

Application of the averaging method to the gyrokinetic plasma

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Abstract: we show that the solution to an oscillatory-singularly perturbed ordinary differential equation may be asymptotically expanded into a sum of oscillating terms. Each of those terms writes as an oscillating operator acting on the solution to a non oscillating ordinary differential equation with an oscillating correction added to it.

The expression of the non oscillating ordinary differential equations are defined by a recurrence relation. We then apply this result to problems where charged particles are submitted to large magnetic field.

1 Introduction and results

Purpose

The goal of this paper is to manage and justify the asymptotic two scale expansion as $\varepsilon \rightarrow 0$ of the solution $\mathbf{X}_\varepsilon(t) = \mathbf{X}_\varepsilon(t; \mathbf{x}, s)$ to the following singularly perturbed dynamical system:

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{a}\left(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon\right) + \frac{1}{\varepsilon}\mathbf{b}\left(t, \mathbf{X}_\varepsilon\right), \quad \mathbf{X}_\varepsilon(s; \mathbf{x}, s) = \mathbf{x}, \quad (1.1)$$

and to apply this to models of charge particles submitted to a strong magnetic field which is possibly non uniform.

More precisely, C_b^m standing for the space of functions that have continuous and bounded derivatives until the order m , we assume that

$$\mathbf{a}(\cdot, \cdot, \cdot) \in (C_b^{m+1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d))^d, \quad \theta \mapsto \mathbf{a}(t, \theta, \mathbf{x}) \text{ is } 2\pi\text{-periodic for every } t \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^d, \quad (1.2)$$

$$\mathbf{b}(\cdot, \cdot) \in (C_b^{m+2}(\mathbb{R} \times \mathbb{R}^d))^d, \quad (1.3)$$

for some $m \geq 0$. In particular, it implies that $|\nabla_{\mathbf{x}} \cdot \mathbf{b}| \leq C$, uniformly on $\mathbb{R} \times \mathbb{R}^d$. We also suppose that the solution $\mathbf{Z}(t, \theta; \mathbf{z})$ to

$$\frac{\partial \mathbf{Z}}{\partial \theta} = \mathbf{b}(t, \mathbf{Z}), \quad \mathbf{Z}(t, 0; \mathbf{z}) = \mathbf{z}, \quad (1.4)$$

is known and 2π -periodic in θ for every $t \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^d$.

Under those assumptions, we prove that $\mathbf{X}_\varepsilon(\cdot; \mathbf{x}, s) : \mathbb{R} \rightarrow \mathbb{R}^d$ admits the following expansion:

$$\mathbf{X}_\varepsilon(t; \mathbf{x}, s) = \mathbf{X}^0\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s\right) + \varepsilon \mathbf{X}^1\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s\right) + \varepsilon^2 \mathbf{X}^2\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s\right) + \dots, \quad (1.5)$$

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as $\varepsilon \rightarrow 0$, where the $\mathbf{X}^i(t, \theta; \mathbf{x}, s)$ are 2π -periodic in θ . The initial condition in (1.1) is assigned to \mathbf{X}^0 by:

$$\mathbf{X}^0(s, 0; \mathbf{x}, s) = \mathbf{x} \text{ and } \mathbf{X}^i(s, 0; \mathbf{x}, s) = 0 \text{ for } i \geq 1. \quad (1.6)$$

Motivations

The target we have in mind while writing this paper is the conception of pic-methods for plasmas submitted to large magnetic field. Direct applications of this are the simulation of magnetic confinement fusion or isotope resonant separation experiments.

A plasma submitted to a strong magnetic field could be modelled by several Vlasov equations, according to experimental conditions. For instance if we are interested in describing the global behaviour of a plasma in a magnetic confinement fusion experiment, the following equation

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \left((\mathbf{E}(t, \mathbf{x}) + \frac{\mathcal{N}}{\varepsilon}) + \mathbf{v} \times (\mathbf{B}(t, \mathbf{x}) + \frac{\mathcal{M}}{\varepsilon}) \right) \cdot \nabla_{\mathbf{v}} f^\varepsilon = 0, \quad f^\varepsilon|_{t=s} = f_0, \quad (1.7)$$

for a small parameter ε would be a relevant model equation. In equation (1.7), $f^\varepsilon(t, \mathbf{x}, \mathbf{v})$ is at time t , the particle density of the plasma in position \mathbf{x} and with velocity \mathbf{v} . The vector $\mathcal{M} \in \mathbb{S}^2$ gives the direction of the strong magnetic field. This direction may depend on t and \mathbf{x} . The vector $\mathcal{N} \in \mathbb{S}^2$ with $\mathcal{N} \perp \mathcal{M}$ is the direction of a strong electric field (\mathcal{N} may also be 0). The vector fields \mathbf{E} and \mathbf{B} are electric and magnetic fields that may contain a self consistent part. We shall call the scaling leading to this equation the ‘‘Guiding Centre Regime’’.

If now we are interested in understanding what happens close to the plasma boundary in a tokamak, we would prefer to use the so called ‘‘Finite Larmor Radius Regime’’. In other words, we would consider an equation of the type:

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \frac{\mathbf{v}_{\perp}}{\varepsilon} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \left(\mathbf{E}(t, \mathbf{x}) + \mathbf{v} \times \frac{\mathcal{M}}{\varepsilon} \right) \cdot \nabla_{\mathbf{v}} f^\varepsilon = 0, \quad f^\varepsilon|_{t=s} = f_0, \quad (1.8)$$

where the meaning of $f^\varepsilon(t, \mathbf{x}, \mathbf{v})$ is the same as above and where $\mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathcal{M})\mathcal{M}$, $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Lastly, for isotope resonant separation we would be interested in regarding

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \left(\mathbf{E}(t, \frac{t-s}{\varepsilon}, \mathbf{x}) + \mathbf{v} \times (\mathbf{B}(t, \frac{t-s}{\varepsilon}, \mathbf{x}) + \frac{\mathcal{M}}{\varepsilon}) \right) \cdot \nabla_{\mathbf{v}} f^\varepsilon = 0, \quad f^\varepsilon|_{t=s} = f_0, \quad (1.9)$$

where the electric and magnetic fields $\mathbf{E}(t, \frac{t-s}{\varepsilon}, \mathbf{x})$ and $\mathbf{B}(t, \frac{t-s}{\varepsilon}, \mathbf{x})$ oscillate with the same frequency $1/(2\pi\varepsilon)$ as the cyclotron frequency of the plasma particles.

We refer to Frénod and Sonnendrücker [10, 11, 12, 13, 14], Frénod, Raviart and Sonnendrücker [9] and Frénod and Watbled [15] where those models are explained and asymptotically analysed. In particular, they give indication on how to get the self consistent part of the electric field in some examples and under suitable assumptions. Complementary works concerning the asymptotic behaviour of those kind of equations are also led in Golse and Saint Raymond [16, 17], Saint Raymond [32], Brenier [4], Grenier [19, 20], Jabin [21], Schochet [35], Joly, Métivier and Rauch [24]. We also refer to mathematical or physical works where similar methods are used: [22, 23, 36, 7, 8, 27, 26, 6, 5, 28, 18].

It is relatively easy to see that equations (1.7), (1.8) and (1.9) enter the framework of a singularly perturbed convection equation reading:

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{a} \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon} \mathbf{b} \cdot \nabla u_\varepsilon = 0, \quad u_\varepsilon|_{t=s} = u_0, \quad (1.10)$$

for $\mathbf{x} \in \mathbb{R}^d$ and $t > s$. At least formally, the solution $u_\varepsilon(t, \mathbf{x})$ is given by

$$u_\varepsilon(t, \mathbf{x}) = u_0(\mathbf{X}_\varepsilon(s; \mathbf{x}, t)). \quad (1.11)$$

Numerical methods to solve equation (1.7), (1.8) or (1.9), or similar ones, will be applications of generic methods solving (1.10). Since u_ε contains high frequency oscillations, using in state any well known numerical method to solve (1.10) would compel to use a very small time step.

Yet, in order to relax this constraint, we may follow, at least, two strategies. The first one could be based on the work presented in Frénod, Raviart and Sonnendrücker [9] where we proved that u_ε writes $u_\varepsilon(t, \mathbf{x}) = \sum_{i \geq 0} \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, and where we determined the equations satisfied by every terms U^i . Since those equations are independent of ε , the computation of the first terms U^i could be led using a standard numerical method. Then an approximation of u_ε would be given by: $u_\varepsilon(t, \mathbf{x}) \simeq \sum_{i=0}^k \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, for some $k \in \mathbb{N}$.

The second strategy consists in using an approximation $\mathbf{X}_\varepsilon(t; \mathbf{x}, s) \simeq \sum_{i=0}^k \varepsilon^i \mathbf{X}^i(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)$, of (1.5), for some $k \in \mathbb{N}$, after computing $\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^k$. Then the expression (1.11) yields an approximation of u_ε : $u_\varepsilon(t, \mathbf{x}) \simeq u_0(\sum_{i=0}^k \varepsilon^i \mathbf{X}^i(s, -\frac{t-s}{\varepsilon}; \mathbf{x}, t))$. This motivates the present work.

We give now some references concerning perturbed ordinary differential equations. First, we think to the Lindstedt-Poincaré method explained in Poincaré [31] where the steady state periodic solutions to a perturbed second order ordinary differential equation is studied. Then the Krylov-Bogoliubov-Mitropolsky method, see [3] and [25], allows to describe the transitory behaviour of the solution to a perturbed ordinary differential equation to a periodic solution.

We also cite the works of Verhulst [42] and Sanders and Verhulst [34] (see also Mickens [29]) where the Method of Averaging is developed to treat perturbed ordinary differential equations and adiabatic invariants in Hamiltonian systems.

Lastly, we mention the work initiated by Tikhonov [38, 39, 40] and developed by Vasilieva and others (see the review paper of Vasilieva [41] and the references in it).

This work is also related to homogenisation methods based on weak-* convergence and two scale convergence. The most important ideas about those tools may be found in Tartar [37], Bensoussan, Lions and Papanicolaou [2], Sanchez-Palencia [33], N'Guetseng [30], Allaire [1] and Frénod, Raviart and Sonnendrücker [9].

Theorems

We first justify the expansion (1.5) until order 0.

THEOREM 1.1 *Setting*

$$\tilde{\mathbf{a}}^0(t, \mathbf{y}^0) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\alpha}^0(t, \theta, \mathbf{y}^0) d\theta, \quad (1.12)$$

with

$$\tilde{\alpha}^0(t, \theta, \mathbf{y}^0) = \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\}^{-1} \{\mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0)) - \frac{\partial \mathbf{Z}}{\partial t}(t, \theta; \mathbf{y}^0)\}, \quad (1.13)$$

under assumptions (1.2), (1.3) with $m = 0$ and (1.4), for any $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$, $T \in \mathbb{R}$ and any $\varepsilon > 0$, the solution $\mathbf{X}_\varepsilon(\cdot; \mathbf{x}, s)$ of (1.1) exists on $[s, s+T]$, is unique and the sequence $(\mathbf{X}_\varepsilon(\cdot; \mathbf{x}, s))$ satisfies:

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [s, s+T]} |\mathbf{X}_\varepsilon(t; \mathbf{x}, s) - \mathbf{X}^0(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)| = 0, \quad (1.14)$$

$|\cdot|$ standing for the Euclidean norm on \mathbb{R}^d , where \mathbf{X}^0 satisfies

$$\mathbf{X}^0(t, \theta; \mathbf{x}, s) = \mathbf{Z}(t, \theta, \mathbf{Y}^0(t; \mathbf{x}, s)), \quad (1.15)$$

and where \mathbf{Y}^0 is the solution to

$$\frac{d\mathbf{Y}^0}{dt} = \tilde{\mathbf{a}}^0(t, \mathbf{Y}^0), \quad \mathbf{Y}^0(s; \mathbf{x}, s) = \mathbf{x}. \quad (1.16)$$

In the Theorem above, $\nabla_{\mathbf{z}}\mathbf{Z}(t, \theta; \mathbf{z})$ stands for the Jacobian matrix of $\mathbf{z} \mapsto \mathbf{Z}(t, \theta; \mathbf{z})$.

