t\bigg),t\bigg).\tag{3.3}$$

Setting

$$\tilde{\mathbf{a}}_\varepsilon(\mathbf{x},t)=\alpha\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)=\nabla X\left(\frac{t}{\varepsilon};\mathbf{x},t\right)^{-1}\left(\mathbf{a}\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)-\frac{\partial X}{\partial t}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)\right),\tag{3.4}$$

we first have:

LEMMA 3.2. – *The function v_ε is the solution of the initial value problem*

$$\begin{aligned}&\frac{\partial v_\varepsilon}{\partial t}+\tilde{\mathbf{a}}_\varepsilon\cdot\nabla v_\varepsilon=0,\quad \mathbf{x}\in\mathbb{R}^d,\ t>0,\\&v_\varepsilon(\mathbf{x},0)=u_0(\mathbf{x}).\end{aligned}\tag{3.5}$$

Proof. – We infer from (3.3)

$$\frac{\partial v_\varepsilon}{\partial t}(\mathbf{x},t)=\frac{\partial u_\varepsilon}{\partial t}\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)+\left(\frac{\partial X}{\partial t}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)+\frac{1}{\varepsilon}\frac{\partial X}{\partial\theta}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)\right)\cdot\nabla u_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right),$$

that is

$$\frac{\partial v_\varepsilon}{\partial t}(\mathbf{x},t)=\left(\frac{\partial u_\varepsilon}{\partial t}+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u_\varepsilon\right)\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)+\frac{\partial X}{\partial t}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)\cdot\nabla u_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right).$$

On the other hand

$$\nabla v_\varepsilon(\mathbf{x},t)=\nabla X\left(\frac{t}{\varepsilon};\mathbf{x},t\right)^T\nabla u_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right),$$

hence

$$\begin{aligned}&\left(\mathbf{a}\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)-\frac{\partial X}{\partial t}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)\right)\cdot\nabla u_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)\\&\quad=\left(\nabla X\left(\frac{t}{\varepsilon};\mathbf{x},t\right)^{-1}\left(\mathbf{a}\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)-\frac{\partial X}{\partial t}\left(\frac{t}{\varepsilon};\mathbf{x},t\right)\right)\right)\cdot\nabla v_\varepsilon(\mathbf{x},t)=(\tilde{\mathbf{a}}_\varepsilon\cdot\nabla v_\varepsilon)(\mathbf{x},t).\end{aligned}$$

We infer that

$$\left(\frac{\partial v_\varepsilon}{\partial t}+\tilde{\mathbf{a}}_\varepsilon\cdot\nabla v_\varepsilon\right)(\mathbf{x},t)=\left(\frac{\partial u_\varepsilon}{\partial t}+\mathbf{a}\cdot\nabla u_\varepsilon+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u_\varepsilon\right)\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)=0,$$

whence the result. □

Theorem 1.4 then yields:

COROLLARY 3.3. – *We assume that $u_0\in L^p(\mathbb{R}^d)$, $1<p\leqslant\infty$. Then v_ε two scale converges to V^0 and in $L^\infty(0,T;L^p(\mathbb{R}^d))$ weak-$*$. Moreover, if $u_0\in L^2(\mathbb{R}^d\cap L^{2q}(\mathbb{R}^d))$ for $q>1$, v_ε converges to V^0 in $L^2(Q_T)$ strong.*

Proof. – Let again $\psi=\psi(\mathbf{x},t,\theta)$ be a, 2π-periodic in θ, C^1 function such that $\psi(\cdot,\cdot,\theta)\in C^1_c(\mathbb{R}^d\times[0,T))$ and let $\psi_\varepsilon(\mathbf{x},t)=\psi(\mathbf{x},t,\frac{t}{\varepsilon})$; we compute

$$\int_{Q_T}v_\varepsilon\psi_\varepsilon\,\mathrm{d}\mathbf{x}\,\mathrm{d}t=\int_{Q_T}u_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)\psi\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

With the change of variable $\mathbf{y} = X(\frac{t}{\varepsilon}; \mathbf{x}, t)$, we obtain:

$$\int_{Q_T} v_\varepsilon \psi_\varepsilon \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T} u_\varepsilon(\mathbf{y}, t)\psi\left(X\left(-\frac{t}{\varepsilon}; \mathbf{y}, t\right), t, \frac{t}{\varepsilon}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}t$$

and using the two scale convergence of u_ε to U^0, we find that

$$\begin{aligned}\int_{Q_T} v_\varepsilon \psi_\varepsilon \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to \frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T} U^0(\mathbf{y}, t, \theta)\psi\big(X(-\theta; \mathbf{y}, t), t, \theta\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\,\mathrm{d}\theta \\ = \frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T} V^0\big(X(-\theta; \mathbf{y}, t), t\big)\psi\big(X(-\theta; \mathbf{y}, t), t, \theta\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\,\mathrm{d}\theta \\ = \frac{1}{2\pi}\int_0^{2\pi}\int_{Q_T} V^0(\mathbf{x}, t)\psi(\mathbf{x}, t, \theta)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\,\mathrm{d}\theta.\end{aligned}$$

In other words, v_ε two-scale converges to V^0 and also in $L^\infty(0, T; L^p(\mathbb{R}^d))$ weak-*, as V^0 does not depend on θ.

