Application of the averaging method to the gyrokinetic plasma
Long version

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Abstract: we show that the solution to an oscillatory-singularly perturbed ordinary differential equation may be asymptotically expanded into a sum of oscillating terms. Each of those terms writes as an oscillating operator acting on the solution to a non oscillating ordinary differential equation with an oscillating correction added to it.
The expression of the non oscillating ordinary differential equations are defined by a recurrence relation.
We then apply this result to problems where charged particles are submitted to large magnetic field.

1 Introduction and results

Purpose
The goal of this paper is to manage and justify the asymptotic two scale expansion as $\varepsilon \to 0$ of the solution $\mathbf{X}_\varepsilon(t) = \mathbf{X}_\varepsilon(t; \mathbf{x}, s)$ to the following singularly perturbed dynamical system:

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{a}(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{b}(t, \mathbf{X}_\varepsilon), \quad \mathbf{X}_\varepsilon(s; \mathbf{x}, s) = \mathbf{x}, \tag{1.1}$$

and to apply this to models of charge particles submitted to a strong magnetic field which is possibly non uniform.

More precisely, C_b^k standing for the space of functions that have continuous and bounded derivatives until the order k, we assume that

$$\mathbf{a}(\cdot,\cdot,\cdot) \in (C_b^{k+1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d, \ \theta \mapsto \mathbf{a}(t,\theta,\mathbf{x}) \text{ is } 2\pi\text{-periodic for every } t \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^d, \tag{1.2}$$

$$\mathbf{b}(\cdot,\cdot) \in (C_b^{k+2}(\mathbb{R}\times\mathbb{R}^d))^d, \tag{1.3}$$

for some $k \geq 0$. In particular, it implies that $|\nabla_x \cdot \mathbf{b}| \leq C$, uniformly on $\mathbb{R}\times\mathbb{R}^d$. We also suppose that the solution $\mathbf{Z}(t, \theta; \mathbf{z})$ to

$$\frac{\partial \mathbf{Z}}{\partial \theta} = \mathbf{b}(t, \mathbf{Z}), \quad \mathbf{Z}(t, 0; \mathbf{z}) = \mathbf{z}, \tag{1.4}$$

is known and 2π-periodic in θ for every $t \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^d$.

Under those assumptions, we prove that $\mathbf{X}_\varepsilon(\cdot; \mathbf{x}, s) : \mathbb{R} \to \mathbb{R}^d$ admits the following expansion:

$$\mathbf{X}_\varepsilon(t; \mathbf{x}, s) = \mathbf{X}^0(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \varepsilon\mathbf{X}^1(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \varepsilon^2\mathbf{X}^2(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \dots, \tag{1.5}$$

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as $\varepsilon \to 0$, where the $\mathbf{X}^i(t,\theta;\mathbf{x},s)$ are $2\pi-$periodic in θ. The initial condition in (1.1) is assigned to $\mathbf{X}^0$ by:

$$\mathbf{X}^0(s,0;\mathbf{x},s) = \mathbf{x} \text{ and } \mathbf{X}^i(s,0;\mathbf{x},s) = 0 \text{ for } i \geq 1. \tag{1.6}$$

Motivations

The target we have in mind while writing this paper is the conception of pic-methods for plasmas submitted to large magnetic field. Direct applications of this are the simulation of magnetic confinement fusion or isotope resonant separation experiments.

A plasma submitted to a strong magnetic field could be modelled by several Vlasov equations, according to experimental conditions. For instance if we are interested in describing the global behaviour of a plasma in a magnetic confinement fusion experiment, the following equation

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \Big((\mathbf{E}(t,\mathbf{x}) + \frac{\mathcal{N}}{\varepsilon}) + \mathbf{v}\times(\mathbf{B}(t,\mathbf{x}) + \frac{\mathcal{M}}{\varepsilon})\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.7}$$

for a small parameter ε would be a relevant model equation. In equation (1.7), $f^\varepsilon(t,\mathbf{x},\mathbf{v})$ is at time t, the particle density of the plasma in position $\mathbf{x}$ and with velocity $\mathbf{v}$. The vector $\mathcal{M}\in\mathbb{S}^2$ gives the direction of the strong magnetic field. This direction may depend on t and $\mathbf{x}$. the vector $\mathcal{N}\in\mathbb{S}^2$ with $\mathcal{N}\perp\mathcal{M}$ is the direction of a strong electric field ($\mathcal{N}$ may also be 0). The vector fields $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields that may contain a self consistent part. We shall call the scaling leading to this equation the "Guiding Centre Regime".

If now we are interested in understanding what happens close to the plasma boundary in a tokamak, we would prefer to use the so called "Finite Larmor Radius Regime". In other words, we would consider an equation of the type:

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon + \frac{\mathbf{v}_\perp}{\varepsilon}\cdot\nabla_x f^\varepsilon + \Big(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon}\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.8}$$

where the meaning of $f^\varepsilon(t,\mathbf{x},\mathbf{v})$ is the same as above and where $\mathbf{v}_\parallel = (\mathbf{v}\cdot\mathcal{M})\mathcal{M}$, $\mathbf{v}_\perp = \mathbf{v}-\mathbf{v}_\parallel$.

Lastly, for isotope resonant separation we would be interested in regarding

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \Big(\mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{x}) + \mathbf{v}\times(\mathbf{B}(t,\frac{t-s}{\varepsilon},\mathbf{x}) + \frac{\mathcal{M}}{\varepsilon})\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.9}$$

where the electric and magnetic fields $\mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{x})$ and $\mathbf{B}(t,\frac{t-s}{\varepsilon},\mathbf{x})$ oscillate with the same frequency $1/(2\pi\varepsilon)$ as the cyclotron frequency of the plasma particles.

We refer to Frénod and Sonnendrücker [10, 11, 12, 13, 14], Frénod, Raviart and Sonnendrücker [9] and Frénod and Watbled [15] where those models are explained and asymptotically analysed. In particular, they give indication on how to get the self consistent part of the electric field in some examples and under suitable assumptions. Complementary works concerning the asymptotic behaviour of those kind of equations are also led in Golse and Saint Raymond [16, 17], Saint Raymond [32], Brenier [4], Grenier [19, 20], Jabin [21], Schochet [35], Joly, Métivier and Rauch [24]. We also refer to mathematical or physical works where similar methods are used: [22, 23, 36, 7, 8, 27, 26, 6, 5, 28, 18].

It is relatively easy to see that equations (1.7), (1.8) and (1.9) enter the framework of a singularly perturbed convection equation reading:

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{a}\cdot\nabla u_\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u_\varepsilon = 0, \quad u_{\varepsilon\,|t=s} = u_0, \tag{1.10}$$

for $\mathbf{x}\in\mathbb{R}^d$ and $t>s$. At least formally, the solution $u_\varepsilon(t,\mathbf{x})$ is given by

$$u_\varepsilon(t,\mathbf{x}) = u_0(\mathbf{X}_\varepsilon(s;\mathbf{x},t)). \tag{1.11}$$

Numerical methods to solve equation (1.7), (1.8) or (1.9), or similar ones, will be applications of generic methods solving (1.10). Since u_ε contains high frequency oscillations, using in state any well known numerical method to solve (1.10) would compel to use a very small time step.