In order to set this Theorem and to justify the expansion (1.5) for higher orders, using a Van der Pol transformation, we define \mathbf{Y}_ε being such that

$$\mathbf{X}_\varepsilon(t; \mathbf{x}, s) = \mathbf{Z}(t, \frac{t-s}{\varepsilon}; \mathbf{Y}_\varepsilon(t, \mathbf{x}, s)) = [\mathbf{Z}(\mathbf{Y}_\varepsilon)]_\varepsilon, \quad (1.17)$$

where for any function f we write $[f(\mathbf{Y}_\varepsilon, \dots, \mathbf{Y}_\varepsilon^k)]_\varepsilon$ for $f(t, \frac{t-s}{\varepsilon}, \mathbf{Y}_\varepsilon(t; \mathbf{x}, s), \dots, \mathbf{Y}_\varepsilon^k(t; \mathbf{x}, s))$ and $[f(\mathbf{Y}^0, \dots, \mathbf{Y}^k)]_\varepsilon$ for $f(t, \frac{t-s}{\varepsilon}, \mathbf{Y}^0(t; \mathbf{x}, s), \dots, \mathbf{Y}^k(t; \mathbf{x}, s))$. It is an easy game to show that

$$\frac{d\mathbf{Y}_\varepsilon}{dt} = \tilde{\alpha}^0(t, \frac{t-s}{\varepsilon}, \mathbf{Y}_\varepsilon) = [\tilde{\alpha}^0(\mathbf{Y}_\varepsilon)]_\varepsilon; \quad \mathbf{Y}_\varepsilon(s; \mathbf{x}, s) = \mathbf{x}. \quad (1.18)$$

Indeed, derivating (1.17) and using (1.18) we get

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \frac{1}{\varepsilon} \left[\frac{\partial \mathbf{Z}}{\partial \theta}(\mathbf{Y}_\varepsilon) \right]_\varepsilon + \left[\frac{\partial \mathbf{Z}}{\partial t}(\mathbf{Y}_\varepsilon) \right]_\varepsilon + \{ \nabla_{\mathbf{z}} \mathbf{Z}(\mathbf{Y}_\varepsilon) \}_\varepsilon \{ \tilde{\alpha}^0(\mathbf{Y}_\varepsilon) \}_\varepsilon = [\mathbf{a}(\mathbf{X}_\varepsilon)]_\varepsilon + \frac{1}{\varepsilon} \mathbf{b}(\mathbf{X}_\varepsilon), \quad (1.19)$$

and we have the initial condition $\mathbf{X}_\varepsilon(s; \mathbf{x}, s) = \mathbf{x}$.

We also use the following notation. For a vector field \mathbf{Z} , The i -th component of $\{ \nabla_{\mathbf{x}}^k \mathbf{Z} \} \{ \mathbf{x}^0, \dots, \mathbf{x}^k \}$, for $i = 1, \dots, d$, is given by:

$$\left(\{ \nabla_{\mathbf{x}}^k \mathbf{Z} \} \{ \mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^k \} \right)_i = \sum_{l_1, \dots, l_k=1}^d \frac{\partial^k \mathbf{Z}_i}{\partial x_{l_1} \dots \partial x_{l_k}} \mathbf{x}_{l_1}^0 \dots \mathbf{x}_{l_k}^k. \quad (1.20)$$

In order to simplify we shall sometimes denote $\{ \nabla_{\mathbf{x}}^k \mathbf{Z} \} \{ \mathbf{x}^0, \mathbf{x}^0, \dots, \mathbf{x}^0 \}$ by $\{ \nabla_{\mathbf{x}}^k \mathbf{Z} \} \{ \mathbf{x}^0 \}^k$.

Now for $k \geq 0$ we recursively define

$$\tilde{\mathbf{A}}^k(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^k) = \frac{1}{\theta} \int_0^\theta \tilde{\alpha}^k(t, \sigma, \mathbf{y}^0, \dots, \mathbf{y}^k) d\sigma - \tilde{\mathbf{a}}^k(t, \mathbf{y}^0, \dots, \mathbf{y}^k), \quad (1.21)$$

where $\tilde{\alpha}^0$ and $\tilde{\mathbf{a}}^0$ are given by (1.13) and (1.12) and where for $k \geq 1$ we have:

$$\begin{aligned} \tilde{\alpha}^k(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^k) &= \{ \nabla_{\mathbf{y}^0} \tilde{\alpha}^0(t, \theta; \mathbf{y}^0) \} \{ \mathbf{y}^k + \theta \tilde{\mathbf{A}}^{k-1}(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^{k-1}) \} + \\ &\frac{1}{2} \{ \nabla_{\mathbf{y}^0}^2 \tilde{\alpha}^0(t, \theta; \mathbf{y}^0) \} \left(\sum_{j=1}^{k-1} \{ \mathbf{y}^j + \theta \tilde{\mathbf{A}}^{j-1}(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^{j-1}) \}, \mathbf{y}^{k-j} + \theta \tilde{\mathbf{A}}^{k-j-1}(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^{k-j-1}) \right) + \\ &\dots + \frac{1}{k!} \{ \nabla_{\mathbf{y}^0}^k \tilde{\alpha}^0(t, \theta; \mathbf{y}^0) \} \{ \mathbf{y}^1 + \theta \tilde{\mathbf{A}}^0(t, \theta, \mathbf{y}^0) \}^k - \\ &\theta \left(\sum_{j=0}^{k-1} \{ \nabla_{\mathbf{y}^j} \tilde{\mathbf{A}}^{k-1}(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^{k-1}) \} \{ \tilde{\mathbf{a}}^j(t, \mathbf{y}^0, \dots, \mathbf{y}^j) \} + \frac{\partial \tilde{\mathbf{A}}^{k-1}}{\partial t}(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^{k-1}) \right), \end{aligned} \quad (1.22)$$

and

$$\tilde{\mathbf{a}}^k(t, \mathbf{y}^0, \dots, \mathbf{y}^k) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\alpha}^k(t, \theta, \mathbf{y}^0, \dots, \mathbf{y}^k) d\theta. \quad (1.23)$$

With those notations, we can state the asymptotic expansion of \mathbf{Y}_ε ,

$$\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \varepsilon(\mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0]_\varepsilon) + \varepsilon^2(\mathbf{Y}^2 + [\theta \tilde{\mathbf{A}}^1]_\varepsilon) + \dots \quad (1.24)$$

In other words, defining for $k \geq 1$,

$$\begin{aligned} \mathbf{Y}_\varepsilon^k &= \frac{1}{\varepsilon^k} (\mathbf{Y}_\varepsilon - \mathbf{Y}^0 - \varepsilon(\mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0]_\varepsilon) - \dots - \varepsilon^{k-1}(\mathbf{Y}^{k-1} + [\theta \tilde{\mathbf{A}}^{k-2}]_\varepsilon)) - [\theta \tilde{\mathbf{A}}^{k-1}]_\varepsilon, \\ &= \frac{1}{\varepsilon} (\mathbf{Y}_\varepsilon^{k-1} - \mathbf{Y}^{k-1}) - [\theta \tilde{\mathbf{A}}^{k-1}]_\varepsilon, \end{aligned} \quad (1.25)$$

we have the following Theorem.

THEOREM 1.2 Under assumptions (1.2), (1.3) and (1.4) for any $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $T \in \mathbb{R}$, the sequences $(\mathbf{Y}_\varepsilon^k(\cdot; \mathbf{x}, s))$, for $k = 0, \dots, m$, are bounded in $L^\infty([s, s+T])$ and we have

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [s, s+T]} |\mathbf{Y}_\varepsilon(t; \mathbf{x}, s) - \mathbf{Y}^0(t; \mathbf{x}, s)| = 0, \text{ and } \lim_{\varepsilon \rightarrow 0} \sup_{t \in [s, s+T]} |\mathbf{Y}_\varepsilon^k(t; \mathbf{x}, s) - \mathbf{Y}^k(t; \mathbf{x}, s)| = 0, \quad (1.26)$$

for $k \geq 1$, where \mathbf{Y}^0 is solution to (1.16) and where \mathbf{Y}^k is the solution to

$$\frac{d\mathbf{Y}^k}{dt} = \tilde{\mathbf{a}}^k(t, \mathbf{Y}^0, \dots, \mathbf{Y}^k), \quad \mathbf{Y}^k(s; \mathbf{x}, s) = 0. \quad (1.27)$$

As a consequence of this Theorem and of (1.17), defining $(\mathbf{X}_\varepsilon^k(t; \mathbf{x}, s))$ by:

$$\mathbf{X}_\varepsilon^k = \frac{1}{\varepsilon^k} (\mathbf{X}_\varepsilon - [\mathbf{X}^0]_\varepsilon - \dots - \varepsilon^{k-1} [\mathbf{X}^{k-1}]_\varepsilon) = \frac{1}{\varepsilon} (\mathbf{X}_\varepsilon^{k-1} - [\mathbf{X}^{k-1}]_\varepsilon), \quad (1.28)$$

for $k \geq 1$, we have the rigorous justification of asymptotic expansion (1.5).

THEOREM 1.3 Under assumptions (1.2), (1.3) and (1.4) for any $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $T \in \mathbb{R}$, the sequences $(\mathbf{X}_\varepsilon^k(\cdot; \mathbf{x}, s))$, for $k = 1, \dots, m$, are bounded in $L^\infty([s, s+T])$ and we have:

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [s, s+T]} |\mathbf{X}_\varepsilon^k(t; \mathbf{x}, s) - \mathbf{X}^k(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)| = 0, \quad (1.29)$$

where \mathbf{X}^k writes

$$\begin{aligned} \mathbf{X}^k(t, \theta; \mathbf{x}, s) &= \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{Y}^0(t; \mathbf{x}, s))\} \{\mathbf{Y}^k(t; \mathbf{x}, s) + \theta \tilde{\mathbf{A}}^{k-1}(t, \theta, \mathbf{Y}^0(t; \mathbf{x}, s), \dots, \mathbf{Y}^{k-1}(t; \mathbf{x}, s))\} + \\ &\frac{1}{2} \{\nabla_z^2 \mathbf{Z}(t, \theta; \mathbf{Y}^0(t; \mathbf{x}, s))\} \left(\sum_{j=1}^{k-1} \{\mathbf{Y}^j(t; \mathbf{x}, s) + \theta \tilde{\mathbf{A}}^{j-1}(t, \theta, \mathbf{Y}^0(t; \mathbf{x}, s), \dots, \mathbf{Y}^{j-1}(t; \mathbf{x}, s))\}, \right. \\ &\quad \left. \mathbf{Y}^{k-j}(t; \mathbf{x}, s) + \theta \tilde{\mathbf{A}}^{k-j-1}(t, \theta, \mathbf{Y}^0(t; \mathbf{x}, s), \dots, \mathbf{Y}^{k-j-1}(t; \mathbf{x}, s)) \right) + \\ &\dots + \frac{1}{k!} \{\nabla_z^k \mathbf{Z}(t, \theta; \mathbf{Y}^0(t; \mathbf{x}, s))\} \{\mathbf{Y}^1(t; \mathbf{x}, s) + \theta \tilde{\mathbf{A}}^0(t, \theta, \mathbf{Y}^0(t; \mathbf{x}, s))\}^k, \quad (1.30) \end{aligned}$$

with \mathbf{Y}^k solution to (1.27).