Finally, if $u_0 \in L^2(\mathbb{R}^d) \cap L^{2q}(\mathbb{R}^d)$, the strong convergence of v_ε to V^0 in $L^2(Q_T)$ is nothing else than property (1.25). □

Let us now move to the higher-order approximations. Before proving Theorem 1.5, let us prove a few technical lemmas:

LEMMA 3.4. – *When*

$$U(\mathbf{x}, t, \theta) = W\big(X(-\theta; \mathbf{x}, t), t, \theta\big),$$

we have:

$$\begin{aligned}&\int_0^\theta \left(\frac{\partial U}{\partial t} + \mathbf{a}\cdot\nabla U\right)\big(X(\sigma; \mathbf{x}, t), t, \sigma\big)\,\mathrm{d}\sigma \\ &\quad = \int_0^\theta \bigg[\frac{\partial W}{\partial t}(\mathbf{x}, t, \sigma) \\ &\qquad + \left(\nabla X(\sigma; \mathbf{x}, t)^{-1}\left(\mathbf{a}\big(X(\sigma; \mathbf{x}, t), t\big) - \frac{\partial X}{\partial t}(\sigma; \mathbf{x}, t)\right)\right)\cdot\nabla W(\mathbf{x}, t, \sigma)\bigg]\,\mathrm{d}\sigma.\end{aligned}$$

The proof can be adapted from the proof of Lemma 2.3. In particular, if

$$U(\mathbf{x}, t, \theta) = V\big(X(-\theta; \mathbf{x}, t), t\big),$$

we get

$$\int_0^\theta \left(\frac{\partial U}{\partial t}(\mathbf{x}, t) + \mathbf{a}\cdot\nabla U\right)(X(\sigma; \mathbf{x}, t), t, \sigma)\,\mathrm{d}\sigma$$

$$= \theta \frac{\partial V}{\partial t} + \int_0^\theta \left(\nabla X(\sigma; \mathbf{x}, t)^{-1} \left(\mathbf{a}\big(X(\sigma; \mathbf{x}, t), t\big) - \frac{\partial X}{\partial t}(\sigma; \mathbf{x}, t) \right) \mathrm{d}\sigma \right) \cdot \nabla V(\mathbf{x}, t).$$

LEMMA 3.5. – *For $k \geqslant 1$ the function v_ε^k defined by* (1.27) *is the solution of the initial value problem*

$$\frac{\partial v_\varepsilon^k}{\partial t} + \tilde{\mathbf{a}}_\varepsilon \cdot \nabla v_\varepsilon^k = -\left(\left(\frac{\partial W^k}{\partial t} \right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon \cdot \left(\nabla W^k\right)_\varepsilon \right), \qquad v_\varepsilon^k(\mathbf{x}, 0) = 0, \tag{3.6}$$

where W^k is defined by (1.22) *and $\tilde{\mathbf{a}}_\varepsilon$ is defined by* (3.4)*.*

Proof. – Let us prove this lemma by induction. The result for $k = 0$ is Lemma 3.2. Then for $k \geqslant 1$, we have:

$$v_\varepsilon^k = \frac{1}{\varepsilon}\left(v_\varepsilon^{k-1} - V^{k-1}\right) - W_\varepsilon^k.$$

Hence

$$\frac{\partial v_\varepsilon^k}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial v_\varepsilon^{k-1}}{\partial t} - \frac{\partial V^{k-1}}{\partial t} \right) - \left(\frac{\partial W^k}{\partial t} \right)_\varepsilon - \frac{1}{\varepsilon}\left(\frac{\partial W^k}{\partial \theta} \right)_\varepsilon,$$

and

$$\nabla v_\varepsilon^k = \frac{1}{\varepsilon}\left(\nabla v_\varepsilon^{k-1} - \nabla V^{k-1}\right) - \left(\nabla W^k\right)_\varepsilon.$$

It follows that

$$\begin{aligned} \frac{v_\varepsilon^k}{t} + \tilde{\mathbf{a}}_\varepsilon \cdot \nabla v_\varepsilon^k = {} & \frac{1}{\varepsilon}\left(\frac{\partial v_\varepsilon^{k-1}}{\partial t} + \tilde{\mathbf{a}}_\varepsilon \cdot \nabla v_\varepsilon^{k-1} - \frac{\partial V^{k-1}}{\partial t} - \tilde{\mathbf{a}}_\varepsilon \cdot \nabla V^{k-1} \right) \\ & - \left[\left(\frac{\partial W^k}{\partial t} \right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon \cdot \left(\nabla W^k\right)_\varepsilon \right] - \frac{1}{\varepsilon}\left(\frac{\partial W^k}{\partial \theta} \right)_\varepsilon \\ = {} & -\left[\left(\frac{\partial W^k}{\partial t} \right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon \cdot \left(\nabla W^k\right)_\varepsilon \right] \\ & - \frac{1}{\varepsilon}\left(\frac{\partial V^{k-1}}{\partial t} + \left(\frac{\partial W^{k-1}}{\partial t} \right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon \cdot \left[\nabla V^{k-1} + \left(\nabla W^{k-1}\right)_\varepsilon \right] \right) - \frac{1}{\varepsilon}\left(\frac{\partial W^k}{\partial \theta} \right)_\varepsilon . \end{aligned} \tag{3.7}$$