Yet, in order to relax this constraint, we may follow, at least, two strategies. The first one could be based on the work presented in Frénod, Raviart and Sonnendrücker [9] where we proved that u_ε writes $u_\varepsilon(t,\mathbf{x}) = \sum_{i\geq 0} \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, and where we determined the equations satisfied by every terms U^i. Since those equations are independent of ε, the computation of the first terms U^i could be led using a standard numerical method. Then an approximation of u_ε would be given by: $u_\varepsilon(t,\mathbf{x}) \simeq \sum_{i=0}^k \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, for some $k\in\mathbb{N}$.

The second strategy consists in using an approximation $\mathbf{X}_\varepsilon(t;\mathbf{x},s) \simeq \sum_{i=0}^k \varepsilon^i \mathbf{X}^i(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)$, of (1.5), for some $k\in\mathbb{N}$, after computing $\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^k$. Then the expression (1.11) yields an approximation of u_ε: $u_\varepsilon(t,\mathbf{x}) \simeq u_0\big(\sum_{i=0}^k \varepsilon^i \mathbf{X}^i(s, -\frac{t-s}{\varepsilon}; \mathbf{x}, t)\big)$. This motivates the present work.

We give Now some references concerning perturbed ordinary differential equations. First, we think to the Lindstedt-Poincaré method explained in Poincaré [31] where the steady state periodic solutions to a perturbed second order ordinary differential equation is studied. Then the Krylov-Bogoliubov-Mitropolsky method, see [3] and [25], allows to discribe the transitory behaviour of the solution to a perturbed ordinary differential equation to a periodic solution.

We also cite the works of Verhulst [42] and Sanders and Verhulst [34] (see also Mickens [29]) where the Method of Averaging is developed to treat perturbed ordinary differential equations and adiabatic invariants in Hamiltonian systems.

Lastly, we mention the work initiated by Tikhonov [38, 39, 40] and developed by Vasilieva and others (see the review paper of Vasilieva [41] and the references in it).

Here, in order to characterise and justify the two scale expansion (1.5), we revisit the single-phase averaging in a framework adapted to our motivations. For this, we use classical methods for ordinary differential equations, and homogenisation methods based on weak$-*$ convergence and two scale convergence. The most important ideas about those tools may be found in Tartar [37], Bensoussan, Lions and Papanicolaou [2], Sanchez-Palencia [33], N'Guetseng [30], Allaire [1] and Frénod, Raviart and Sonnendrücker [9].

Theorems

The main results are the following. First, we give the characterisation of the terms of the expansion.

THEOREM 1.1 *Setting*

$$\tilde{\mathbf{a}}^0(t,\mathbf{y}^0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\alpha}^0(t,\theta,\mathbf{y}^0)\,d\theta, \tag{1.12}$$

with

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0) = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}^{-1}\big\{\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0)) - \frac{\partial \mathbf{Z}}{\partial t}(t,\theta;\mathbf{y}^0)\big\}, \tag{1.13}$$

under assumptions (1.2), (1.3) for $k=0$ and (1.4) the first term $\mathbf{X}^0$ of (1.5) satisfies

$$\mathbf{X}^0(t,\theta;\mathbf{x},s) = \mathbf{Z}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s)), \tag{1.14}$$

where $\mathbf{Y}^0$ is the solution to

$$\frac{d\mathbf{Y}^0}{dt} = \tilde{\mathbf{a}}^0(t,\mathbf{Y}^0), \quad \mathbf{Y}^0(s;\mathbf{x},s) = \mathbf{x}. \tag{1.15}$$

In the Theorem above, $\nabla_z \mathbf{Z}(t,\theta;\mathbf{z})$ stands for the Jacobian matrix of $\mathbf{z}\mapsto \mathbf{Z}(t,\theta;\mathbf{z})$.

In order to set the results for next terms of the asymptotic expansion, we need to introduce some notations. For a vector field $\mathbf{b}$, the $i-$th component of $\{\nabla_x^k \mathbf{b}\}\{\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^k\}$, for $i = 1, \dots, d$, is given by:

$$\big(\{\nabla_x^k \mathbf{b}\}\{\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^k\}\big)_i = \sum_{l_1,\dots,l_k=1}^{d} \frac{\partial^k \mathbf{b}_i}{\partial x_{l_1} \dots \partial x_{l_k}} \mathbf{x}^0_{l_1} \dots \mathbf{x}^k_{l_k}. \tag{1.16}$$

In order to simplify we shall sometimes denote $\{\nabla_x^k \mathbf{b}\}\{\mathbf{x}^0, \mathbf{x}^0, \dots, \mathbf{x}^0\}$ by $\{\nabla_x^k \mathbf{b}\}\{\mathbf{x}^0\}^k$.

Now for $k \geq 0$ we recursively define

$$\tilde{\mathbf{A}}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) = \frac{1}{\theta}\int_0^\theta \tilde{\alpha}^k(t,\sigma,\mathbf{y}^0,\dots,\mathbf{y}^k)\,d\sigma - \tilde{\mathbf{a}}^k(t,\mathbf{y}^0,\dots,\mathbf{y}^k), \tag{1.17}$$

where $\tilde{\alpha}^0$ and $\tilde{\mathbf{a}}^0$ are given by (1.13) and (1.12) and where for $k \geq 1$ we have:

$$\begin{aligned}\tilde{\alpha}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) = &\big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}^{-1}\Big\{\overline{S^k}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) - \{\frac{\partial \nabla_z \mathbf{Z}}{\partial t}(t,\theta;\mathbf{y}^0)\}\\ &\{\mathbf{y}^k + \theta\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\} - \{\nabla_z^2 \mathbf{Z}(t,\theta;\mathbf{y}^0)\}\{\tilde{\mathbf{a}}^0(t,\mathbf{y}^0), \mathbf{y}^k + \theta\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\}\Big\} -\\ &\theta\Big(\sum_{j=0}^{k-1}\{\nabla_{y^j}\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\}\{\tilde{\mathbf{a}}^j((t,\mathbf{y}^0,\dots,\mathbf{y}^j)\} + \frac{\partial \tilde{\mathbf{A}}^{k-1}}{\partial t}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\Big),\end{aligned} \tag{1.18}$$

and

$$\tilde{\mathbf{a}}^k(t,\mathbf{y}^0,\dots,\mathbf{y}^k) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\alpha}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k)\,d\theta. \tag{1.19}$$