We end this result list by giving explicit expressions. Using (1.13) in (1.22) we have the following expression of $\tilde{\alpha}^1$:

$$\begin{aligned} \tilde{\alpha}^1(t, \theta, \mathbf{y}^0, \mathbf{y}^1) &= \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\}^{-1} \left\{ \{\nabla_x \mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0))\} \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\} \{\mathbf{y}^1 + \theta \tilde{\mathbf{A}}^0(t, \theta; \mathbf{y}^0)\} - \right. \\ &\quad \left. \left\{ \frac{\partial \nabla_z \mathbf{Z}}{\partial t}(t, \theta; \mathbf{y}^0) \right\} \{\mathbf{y}^1 + \theta \tilde{\mathbf{A}}^0(t, \theta; \mathbf{y}^0)\} - \{\nabla_z^2 \mathbf{Z}(t, \theta; \mathbf{y}^0)\} \{\tilde{\alpha}^0(t, \mathbf{y}^0), \mathbf{y}^1 + \theta \tilde{\mathbf{A}}^0(t, \theta; \mathbf{y}^0)\} \right\} - \\ &\quad \theta \left(\{\nabla_{y^0} \tilde{\mathbf{A}}^0(t, \theta; \mathbf{y}^0)\} \{\tilde{\mathbf{a}}^0(t, \mathbf{y}^0)\} + \frac{\partial \tilde{\mathbf{A}}^0}{\partial t}(t, \theta; \mathbf{y}^0) \right). \quad (1.31) \end{aligned}$$

On another hand, if \mathbf{b} is linear and independent of t , $\tilde{\alpha}^2$ is given by

$$\begin{aligned} \tilde{\alpha}^2(t, \theta, \mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2) &= \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\}^{-1} \left\{ \{\nabla_x \mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0))\} \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\} \{\mathbf{y}^2 + \theta \tilde{\mathbf{A}}^1(t, \theta; \mathbf{y}^0, \mathbf{y}^1)\} + \right. \\ &\quad \left. \frac{1}{2} \{\nabla_x^2 \mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0))\} \left\{ \{\nabla_z \mathbf{Z}(t, \theta; \mathbf{y}^0)\} \{\mathbf{y}^1 + \theta \tilde{\mathbf{A}}^0(t, \theta; \mathbf{y}^0)\} \right\}^2 - \right. \\ &\quad \left. \theta \left(\{\nabla_{y^0} \tilde{\mathbf{A}}^1(t, \theta; \mathbf{y}^0, \mathbf{y}^1)\} \{\tilde{\mathbf{a}}^0(t, \mathbf{y}^0)\} + \{\nabla_{y^1} \tilde{\mathbf{A}}^1(t, \theta; \mathbf{y}^0, \mathbf{y}^1)\} \{\tilde{\mathbf{a}}^1(t, \mathbf{y}^0, \mathbf{y}^1)\} + \frac{\partial \tilde{\mathbf{A}}^1}{\partial t}(t, \theta; \mathbf{y}^0, \mathbf{y}^1) \right) \right). \quad (1.32) \end{aligned}$$

The paper is organised as follow: in the second section we briefly prove the Theorems. Section 3 is then devoted to applications to models describing magnetic confinement fusion and isotope resonant separation experiments. Among those applications, one involves a non uniform strong magnetic field, in this case the computations are led using Maple.

REMARK 1.1 With very little changes we can apply the previous results to the case when $\mathbf{b} \equiv \mathbf{b}(t, \theta, \mathbf{x})$ also depends on θ , as soon as the regularity of \mathbf{b} is enough and the assumption (1.4) is realized.

2 Proof of the Theorems

2.1 Sketch of the proof of Theorem 1.2

The key point to prove the results of this paper is Theorem 1.2, in the same spirit as in Schochet [35]. This result is classical and known as Single Phase Averaging. We do not give a detailed proof of it but only a formal sketch of it using expansion (1.24) of \mathbf{Y}_ε .

First, because of the definition (1.13) of $\tilde{\alpha}^0$ and the assumptions (1.2) - (1.4) we deduce that the function $(t, \mathbf{y}^0) \mapsto \tilde{\alpha}^0(t, \frac{t-s}{\varepsilon}, \mathbf{y}^0)$ is regular enough to ensure that, for any $\varepsilon > 0$, $\mathbf{Y}_\varepsilon(\cdot; \mathbf{x}, s)$ exists, is unique on $[s, s+T]$ and remains in a bounded set of \mathbb{R}^d independent of ε . Then expanding $\tilde{\alpha}^0(\mathbf{Y}_\varepsilon)$, using $\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \sum_{j \geq 1} \varepsilon^j (\mathbf{Y}^j + [\theta \tilde{\mathbf{A}}^{j-1}]_\varepsilon)$, we obtain

$$\begin{aligned} \tilde{\alpha}^0(\mathbf{Y}_\varepsilon) &= \tilde{\alpha}^0(\mathbf{Y}^0) + \varepsilon \{ \nabla_{\mathbf{y}^0} \tilde{\alpha}^0(\mathbf{Y}^0) \} \{ \mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0]_\varepsilon \} + \varepsilon^2 \left(\{ \nabla_{\mathbf{y}^0} \tilde{\alpha}^0(\mathbf{Y}^0) \} \{ \mathbf{Y}^2 + [\theta \tilde{\mathbf{A}}^1]_\varepsilon \} + \right. \\ &\quad \left. \frac{1}{2} \{ \nabla_{\mathbf{y}^0}^2 \tilde{\alpha}^0(\mathbf{Y}^0) \} \{ \mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0]_\varepsilon \}^2 \right) + \dots + \varepsilon^k \left(\{ \nabla_{\mathbf{y}^0} \tilde{\alpha}^0(\mathbf{Y}^0) \} \{ \mathbf{Y}^k + [\theta \tilde{\mathbf{A}}^{k-1}]_\varepsilon \} + \right. \\ &\quad \left. \frac{1}{2} \{ \nabla_{\mathbf{y}^0}^2 \tilde{\alpha}^0(\mathbf{Y}^0) \} \left(\sum_{j=1}^{k-1} \{ \mathbf{Y}^j + [\theta \tilde{\mathbf{A}}^{j-1}]_\varepsilon, \mathbf{Y}^{k-j} + [\theta \tilde{\mathbf{A}}^{k-j-1}]_\varepsilon \} \right) + \right. \\ &\quad \left. \dots + \frac{1}{k!} \{ \nabla_{\mathbf{y}^0}^k \tilde{\alpha}^0(\mathbf{Y}^0) \} \{ \mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0]_\varepsilon \}^k \right) + \dots \end{aligned} \quad (2.1)$$

Plugging the expansion (1.24) in the dynamical system (1.18) and identifying the terms of the same order, we have

$$\frac{\partial \mathbf{Y}^0}{\partial t} + \frac{\partial [\theta \tilde{\mathbf{A}}^0]}{\partial \theta} = \tilde{\alpha}^0(t, \theta, \mathbf{Y}^0), \quad (2.2)$$

at the order -1 , and at the order $k-1$ for $k \geq 1$ we have

$$\frac{\partial \mathbf{Y}^k}{\partial t} + \frac{\partial [\theta \tilde{\mathbf{A}}^k]}{\partial \theta} = \tilde{\alpha}^k(t, \theta, \mathbf{Y}^0, \dots, \mathbf{Y}^k). \quad (2.3)$$

Then equation (1.16) for \mathbf{Y}^0 , definition (1.21) of $\tilde{\mathbf{A}}^k$ and equation (1.27) for \mathbf{Y}^k follow, ending the sketch of the proof of Theorem 1.2. \blacksquare

2.2 Proof of Theorem 1.1

Existence and uniqueness of \mathbf{X}_ε are consequences of existence and uniqueness of \mathbf{Y}_ε . Then the regularity of \mathbf{Z} and Theorem 1.2 allows to pass to the limit in (1.17) in order to deduce (1.14) and (1.16) and thus prove the Theorem. \blacksquare

2.3 Proof of Theorem 1.3

From Theorem 1.2, we deduce

$$\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \varepsilon(\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon) + \cdots + \varepsilon^{k-1}(\mathbf{Y}^{k-1} + [\theta\tilde{\mathbf{A}}^{k-2}]_\varepsilon) + \varepsilon^k(\mathbf{Y}_\varepsilon^k + [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon), \quad (2.4)$$

and then, from (1.17) we deduce

$$\begin{aligned} \mathbf{X}_\varepsilon = [\mathbf{Z}(\mathbf{Y}_\varepsilon)]_\varepsilon &= [\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon + \varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon \{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\} + \varepsilon^2 \left(\{\nabla_z \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon \{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon\} + \right. \\ &\quad \left. \frac{1}{2} \{\nabla_z^2 \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon \{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}^2 \right) + \cdots + \varepsilon^k \left(\{\nabla_z \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon \{\mathbf{Y}_\varepsilon^k + [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon\} + \right. \\ &\quad \left. \frac{1}{2} \{\nabla_z^2 \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon \left(\sum_{j=1}^{k-1} \{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon, \mathbf{Y}^{k-j} + [\theta\tilde{\mathbf{A}}^{k-j-1}]_\varepsilon \} \right) + \right. \\ &\quad \left. \cdots + \frac{1}{k!} \{\nabla_z^k \mathbf{Z}(\mathbf{Y}^0)\} \{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}^k \right) + o(\varepsilon^k). \end{aligned} \quad (2.5)$$

Comparing this expansion with

$$\mathbf{X}_\varepsilon = [\mathbf{X}^0]_\varepsilon + \varepsilon[\mathbf{X}^1]_\varepsilon + \cdots + \varepsilon^{k-1}[\mathbf{X}^{k-1}]_\varepsilon + \varepsilon^k \mathbf{X}_\varepsilon^k, \quad (2.6)$$

obtained from (1.28) and making the process $\varepsilon \rightarrow 0$ give finally Theorem 1.3. \blacksquare

3 Application to the gyrokinetic plasma

The dynamical systems associated with equations (1.7), (1.8) and (1.9) are in the form of (1.5), i.e.:

$$\frac{d}{dt} \begin{pmatrix} \mathbf{X}_\varepsilon \\ \mathbf{V}_\varepsilon \end{pmatrix} = \mathbf{a}\left(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon, \mathbf{V}_\varepsilon\right) + \frac{1}{\varepsilon} \mathbf{b}\left(t, \mathbf{X}_\varepsilon, \mathbf{V}_\varepsilon\right), \quad \begin{pmatrix} \mathbf{X}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) \\ \mathbf{V}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} \quad (3.1)$$

with variable $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ in place of \mathbf{x} and with ad-hoc fields \mathbf{a} and \mathbf{b} . Then we can apply our result saying that the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ can be expanded as

$$\begin{aligned} \mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s) &= \mathbf{X}^0\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \varepsilon \mathbf{X}^1\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \varepsilon^2 \mathbf{X}^2\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \cdots, \\ \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s) &= \mathbf{V}^0\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \varepsilon \mathbf{V}^1\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \varepsilon^2 \mathbf{V}^2\left(t, \frac{t-s}{\varepsilon}; \mathbf{x}, \mathbf{v}, s\right) + \cdots, \end{aligned} \quad (3.2)$$

and that this expansion may be justified until any order if the regularity of the fields is enough.

In this section, we first lead the computations in the case of Isotope Resonant Separation Regime with the restriction that \mathcal{M} is constant and $\mathbf{B}(t, \theta, \mathbf{x}) = 0$ until the order 1. Then, with the same restrictions, we treat the case of the Guiding Centre Regime until the order 2 and we give the result for the Finite Larmor Radius Regime until the order 0. The forth example concerns the Guiding Centre Regime with a variable strong magnetic field and a constant \mathbf{B} until the order 1.

From the physical point of view, the last example is relevant to understand the behaviour of a plasma in a tokamak. This example shows that the generic computations made before, coupled with the use of Maple, enable to deduce the result relatively comfortably.

3.1 Isotope Resonant Separation Regime with constant strong magnetic field

In the case of Isotope Resonance Separation Regime, i.e. of equation (1.9), \mathbf{a} and \mathbf{b} are:

$$\mathbf{a}(t, \theta, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t, \theta, \mathbf{x}) + \mathbf{v} \times \mathbf{B}(t, \theta, \mathbf{x}) \end{pmatrix}, \quad \mathbf{b}(t, \mathbf{x}, \mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{v} \times \mathcal{M} \end{pmatrix}. \quad (3.3)$$

For simplicity, we restrict to the case when $\mathbf{B}(t, \theta, \mathbf{x}) = 0$ and when $\mathcal{M} = \mathbf{e}_1$ is a constant vector, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ being the frame of \mathbb{R}^3 . Then $\mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})$ and $\{\nabla_{\mathbf{z}, \mathbf{w}} \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})\}^{-1}$ are given by

$$\mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w}) = \begin{pmatrix} \mathbf{z} \\ R(\theta)\mathbf{w} \end{pmatrix}, \quad \{\nabla_{\mathbf{z}, \mathbf{w}} \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})\}^{-1} = \begin{pmatrix} I & 0 \\ 0 & R(-\theta) \end{pmatrix}, \quad (3.4)$$

where $R(\theta)$ is the matrix of the rotation of angle $-\theta$ around \mathcal{M} ,

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}. \quad (3.5)$$

We have the following result.