On the other hand

$$W^k(\mathbf{x}, \mathbf{v}, \theta) = -\int_0^\theta \left(\frac{\partial U^{k-1}}{\partial t} + \mathbf{a} \cdot \nabla U^{k-1} \right)\big(X(\sigma; \mathbf{x}, t), t, \sigma\big)\, \mathrm{d}\sigma,$$

and using the definition of U^{k-1} and Lemma 3.4 yields:

$$\begin{aligned} & \frac{\partial W^k}{\partial \theta}(\mathbf{x}, t, \theta) \\ & \quad = -\frac{\partial V^{k-1}}{\partial t}(\mathbf{x}, t) - \nabla X(\theta; \mathbf{x}, t)^{-1} \left(\mathbf{a}\big(X(\theta; \mathbf{x}, t), t\big) - \frac{\partial X}{\partial t}(\theta; \mathbf{x}, t) \right) \cdot \nabla V^{k-1}(\mathbf{x}, t) \end{aligned}$$

$$-\frac{\partial W^{k-1}}{\partial t}(\mathbf{x},t,\theta)-\nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big)-\frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\cdot\nabla W^{k-1}(\mathbf{x},t,\theta),$$

from which we get:

$$\left(\frac{\partial W^k}{\partial\theta}\right)_\varepsilon=-\frac{\partial V^{k-1}}{\partial t}-\tilde{\mathbf{a}}_\varepsilon\cdot\nabla V^{k-1}-\frac{\partial W^{k-1}}{\partial t}-\tilde{\mathbf{a}}_\varepsilon\cdot\nabla W^{k-1}.$$

Hence the last line in (3.7) vanishes, which yields the result. □

We are now ready to prove Theorem 1.5:

Proof of Theorem 1.5. – The proof consists of three stages.
First stage: A priori estimates. We need to prove that

$$v_\varepsilon^k \text{ and } u_\varepsilon^k \text{ remain bounded in } L^\infty\big(0,T;L^p\big(\mathbb{R}^d\big)\big). \tag{3.8}$$

Let us denote by:

$$f_\varepsilon^k=-\left[\left(\frac{\partial W^k}{\partial t}\right)_\varepsilon+\tilde{\mathbf{a}}_\varepsilon\cdot\big(\nabla W^k\big)_\varepsilon\right], \tag{3.9}$$

and let us notice that, proceeding as in the proof of Lemma 2.5, we get $\nabla\cdot\tilde{\mathbf{a}}_\varepsilon(\mathbf{x},t)=\nabla\cdot\mathbf{a}(X(\frac{t}{\varepsilon};\mathbf{x},t),t)$. With this notation, multiplying the first equation in (3.6) by $p(v_\varepsilon^k)^{p-1}$ and integrating in space, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x}&=\int_{\mathbb{R}^d}(\nabla\cdot\tilde{\mathbf{a}}_\varepsilon)\big(v_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x}+p\int_{\mathbb{R}^d}f_\varepsilon^k\big(v_\varepsilon^k\big)^{p-1}\,\mathrm{d}\mathbf{x}\\&\leqslant\|\nabla\cdot\mathbf{a}\|_\infty\int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x}+\frac{1}{p}\int_{\mathbb{R}^d}\big(f_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x}+\frac{p-1}{p}\int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x},\end{aligned} \tag{3.10}$$

using the Young inequality $|f||v|^{p-1}\leqslant\frac{1}{p}|f|^p+\frac{p-1}{p}|v|^p$. Then applying the Gronwall lemma we get:

$$\int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^p\,\mathrm{d}\mathbf{x}\leqslant\frac{\frac{1}{p}\sup_t\|f\|_p^p}{\sup_t\|\nabla\cdot\mathbf{a}\|_\infty+\frac{p-1}{p}}\mathrm{e}^{T(\sup_t\|\nabla\cdot\mathbf{a}\|_\infty+\frac{p-1}{p})}.$$

Hence, if W^k is a “smooth enough” function, for example $\partial W^k/\partial t$, $\nabla W^k\in L^\infty((0,T)\times(0,2\pi);L^p(\mathbb{R}^d))$, we find that v_ε^k is bounded in $L^\infty(0,T;L^p(\mathbb{R}^d))$. On the other hand it easily follows from (1.26), (1.27) using (1.21) that $u_\varepsilon^k(\mathbf{x},t)=v_\varepsilon^k(X(\frac{t}{\varepsilon};\mathbf{x},t),t)+W_\varepsilon^k(X(\frac{t}{\varepsilon};\mathbf{x},t),t)$. Then, as W_ε^k remains bounded in $L^\infty(0,T;L^p(\mathbb{R}^d))$ provided $W^k\in L^\infty((0,T)\times(0,2\pi);L^p(\mathbb{R}^d))$, we infer that u_ε^k also remains bounded in $L^\infty(0,T;L^p(\mathbb{R}^d))$. Property (3.8) follows.