In formula (1.18) the $\overline{S^k}$ are given by

$$\overline{S^1}(t,\theta,\mathbf{y}^0,\mathbf{y}^1) = \{\nabla_x \mathbf{a}\}\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\} + \frac{1}{2}\{\nabla_x^2 \mathbf{b}\}\Big\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}\Big\}^2, \tag{1.20}$$

$$\begin{aligned}\overline{S^2}(t,\theta,\mathbf{y}^0,\mathbf{y}^1,\mathbf{y}^2) = &\;\{\nabla_x \mathbf{a}\}\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^2 + \theta\tilde{\mathbf{A}}^1\} + \frac{1}{2}\{\nabla_x^2 \mathbf{a}\}\Big\{\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{y}^0)\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}\Big\}^2 +\\ &\{\nabla_x^2 \mathbf{b}\}\Big\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}, \{\nabla_z \mathbf{Z}(t,\theta;\mathbf{y}^0)\}\{\mathbf{y}^2 + \theta\tilde{\mathbf{A}}^1\}\Big\} + \frac{1}{6}\{\nabla_x^3 \mathbf{b}\}\Big\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}\Big\}^3,\end{aligned} \tag{1.21}$$

and for $k \geq 3$,

$$\begin{aligned}\overline{S^k}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) = &\{\nabla_x \mathbf{a}\}\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^k + \theta\tilde{\mathbf{A}}^{k-1}\} + \frac{1}{2}\{\nabla_x^2 \mathbf{a}\}\Big[\sum_{j=1}^{k-1}\Big\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^j + \theta\tilde{\mathbf{A}}^{j-1}\},\\ &\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^{k-j} + \theta\tilde{\mathbf{A}}^{k-j-1}\}\Big\}\Big] + \dots + \frac{1}{k!}\{\nabla_x^k \mathbf{a}\}\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}\}^k + \frac{1}{2}\{\nabla_x^2 \mathbf{b}\}\Big[\sum_{j=1}^{k}\Big\{\{\nabla_z \mathbf{Z}\}\\ &\{\mathbf{y}^j + \theta\tilde{\mathbf{A}}^{j-1}\}, \{\nabla_z \mathbf{Z}\}\{\mathbf{y}^{k+1-j} + \theta\tilde{\mathbf{A}}^{k-j}\}\Big\}\Big] + \dots + \frac{1}{(k+1)!}\{\nabla_x^{k+1} \mathbf{b}\}\{\{\nabla_z \mathbf{Z}\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0\}\}^{k+1},\end{aligned} \tag{1.22}$$

where $\mathbf{Z}$ is evaluated in $(t,\theta;\mathbf{y}^0)$, $\mathbf{a}$ in $(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0))$, $\mathbf{b}$ in $(t,\mathbf{Z}(t,\theta;\mathbf{y}^0))$ and $\tilde{\mathbf{A}}^i$ in $(t,\theta,\mathbf{y}^0,\dots \mathbf{y}^i)$.

We also set:

$$\overline{S^0}(t,\theta,\mathbf{y}^0) = \mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0)). \tag{1.23}$$

Once those notations introduced, we can state the following Theorem.

THEOREM 1.2 *Under assumptions (1.2), (1.3) and (1.4), the term* $\mathbf{X}^k$ *of (1.5) satisfies*

$$\mathbf{X}^k(t,\theta;\mathbf{x},s) = \{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\}\Big\{\mathbf{Y}^k(t;\mathbf{x},s) + \theta\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^{k-1}(t;\mathbf{x},s))\Big\}, \tag{1.24}$$

and $\mathbf{Y}^k$ *is the solution to*

$$\frac{d\mathbf{Y}^k}{dt} = \tilde{\mathbf{a}}^k(t,\mathbf{Y}^0,\dots,\mathbf{Y}^k), \quad \mathbf{Y}^0(s;\mathbf{x},s) = 0. \tag{1.25}$$

Now we give the first approximation result justifying the expansion (1.5) until the first term $\mathbf{X}^0$.

THEOREM 1.3 *Under assumptions (1.2), (1.3) with $k=0$ and (1.4), for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$, $T\in\mathbb{R}$ and any $\varepsilon>0$, the solution $\mathbf{X}_\varepsilon(\cdot;\mathbf{x},s)$ of (1.1) exists on $[s,s+T]$, is unique and the sequence $(\mathbf{X}_\varepsilon(\cdot;\mathbf{x},s))$ satisfies:*

$$\lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\left|\mathbf{X}_\varepsilon(t;\mathbf{x},s)-\mathbf{X}^0(t,\frac{t-s}{\varepsilon};\mathbf{x},s)\right|=0, \tag{1.26}$$

$|\cdot|$ standing for the Euclidean norm on $\mathbb{R}^d$, where $\mathbf{X}^0$ is characterised by Theorem 1.1.

In order to justify the expansion (1.5) for higher orders, we set $\mathbf{X}^0_\varepsilon(t;\mathbf{x},s)=\mathbf{X}_\varepsilon(t;\mathbf{x},s)$ and $\mathbf{Y}^0_\varepsilon$ which is such that

$$\mathbf{X}^0_\varepsilon(t;\mathbf{x},s)=\mathbf{Z}(t,\frac{t-s}{\varepsilon};\mathbf{Y}^0_\varepsilon(t,\mathbf{x},s))=[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon, \tag{1.27}$$

where for any function f we write $[f(\mathbf{Y}^0_\varepsilon,\dots,\mathbf{Y}^k_\varepsilon)]_\varepsilon$ for $f(t,\frac{t-s}{\varepsilon},\mathbf{Y}^0_\varepsilon(t;\mathbf{x},s),\dots,\mathbf{Y}^k_\varepsilon(t;\mathbf{x},s))$ and $[f(\mathbf{Y}^0,\dots,\mathbf{Y}^k)]_\varepsilon$ for $f(t,\frac{t-s}{\varepsilon},\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^k(t;\mathbf{x},s))$.

We then define the sequence $(\mathbf{X}^k_\varepsilon(t;\mathbf{x},s))$ by:

$$\mathbf{X}^k_\varepsilon=\frac{1}{\varepsilon^k}\big(\mathbf{X}_\varepsilon-[\mathbf{X}^0]_\varepsilon-\dots-\varepsilon^{k-1}[\mathbf{X}^{k-1}]_\varepsilon\big)=\frac{1}{\varepsilon}\big(\mathbf{X}^{k-1}_\varepsilon-[\mathbf{X}^{k-1}]_\varepsilon\big), \tag{1.28}$$

for $k\geq 1$. We also define the sequence $(\mathbf{Y}^k_\varepsilon(t;\mathbf{x},s))$ by:

$$\mathbf{Y}^1_\varepsilon=\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}^{-1}_\varepsilon\big\{\frac{1}{\varepsilon}\big([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big)\big\}-\frac{t-s}{\varepsilon}[\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon \tag{1.29}$$

and

$$\mathbf{Y}^k_\varepsilon=\frac{1}{\varepsilon}\big(\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}\big)-\frac{t-s}{\varepsilon}[\tilde{\mathbf{A}}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon, \tag{1.30}$$

for $k\geq 2$. We easily see that, for $k\geq 1$, we have

$$\mathbf{X}^k_\varepsilon=\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^k_\varepsilon+\frac{t-s}{\varepsilon}[\tilde{\mathbf{A}}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon\big\}. \tag{1.31}$$

And then we can state the last main Theorem:

THEOREM 1.4 *Under assumptions (1.2), (1.3) and (1.4) for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$ and $T\in\mathbb{R}$, the sequences $(\mathbf{X}^k_\varepsilon(\cdot;\mathbf{x},s))$ and $(\mathbf{Y}^k_\varepsilon(\cdot;\mathbf{x},s))$ are bounded in $L^\infty([s,s+T])$ and we have for any $k\geq 1$:*

$$\lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\left|\mathbf{X}^k_\varepsilon(t;\mathbf{x},s)-\mathbf{X}^k(t,\frac{t-s}{\varepsilon};\mathbf{x},s)\right|=0,\ \textit{and}\ \lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\left|\mathbf{Y}^k_\varepsilon(t;\mathbf{x},s)-\mathbf{Y}^k(t;\mathbf{x},s)\right|=0, \tag{1.32}$$

where $\mathbf{X}^k$ and $\mathbf{Y}^k$ are given by Theorem 1.2

The paper is organised as follow: In the second section we characterise the terms of the expansion and we prove Theorems 1.1 and 1.2. Then we prove the approximation results. The last section is devoted to applications to models describing magnetic confinement fusion and isotope resonant separation experiments. Among those applications, one involves a non uniform strong magnetic field, in this case the computations are led using Maple.