THEOREM 3.1 *If $\mathbf{E}(t, \theta, \mathbf{x})$ is $C_b^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3)$ and 2π -periodic in θ , the first term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon) + \frac{1}{\varepsilon} \mathbf{V}_\varepsilon \times \mathcal{M}, \quad \mathbf{X}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{V}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}, \quad (3.6)$$

is given by

$$\mathbf{X}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \quad \mathbf{V}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = R(\theta)\mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \quad (3.7)$$

where $(\mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s))$ is solution to

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}_\parallel^0, \quad \frac{d\mathbf{U}^0}{dt} = \frac{1}{2\pi} \int_0^{2\pi} R(-\theta)\mathbf{E}(t, \theta, \mathbf{Y}^0) d\theta, \quad \mathbf{Y}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{U}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}. \quad (3.8)$$

Proof. In view of (1.13) and (1.12), here we have

$$\tilde{\alpha}^0(t, \theta, \mathbf{y}^0, \mathbf{u}^0) = \begin{pmatrix} R(\theta)\mathbf{u}^0 \\ R(-\theta)\mathbf{E}(t, \theta, \mathbf{y}^0) \end{pmatrix}, \quad \tilde{\mathbf{a}}^0(t, \mathbf{y}^0, \mathbf{u}^0) = \begin{pmatrix} \mathbf{u}_\parallel^0 \\ \frac{1}{2\pi} \int_0^{2\pi} R(-\theta)\mathbf{E}(t, \theta, \mathbf{y}^0) d\theta \end{pmatrix}. \quad (3.9)$$

Then the proof of the Theorem is straightforward. ■

In order to obtain the system satisfied by the second term $(\mathbf{Y}^1, \mathbf{U}^1)$ of the expansion we notice that $\mathcal{R}(\theta) = -R(\frac{\pi}{2} + \theta) + R(\frac{\pi}{2})$ is such that $\int_0^\theta R(\sigma) d\sigma = \theta P + \mathcal{R}(\theta)$, with P the matrix of the orthogonal projection onto \mathcal{M} . We have

$$\mathcal{R}(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin \theta & 1 - \cos \theta \\ 0 & \cos \theta - 1 & \sin \theta \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.10)$$

Then, from (1.21), we get:

$$\theta \tilde{\mathbf{A}}^0(t, \theta, \mathbf{y}^0, \mathbf{u}^0) = \begin{pmatrix} \mathcal{R}(\theta) \mathbf{u}^0 \\ \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \mathbf{E}(t, \sigma, \mathbf{y}^0)) \end{pmatrix}, \quad (3.11)$$

where we denote $\int_0^\theta f(\sigma) d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} f(\sigma) d\sigma$ by $(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma)(f(\sigma))$. We also get

$$\begin{aligned} \tilde{\alpha}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) = & \begin{pmatrix} R(\theta) (\mathbf{u}^1 + \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \mathbf{E}(t, \sigma, \mathbf{y}^0))) \\ R(-\theta) \nabla_x \mathbf{E}(t, \theta, \mathbf{y}^0) (\mathbf{y}^1 + \mathcal{R}(\theta) \mathbf{u}^0) \end{pmatrix} - \\ & \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) \left[\begin{pmatrix} R(\sigma) \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\varsigma) \mathbf{E}(t, \varsigma, \mathbf{y}^0) d\varsigma \right) \\ R(-\sigma) \nabla_x \mathbf{E}(t, \sigma, \mathbf{y}^0) \mathbf{u}_\parallel^0 \end{pmatrix} + \begin{pmatrix} 0 \\ R(-\sigma) \frac{\partial \mathbf{E}}{\partial t}(t, \sigma, \mathbf{y}^0) \end{pmatrix} \right] \end{pmatrix} \quad (3.12)$$

Then using $(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma)(R(\sigma)) = \mathcal{R}(\theta)$ and $\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\theta) d\theta = R(\frac{\pi}{2}) - P$, we obtain

$$\begin{aligned} \tilde{\mathbf{a}}^1(t, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) = & \begin{pmatrix} \mathbf{u}_\parallel^1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(\theta - \sigma) \mathbf{E}(t, \sigma, \mathbf{y}^0)) d\theta \\ \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\theta) \nabla_x \mathbf{E}(t, \theta, \mathbf{y}^0) d\theta \right) \mathbf{y}^1 + \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\theta) \nabla_x \mathbf{E}(t, \theta, \mathbf{y}^0) \mathcal{R}(\theta) d\theta \right) \mathbf{u}^0 \end{pmatrix} - \\ & \begin{pmatrix} (R(\frac{\pi}{2}) - P) \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\varsigma) \mathbf{E}(t, \varsigma, \mathbf{y}^0) d\varsigma \right) \\ \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \nabla_x \mathbf{E}(t, \sigma, \mathbf{y}^0)) d\theta \right) \mathbf{u}_\parallel^0 \end{pmatrix} - \\ & \begin{pmatrix} 0 \\ \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \frac{\partial \mathbf{E}}{\partial t}(t, \sigma, \mathbf{y}^0)) d\theta \right) \end{pmatrix}. \quad (3.13) \end{aligned}$$

Hence we can state the following Theorem.

THEOREM 3.2 *If $\mathbf{E}(t, \theta, \mathbf{x})$ is $C_b^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3)$ and 2π -periodic in θ , the second term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ to (3.6) is given by*

$$\begin{aligned} \mathbf{X}^1(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \mathbf{Y}^1(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \\ \mathbf{V}^1(t, \theta; \mathbf{x}, \mathbf{v}, s) &= R(\theta) \mathbf{U}^1(t; \mathbf{x}, \mathbf{v}, s) + R(\theta) \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \mathbf{E}(t, \sigma, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))), \end{aligned} \quad (3.14)$$

where $(\mathbf{Y}^1, \mathbf{U}^1)$ is solution to

$$\begin{aligned} \frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}_{\parallel}^1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) \left(R(\theta - \sigma) \mathbf{E}(t, \sigma, \mathbf{Y}^0) \right) d\theta - \\ &\quad \left(R\left(\frac{\pi}{2}\right) - P \right) \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\varsigma) \mathbf{E}(t, \varsigma, \mathbf{Y}^0) d\varsigma \right), \\ \frac{d\mathbf{U}^1}{dt} &= \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\theta) \nabla_x \mathbf{E}(t, \theta, \mathbf{Y}^0) d\theta \right) \mathbf{Y}^1 + \left(\frac{1}{2\pi} \int_0^{2\pi} R(-\theta) \nabla_x \mathbf{E}(t, \theta, \mathbf{Y}^0) \mathcal{R}(\theta) d\theta \right) \mathbf{U}^0 - \\ &\quad \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) \left(R(-\sigma) \nabla_x \mathbf{E}(t, \sigma, \mathbf{Y}^0) \right) d\theta \right) \mathbf{U}_{\parallel}^0 - \\ &\quad \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) \left(R(-\sigma) \frac{\partial \mathbf{E}}{\partial t}(t, \sigma, \mathbf{Y}^0) \right) d\theta, \\ \mathbf{Y}^1(s; \mathbf{x}, \mathbf{v}, s) &= 0, \quad \mathbf{U}^1(s; \mathbf{x}, \mathbf{v}, s) = 0. \end{aligned} \tag{3.15}$$

3.2 Guiding Centre Regime with constant strong magnetic field

In the case of the Guiding Centre Regime, i.e. of equation (1.7), we have

$$\mathbf{a}(t, \theta, \mathbf{x}, \mathbf{v}) = \mathbf{a}(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t, \mathbf{x}) + \mathbf{v} \times \mathbf{B}(t, \mathbf{x}) \end{pmatrix}, \quad \text{and } \mathbf{b}(t, \mathbf{x}, \mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} 0 \\ \mathcal{N} + \mathbf{v} \times \mathcal{M} \end{pmatrix}, \tag{3.16}$$

As previously, we make the restriction $\mathbf{B}(t, \theta, \mathbf{x}) = \mathcal{N} = 0$ and $\mathcal{M} = \mathbf{e}_1$. Since this situation is similar to the previous one with the only difference that $\mathbf{E}(t, \mathbf{x})$ does not depend on θ we can directly deduce the following Theorem.

THEOREM 3.3 *If $\mathbf{E}(t, \mathbf{x})$ is $C_b^2(\mathbb{R} \times \mathbb{R}^3)$, the first and second terms of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t, \mathbf{X}_\varepsilon) + \frac{1}{\varepsilon} \mathbf{V}_\varepsilon \times \mathcal{M}, \quad \mathbf{X}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{V}_\varepsilon(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}, \tag{3.17}$$

are given by

$$\mathbf{X}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \quad \mathbf{V}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = R(\theta) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \tag{3.18}$$

and

$$\begin{aligned} \mathbf{X}^1(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \mathbf{Y}^1(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \\ \mathbf{V}^1(t, \theta; \mathbf{x}, \mathbf{v}, s) &= R(\theta) \mathbf{U}^1(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta) \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s)), \end{aligned} \tag{3.19}$$

where $(\mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s))$ is solution to

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}_{\parallel}^0, \quad \frac{d\mathbf{U}^0}{dt} = \mathbf{E}_{\parallel}(t, \mathbf{Y}^0), \quad \mathbf{Y}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{U}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}, \tag{3.20}$$

and where $(\mathbf{Y}^1, \mathbf{U}^1)$ is solution to

$$\begin{aligned} \frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}_{\parallel}^1 + \left(R\left(\frac{\pi}{2}\right) - P \right) \mathbf{E}(t, \mathbf{Y}^0) = \mathbf{U}_{\parallel}^1 + \mathbf{E}(t, \mathbf{Y}^0) \times \mathcal{M}, \\ \frac{d\mathbf{U}^1}{dt} &= P \nabla_x \mathbf{E}(t, \mathbf{Y}^0) \mathbf{Y}^1 + \frac{1}{2} \text{tr}((I - P) \nabla_x \mathbf{E}(t, \mathbf{Y}^0)) \left(R\left(-\frac{\pi}{2}\right) - P \right) \mathbf{U}^0 + \\ &\quad \frac{1}{2} \text{tr}((R(-\frac{\pi}{2}) - P) \nabla_x \mathbf{E}(t, \mathbf{Y}^0)) (I - P) \mathbf{U}^0 - \left(R\left(-\frac{\pi}{2}\right) - P \right) \nabla_x \mathbf{E}(t, \mathbf{Y}^0) \mathbf{U}_{\parallel}^0 - \\ &\quad \left(R\left(-\frac{\pi}{2}\right) - P \right) \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{Y}^0), \end{aligned} \tag{3.21}$$

$$\mathbf{Y}^1(s; \mathbf{x}, \mathbf{v}, s) = 0, \quad \mathbf{U}^1(s; \mathbf{x}, \mathbf{v}, s) = 0.$$

In the computations leading to this Theorem, we use, among other formula,

$$\left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (\mathcal{R}(-\sigma)) = -\mathcal{R}(-\theta), \quad \frac{1}{2\pi} \int_0^{2\pi} -\mathcal{R}(-\theta) d\theta = R(-\frac{\pi}{2}) - P, \quad -R(\theta)\mathcal{R}(-\theta) = \mathcal{R}(\theta), \quad (3.22)$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} R(-\theta) \nabla_x \mathbf{E}(t, \mathbf{Y}^0) \mathcal{R}(\theta) d\theta &= \frac{1}{2} \text{tr}((I - P) \nabla_x \mathbf{E}(t, \mathbf{Y}^0)) (R(-\frac{\pi}{2}) - P) + \\ &\frac{1}{2} \text{tr}((R(-\frac{\pi}{2}) - P) \nabla_x \mathbf{E}(t, \mathbf{Y}^0)) (I - P). \end{aligned} \quad (3.23)$$

Now we turn to the characterisation of $(\mathbf{X}^2, \mathbf{V}^2)$. For this purpose we notice that here

$$\begin{aligned} \tilde{\alpha}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) &= \\ &\left(\begin{array}{c} R(\theta) \mathbf{u}^1 + \mathcal{R}(\theta) \mathbf{E}(t, \mathbf{y}^0) \\ R(-\theta) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathbf{y}^1 + R(-\theta) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathcal{R}(\theta) \mathbf{u}^0 + \mathcal{R}(-\theta) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathbf{u}_\parallel^0 + \mathcal{R}(-\theta) \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{y}^0) \end{array} \right). \end{aligned} \quad (3.24)$$