Thanks to (3.8) and Theorem 3.1, there exists a subsequence extracted from the sequence (u_ε^k), still denoted by (u_ε^k), and a function $U^k\in L^\infty(0,T;L_\sharp^\infty(\mathbb{R};L^p(\mathbb{R}^d)))$ such that

$$u_\varepsilon^k\to U^k\quad\text{two-scale},$$

and, in the same way, there exists a subsequence (v_ε^k), still denoted by (v_ε^k), and a function $V^k \in L^\infty(0,T;L^\infty_\sharp(\mathbb{R};L^p(\mathbb{R}^d)))$ such that

$$v_\varepsilon^k \to V^k \quad \text{two-scale.}$$

Second stage: Characterization of U^k and V^k. We start by the characterization of V^k. Let $\psi = \psi(\mathbf{x},t,\theta)$ a test function; we infer from (3.6)

$$\int_{Q_T} v_\varepsilon^k\left(\frac{\partial \psi_\varepsilon}{\partial t} + \nabla\cdot(\tilde{\mathbf{a}}_\varepsilon \psi_\varepsilon)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T}\left(\left(\frac{\partial W^k}{\partial t}\right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon\cdot\left(\nabla W^k\right)_\varepsilon\right)\psi_\varepsilon\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

that is

$$\int_{Q_T} v_\varepsilon^k\left(\left(\frac{\partial \psi}{\partial t}\right)_\varepsilon + \frac{1}{\varepsilon}\left(\frac{\partial \psi_\varepsilon}{\partial \theta}\right)_\varepsilon + \nabla\cdot(\tilde{\mathbf{a}}_\varepsilon \psi_\varepsilon)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T}\left(\left(\frac{\partial W^k}{\partial t}\right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon\cdot\left(\nabla W^k\right)_\varepsilon\right)\psi_\varepsilon\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

If we multiply by ε and let ε tend to 0 and take into account the two scale convergence of v_ε^k to V^k, we obtain:

$$\int_0^{2\pi}\int_{Q_T} V^k\frac{\partial \psi}{\partial \theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}\theta = 0,$$

that is

$$\frac{\partial V^k}{\partial \theta} = 0.$$

Thus $V^k = V^k(\mathbf{x},t)$ is independent of θ.

Let now $\varphi = \varphi(\mathbf{x},t) \in C_c^1(\mathbb{R}^d\times[0,T))$ be a test function independent of θ. We then find

$$\int_{Q_T} v_\varepsilon^k\left(\frac{\partial \varphi}{\partial t} + \nabla\cdot(\tilde{\mathbf{a}}_\varepsilon \varphi)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T}\left(\left(\frac{\partial W^k}{\partial t}\right)_\varepsilon + \tilde{\mathbf{a}}_\varepsilon\cdot\left(\nabla W^k\right)_\varepsilon\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

and passing to the limit in the standard manner

$$\begin{aligned}
&\int_{Q_T} V^k\left(\frac{\partial \varphi}{\partial t} + \nabla\cdot\left(\tilde{\mathbf{a}}^0\varphi\right)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{Q_T}\left(\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial W^k}{\partial t}(\mathbf{x},t,\theta)\right.\right. \\
&\qquad \left.\left. + \left(\nabla X(\theta;\mathbf{x},t)^{-1}\left(\mathbf{a}\big(X(\theta;\mathbf{x},t),t\big) - \frac{\partial X}{\partial t}(\theta;\mathbf{x},t)\right)\right)\cdot\nabla W^k(\mathbf{x},t,\theta)\right]\mathrm{d}\theta\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

This means that V^k is a weak solution of (1.23) for $k=1$. We have thus characterized V^k.

It remains to characterize U^k. We infer from (1.26) and (1.27), using (1.21), that

$$u_\varepsilon^k(\mathbf{x},t) = \left(v_\varepsilon^k + W_\varepsilon^k\right)\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right),t\right),$$

that is

$$u_\varepsilon^k(\mathbf{x},t) = v_\varepsilon^k\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right),t\right) + W^k\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right),t,\frac{t}{\varepsilon}\right).$$

Let then $\psi = \psi(\mathbf{x},t,\theta)$ be the usual test function; we have:

$$\int_{Q_T} u_\varepsilon^k \psi_\varepsilon \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T} \left(v_\varepsilon^k(\mathbf{y},t) + W^k\left(\mathbf{y},t,\frac{t}{\varepsilon}\right)\right)\psi_\varepsilon\left(X\left(\frac{t}{\varepsilon};\mathbf{y},t\right),t,\frac{t}{\varepsilon}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}t.$$

Using the two scale convergence of u_ε^k and v_ε^k to U^k and V^k respectively, we obtain:

$$\int_0^{2\pi}\int_{Q_T} U^k\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\,\mathrm{d}\theta = \int_0^{2\pi}\int_{Q_T}\big(V^k(\mathbf{y},t) + W^k(\mathbf{y},t,\theta)\big)\psi\big(X(\theta;\mathbf{y},t),t,\theta\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\,\mathrm{d}\theta,$$

hence

$$U^k(\mathbf{x},t,\theta) = V^k\big(X(-\theta;\mathbf{y},t),t\big) + W^k\big(X(-\theta;\mathbf{y},t),t,\theta\big),$$

which is the wanted characterization of U^k. Moreover, since the solution of (1.22) is unique, we can deduce that the whole sequences u_ε^k and v_ε^k converge two scale and not only subsequences.