REMARK 1.1 With very little changes we can apply the previous results to the case when $\mathbf{b}\equiv\mathbf{b}(t,\theta,\mathbf{x})$ also depends on θ, as soon as the regularity of $\mathbf{b}$ is enough and the assumption (1.4) is realized.

2 Asymptotic expansion

In this section, we assume that the asymptotic expansion (1.5) of the solution $\mathbf{X}_\varepsilon$ to (1.1) holds true and we deduce the form of each $\mathbf{X}^i$. This yields Theorems 1.1 and 1.2.

2.1 Equation for each order

Expanding $\mathbf{a}(\mathbf{X}_\varepsilon)=\mathbf{a}(t,\theta,\mathbf{X}_\varepsilon)$, using $\mathbf{X}_\varepsilon=\mathbf{X}^0+\sum_{j\geq 1}\varepsilon^j\mathbf{X}^j$, we obtain

$$\begin{aligned}\mathbf{a}(\mathbf{X}_\varepsilon)=\mathbf{a}(\mathbf{X}^0)+\varepsilon\{\nabla_x\mathbf{a}(\mathbf{X}^0)\}\mathbf{X}^1+\varepsilon^2\big(\{\nabla_x\mathbf{a}(\mathbf{X}^0)\}\mathbf{X}^2+\frac{1}{2}\{\nabla_x^2\mathbf{a}(\mathbf{X}^0)\}\{\mathbf{X}^1\}^2\big)+\dots+\\ \varepsilon^k\big(\{\nabla_x\mathbf{a}(\mathbf{X}^0)\}\mathbf{X}^k+\frac{1}{2}\{\nabla_x^2\mathbf{a}(\mathbf{X}^0)\}\big[\sum_{j=1}^{k-1}\{\mathbf{X}^j,\mathbf{X}^{k-j}\}\big]+\dots+\frac{1}{k!}\{\nabla_x^k\mathbf{a}(\mathbf{X}^0)\}\{\mathbf{X}^1\}^k\big)+\dots\end{aligned} \tag{2.1}$$

Making the same for $\mathbf{b}$ and then plugging the expansion (1.5) in the dynamical system (1.1) we get:

$$\begin{aligned}\frac{1}{\varepsilon}\Big(\frac{\partial\mathbf{X}^0}{\partial\theta}-\mathbf{b}(t,\mathbf{X}^0)\Big)+\frac{\partial\mathbf{X}^1}{\partial\theta}+\frac{\partial\mathbf{X}^0}{\partial t}-\{\nabla_x\mathbf{b}(t,\mathbf{X}^0)\}\mathbf{X}^1-\mathbf{a}(t,\frac{t-s}{\varepsilon},\mathbf{X}^0)+\\ \sum_{k\geq 2}\varepsilon^{k-1}\Big(\frac{\partial\mathbf{X}^k}{\partial\theta}+\frac{\partial\mathbf{X}^{k-1}}{\partial t}-\{\nabla_x\mathbf{b}(t,\mathbf{X}^0)\}\mathbf{X}^k-S^{k-1}(t,\frac{t-s}{\varepsilon},\mathbf{X}^0,\dots,\mathbf{X}^{k-1})\Big)=0,\end{aligned} \tag{2.2}$$

where

$$S^1(t,\theta,\mathbf{x}^0,\mathbf{x}^1)=\{\nabla_x\mathbf{a}(t,\theta,\mathbf{x}^0)\}\mathbf{x}^1+\frac{1}{2}\{\nabla_x^2\mathbf{b}(t,\mathbf{x}^0)\}\{\mathbf{x}^1\}^2, \tag{2.3}$$

$$\begin{aligned}S^2(t,\theta,\mathbf{x}^0,\mathbf{x}^1,\mathbf{x}^2)=\{\nabla_x\mathbf{a}(t,\theta,\mathbf{x}^0)\}\{\mathbf{x}^2\}+\\ \frac{1}{2}\{\nabla_x^2\mathbf{a}(t,\theta,\mathbf{x}^0)\}\{\mathbf{x}^1\}^2+\{\nabla_x^2\mathbf{b}(t,\mathbf{x}^0)\}\{\mathbf{x}^1,\mathbf{x}^2\}+\frac{1}{6}\{\nabla_x^3\mathbf{b}(t,\mathbf{x}^0)\}\{\mathbf{x}^1\}^3,\end{aligned} \tag{2.4}$$

and for $k\geq 3$,

$$\begin{aligned}S^k(t,\theta,\mathbf{x}^0,\dots,\mathbf{x}^k)=\{\nabla_x\mathbf{a}(t,\theta,\mathbf{x}^0)\}\{\mathbf{x}^k\}+\frac{1}{2}\{\nabla_x^2\mathbf{a}(t,\theta,\mathbf{x}^0)\}\sum_{j=1}^{k-1}\{\mathbf{x}^j,\mathbf{x}^{k-j}\}+\dots+\\ \frac{1}{k!}\{\nabla_x^k\mathbf{a}(t,\theta,\mathbf{x}^0)\}\{\mathbf{x}^1\}^k+\frac{1}{2}\{\nabla_x^2\mathbf{b}(t,\mathbf{x}^0)\}\sum_{j=1}^{k}\{\mathbf{x}^j,\mathbf{x}^{k+1-j}\}+\dots+\frac{1}{(k+1)!}\{\nabla_x^{k+1}\mathbf{b}(t,\mathbf{x}^0)\}\{\mathbf{x}^1\}^{k+1}.\end{aligned} \tag{2.5}$$

We also set

$$S^0(t,\theta,\mathbf{x}^0)=\mathbf{a}(t,\theta,\mathbf{x}^0). \tag{2.6}$$

Hence, identifying the terms of the same order in ε in equation (2.2), we have

$$\frac{\partial\mathbf{X}^0}{\partial\theta}=\mathbf{b}(t,\mathbf{X}^0), \tag{2.7}$$

at the order -1, and at the order $k-1$ for $k\geq 1$ we have

$$\frac{\partial\mathbf{X}^k}{\partial\theta}-\{\nabla_x\mathbf{b}(t,\mathbf{X}^0)\}\mathbf{X}^k=S^{k-1}(t,\theta,\mathbf{X}^0,\dots,\mathbf{X}^{k-1})-\frac{\partial\mathbf{X}^{k-1}}{\partial t}. \tag{2.8}$$

2.2 $\mathbf{X}^0$ – $\mathbf{Y}^0$ link

Equation (2.7) can be solved easily in the class of $2\pi-$periodic functions in θ. It gives $\mathbf{X}^0(t,\theta;\mathbf{x},s)=\mathbf{Z}(t,\theta;\mathbf{Y}^0(t,\mathbf{x},s))$, i.e. (1.14), where $\mathbf{Z}$ is defined by (1.4) and where $\mathbf{Y}^0(t;\mathbf{x},s)=\mathbf{X}^0(t,0;\mathbf{x},s)$. Taking now into account the initial data (1.6), we notice that $\mathbf{Y}^0(s;\mathbf{x},s)=\mathbf{X}^0(s,0;\mathbf{x},s)=\mathbf{x}$, which is the initial condition of equation (1.15).