In order to get now $\theta \tilde{\mathbf{A}}^1$ we need first to compute

$$\left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (\mathcal{R}(\sigma)) = I - R(\theta), \quad \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (\mathcal{R}(-\sigma)) = I - R(-\theta). \quad (3.25)$$

Secondly,

$$\begin{aligned} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathcal{R}(\sigma)) &= \\ P \nabla_x \mathbf{E}(t, \mathbf{y}^0) (I - R(\theta)) + (I - P) \nabla_x \mathbf{E}(t, \mathbf{y}^0) &\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{\sin^2 \theta}{2} & \sin \theta - \frac{\sin 2\theta}{4} \\ 0 & \frac{\sin 2\theta}{4} - \sin \theta & \frac{\sin^2 \theta}{2} \end{array} \right) + \\ (R(-\frac{\pi}{2}) - P) \nabla_x \mathbf{E}(t, \mathbf{y}^0) &\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -\frac{\sin 2\theta}{4} & 1 - \cos \theta - \frac{\sin^2 \theta}{2} \\ 0 & \frac{\sin^2 \theta}{2} + \cos \theta - 1 & \frac{-\sin 2\theta}{4} \end{array} \right), \end{aligned} \quad (3.26)$$

which also reads

$$\begin{aligned} \left(\int_0^\theta d\sigma - \frac{\theta}{2\pi} \int_0^{2\pi} d\sigma \right) (R(-\sigma) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathcal{R}(\sigma)) &= P \nabla_x \mathbf{E}(t, \mathbf{y}^0) (I - R(\theta)) + \\ \frac{1}{2} (-\mathcal{R}(-\theta) + R(\frac{\pi}{2}) - P) \nabla_x \mathbf{E}(t, \mathbf{y}^0) &(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P) + \frac{1}{2} (R(-\frac{\pi}{2}) - P) \nabla_x \mathbf{E}(t, \mathbf{y}^0) (R(\frac{\pi}{2}) - P). \end{aligned} \quad (3.27)$$

Hence integrating (3.24) we have

$$\theta \tilde{\mathbf{A}}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) = \left(\begin{array}{c} R(\theta) \mathbf{u}^1 + (I - R(\theta)) \mathbf{E}(t, \mathbf{y}^0) \\ -\mathcal{R}(-\theta) \nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathbf{y}^1 - (I - R(-\theta)) \left(\nabla_x \mathbf{E}(t, \mathbf{y}^0) \mathbf{u}_\parallel^0 + \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{y}^0) \right) \end{array} \right) +$$

$$\begin{aligned} & \left(\begin{array}{c} 0 \\ (P\nabla_x \mathbf{E}(t, \mathbf{y}^0)(I - R(\theta)) + \frac{1}{2}(-\mathcal{R}(-\theta) + R(\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{y}^0)(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P))\mathbf{u}^0 \end{array} \right) + \\ & \left(\begin{array}{c} 0 \\ (\frac{1}{2}(R(-\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{y}^0)(R(\frac{\pi}{2}) - P)\mathbf{u}^0) \end{array} \right). \end{aligned} \quad (3.28)$$

In order to obtain the expression of $\tilde{\alpha}^2$, we need to compute

$$\begin{aligned} & \{\nabla_{z,w} \mathbf{Z}(t, \theta; \mathbf{y}^0, \mathbf{u}^0)\}^{-1} \{\nabla_{x,v} \mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0, \mathbf{u}^0))\} \{\nabla_{z,w} \mathbf{Z}(t, \theta; \mathbf{y}^0, \mathbf{u}^0)\} \\ & \left\{ \begin{pmatrix} \mathbf{y}^2 \\ \mathbf{u}^2 \end{pmatrix} + \theta \tilde{\mathbf{A}}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) \right\} = \\ & \begin{pmatrix} 0 & R(\theta) \\ R(-\theta)\nabla_x \mathbf{E}(t, \mathbf{y}^0) & 0 \end{pmatrix} \left(\begin{pmatrix} \mathbf{y}^2 \\ \mathbf{u}^2 \end{pmatrix} + \theta \tilde{\mathbf{A}}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) \right) = \\ & \left(\begin{array}{c} R(\theta)\mathbf{u}^2 + \mathcal{R}(\theta)\nabla_x \mathbf{E}(t, \mathbf{y}^0)\mathbf{y}^1 - (R(\theta) - I)\left(\nabla_x \mathbf{E}(t, \mathbf{y}^0)\mathbf{u}^0 + \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{y}^0)\right) + \\ (P\nabla_x \mathbf{E}(t, \mathbf{y}^0)(I - R(\theta)) + \frac{1}{2}(R(\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{y}^0)(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P) + \\ \frac{1}{2}(R(\theta - \frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{y}^0)(R(\frac{\pi}{2}) - P))\mathbf{u}^0 \\ \\ R(-\theta)\nabla_x \mathbf{E}(t, \mathbf{y}^0)(\mathbf{y}^2 + \mathcal{R}(\theta)\mathbf{u}^1 + (I - R(\theta))\mathbf{E}(t, \mathbf{y}^0)) \end{array} \right), \end{aligned} \quad (3.29)$$

$$\begin{aligned} & \{\nabla_{x,v}^2 \mathbf{a}(t, \theta, \mathbf{Z}(t, \theta; \mathbf{y}^0, \mathbf{u}^0))\} \left\{ \left\{ \nabla_{z,w} \mathbf{Z}(t, \theta; \mathbf{y}^0, \mathbf{u}^0) \right\} \left\{ \begin{pmatrix} \mathbf{y}^1 \\ \mathbf{u}^1 \end{pmatrix} + \theta \tilde{\mathbf{A}}^0(t, \theta, \mathbf{y}^0, \mathbf{u}^0) \right\} \right\}^2 = \\ & \left(\begin{array}{c} 0 \\ \left\{ \nabla_x^2 \mathbf{E}(t, \mathbf{y}^0) \right\}^2 \left(\mathbf{y}^1, \mathbf{y}^1 \right) + 2\{\mathbf{y}^1, \mathcal{R}(\theta)\mathbf{u}^0\} + \{\mathcal{R}(\theta)\mathbf{u}^0, \mathcal{R}(\theta)\mathbf{u}^0\} \end{array} \right). \end{aligned} \quad (3.30)$$

A direct but heavy computation also leads the derivatives $\nabla_{\mathbf{y}^0, \mathbf{u}^0} \tilde{\mathbf{A}}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) \tilde{\mathbf{a}}^0(t, \mathbf{y}^0, \mathbf{u}^0)$, $\nabla_{\mathbf{y}^1, \mathbf{u}^1} \tilde{\mathbf{A}}^1(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1) \tilde{\mathbf{a}}^1(t, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1)$, and $\frac{\partial \tilde{\mathbf{A}}^1}{\partial t}(t, \theta, \mathbf{y}^0, \mathbf{u}^0, \mathbf{y}^1, \mathbf{u}^1)$.

Then in view of (1.32), we get $\tilde{\alpha}^2$, which, integrating with respect to θ leads to $\tilde{\mathbf{a}}^2$ and to the equation satisfied by $(\mathbf{X}^2, \mathbf{V}^2)$. Then we have the following Theorem.

THEOREM 3.4 *If $\mathbf{E}(t, \mathbf{x})$ is $C_b^3(\mathbb{R} \times \mathbb{R}^3)$, the third term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ to (3.17) is given by*

$$\begin{aligned} \mathbf{X}^2(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \mathbf{Y}^2(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta)\mathbf{U}^1 + (I - R(\theta))\mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s)), \\ \mathbf{V}^2(t, \theta; \mathbf{x}, \mathbf{v}, s) &= R(\theta)\mathbf{U}^2(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta)\nabla_x \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))\mathbf{Y}^1(t; \mathbf{x}, \mathbf{v}, s) - \\ & (R(\theta) - I)\nabla_x \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))\left(\mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s) + \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))\right) + \\ & \left(P\nabla_x \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))(I - R(\theta)) + \frac{1}{2}(R(\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P) + \right. \\ & \left. \frac{1}{2}(R(\theta - \frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s))(R(\frac{\pi}{2}) - P) \right) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s). \end{aligned} \quad (3.31)$$

Moreover $(\mathbf{Y}^2, \mathbf{U}^2)$ is solution to

$$\begin{aligned} \frac{d\mathbf{Y}^2}{dt} = & \mathbf{U}_{\parallel}^2 + (R(\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\mathbf{Y}^1 + (I - P)\left(\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\mathbf{U}_{\parallel}^0 + \frac{\partial \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)\right) + \\ & \left(P\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(I - P) + (R(\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(R(\frac{\pi}{2}) - P)\right)\mathbf{U}^0 - \\ & \frac{1}{2}\text{tr}((I - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(I - P)\mathbf{U}^0 - \frac{1}{2}\text{tr}((R(-\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(R(\frac{\pi}{2}) - P)\mathbf{U}^0, \end{aligned} \quad (3.32)$$

$$\begin{aligned} \frac{d\mathbf{U}^2}{dt} = & P\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\mathbf{Y}^2 + \\ & \frac{1}{2}\text{tr}((I - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(R(-\frac{\pi}{2}) - P)\mathbf{U}^1 + \frac{1}{2}\text{tr}((R(-\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(I - P)\mathbf{U}^1 + \\ & P\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\mathbf{E}(t, \mathbf{Y}^0) - \left(P\nabla_x \mathbf{E}(t, \mathbf{Y}^0)P + \frac{1}{2}\text{tr}((I - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(I - P) + \right. \\ & \left. \frac{1}{2}\text{tr}((R(-\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0))(R(-\frac{\pi}{2}) - P)\right)\mathbf{E}(t, \mathbf{Y}^0) + \\ & \frac{1}{2}\left\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\right\}^2 \left(\{\mathbf{Y}^1, \mathbf{Y}^1\} + 2\{\mathbf{Y}^1, \mathbf{U}_{\parallel}^0\} + \{(I - P)\mathbf{U}^0, (I - P)\mathbf{U}^0\} + \right. \\ & \left. \{(R(-\frac{\pi}{2}) - P)\mathbf{U}^0, (R(-\frac{\pi}{2}) - P)\mathbf{U}^0\}\right) + \\ & (P - R(-\frac{\pi}{2}))\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\}\{\mathbf{Y}^1, \mathbf{U}_{\parallel}^0\} + (I - P)\left(\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\}\{\mathbf{U}_{\parallel}^0, \mathbf{U}_{\parallel}^0\} + \frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)\mathbf{U}_{\parallel}^0\right) - \\ & P\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\}\{(I - P)\mathbf{U}^0, \mathbf{U}_{\parallel}^0\} - \\ & \frac{1}{4}(I - P)\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\}\{(I - P)\mathbf{U}^0, \mathbf{U}_{\parallel}^0\} - \frac{3}{4}(R(-\frac{\pi}{2}) - P)\{\nabla_x^2 \mathbf{E}(t, \mathbf{Y}^0)\}\{(R(\frac{\pi}{2}) - P)\mathbf{U}^0, \mathbf{U}_{\parallel}^0\} - \\ & \left(- (I - P)\left(\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\right) + P\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(I - P) + \right. \\ & \left. \frac{1}{4}(I - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(I - P) + \frac{3}{4}(R(-\frac{\pi}{2}) - P)\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(R(\frac{\pi}{2}) - P)\right)\mathbf{E}_{\parallel}(t, \mathbf{Y}^0) + \\ & (P - R(-\frac{\pi}{2}))\nabla_x \mathbf{E}(t, \mathbf{Y}^0)\mathbf{U}_{\parallel}^1 + (P - R(-\frac{\pi}{2}))\nabla_x \mathbf{E}(t, \mathbf{Y}^0)(R(\frac{\pi}{2}) - P)\mathbf{E}(t, \mathbf{Y}^0) + \\ & (P - R(-\frac{\pi}{2}))\frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)\mathbf{Y}^1 - (I - P)\left(\frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)\mathbf{U}_{\parallel}^0 + \frac{\partial^2 \mathbf{E}}{\partial t^2}(t, \mathbf{Y}^0)\right) - P\frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)(I - P) - \\ & \left(\frac{1}{4}(I - P)\frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)(I - P) + \frac{3}{4}(R(-\frac{\pi}{2}) - P)\frac{\partial \nabla_x \mathbf{E}}{\partial t}(t, \mathbf{Y}^0)(R(\frac{\pi}{2}) - P)\right)\mathbf{U}^0. \end{aligned} \quad (3.33)$$

3.3 Finite Larmor Radius Regime with constant strong magnetic field

In the case of Finite Larmor Radius Regime (1.8) with $\mathcal{M} = \mathbf{e}_1$, we have

$$\mathbf{a}(t, \theta, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v}_{\parallel} \\ \mathbf{E}(t, \mathbf{x}) \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t, \mathbf{x}, \mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} \mathbf{v}_{\perp} \\ \mathbf{v} \times \mathcal{M} \end{pmatrix}, \quad (3.34)$$

$$\mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w}) = \begin{pmatrix} \mathbf{z} + \mathcal{R}(\theta)\mathbf{w} \\ R(\theta)\mathbf{w} \end{pmatrix}, \quad \{\nabla_{\mathbf{z}, \mathbf{w}} \mathbf{Z}(t, \theta; \mathbf{x}, \mathbf{v})\}^{-1} = \begin{pmatrix} I & \mathcal{R}(-\theta) \\ 0 & R(-\theta) \end{pmatrix}. \quad (3.35)$$

with R and \mathcal{R} defined by (3.5) and (3.10).