Third stage: Strong convergence. We assume now that $u_0 \in L^2(\mathbb{R}^d)$. We start by verifying that

$$\lim_{\varepsilon\to 0}\|v_\varepsilon^k\|_{L^2(Q_T)} = \|V^k\|_{L^2(Q_T)}. \tag{3.11}$$

Multiplying the first equation of (3.6) by $4(v_\varepsilon^k)^3$ and integrating by parts, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^4\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^d}(\nabla\cdot\tilde{\mathbf{a}}_\varepsilon)\big(v_\varepsilon^k\big)^4\,\mathrm{d}\mathbf{x} + 4\int_{\mathbb{R}^d} f_\varepsilon^k\big(v_\varepsilon^k\big)^3\,\mathrm{d}\mathbf{x}\\ &\leqslant \int_{\mathbb{R}^d}(\nabla\cdot\tilde{\mathbf{a}}_\varepsilon)\big(v_\varepsilon^k\big)^4\,\mathrm{d}\mathbf{x} + 2\left(\int_{\mathbb{R}^d}\big(f_\varepsilon^k v_\varepsilon^k\big)^2\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^4\,\mathrm{d}\mathbf{x}\right)\\ &\leqslant \int_{\mathbb{R}^d}(\nabla\cdot\tilde{\mathbf{a}}_\varepsilon + 2)\big(v_\varepsilon^k\big)^4\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\big(f_\varepsilon^k\big)^2\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\big(v_\varepsilon^k\big)^2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

We already know from the first stage of the proof (applied for $p=2$) that v_ε^k is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d))$. This, in addition to the smoothness of f_ε^k which comes from the smoothness of W_ε^k implies that the right-hand side of the previous inequality is bounded. Hence we get here in the same way as in the first stage of this proof, using the Gronwall lemma that $(v_\varepsilon^k)^2$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d))$, which implies that it two-scale converges. It is then an easy game to show, as we have done previously that its two-scale limit is nothing else than $(V^k)^2$. Property (3.11) then follows. Hence we have:

$$v_\varepsilon^k \to V^k \quad \text{in } L^2(Q_T) \text{ weak},$$
$$\lim_{\varepsilon\to 0}\|v_\varepsilon^k\|_{L^2(Q_T)} = \|V^k\|_{L^2(Q_T)}.$$

This implies that

$$v_\varepsilon^k \to V^k \quad \text{in } L^2(Q_T) \text{ strong}.$$

As a consequence of this result, we obtain that

$$\lim_{\varepsilon\to 0}\int_{Q_T}\left|u_\varepsilon^k(\mathbf{x},t)-U^k\left(\mathbf{x},t,\frac{t}{\varepsilon}\right)\right|^2\mathrm{d}\mathbf{x}\,\mathrm{d}t=0.$$

Indeed, we can write due to (1.26) and (1.27)

$$u_\varepsilon^k(\mathbf{x},t)-V^k\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right),t\right)-W^k\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right),t,\frac{t}{\varepsilon}\right)=\left(v_\varepsilon^k-V^k\right)\left(X\left(-\frac{t}{\varepsilon};\mathbf{x},t\right)\right),$$

and the conclusion follows. □

4. Application to the Vlasov equation

The results presented here can be applied to the following dimensionless Vlasov equations introduced at the beginning of the article. In addition, we decompose here the electric and magnetic fields into a large part which tends to infinity and part which does not depend on ε,

$$\frac{\partial f^\varepsilon}{\partial t}+\mathbf{v}\cdot\nabla_x f^\varepsilon+\left(\left(\mathbf{E}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathcal{N}(\mathbf{x},t)\right)+\mathbf{v}\times\left(\mathbf{B}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathcal{M}(\mathbf{x},t)\right)\right)\cdot\nabla_v f^\varepsilon=0, \tag{4.1}$$

$$f^\varepsilon_{t=0}=f_0, \tag{4.2}$$

and

$$\begin{aligned}&\frac{\partial f^\varepsilon}{\partial t}+\mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon+\frac{1}{\varepsilon}\mathbf{v}_\perp\cdot\nabla_x f^\varepsilon\\&\quad+\left(\left(\mathbf{E}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathcal{N}(\mathbf{x},t)\right)+\mathbf{v}\times\left(\mathbf{B}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathcal{M}(\mathbf{x},t)\right)\right)\cdot\nabla_v f^\varepsilon=0,\end{aligned} \tag{4.3}$$

$$f^\varepsilon_{t=0}=f_0, \tag{4.4}$$

the latter corresponding to the Finite Larmor Radius Regime, in the case when $\mathcal{M}$ and $\mathcal{N}$ are smoothly varying fields satisfying:

$$\left\|\mathcal{M}(\mathbf{x},t)\right\|=1,\quad \mathcal{M}(\mathbf{x},t)\perp\mathcal{N}(\mathbf{x},t),$$

for every $\mathbf{x}\in\mathbb{R}^3$, and $t>0$. In equation (4.3) we denote $\mathbf{v}_\parallel=(\mathbf{v}\cdot\mathcal{M})\mathcal{M}$ and $\mathbf{v}_\perp=\mathbf{v}-\mathbf{v}_\parallel$.

However, the computations being particularly tedious, we just give partial results with very few details concerning

$$\frac{\partial f^\varepsilon}{\partial t}+\mathbf{v}\cdot\nabla_x f^\varepsilon+\left(\mathbf{E}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathbf{e}_2+\frac{1}{\varepsilon}\mathbf{v}\times\mathbf{e}_1\right)\cdot\nabla_v f^\varepsilon=0, \tag{4.5}$$

$$f^\varepsilon_{t=0}=f_0, \tag{4.6}$$

i.e. the case of large and constant electric and magnetic fields having the same norm and being orthogonal to each other, we assume here without loss of generality that the magnetic field is along $\mathbf{e}_1$ and the electric field is along $\mathbf{e}_2$.