2.3 First $\mathbf{X}^k$ – $\mathbf{Y}^k$ link

In order to treat equation (2.8) we first notice that since

$$\frac{\partial \nabla_z \mathbf{Z}}{\partial \theta}(t,\theta;\mathbf{z}) = \nabla_z\big[\mathbf{b}(t,\mathbf{Z}(t,\theta;\mathbf{z}))\big] = \{\nabla_x \mathbf{b}(t,\mathbf{Z}(t,\theta;\mathbf{z})\}\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{z})\},\quad \nabla_z \mathbf{Z}(t,0;\mathbf{z}) = I, \tag{2.9}$$

the Jacobian matrix $\nabla_z \mathbf{Z}(t,\theta;\mathbf{z})$ is the Wronskian matrix of

$$\frac{\partial \tilde{\mathbf{Z}}}{\partial \theta} = \{\nabla_x \mathbf{b}(t,\mathbf{Z}(t,\theta;\mathbf{z}))\}\tilde{\mathbf{Z}},\ \ \mathbf{Z}(t,0;\mathbf{z},\tilde{\mathbf{z}}) = \tilde{\mathbf{z}},\quad \text{i.e. } \tilde{\mathbf{Z}}(t,\theta;\mathbf{z},\tilde{\mathbf{z}}) = \{\nabla_z \mathbf{Z}(t,\theta;\mathbf{z})\}\tilde{\mathbf{z}}. \tag{2.10}$$

Applying this, we shall deduce the form of $\mathbf{X}^k$ for $k \geq 1$. Defining $\tilde{\mathbf{X}}^k$ by

$$\tilde{\mathbf{X}}^k(t,\theta;\mathbf{x},s) = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\big\}^{-1}\mathbf{X}^k(t,\theta;\mathbf{x},s), \tag{2.11}$$

we have $\big(\frac{\partial}{\partial\theta} - \{\nabla_x \mathbf{b}(t,\mathbf{Z}(t,\theta;\mathbf{Y}^0))\}\big)\big(\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\}\tilde{\mathbf{X}}^k\big) = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\big\}\big\{\frac{\partial \tilde{\mathbf{X}}^k}{\partial \theta}\big\}$. Hence the equation (2.8) satisfied by $\mathbf{X}^k$ gives

$$\frac{\partial \tilde{\mathbf{X}}^k}{\partial \theta} = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\big\}^{-1}\big\{S^{k-1}(t,\theta,\mathbf{X}^0,\dots,\mathbf{X}^{k-1}) - \frac{\partial \mathbf{X}^{k-1}}{\partial t}\big\}, \tag{2.12}$$

which may be integrated directly and which yields

$$\tilde{\mathbf{X}}^k(t,\theta;\mathbf{x},s) = \tilde{\mathbf{X}}^k(t,0;\mathbf{x},s) + \int_0^\theta \big\{\nabla_z \mathbf{Z}(t,\sigma;\mathbf{Y}^0)\big\}^{-1}\big\{S^{k-1}(t,\sigma,\mathbf{X}^0,\dots,\mathbf{X}^{k-1}) - \frac{\partial \mathbf{X}^{k-1}}{\partial t}\big\}d\sigma. \tag{2.13}$$

In view of (2.11) we have $\tilde{\mathbf{X}}^k(t,0;\mathbf{x},s) = \mathbf{X}^k(t,0;\mathbf{x},s)$. Then setting for $k \geq 1$,

$$\mathbf{Y}^k(t;\mathbf{x},s) = \mathbf{X}^k(t,0;\mathbf{x},s) = \tilde{\mathbf{X}}^k(t,0;\mathbf{x},s), \tag{2.14}$$

and for $k \geq 0$,

$$\begin{aligned} A^k(t,\theta,\mathbf{x},s) = \frac{1}{\theta}\int_0^\theta &\big\{\nabla_z \mathbf{Z}(t,\sigma;\mathbf{Y}^0(t;\mathbf{x},s))\big\}^{-1} \\ &\big\{S^k(t,\sigma,\mathbf{X}^0(t,\sigma;\mathbf{x},s),\dots,\mathbf{X}^k(t,\sigma;\mathbf{x},s)) - \frac{\partial \mathbf{X}^k}{\partial t}(t,\sigma;\mathbf{x},s)\big\}d\sigma, \end{aligned} \tag{2.15}$$

we obtain the form of $\mathbf{X}^k$ for $k \geq 1$:

$$\mathbf{X}^k(t,\theta;\mathbf{x},s) = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\big\}\big\{\mathbf{Y}^k(t;\mathbf{x},s) + \theta A^{k-1}(t,\theta,\mathbf{x},s)\big\}. \tag{2.16}$$

Now we need to express A^{k-1}. On this topic, we can prove the following result.

LEMMA 2.1 *for $k \geq 0$ we have*

$$A^k(t,\theta,\mathbf{x},s) = \frac{1}{\theta}\int_0^\theta \alpha^k(t,\sigma,\mathbf{x},s)d\sigma - \frac{\partial \mathbf{Y}^k}{\partial t}(t;\mathbf{x},s), \tag{2.17}$$

where $\alpha^0(t,\theta,\mathbf{x},s) = \tilde{\alpha}^0(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s))$ with $\tilde{\alpha}^0$ defined by (1.13) and where α^k is given by

$$\begin{aligned} \alpha^k(t,\theta,\mathbf{x},s) = &\big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\big\}^{-1}\big\{S^k(t,\theta,\mathbf{X}^0(t,\theta;\mathbf{x},s),\dots,\mathbf{X}^k(t,\theta;\mathbf{x},s))\big\} - \\ &\big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\big\}^{-1}\big(\big\{\frac{\partial \nabla_z \mathbf{Z}}{\partial t}(t,\theta;\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^k + \theta A^{k-1}(t,\theta,\mathbf{x},s)\big\} + \\ &\big\{\nabla_z^2 \mathbf{Z}(t,\theta;\mathbf{Y}^0)\big\}\big\{\frac{\partial \mathbf{Y}^0}{\partial t}, \mathbf{Y}^k + \theta A^{k-1}(t,\theta,\mathbf{x},s)\big\}\big) - \theta\frac{\partial A^{k-1}}{\partial t}(t,\theta,\mathbf{x},s), \end{aligned} \tag{2.18}$$

for $k \geq 1$.

Proof. Since $\mathbf{X}^0$ is given in terms of $\mathbf{Y}^0$ by (1.14), we have

$$\frac{\partial \mathbf{X}^0}{\partial t} = \frac{\partial \mathbf{Z}}{\partial t}(t,\theta;\mathbf{Y}^0) + \{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\}\{\frac{\partial \mathbf{Y}^0}{\partial t}\}. \tag{2.19}$$

Hence, replacing in (2.15) $\frac{\partial \mathbf{X}^0}{\partial t}$ by its expression we obtain

$$A^0(t,\theta,\mathbf{x},s) = \frac{1}{\theta}\int_0^\theta \{\nabla_z \mathbf{Z}(t,\sigma;\mathbf{Y}^0(t;\mathbf{x},s))\}^{-1} = \frac{1}{\theta}\int_0^\theta \tilde{\alpha}^0(t,\sigma,\mathbf{Y}^0(t;\mathbf{x},s))d\sigma - \frac{\partial \mathbf{Y}^0}{\partial t}(t;\mathbf{x},s), \tag{2.20}$$

proving the Lemma for $k=0$.