Here we give the result for this case only for the order 0. We have

$$\tilde{\alpha}^0(t, \theta, \mathbf{y}^0, \mathbf{u}^0) = \begin{pmatrix} \mathbf{u}_{\parallel}^0 + \mathcal{R}(-\theta)\mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta)\mathbf{u}^0, t) \\ R(-\theta)\mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta)\mathbf{u}^0, t) \end{pmatrix}, \quad (3.36)$$

and

$$\tilde{\mathbf{a}}^0(t, \mathbf{y}^0, \mathbf{u}^0) = \begin{pmatrix} \mathbf{u}_{\parallel}^0 + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(-\theta) \mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta) \mathbf{u}^0, t) d\theta \\ \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(-\theta) \mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta) \mathbf{u}^0, t) d\theta \end{pmatrix}. \quad (3.37)$$

Hence we have the following Theorem.

THEOREM 3.5 *If we assume that $\mathbf{E}(t, \mathbf{x})$ is $C_b^1(\mathbb{R} \times \mathbb{R}^3)$, the first term of the expansion (3.2) of the solution $(\mathbf{X}_{\varepsilon}(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_{\varepsilon}(t; \mathbf{x}, \mathbf{v}, s))$ to*

$$\frac{d\mathbf{X}_{\varepsilon}}{dt} = \mathbf{V}_{\varepsilon \parallel} + \frac{1}{\varepsilon} \mathbf{V}_{\varepsilon \perp}, \quad \frac{d\mathbf{V}_{\varepsilon}}{dt} = \mathbf{E}(t, \mathbf{X}_{\varepsilon}) + \frac{1}{\varepsilon} \mathbf{V}_{\varepsilon} \times \mathcal{M}, \quad \mathbf{X}_{\varepsilon}(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{V}_{\varepsilon}(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}, \quad (3.38)$$

is given by

$$\mathbf{X}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s) + \mathcal{R}(\theta) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \quad \mathbf{V}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \mathcal{R}(\theta) \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \quad (3.39)$$

where $(\mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s))$ is solution to

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}_{\parallel}^0 + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(-\theta) \mathbf{E}(\mathbf{Y}^0 + \mathcal{R}(\theta) \mathbf{U}^0, t) d\theta, \quad \frac{d\mathbf{U}^0}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(-\theta) \mathbf{E}(\mathbf{Y}^0 + \mathcal{R}(\theta) \mathbf{U}^0, t) d\theta, \quad (3.40)$$

with the initial conditions

$$\mathbf{Y}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{U}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}. \quad (3.41)$$

3.4 Guiding Centre Regime with variable strong magnetic field

Here we study a situation representative of what happens inside a tokamak, i.e., the Guiding Centre Regime with a variable \mathcal{M} , with $\mathcal{N} = 0$ and $\mathbf{B} = \mathbf{e}_3$. In other words we consider the following system:

$$\frac{d\mathbf{X}_{\varepsilon}}{dt} = \mathbf{V}_{\varepsilon}, \quad \frac{d\mathbf{V}_{\varepsilon}}{dt} = \mathbf{E}(t, \mathbf{X}_{\varepsilon}) + \mathbf{V}_{\varepsilon} \times \mathbf{e}_3 + \frac{1}{\varepsilon} \mathbf{V}_{\varepsilon} \times \mathcal{M}(\mathbf{X}_{\varepsilon}), \quad \mathbf{X}_{\varepsilon}(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{V}_{\varepsilon}(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}, \quad (3.42)$$

where

$$\mathcal{M}(\mathbf{x}) = \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = \rho^T(\mathbf{x}) \mathbf{e}_1, \quad \text{with } \rho(\mathbf{x}) = \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{pmatrix} -x_2 & x_1 & 0 \\ -x_1 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.43)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ in the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of \mathbb{R}^3 . In this case

$$\mathbf{a}(t, \theta, \mathbf{x}, \mathbf{v}) = \mathbf{a}(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t, \mathbf{x}) + \mathbf{v} \times \mathbf{e}_3 \end{pmatrix}, \quad \text{and } \mathbf{b}(t, \mathbf{x}, \mathbf{v}) = \mathbf{b}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{v} \times \mathcal{M}(\mathbf{x}) \end{pmatrix}, \quad (3.44)$$

and thus, $R(\theta)$ being defined by (3.5),

$$\mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w}) = \begin{pmatrix} \mathbf{z} \\ \rho^T(\mathbf{z}) R(\theta) \rho(\mathbf{z}) \mathbf{w} \end{pmatrix}, \quad \text{where } \rho^T R \rho = \begin{pmatrix} \frac{z_2^2 + z_1^2 \cos(\theta)}{z_1^2 + z_2^2} & \frac{z_1 z_2 (\cos(\theta) - 1)}{z_1^2 + z_2^2} & -\frac{z_1 \sin(\theta)}{\sqrt{z_1^2 + z_2^2}} \\ \frac{z_1 z_2 (\cos(\theta) - 1)}{z_1^2 + z_2^2} & \frac{z_1^2 + z_2^2 \cos(\theta)}{z_1^2 + z_2^2} & -\frac{z_2 \sin(\theta)}{\sqrt{z_1^2 + z_2^2}} \\ \frac{z_1 \sin(\theta)}{\sqrt{z_1^2 + z_2^2}} & \frac{z_2 \sin(\theta)}{\sqrt{z_1^2 + z_2^2}} & \cos(\theta) \end{pmatrix}, \quad (3.45)$$

and we have the following Theorem.

THEOREM 3.6 *If $\mathbf{E}(t, \mathbf{x})$ is $C_b^2(\mathbb{R} \times \mathbb{R}^3)$, the first term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}_\varepsilon(t; \mathbf{x}, \mathbf{v}, s))$ to (3.42) is given by*

$$\mathbf{X}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \quad \mathbf{V}^0(t, \theta; \mathbf{x}, \mathbf{v}, s) = \rho^T(\mathbf{x})R(\theta)\rho(\mathbf{x})\mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s), \quad (3.46)$$

where $(\mathbf{Y}^0(t; \mathbf{x}, \mathbf{v}, s), \mathbf{U}^0(t; \mathbf{x}, \mathbf{v}, s))$ is solution to

$$\begin{aligned} \frac{d\mathbf{Y}^0}{dt} &= \bar{A}(\mathbf{Y}^0)\mathbf{U}^0, \quad \frac{d\mathbf{U}^0}{dt} = \bar{\beta}(\mathbf{Y}^0, \mathbf{U}^0) + \bar{A}(\mathbf{Y}^0)\mathbf{E}_\parallel(t, \mathbf{Y}^0) + \mathbf{U}^0 \times \bar{A}(\mathbf{Y}^0)\mathbf{e}_3, \\ \text{with } \bar{A}(\mathbf{y}) &= \frac{1}{y_1^2 + y_2^2} \begin{pmatrix} y_2^2 & -y_1y_2 & 0 \\ -y_1y_2 & y_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } \bar{\beta}(\mathbf{y}, \mathbf{u}) = \begin{pmatrix} (y_2u_1 - y_1u_2)u_2 \\ \frac{y_1^2 + y_2^2}{(y_1u_2 - y_2u_1)u_1} \\ \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} \\ 0 \end{pmatrix}, \end{aligned} \quad (3.47)$$

$$\mathbf{Y}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{x}, \quad \mathbf{U}^0(s; \mathbf{x}, \mathbf{v}, s) = \mathbf{v}.$$

The term $\mathbf{X}^1(t, \theta; \mathbf{x}, \mathbf{v}, s)$ is given by

$$\begin{aligned} \mathbf{X}^1_1(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \frac{1}{\Omega^2} \left(\mathbf{Y}^0_1 \Omega (\cos(\theta) - 1) \mathbf{U}^0_3 + \mathbf{Y}^0_1 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \sin(\theta) + \mathbf{Y}^1_1 \Omega^2 \right), \\ \mathbf{X}^1_2(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \frac{1}{\Omega^2} \left(\mathbf{Y}^0_2 \Omega (\cos(\theta) - 1) \mathbf{U}^0_3 + \mathbf{Y}^0_2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \sin(\theta) + \mathbf{Y}^1_2 \Omega^2 \right), \\ \mathbf{X}^1_3(t, \theta; \mathbf{x}, \mathbf{v}, s) &= \frac{1}{\Omega} \left((-\mathbf{Y}^0_1 \mathbf{U}^0_1 - \mathbf{Y}^0_2 \mathbf{U}^0_2) (\cos(\theta) - 1) + \mathbf{Y}^1_3 \Omega + \sin(\theta) \mathbf{U}^0_3 \Omega \right), \end{aligned} \quad (3.48)$$

where $\Omega = \sqrt{\mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_2{}^2}$ and where $\mathbf{Y}^1(t; \mathbf{x}, \mathbf{v}, s)$ is solution to

$$\begin{aligned} \frac{d\mathbf{Y}^1_1}{dt} &= \left(\left(-\Omega^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \right) \mathbf{Y}^1_1 + \left((2\mathbf{U}^0_1 \mathbf{Y}^0_2 - \mathbf{U}^0_2 \mathbf{Y}^0_1) \Omega^2 + \right. \right. \\ &\quad \left. \left. 2\mathbf{Y}^0_2{}^2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \right) \mathbf{Y}^1_2 + \mathbf{Y}^0_2{}^2 \mathbf{U}^1_1 \Omega^2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_2 - \Omega^3 \mathbf{Y}^0_1 \mathbf{E}_3 - \mathbf{Y}^0_2 \mathbf{U}^0_3 \Omega^3 - \right. \\ &\quad \left. 2\mathbf{Y}^0_2 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \Omega \right) / \Omega^4, \end{aligned}$$

$$\begin{aligned} \frac{d\mathbf{Y}^1_2}{dt} &= \left(\left((2\mathbf{U}^0_2 \mathbf{Y}^0_1 - \mathbf{U}^0_1 \mathbf{Y}^0_2) \Omega^2 - 2\mathbf{Y}^0_1{}^2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \right) \mathbf{Y}^1_1 + \left(-\Omega^2 \mathbf{Y}^0_1 \mathbf{U}^0_1 - \right. \right. \\ &\quad \left. \left. 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \right) \mathbf{Y}^1_2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_1 + \mathbf{Y}^0_1{}^2 \mathbf{U}^1_2 \Omega^2 - \right. \\ &\quad \left. \Omega^3 \mathbf{Y}^0_2 \mathbf{E}_3 + \Omega^3 \mathbf{Y}^0_1 \mathbf{U}^0_3 + 2\mathbf{Y}^0_1 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \Omega \right) / \Omega^4, \end{aligned}$$

$$\begin{aligned} \frac{d\mathbf{Y}^1_3}{dt} &= \left(\Omega^2 \mathbf{Y}^0_1 \mathbf{E}_1 + \Omega^2 \mathbf{Y}^0_2 \mathbf{E}_2 + \left(\mathbf{U}^0_2 \mathbf{Y}^0_1 + \mathbf{U}^0_1{}^2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2{}^2 \right) \Omega^2 - \right. \\ &\quad \left. \left(\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2 \right)^2 \right) / \Omega^3, \end{aligned}$$

$$\mathbf{Y}^1(s; \mathbf{x}, \mathbf{v}, s) = 0, \quad (3.49)$$

with \mathbf{E} being evaluated in \mathbf{Y}^0 . The result concerning $\mathbf{V}^1(t, \theta; \mathbf{x}, \mathbf{v}, s)$ is given in the appendix A.