Our second application concerns the following dimensionless Vlasov equation

$$\frac{\partial f^\varepsilon}{\partial t}+\mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon+\frac{1}{\varepsilon}\mathbf{v}_\perp\cdot\nabla_x f^\varepsilon+\left(\mathbf{E}(\mathbf{x},t)+\frac{1}{\varepsilon}\mathbf{v}\times\mathbf{e}_1\right)\cdot\nabla_v f^\varepsilon=0, \tag{4.7}$$

$$f^\varepsilon_{t=0}=f_0, \tag{4.8}$$

which is the case of a large and constant magnetic field in the other scaling we proposed in the introduction. In the above equations $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the frame of $\mathbb{R}^3$ and where for any vector $\mathbf{v}$ we denote $\mathbf{v}_\parallel = v_1\mathbf{e}_1$ and $\mathbf{v}_\perp = v_2\mathbf{e}_2 + v_3\mathbf{e}_3$.

In both cases, the solution f^ε can be expanded as

$$f^\varepsilon(\mathbf{x}, \mathbf{v}, t) = F^0\left(\mathbf{x}, \mathbf{v}, t, \frac{t}{\varepsilon}\right) + \varepsilon F^1\left(\mathbf{x}, \mathbf{v}, t, \frac{t}{\varepsilon}\right) + \cdots, \tag{4.9}$$

this expansion being justified, using two scale convergence, as soon as enough regularity is assumed on f_0 and $\mathbf{E}$.

In our first application, equation (4.5) is of the form (1.8) with variable $(\mathbf{x}, \mathbf{v})$ in place of $\mathbf{x}$ and with

$$\mathbf{a}(\mathbf{x}, \mathbf{v}, t) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(\mathbf{x}, t) \end{pmatrix} \quad \text{and} \quad \mathbf{b}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{e}_2 + \mathbf{v} \times \mathbf{e}_1 \end{pmatrix}.$$

Then X does not depend on t and

$$X(\theta; \mathbf{x}, \mathbf{v}, t) = \begin{pmatrix} \mathbf{x} \\ R(\theta)(\mathbf{v} + \mathbf{e}_3) - \mathbf{e}_3 \end{pmatrix}, \qquad \nabla X(\theta; \mathbf{x}, \mathbf{v}, t)^{-1} = \begin{pmatrix} I & 0 \\ 0 & R(-\theta) \end{pmatrix},$$

with

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.10}$$

Hence,

$$\alpha(\mathbf{x}, \mathbf{v}, t, \theta) = \begin{pmatrix} R(\theta)(\mathbf{v} + \mathbf{e}_3) - \mathbf{e}_3 \\ R(-\theta)\mathbf{E}(\mathbf{x}, t) \end{pmatrix}, \qquad \mathbf{a}^0(\mathbf{x}, \mathbf{v}) = \tilde{\mathbf{a}}^0(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v}_\parallel - \mathbf{e}_3 \\ \mathbf{E}_\parallel(\mathbf{x}, t) \end{pmatrix}, \tag{4.11}$$

and we have the following theorem:

THEOREM 4.1. – *The first term F^0 of the expansion of the solution of equation* (4.5) *is given by*

$$F^0(\mathbf{x}, \mathbf{v}, t, \theta) = G^0\big(\mathbf{x}, R(-\theta)(\mathbf{v} + \mathbf{e}_3) - \mathbf{e}_3, t\big), \tag{4.12}$$

where the function $G^0(\mathbf{x}, \mathbf{v}, t)$ is solution to

$$\frac{\partial G^0}{\partial t} + (\mathbf{v}_\parallel - \mathbf{e}_3) \cdot \nabla_x G^0 + \mathbf{E}_\parallel \cdot \nabla_v G^0 = 0, \tag{4.13}$$

$$G^0_{t=0} = f_0. \tag{4.14}$$

Notice, by the way, that the term $-\mathbf{e}_3$ appearing in the kinetic velocity in (4.13) is nothing but the well known and so called $(E \times B/|B|^2)$-drift.

In order to deduce the equation for F^1 we turn to Theorems 1.3 and 2.6. Noticing

$$\int_0^\theta R(-\theta)\,\mathrm{d}\theta = \theta P + R\left(\frac{\pi}{2} - \theta\right) - R\left(\frac{\pi}{2}\right), \quad \text{with } P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.15}$$

we get:

$$W^1(\mathbf{x},\mathbf{v},t,\theta)=\left(R\left(\frac{\pi}{2}+\theta\right)-R\left(\frac{\pi}{2}\right)\right)(\mathbf{v}+\mathbf{e}_3)\cdot\nabla_{x_\perp}G^0\big(\mathbf{x},R(-\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3,t\big)$$
$$-\left(R\left(\frac{\pi}{2}-\theta\right)-R\left(\frac{\pi}{2}\right)\right)\mathbf{E}(\mathbf{x},t)\cdot\nabla_{v_\perp}G^0\big(\mathbf{x},R(-\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3,t\big).$$