Now, for $k \geq 1$, $\mathbf{X}^k$ expresses in terms of $\mathbf{Y}^k$ via (2.16). Then

$$\frac{\partial \mathbf{X}^k}{\partial t} = \{\frac{\partial \nabla_z \mathbf{Z}}{\partial t}(t,\theta;\mathbf{Y}^0)\}\{\mathbf{Y}^k + \theta A^{k-1}(t,\theta,\mathbf{x},s)\} + \\ \{\nabla_z^2 \mathbf{Z}(t,\theta;\mathbf{Y}^0)\}\{\frac{\partial \mathbf{Y}^0}{\partial t}, \mathbf{Y}^k + \theta A^{k-1}(t,\theta,\mathbf{x},s)\} + \{\nabla_z \mathbf{Z}(t,\theta;\mathbf{Y}^0)\}\{\frac{\partial \mathbf{Y}^k}{\partial t} + \theta\frac{\partial A^{k-1}}{\partial t}(t,\theta,\mathbf{x},s)\}, \tag{2.21}$$

and using this expression in (2.15) gives the Lemma for $k \geq 1$. ■

2.4 First equation for $\mathbf{Y}^k$

We shall take into account the $2\pi-$periodicity of the $\mathbf{X}^k$ with respect to the variable θ.

LEMMA 2.2 *Equation (2.8) is solvable in the class of $2\pi-$periodic functions in θ if and only if*

$$\frac{\partial \mathbf{Y}^{k-1}}{\partial t}(t;\mathbf{x},s) = \frac{1}{2\pi}\int_0^{2\pi} \alpha^{k-1}(t,\theta,\mathbf{x},s)d\theta. \tag{2.22}$$

Proof. We have seen that the solution $\mathbf{X}^k$ to (2.8) has an expression given by (2.16). Since $\nabla_z \mathbf{Z}$ is $2\pi-$periodic in θ and $\mathbf{Y}^k$ does not depend on θ, this expression of $\mathbf{X}^k$ is $2\pi-$periodic in θ if and only if $(\theta+2\pi)A^{k-1}(t,\theta+2\pi,\mathbf{x},s) = \theta A^{k-1}(t,\theta,\mathbf{x},s)$, for every θ. Since A^{k-1} is given by (2.17) we finally obtain (2.22), ending the proof of the Lemma. ■

2.5 Proof of Theorem 1.1

We already proved formula (1.14) and initial data of (1.15). Now applying Lemma 2.2 for $k=1$, we obtain

$$\frac{\partial \mathbf{Y}^0}{\partial t}(t;\mathbf{x},s) = \frac{1}{2\pi}\int_0^{2\pi} \alpha^0(t,\theta,\mathbf{x},s)d\theta. \tag{2.23}$$

As we saw in Lemma 2.1 that $\alpha^0(t,\theta,\mathbf{x},s) = \tilde{\alpha}^0(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s))$ with $\tilde{\alpha}^0$ defined by (1.13), equation (2.23) directly yields dynamical system (1.15). This ends the proof of Theorem 1.1. ■

2.6 Final $\mathbf{X}^k$ – $\mathbf{Y}^k$ link

Since Theorem 1.1 is true we can replace $\partial \mathbf{Y}^0/\partial t$ by $\tilde{\mathbf{a}}^0(t,\mathbf{Y}^0)$ in the expression of α^k (2.18). Using the equation (2.22) we find for $\mathbf{Y}^k$ in the expression of A^k (2.17), we deduce

$$A^{k-1}(t,\theta,\mathbf{x},s) = \frac{1}{\theta}\int_0^\theta \alpha^{k-1}(t,\sigma,\mathbf{x},s)d\sigma - \frac{1}{2\pi}\int_0^{2\pi} \alpha^{k-1}(t,\sigma,\mathbf{x},s)d\sigma, \tag{2.24}$$

which may also be used in (2.18).

Hence, since $\alpha^0(t,\theta,\mathbf{x},s)=\tilde{\alpha}^0(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s))$ with $\tilde{\alpha}^0$ defined by (1.13), using a simple induction procedure involving (2.16), (2.17) and (2.18) we can show that

$$\alpha^k(t,\theta,\mathbf{x},s)=\tilde{\alpha}^k(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^k(t;\mathbf{x},s)), \tag{2.25}$$

$$A^k(t,\theta;\mathbf{x},s)=\tilde{\mathbf{A}}^k(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^k(t;\mathbf{x},s)), \tag{2.26}$$

for any $k\geq 0$, for functions $\tilde{\alpha}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k)$ and $\tilde{\mathbf{A}}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k)$ we now need to characterise.

In view of the definition of S^k and $\overline{S^k}$, it is not difficult to see that for any $k\geq 0$,

$$S^k(t,\theta,\mathbf{X}^0(t,\theta;\mathbf{x},s),\dots,\mathbf{X}^{k-1}(t,\theta;\mathbf{x},s))=\overline{S^k}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^k(t;\mathbf{x},s)). \tag{2.27}$$

It is not more difficult to see that for any $k\geq 0$,

$$\frac{\partial A^{k-1}}{\partial t}=\sum_{j=0}^{k-1}\{\nabla_{y^j}\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})\}\{\frac{d\mathbf{Y}^i}{dt}\}+\frac{\partial\tilde{\mathbf{A}}^{k-1}}{\partial t}(t,\theta,\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1}), \tag{2.28}$$

where, in view of (2.22), we can replace $\frac{d\mathbf{Y}^i}{dt}$ by $\tilde{\mathbf{a}}^i(t,\mathbf{Y}^0,\dots,\mathbf{Y}^i)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{\alpha}^i(t,\theta,\mathbf{Y}^0,\dots,\mathbf{Y}^i)\,d\theta$.

Hence, injecting those formula in (2.18) and using a second induction procedure, we then deduce that $\tilde{\alpha}^k$ and $\tilde{\mathbf{A}}^k$ appearing in (2.25) and (2.26) are nothing but the $\tilde{\alpha}^k$ and $\tilde{\mathbf{A}}^k$ given by (1.18) and (1.17) with $\tilde{\mathbf{a}}^k$ define by (1.19).

Hence, we proved that formula (1.24) holds true.

2.7 Final equation for $\mathbf{Y}^k$, proof of Theorem 1.2

Now, equation (2.22) gives that $\mathbf{Y}^k$ is the unique solution to (1.25), the initial condition being an easy consequence of (2.14) and (1.6). This ends the proof of Theorem 1.2. ■

3 Approximation results

The proof of the approximation results (Theorems 1.3 and 1.4) are essentially based on showing that for each k the sequence $(\mathbf{Y}^k_\varepsilon)$ converges towards $\mathbf{Y}^k$. As we have to manage three definitions: (1.27) of $\mathbf{Y}^0_\varepsilon$, (1.29) of $\mathbf{Y}^1_\varepsilon$ and (1.30) of $\mathbf{Y}^k_\varepsilon$ for $k\geq 2$, we share the proof into three steps. We first prove the order 0 approximation. Then we obtain the result for the order 1. After this, using an induction procedure, we get the order k approximation result for $k\geq 2$.