This Theorem is the consequence of the following computations. Setting $A = \rho^T R \rho$ we have

$$\{\nabla_{z,w} \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})\} = \begin{pmatrix} I & 0 \\ \nabla_z(A\mathbf{w}) & A \end{pmatrix} \quad \text{and} \quad \{\nabla_{z,w} \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})\}^{-1} = \begin{pmatrix} I & 0 \\ A^T \nabla_z(A\mathbf{w}) & A^T \end{pmatrix}. \quad (3.50)$$

Then, since in this example $\frac{\partial \mathbf{Z}}{\partial t} = 0$, and

$$\mathbf{a}(t, \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})) = \begin{pmatrix} A\mathbf{w} \\ \mathbf{E} + A\mathbf{w} \times \mathbf{e}_3 \end{pmatrix}, \quad (3.51)$$

applying (1.12) and (1.13) we get

$$\tilde{\alpha}^0(t, \theta, \mathbf{z}, \mathbf{w}) = \begin{pmatrix} A\mathbf{w} \\ -A^T \nabla_z (A\mathbf{w})(A\mathbf{w}) + A^T \mathbf{E} + \mathbf{w} \times A^T \mathbf{e}_3 \end{pmatrix}, \quad (3.52)$$

and since $\int_0^{2\pi} A d\theta = \int_0^{2\pi} A^T d\theta = 2\pi \bar{A}$, and $\int_0^{2\pi} -A^T \nabla_z (A\mathbf{w})(A\mathbf{w}) = \bar{\beta}$, we finally get the first part of the Theorem.

The second part of the Theorem is obtained following the computation program described in the previous sections using Maple.

A Appendix : \mathbf{V}^1 for the Guiding Centre Regime with variable strong magnetic field

The computations of this appendix have been realized using Maple5.5, on computers of Medicis Centre, Ecole Polytechnique, Palaiseau, France.

The electric field \mathbf{E} and its derivatives being evaluated in \mathbf{Y}^0 , and Ω standing for $\sqrt{(\mathbf{Y}^0_1)^2 + (\mathbf{Y}^0_2)^2}$, the term $\mathbf{V}^1(t, \theta; \mathbf{x}, \mathbf{v}, s)$ of the expansion (3.2) of the solution to (3.42) is given by :

$$\begin{aligned} \mathbf{V}^1_1(t; \mathbf{x}, \mathbf{v}, s) = & \left(\left((2\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^2 - 2\mathbf{Y}^0_1{}^2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \right) \mathbf{Y}^1_1 + \right. \\ & \left(\mathbf{Y}^0_1 \mathbf{U}^0_2 \Omega^2 - 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \right) \mathbf{Y}^1_2 + \mathbf{Y}^0_1{}^2 \mathbf{U}^1_1 \Omega^2 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_2 + \Omega^3 \mathbf{Y}^0_1 \mathbf{E}_3 + \\ & \left. \Omega^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 + \mathbf{Y}^0_2 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \right) \cos(\theta) + \left((-\Omega^3 \mathbf{U}^0_3 + \Omega \mathbf{U}^0_3 \mathbf{Y}^0_1{}^2) \mathbf{Y}^1_1 + \right. \\ & \left. \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2 \Omega \mathbf{Y}^1_2 - \mathbf{Y}^0_1 \Omega^3 \mathbf{U}^1_3 + \Omega^2 \mathbf{Y}^0_1{}^2 \mathbf{E}_1 + \Omega^2 \mathbf{Y}^0_1 \mathbf{E}_2 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \Omega^4 + \right. \\ & \left. \mathbf{U}^0_2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega^2 \right) \sin(\theta) + \left(-\Omega^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \right) \mathbf{Y}^1_1 + \\ & \left((2\mathbf{U}^0_1 \mathbf{Y}^0_2 - \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega^2 + 2\mathbf{Y}^0_2{}^2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \right) \mathbf{Y}^1_2 + \mathbf{Y}^0_2{}^2 \mathbf{U}^1_1 \Omega^2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_2 - \\ & \left. \Omega^3 \mathbf{Y}^0_1 \mathbf{E}_3 - \Omega^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 - \mathbf{Y}^0_2 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega \right) / \Omega^4, \quad (\text{A.1}) \end{aligned}$$

$$\begin{aligned} \mathbf{V}^1_2(t; \mathbf{x}, \mathbf{v}, s) = & \left(\left((\mathbf{U}^0_1 \mathbf{Y}^0_2 \Omega^2 - 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)) \mathbf{Y}^1_1 + \left((\mathbf{Y}^0_1 \mathbf{U}^0_1 + 2\mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^2 - \right. \right. \right. \\ & \left. \left. 2\mathbf{Y}^0_2{}^2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \right) \mathbf{Y}^1_2 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_1 + \mathbf{Y}^0_2{}^2 \mathbf{U}^1_2 \Omega^2 + \Omega^3 \mathbf{Y}^0_2 \mathbf{E}_3 - \Omega^3 \mathbf{Y}^0_1 \mathbf{U}^0_3 - \right. \\ & \left. \mathbf{Y}^0_1 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega \right) \cos(\theta) + \left(\mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2 \Omega \mathbf{Y}^1_1 + (-\Omega^3 \mathbf{U}^0_3 + \Omega \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2) \mathbf{Y}^1_2 - \right. \\ & \left. \mathbf{Y}^0_2 \Omega^3 \mathbf{U}^1_3 + \Omega^2 \mathbf{Y}^0_1 \mathbf{Y}^0_2 \mathbf{E}_1 + \Omega^2 \mathbf{Y}^0_2{}^2 \mathbf{E}_2 - \mathbf{U}^0_1 \Omega^4 - \mathbf{U}^0_1 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega^2 \right) \sin(\theta) + \\ & \left((2\mathbf{Y}^0_1 \mathbf{U}^0_2 - \mathbf{U}^0_1 \mathbf{Y}^0_2) \Omega^2 - 2\mathbf{Y}^0_1{}^2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \right) \mathbf{Y}^1_1 + \\ & \left(-\Omega^2 \mathbf{Y}^0_1 \mathbf{U}^0_1 - 2\mathbf{Y}^0_1 \mathbf{Y}^0_2 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \right) \mathbf{Y}^1_2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_1 + \mathbf{Y}^0_1{}^2 \mathbf{U}^1_2 \Omega^2 - \\ & \left. \Omega^3 \mathbf{Y}^0_2 \mathbf{E}_3 + \Omega^3 \mathbf{Y}^0_1 \mathbf{U}^0_3 + \mathbf{Y}^0_1 \mathbf{U}^0_3 (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega \right) / \Omega^4, \quad (\text{A.2}) \end{aligned}$$

$$\begin{aligned} \mathbf{V}^1_3(t; \mathbf{x}, \mathbf{v}, s) = & \left(\left(\Omega^3 \mathbf{U}^1_3 - \Omega^2 \mathbf{Y}^0_1 \mathbf{E}_1 - \Omega^2 \mathbf{Y}^0_2 \mathbf{E}_2 + (\mathbf{U}^0_1 \mathbf{Y}^0_2 - \mathbf{Y}^0_1 \mathbf{U}^0_2) \Omega^2 - \right. \right. \\ & \left. \left. (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{Y}^0_1 \mathbf{U}^0_2)^2 \right) \cos(\theta) + \left((\mathbf{U}^0_1 \Omega^2 - \mathbf{Y}^0_1 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)) \mathbf{Y}^1_1 + \right. \right. \\ & \left. \left. (\mathbf{U}^0_2 \Omega^2 - \mathbf{Y}^0_2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)) \mathbf{Y}^1_2 + \mathbf{Y}^0_2 \Omega^2 \mathbf{U}^1_2 + \Omega^3 \mathbf{E}_3 + \mathbf{Y}^0_1 \Omega^2 \mathbf{U}^1_1 \right) \sin(\theta) + \right. \\ & \left. \Omega^2 \mathbf{Y}^0_1 \mathbf{E}_1 + \Omega^2 \mathbf{Y}^0_2 \mathbf{E}_2 + (\mathbf{U}^0_1{}^2 + \mathbf{Y}^0_1 \mathbf{U}^0_2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2{}^2) \Omega^2 - (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)^2 \right) / \Omega^3, \quad (\text{A.3}) \end{aligned}$$

where $\mathbf{U}^1(t; \mathbf{x}, \mathbf{v}, s)$ is solution to:

$$\begin{aligned} \frac{d\mathbf{U}^1}{dt} = & 1/4 \left((-4 \mathbf{Y}^0_2 \Omega^8 \mathbf{Y}^0_1 \frac{\partial \mathbf{E}_2}{\partial \mathbf{x}_3} + 4 \mathbf{Y}^0_2{}^2 \Omega^8 \frac{\partial \mathbf{E}_1}{\partial \mathbf{x}_3}) \mathbf{Y}^1_3 + (-4 \Omega^9 \mathbf{Y}^0_2 + (-\mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 + 4 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 - 4 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2) \Omega^7 + \right. \\ & \left. (-\mathbf{Y}^0_1{}^4 \mathbf{U}^0_1 - 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 + 2 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_1 + 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 - \mathbf{U}^0_2 \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3 - 2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 \mathbf{Y}^0_2 - 4 \mathbf{U}^0_2 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1) \Omega^5 + \mathbf{Y}^0_1 (3 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_2{}^4 + \mathbf{Y}^0_1{}^4 + \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^3 \right) \mathbf{E}_3 + \left(4 \Omega^8 \mathbf{Y}^0_2 \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^3 \Omega^6 \mathbf{U}^0_3 + \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^2 (-\mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 + \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 + \mathbf{U}^0_3 \mathbf{Y}^0_1 \mathbf{Y}^0_2) \right. \\ & \left. \Omega^4 + \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^2 \right) \mathbf{U}^1_1 + (-\Omega^7 \mathbf{Y}^0_1 \mathbf{U}^0_3 + \mathbf{Y}^0_1 (8 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 - \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 - \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 + 5 \mathbf{U}^0_3 \mathbf{Y}^0_1 \mathbf{Y}^0_2 + 2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3) \Omega^5 - \mathbf{Y}^0_1{}^2 (8 \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1 + 7 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 6 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 + 7 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_1 - \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2) \Omega^3) \mathbf{Y}^1_2 + \left(\mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^5 \mathbf{U}^0_2{}^3 - 8 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^7 + 7 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^7 + \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^5 \mathbf{U}^0_1{}^3 + 6 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^7 \mathbf{U}^0_2{}^2 + \mathbf{Y}^0_1{}^6 \mathbf{U}^0_1{}^3 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_2{}^4 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^3 + 5 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 + \mathbf{U}^0_3 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_1{}^7 - 5 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 + 3 \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_1{}^2 + 19 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_2{}^2 + 9 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^6 \mathbf{Y}^0_2 - 9 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^6 \mathbf{Y}^0_1 + \mathbf{U}^0_2{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2{}^3 - 16 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^4 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^2 + 3 \mathbf{Y}^0_2{}^4 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^2 \mathbf{U}^0_1 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^2 \mathbf{U}^0_1 + \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 + 3 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2{}^2 - 15 \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 + 3 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2 \mathbf{U}^0_1{}^2 - 5 \mathbf{Y}^0_2{}^6 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1 + 10 \mathbf{Y}^0_2{}^5 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^2 + 8 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^6 - 6 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{U}^0_1 + 19 \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{U}^0_1 + 7 \mathbf{Y}^0_1{}^6 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{U}^0_1 + 3 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{U}^0_2 + 16 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{U}^0_2 \right) \Omega + (-4 \Omega^8 \mathbf{U}^0_2 \mathbf{Y}^0_1 - \mathbf{Y}^0_1{}^2 \Omega^6 \mathbf{Y}^0_2 \mathbf{U}^0_3 + \mathbf{Y}^0_1 (\mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 - \mathbf{U}^0_2 \mathbf{Y}^0_2{}^4 + \mathbf{U}^0_2 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 + \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 + \mathbf{U}^0_2 \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3 + \mathbf{Y}^0_1{}^4 \mathbf{U}^0_1) \Omega^4 - \mathbf{Y}^0_1{}^5 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^2) \mathbf{U}^1_2 + \left(6 \Omega^6 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{E}_1 - 2 \mathbf{Y}^0_1 (-\mathbf{Y}^0_2{}^2 + 2 \mathbf{Y}^0_1{}^2) \Omega^6 \mathbf{E}_2 + 4 \Omega^8 \mathbf{Y}^0_2{}^2 \frac{\partial \mathbf{E}_1}{\partial \mathbf{x}_2} - 4 \Omega^8 \mathbf{Y}^0_1 \frac{\partial \mathbf{E}_2}{\partial \mathbf{x}_2} \mathbf{Y}^0_2 - \Omega^8 \mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_1 (-\mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{U}^0_2 + 3 \mathbf{Y}^0_2 \mathbf{U}^0_3{}^2 + \mathbf{U}^0_1 \mathbf{Y}^0_1 \mathbf{U}^0_2 + 12 \mathbf{Y}^0_2 \mathbf{U}^0_2{}^2 + 2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 - \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 4 \mathbf{Y}^0_2 \mathbf{U}^0_1{}^2 + 3 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_1 \mathbf{U}^0_3{}^2) \Omega^6 + (-\mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2{}^2 + 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2 + 10 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 \mathbf{U}^0_2 + \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3 - 3 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2{}^2 - 8 \mathbf{Y}^0_2 \mathbf{U}^0_2{}^3 \mathbf{Y}^0_1{}^3 - \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 - 3 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1{}^4 + \mathbf{U}^0_2 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 + 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1{}^2 - 9 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^2 - 2 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 2 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 \mathbf{U}^0_1{}^2 - 4 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2 - 6 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 - 8 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^4 - \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 \mathbf{Y}^0_1) \Omega^4 - \mathbf{Y}^0_2 (-8 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^5 - 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2{}^2 - 7 \mathbf{Y}^0_2 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1{}^4 - \mathbf{Y}^0_2{}^4 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1 + 9 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2{}^2 - 2 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^2 - 4 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 \mathbf{U}^0_1{}^2 - 9 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_1 - 14 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^2 - 9 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_2 - 7 \mathbf{Y}^0_1{}^5 \mathbf{U}^0_3{}^2 + 11 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2{}^2 + 12 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_1 + \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 + 7 \mathbf{U}^0_1 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^5 - 14 \mathbf{Y}^0_1 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2 + \mathbf{U}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^2 + 2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 + \mathbf{U}^0_2 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^2 + 10 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^5 + 8 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 \mathbf{U}^0_2{}^2 - 6 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1{}^2) \Omega^2 + 2 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \mathbf{Y}^1_2 + (-6 \mathbf{Y}^0_2{}^5 \mathbf{U}^0_1{}^3 + \mathbf{Y}^0_1{}^4 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^2 - 2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^5 + 5 \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_2{}^3 - 5 \mathbf{U}^0_3 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_1{}^5 - \mathbf{U}^0_1 \mathbf{Y}^0_1{}^6 \mathbf{U}^0_3 + 8 \mathbf{Y}^0_2{}^5 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 - \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 \mathbf{U}^0_2{}^2 + \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2{}^3 - 4 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2{}^2 - 6 \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2{}^2 + 12 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1{}^2 - 5 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_3 \mathbf{Y}^0_2 \mathbf{U}^0_1{}^2 - \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^4 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 - 19 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1{}^2 \mathbf{U}^0_2{}^3 \mathbf{U}^0_3 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 4 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^5 + 14 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_1 + \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2 - 4 \mathbf{Y}^0_1{}^5 \mathbf{U}^0_2 \mathbf{U}^0_1 - 6 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 + 9 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^2 - 7 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 + 12 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_1 - 18 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_2{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 + 18 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2 \mathbf{U}^0_1{}^2 + \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2 \mathbf{U}^0_3 - 5 \mathbf{Y}^0_1{}^5 \mathbf{Y}^0_2 \mathbf{U}^0_1 \mathbf{U}^0_3 - 6 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 \mathbf{U}^0_3 - 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^5 \mathbf{Y}^0_1 \mathbf{U}^0_3 - 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_1 \mathbf{U}^0_3 + 5 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_3 + 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_3 - 2 \mathbf{Y}^0_1{}^4 \mathbf{Y}^0_2 \mathbf{U}^0_1 \mathbf{U}^0_3 + 2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_2 - 4 \mathbf{U}^0_1 \mathbf{U}^0_2 \mathbf{Y}^0_2{}^4 \mathbf{U}^0_3 \mathbf{Y}^0_1 - 17 \mathbf{U}^0_1 \mathbf{U}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^3 + 4 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^6 \mathbf{U}^0_3) \Omega^3 + \left(6 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2{}^4 + 6 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2{}^2 + \mathbf{Y}^0_1{}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 - \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 \mathbf{U}^0_1{}^2 - 6 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 + \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2 - 12 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 \mathbf{U}^0_2 + 6 \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3 - 4 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 + \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 5 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 + 11 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 2 \mathbf{U}^0_1 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 - 3 \mathbf{U}^0_3 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 \mathbf{U}^0_2 - \mathbf{Y}^0_2 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 \mathbf{U}^0_2{}^2 - \mathbf{U}^0_3 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_2{}^2 - 2 \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 - 4 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 \mathbf{Y}^0_1{}^4 - 5 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1{}^3 \mathbf{Y}^0_2 + 2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2 - 2 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 + 2 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 + 6 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 - 4 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 \mathbf{U}^0_3 \mathbf{U}^0_2 \right) \Omega^5 + \left(4 \Omega^7 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_1 + \mathbf{Y}^0_2 (6 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 - 5 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 - 6 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 + 6 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1 - 6 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 - 5 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3) \Omega^5 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 (3 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 \mathbf{Y}^0_2 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 9 \mathbf{Y}^0_2{}^2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 - \mathbf{U}^0_2 \mathbf{Y}^0_2{}^3 \mathbf{Y}^0_1 + 5 \mathbf{Y}^0_1{}^4 \mathbf{U}^0_3 - \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^3 + 3 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^4 + 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3) \Omega^3 \right) \mathbf{E}_1 + \left((\mathbf{Y}^0_1 (-2 \mathbf{U}^0_3 + \mathbf{U}^0_2) \Omega^7 + \mathbf{Y}^0_1 (-\mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 8 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 + 4 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 - \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 - 2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 + 9 \mathbf{U}^0_3 \mathbf{Y}^0_1 \mathbf{Y}^0_2) \Omega^5 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 (7 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 - \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_1 + 6 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3 + 8 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 + 9 \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1) \Omega^3 \right) \mathbf{Y}^1_2 + \left(\mathbf{Y}^0_1{}^2 \Omega^7 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \Omega^5 \right) \mathbf{U}^1_1 + \left(\mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^7 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \Omega^5 \right) \mathbf{U}^1_2 + \left(-\mathbf{Y}^0_1 (\mathbf{Y}^0_2 + 2 \mathbf{Y}^0_1) \Omega^7 + \mathbf{Y}^0_1{}^2 (\mathbf{Y}^0_1 \mathbf{Y}^0_2 + 2 \mathbf{Y}^0_1{}^2 + 3 \right. \end{aligned}$$

$$\begin{aligned}
& \mathbf{U}^0_1 \mathbf{U}^0_2{}^2 + \mathbf{U}^0_3 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3{}^2 - \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 - 4 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1 - 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_3 \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^2 \mathbf{U}^0_1{}^2 \mathbf{U}^0_3 \\
& \mathbf{Y}^0_2 - 2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_1 \mathbf{Y}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3{}^2 + 2 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1 + 3 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1 - 4 \mathbf{Y}^0_2 \mathbf{U}^0_1 \mathbf{U}^0_3{}^2 \\
& \mathbf{Y}^0_1{}^2 + 2 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2 \mathbf{U}^0_1{}^2 - 2 \mathbf{Y}^0_1 \mathbf{U}^0_1{}^3 \mathbf{Y}^0_2{}^2 + 4 \mathbf{U}^0_2 \mathbf{U}^0_3{}^2 \mathbf{Y}^0_1{}^3) \Omega^3 + (-\mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \mathbf{U}^0_3 \Omega^5 + \mathbf{Y}^0_2 (-2 \mathbf{Y}^0_1{}^2 \\
& \mathbf{Y}^0_2 \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^3 \mathbf{U}^0_3 - 2 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_1 + 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1 + 3 \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 - \mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{Y}^0_2{}^2 + 2 \mathbf{U}^0_2 \mathbf{Y}^0_1{}^3 + \mathbf{Y}^0_2{}^3 \mathbf{U}^0_3) \\
& \Omega^3) \mathbf{E}_3 + ((\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^5 - \mathbf{Y}^0_2{}^2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^3) \mathbf{Y}^1_2{}^2 + (-\Omega^6 \mathbf{Y}^0_1 \mathbf{U}^0_3 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \\
& \mathbf{U}^0_3 \Omega^4) \mathbf{U}^1_3 + (-2 \mathbf{Y}^0_1 \mathbf{U}^0_1 - 2 \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^7 \frac{\partial \mathbf{E}_3}{\partial \mathbf{x}_3} + (\mathbf{Y}^0_1{}^2 (-\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \mathbf{U}^0_3 \Omega^4 - \mathbf{Y}^0_1{}^2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (-\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \\
& \mathbf{U}^0_3 \Omega^2) \mathbf{U}^1_1 + (-4 \Omega^7 \mathbf{Y}^0_2 + (2 \mathbf{Y}^0_1{}^2 \mathbf{U}^0_3 + 12 \mathbf{U}^0_1 \mathbf{Y}^0_2{}^2 - 12 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1) \Omega^5 - 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \\
& \mathbf{U}^0_3 + \mathbf{U}^0_2 \mathbf{Y}^0_1 - \mathbf{U}^0_1 \mathbf{Y}^0_2) \Omega^3 + 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_1{}^3 + 2 \mathbf{Y}^0_2{}^3 + \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^2) (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) \Omega) \mathbf{E}_1 + (2 \mathbf{Y}^0_2 \\
& \mathbf{U}^0_2 + 2 \mathbf{Y}^0_1 \mathbf{U}^0_1) \Omega^7 + (\mathbf{Y}^0_1{}^2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \mathbf{U}^0_3 \Omega^4 - \mathbf{Y}^0_1 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (\mathbf{Y}^0_2{}^3 - \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_2 \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^3) \mathbf{U}^0_3 \Omega^2) \\
& \mathbf{U}^1_2 + ((\mathbf{Y}^0_2 \mathbf{U}^0_2 + 4 \mathbf{Y}^0_1 \mathbf{U}^0_1 + 2 \mathbf{U}^0_1 \mathbf{Y}^0_2) \Omega^5 + (-4 \mathbf{Y}^0_1{}^3 \mathbf{U}^0_1 - \mathbf{Y}^0_2{}^3 \mathbf{U}^0_1 - 2 \mathbf{Y}^0_2{}^3 \mathbf{U}^0_2 - 6 \mathbf{Y}^0_1 \mathbf{Y}^0_2{}^2 \mathbf{U}^0_1 + \mathbf{Y}^0_1 \\
& \mathbf{Y}^0_2{}^2 \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_1 - \mathbf{Y}^0_1{}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^3 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_1{}^2 + 2 \mathbf{Y}^0_1 \mathbf{Y}^0_2 - \mathbf{Y}^0_2{}^2) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega) \\
& \mathbf{Y}^1_1{}^2) / \Omega^8,
\end{aligned}$$

$$\mathbf{U}^1(s; \mathbf{x}, \mathbf{v}, s) = 0.$$

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