Now, applying formula (2.25), we get:

$$-\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial W^1}{\partial t}(\mathbf{x},t,\theta)+\alpha(\mathbf{x},\mathbf{v},t,\theta)\cdot\nabla W^1(\mathbf{x},\mathbf{v},t,\theta)\,\mathrm{d}\theta$$
$$=\Bigg[\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}0\\(R(\frac{\pi}{2}-\theta)-R(\frac{\pi}{2}))\frac{\partial\mathbf{E}}{\partial t}\end{pmatrix}$$
$$+\begin{pmatrix}0 & -(R(\frac{\pi}{2}+\theta)-R(\frac{\pi}{2}))\\(R(\frac{\pi}{2}-\theta)-R(\frac{\pi}{2}))(\nabla_x\mathbf{E}) & 0\end{pmatrix}\begin{pmatrix}R(\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3\\R(-\theta)\mathbf{E}\end{pmatrix}$$
$$-\begin{pmatrix}0 & P\\P\nabla_x\mathbf{E} & 0\end{pmatrix}\begin{pmatrix}-(R(\frac{\pi}{2}+\theta)-R(\frac{\pi}{2}))(\mathbf{v}+\mathbf{e}_3)\\(R(\frac{\pi}{2}-\theta)-R(\frac{\pi}{2}))\mathbf{E}\end{pmatrix}\mathrm{d}\theta\Bigg]\cdot\begin{pmatrix}\nabla_xG^0\\\nabla_vG^0\end{pmatrix},$$

which, after a straightforward (and quite long) computation, gives

$$(\mathbf{e}_1\times\mathbf{E})\cdot\nabla_xG^0+\left(\mathbf{e}_1\times\left(\frac{\partial\mathbf{E}_\perp}{\partial t}+v_1\frac{\partial\mathbf{E}_\perp}{\partial x_1}-\frac{\partial\mathbf{E}_\perp}{\partial x_3}\right)\right.$$
$$\left.+\frac{1}{2}\left(\left(-\frac{\partial E_2}{\partial x_3}+\frac{\partial E_3}{\partial x_2}\right)\mathbf{v}_\perp+\left(\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right)(\mathbf{v}_\perp\times\mathbf{e}_1)\right)\right)\cdot\nabla_vG^0+\mathbf{e}_1\times\mathbf{v}_\perp\cdot\nabla_{x_\perp}E_1\frac{\partial G^0}{\partial v_1}.$$

Hence we proved the following theorem:

THEOREM 4.2. – *The second term F^1 of the expansion of the solution of equation* (4.5) *is given by*

$$\begin{aligned}F^1(\mathbf{x},\mathbf{v},t,\theta)=\;&G^1\big(\mathbf{x},R(-\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3,t\big)\\&-\left(R\left(\frac{\pi}{2}-\theta\right)-R\left(\frac{\pi}{2}\right)\right)(\mathbf{v}+\mathbf{e}_3)\cdot\nabla_{x_\perp}G^0\big(\mathbf{x},R(-\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3,t\big)\\&-\left(R\left(\frac{\pi}{2}-\theta\right)-R\left(\frac{\pi}{2}\right)\right)\mathbf{E}(\mathbf{x},t)\cdot\nabla_{v_\perp}G^0\big(\mathbf{x},R(-\theta)(\mathbf{v}+\mathbf{e}_3)-\mathbf{e}_3,t\big),\end{aligned}\tag{4.16}$$

where $G^1(\mathbf{x},\mathbf{v},t)$ is solution to

$$\begin{aligned}&\frac{\partial G^1}{\partial t}+(\mathbf{v}_\parallel-\mathbf{e}_3)\cdot\nabla_xG^1+\mathbf{E}_\parallel\cdot\nabla_vG^1\\&\quad=(\mathbf{e}_1\times\mathbf{E})\cdot\nabla_xG^0+\left(\mathbf{e}_1\times\left(\frac{\partial\mathbf{E}_\perp}{\partial t}+v_1\frac{\partial\mathbf{E}_\perp}{\partial x_1}-\frac{\partial\mathbf{E}_\perp}{\partial x_3}\right)\right.\\&\qquad+\frac{1}{2}\left(\left(-\frac{\partial E_2}{\partial x_3}+\frac{\partial E_3}{\partial x_2}\right)\mathbf{v}_\perp+\left(\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right)(\mathbf{v}_\perp\times\mathbf{e}_1)\right)\\&\qquad\left.+(\mathbf{e}_1\times\mathbf{v}_\perp\cdot\nabla_{x_\perp}E_1)\mathbf{e}_1\right)\cdot\nabla_vG^0,\\&G^1_{t=0}=0.\end{aligned}\tag{4.17}$$

In our second application, namely equation (4.7), we have:

THEOREM 4.3. – *Denoting by* $\mathcal{R}(\theta) = -(R(\frac{\pi}{2}+\theta) - R(\frac{\pi}{2}))$, *the first and the second term of the expansion of the solution of equation* (4.7) *are given by*:

$$F^0(\mathbf{x},\mathbf{v},t,\theta) = G^0\big(\mathbf{x}+\mathcal{R}(-\theta)\mathbf{v}, R(-\theta)\mathbf{v}, t\big), \tag{4.18}$$

$$\begin{aligned} &F^1(\mathbf{x},\mathbf{v},t,\theta)\\ &\quad = G^1\big(\mathbf{x}+\mathcal{R}(-\theta)\mathbf{v}, R(-\theta)\mathbf{v}, t\big)\\ &\qquad - \left[\int_0^\theta \begin{pmatrix} \mathcal{R}(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma-\theta)\mathbf{v},t) \\ R(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma-\theta)\mathbf{v},t)\end{pmatrix} \mathrm{d}\sigma \right.\\ &\qquad \left. - \frac{\theta}{2\pi}\int_0^{2\pi} \begin{pmatrix} \mathcal{R}(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma-\theta)\mathbf{v},t) \\ R(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma-\theta)\mathbf{v},t)\end{pmatrix} \mathrm{d}\sigma\right] \cdot \begin{pmatrix} \nabla_x G^0(\mathbf{x}+\mathcal{R}(-\theta)\mathbf{v}, R(-\theta)\mathbf{v},t) \\ \nabla_v G^0(\mathbf{x}+\mathcal{R}(-\theta)\mathbf{v}, R(-\theta)\mathbf{v},t)\end{pmatrix}, \end{aligned} \tag{4.19}$$

where G^0 *is solution to*

$$\frac{\partial G^0}{\partial t} + \begin{pmatrix} \mathbf{v}_\parallel + \frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\,\mathrm{d}\theta \\ \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\,\mathrm{d}\theta \end{pmatrix} \cdot \begin{pmatrix}\nabla_x G^0\\ \nabla_v G^0\end{pmatrix} = 0, \tag{4.20}$$

$$G^0_{t=0} = f_0, \tag{4.21}$$

and where G^1 *is solution to*:

$$\begin{aligned} &\frac{\partial G^1}{\partial t} + \begin{pmatrix} \mathbf{v}_\parallel + \frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\,\mathrm{d}\theta \\ \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\,\mathrm{d}\theta \end{pmatrix} \cdot \begin{pmatrix}\nabla_x G^1\\ \nabla_v G^1\end{pmatrix}\\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^\theta \begin{pmatrix} \mathcal{R}(-\sigma)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) \\ R(-\sigma)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\end{pmatrix}\mathrm{d}\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} \begin{pmatrix} \mathcal{R}(-\sigma)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) \\ R(-\sigma)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\end{pmatrix}\mathrm{d}\sigma\right.\\ &\qquad + \left(\int_0^\theta \begin{pmatrix} \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)+P \\ R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)\end{pmatrix}\mathrm{d}\sigma\right.\\ &\qquad \left. - \frac{\theta}{2\pi}\int_0^{2\pi} \begin{pmatrix} \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)+P \\ R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)\end{pmatrix}\mathrm{d}\sigma\right)\cdot\\ &\qquad \cdot\begin{pmatrix} \mathbf{v}_\parallel + \mathcal{R}(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t) \\ R(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\end{pmatrix}\\ &\qquad - \frac{1}{2\pi}\left(\int_0^\theta \begin{pmatrix} \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & \mathcal{R}(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)+P \\ R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) & R(-\sigma)\nabla_x\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\mathcal{R}(\sigma)\end{pmatrix}\mathrm{d}\sigma\right)\cdot\\ &\qquad \left.\cdot\left(\int_0^\theta \begin{pmatrix} \mathcal{R}(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) \\ R(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\end{pmatrix}\mathrm{d}\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} \begin{pmatrix} \mathcal{R}(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t) \\ R(-\sigma)\mathbf{E}(\mathbf{x}+\mathcal{R}(\sigma)\mathbf{v},t)\end{pmatrix}\mathrm{d}\sigma\right)\right]\cdot\\ &\qquad \cdot\begin{pmatrix}\nabla_x G^0\\ \nabla_v G^0\end{pmatrix} G^1_{t=0} = 0. \end{aligned} \tag{4.22}$$

The proof of this theorem is straightforward (but tedious) once we notice that equation (4.7) is of the form (1.8) with

$$\mathbf{a}(\mathbf{x},\mathbf{v},t)=\begin{pmatrix}\mathbf{v}_\parallel\\ \mathbf{E}(t,\mathbf{x})\end{pmatrix}\quad\text{and}\quad \mathbf{b}(\mathbf{x},\mathbf{v})=\begin{pmatrix}\mathbf{v}_\perp\\ \mathbf{v}\times\mathbf{e}_1\end{pmatrix}.$$

Hence, since $\int_0^\theta R(\sigma)\,\mathrm{d}\sigma=\theta P+\mathcal{R}(\theta)$, we have:

$$X(\theta;\mathbf{x},\mathbf{v},t)=\begin{pmatrix}\mathbf{x}+\mathcal{R}(\theta)\mathbf{v}\\ R(\theta)\mathbf{v}\end{pmatrix},\qquad \nabla X(\theta;\mathbf{x},\mathbf{v},t)^{-1}=\begin{pmatrix}I & \mathcal{R}(-\theta)\\ 0 & R(-\theta)\end{pmatrix},$$

and then

$$\alpha(\mathbf{x},\mathbf{v},t,\theta)=\begin{pmatrix}\mathbf{v}_\parallel+\mathcal{R}(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\\ R(-\theta)\mathbf{E}(\mathbf{x}+\mathcal{R}(\theta)\mathbf{v},t)\end{pmatrix}.$$

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