3.1 Order 0 approximation, proof of Theorem 1.3

The first thing we have to do is to prove that $\mathbf{X}_\varepsilon$ well exists on a time interval not depending on ε. For this purpose we consider $\mathbf{Y}^0_\varepsilon=\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)$ solution to

$$\frac{d\mathbf{Y}^0_\varepsilon}{dt}=\tilde{\alpha}^0(t,\frac{t-s}{\varepsilon},\mathbf{Y}^0_\varepsilon)=[\tilde{\alpha}^0(\mathbf{Y}^0_\varepsilon)]_\varepsilon;\quad \mathbf{Y}^0_\varepsilon(s;\mathbf{x},s)=\mathbf{x}. \tag{3.1}$$

Because of the definition (1.13) of $\tilde{\alpha}^0$ and the assumption (1.2) with $k=0$, we deduce that the function $(t,\mathbf{y}^0)\mapsto\tilde{\alpha}^0(t,\frac{t-s}{\varepsilon},\mathbf{y}^0)$ is C^1 on $\mathbb{R}\times\mathbb{R}^d$. Hence, for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$, $T\in\mathbb{R}$, $\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)$ exists on $[s,s+T]$ and is unique for any $\varepsilon>0$.

Now we notice that $\mathbf{X}^0_\varepsilon$ defined by $\mathbf{X}^0_\varepsilon(t;\mathbf{x},s)=[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon$, is the unique solution to

$$\frac{d\mathbf{X}^0_\varepsilon}{dt}=\frac{1}{\varepsilon}\big[\frac{\partial\mathbf{Z}}{\partial\theta}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon+\big[\frac{\partial\mathbf{Z}}{\partial t}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon+\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0_\varepsilon)\}_\varepsilon\{\tilde{\alpha}^0(\mathbf{Y}^0_\varepsilon)\big\}_\varepsilon=[\mathbf{a}(\mathbf{X}^0_\varepsilon)]_\varepsilon+\frac{1}{\varepsilon}\mathbf{b}(\mathbf{X}^0_\varepsilon), \tag{3.2}$$

equiped with the initial condition $\mathbf{X}^0_\varepsilon(s;\mathbf{x},s)=\mathbf{x}$. Then for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$, $T\in\mathbb{R}$, and any $\varepsilon>0$, the solution $\mathbf{X}_\varepsilon(\cdot;\mathbf{x},s)$ of (1.1) exists and is unique on $[s,s+T]$. Moreover it is given by

$$\mathbf{X}_\varepsilon(t;\mathbf{x},s)=\mathbf{X}^0_\varepsilon(t;\mathbf{x},s)=\mathbf{Z}(t,\frac{t-s}{\varepsilon};\mathbf{Y}^0_\varepsilon(t;\mathbf{x},s))=[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon. \tag{3.3}$$

Now we shall prove the convergence of $\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)$ to $\mathbf{Y}^0(\cdot;\mathbf{x},s)$ solution to (1.15). We saw that $\nabla_z\mathbf{Z}$ satisfies (2.9). From this we deduce that $\nabla_z\mathbf{Z}(t,\cdot;\mathbf{z})$ is a $2\pi-$periodic continuous function oscillating around the identity matrix. Because of (1.3), this is the same for $\{\nabla_z\mathbf{Z}(t,\cdot;\mathbf{z})\}^{-1}$. Hence, $\{\nabla_z\mathbf{Z}(t,\cdot;\mathbf{z})\}^{-1}$ remains in a bounded set which can be chosen independent of t and $\mathbf{z}$. In a same way, derivating (1.4) with respect to t, we deduce that $\frac{\partial\mathbf{Z}}{\partial t}$ is a $2\pi-$periodic continuous function oscillating around 0, and then, remains in a bounded set.
Hence the function $\tilde{\alpha}^0$ has a bounded range in $\mathbb{R}^d$ and then the ranges of the functions $(t,\mathbf{y}^0)\mapsto\tilde{\alpha}^0(t,\frac{t-s}{\varepsilon},\mathbf{y}^0)$ may be bounded in $\mathbb{R}^d$, independently of ε. Then, we deduce that for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$ and $T\in\mathbb{R}$, the sequences $\big(\frac{d\mathbf{Y}^0_\varepsilon}{dt}(\cdot;\mathbf{x},s)\big)_\varepsilon$ and $\big(\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)\big)_\varepsilon$ are bounded in $L^\infty([s,s+T])$. Applying the Rellich-Kondrachov Compactness Theorem we deduce that there exits a subsequence (still denoted ε) such that

$$\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)\to\mathbf{Y}^0(\cdot;\mathbf{x},s)\text{ in }L^\infty([s,s+T]),\ (\text{and in }C^0([s,s+T])). \tag{3.4}$$

Now, in order to prove that $\mathbf{Y}^0(\cdot;\mathbf{x},s)$ satisfies equation (1.15) we prove the following Proposition.

PROPOSITION 3.1 *for any $\mathbf{x}\in\mathbb{R}^d$, $s\in\mathbb{R}$ and $T\in\mathbb{R}$, we have*

$$\big[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon-\big[\mathbf{Z}(\mathbf{Y}^0)\big]_\varepsilon\to 0\ \textit{in}\ L^\infty([s,s+T]), \tag{3.5}$$

$$\big[\frac{\partial\mathbf{Z}}{\partial t}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon-\big[\frac{\partial\mathbf{Z}}{\partial t}(\mathbf{Y}^0)\big]_\varepsilon\to 0\ \textit{in}\ L^\infty([s,s+T]), \tag{3.6}$$

$$\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0_\varepsilon)\big\}^{-1}_\varepsilon-\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}^{-1}_\varepsilon\to 0\ \textit{in}\ L^\infty([s,s+T]). \tag{3.7}$$

Proof. In view of the assumptions (1.2) with $k=0$, (1.3) and (1.4), using the classical Theorem on continuous and differentiable dependency on initial data and the $2\pi-$periodicity of $\mathbf{Z}(t,\cdot;\mathbf{z})$, we easily obtain that, $|\mathbf{Z}(t,\theta;\mathbf{z})-\mathbf{Z}(t,\theta;\tilde{\mathbf{z}})|\le C|\mathbf{z}-\tilde{\mathbf{z}}|$, for any $t\in[s,s+T]$, $\theta\in\mathbb{R}$, $\mathbf{z}\in\mathbb{R}^d$ and $\tilde{\mathbf{z}}\in\mathbb{R}^d$ and for a constant C which does not depend on θ and t. From this, we immediately deduce that

$$\big\|[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big\|_{L^\infty([s,s+T])}\le C\|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0\|_{L^\infty([s,s+T])}, \tag{3.8}$$

yielding (3.5) as a consequence of (3.4) .

Because of the regularity assumed on $\mathbf{b}$, the same proof may be led to deduce (3.6) and (3.7). Then the Proposition in proved. ■

From this Proposition, we can prove the following Lemma.

LEMMA 3.2 *Under assumptions (1.2), (1.3) for $k=0$ and (1.4), the limit $\mathbf{Y}^0$ of the sequence of solutions $\mathbf{Y}^0_\varepsilon$ to (3.1) is the unique solution to*

$$\frac{d\mathbf{Y}^0}{dt}=\frac{1}{2\pi}\int_0^{2\pi}\tilde{\alpha}^0(t,\theta,\mathbf{Y}^0)\,d\theta=\tilde{\mathbf{a}}^0(t,\mathbf{Y}^0),\ \ \mathbf{Y}^0(s,\mathbf{x},s)=\mathbf{x}. \tag{3.9}$$

Proof. Using the regularity assumption made on $\mathbf{a}$, we deduce that $|\mathbf{a}(t,\theta,\mathbf{z})-\mathbf{a}(t,\theta,\tilde{\mathbf{z}})|\le C|\mathbf{z}-\tilde{\mathbf{z}}|$, for any $t\in[s,s+T]$, $\theta\in\mathbb{R}$, $\mathbf{z}\in\mathbb{R}^d$ and $\tilde{\mathbf{z}}\in\mathbb{R}^d$ and for a constant C which does not depend on θ and t. This yields $\big\|[\mathbf{a}(\mathbf{Z}(\mathbf{Y}^0_\varepsilon))]_\varepsilon-[\mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))]_\varepsilon\big\|_{L^\infty([s,s+T])}\le C\|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0\|_{L^\infty([s,s+T])}$ and finally

$$[\mathbf{a}(\mathbf{Z}(\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)))]_\varepsilon-[\mathbf{a}(\mathbf{Z}(\mathbf{Y}^0(\cdot;\mathbf{x},s)))]_\varepsilon\to 0\text{ in }L^\infty([s,s+T]). \tag{3.10}$$

Now, using convergence (3.10), (3.6) and (3.7) in the definition (1.13) of $\tilde{\alpha}^0$ it is an easy game to show

$$[\tilde{\alpha}^0(\cdot,\cdot,\mathbf{Z}(\cdot,\cdot;\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)))]_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi}\tilde{\alpha}^0(\cdot,\theta,\mathbf{Z}(\cdot,\theta;\mathbf{Y}^0(\cdot;\mathbf{x},s)))d\theta \text{ in } L^\infty([s,s+T]) \text{ weak}-*. \tag{3.11}$$

Hence, passing to the limit in equation (3.1) proves the Lemma. ■

REMARK 3.1 Since the solution to (3.9) is unique, we can deduce that the whole sequences $(\mathbf{Y}^0_\varepsilon)$ and $(\mathbf{X}^0_\varepsilon) = (\mathbf{X}_\varepsilon)$ (and not only subsequences) converge.

Now we have at hand all we need to end the proof of Theorem 1.3. In view of the link (3.3) between $\mathbf{X}_\varepsilon$ and $\mathbf{Y}^0_\varepsilon$ and setting $\mathbf{X}^0$ from $\mathbf{Y}^0$ by (1.14), the convergence (3.5) is nothing but (1.26) of Theorem 1.3. As the equation (3.9) is the one appearing in Theorem 1.1, we may say that $\mathbf{X}^0$ is characterised by Theorem 1.1. This ends the proof of Theorem 1.3 ■

3.2 Order 1 approximation

All along this subsection, we make the assumptions of Theorem 1.4 for $k=1$. This has the following consequences: $\mathbf{a}\in(C^2_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$, $\mathbf{b}\in(C^3_b(\mathbb{R}\times\mathbb{R}^d))^d$, $\mathbf{Z}\in(C^3(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$, $\nabla_z\mathbf{Z}\in(C^2_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^{d^2}$ and $\frac{\partial\mathbf{Z}}{\partial t}\in(C^2_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$. In view of the definitions (1.13) of $\tilde{\alpha}_0$ and (1.12) of $\tilde{\mathbf{a}}^0$, we also have $\tilde{\alpha}^0\in(C^2_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$ and $\tilde{\mathbf{a}}^0\in(C^2_b(\mathbb{R}\times\mathbb{R}^d))^d$; and, since $\tilde{\mathbf{a}}^0(t,\mathbf{y}^0)$ is the mean value of the function $\tilde{\alpha}^0(t,\cdot,\mathbf{y}^0)$, we deduce $(t,\theta,\mathbf{y}^0)\mapsto\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0)$ is $2\pi-$periodic in θ and $(C^2_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$. Having those regularities at hand, we can prove that $\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0$ is of order ε.

PROPOSITION 3.3 *There exits a constant c independent of ε such that*

$$\|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0\|_{L^\infty([s,s+T])} = \|\mathbf{Y}^0_\varepsilon(\cdot;\mathbf{x},s)-\mathbf{Y}^0(\cdot;\mathbf{x},s)\|_{L^\infty([s,s+T])}\leq c\varepsilon. \tag{3.12}$$

Proof. The function $\mathcal{Y}^1_\varepsilon=\frac{1}{\varepsilon}(\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0)-[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon$, defined for $t\in[s,s+T]$ satisfies

$$\frac{d\mathcal{Y}^1_\varepsilon}{dt}=\frac{1}{\varepsilon}\big([\tilde{\alpha}^0(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\tilde{\alpha}^0(\mathbf{Y}^0)]_\varepsilon\big)-\big[\theta\{\nabla_{y^0}\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\}\{\tilde{\mathbf{a}}^0(\mathbf{Y}^0)\}+\frac{\partial\tilde{\mathbf{A}}^0}{\partial t}(\mathbf{Y}^0)\big]_\varepsilon,\quad \mathcal{Y}^1_\varepsilon(s;\mathbf{x},s)=0. \tag{3.13}$$

Because of the regularity of the involved functions, for any $t\in[s,s+T]$, we have $\frac{1}{\varepsilon}\big|[\tilde{\alpha}^0(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\tilde{\alpha}^0(\mathbf{Y}^0)]_\varepsilon\big|\leq\frac{c_1}{\varepsilon}|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0|\leq c_1\big(|\mathcal{Y}^1_\varepsilon|+\big|[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\big|\big)$, $\big|\big[\theta\{\nabla_{y^0}\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\}\{\tilde{\mathbf{a}}^0(\mathbf{Y}^0)\}+\frac{\partial\tilde{\mathbf{A}}^0}{\partial t}(\mathbf{Y}^0)\big]_\varepsilon\big|\leq c_1$, and $\big|[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\big|\leq c_1$, for a constant c_1 which does not depend on ε. Hence making the dot product of (3.13) by $\mathcal{Y}^1_\varepsilon$, we obtain

$$\left|\frac{d(|\mathcal{Y}^1_\varepsilon|^2)}{dt}\right|\leq\frac{c_1}{2}\big((|\mathcal{Y}^1_\varepsilon|+|[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon|)|\mathcal{Y}^1_\varepsilon|+|\mathcal{Y}^1_\varepsilon|\big)\leq c_2\big(|\mathcal{Y}^1_\varepsilon|^2+|\mathcal{Y}^1_\varepsilon|\big)\leq c_3\big(|\mathcal{Y}^1_\varepsilon|^2+1\big), \tag{3.14}$$

for constants c_2 and c_3 independent of ε. From this last inequality, using the Gronwall Lemma, we deduce that $\mathcal{Y}^1_\varepsilon$ remains in a bounded set which does not depend on ε.
Finally, since $\big|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0\big|\leq\varepsilon\big(|\mathcal{Y}^1_\varepsilon|+|[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon|\big)\leq c\varepsilon$, for every $t\in[s,s+T]$, and with c independent of ε, the Proposition is proved. ■

We consider now $\mathbf{Y}^1_\varepsilon$ defined by (1.29). Since, for a constant c independent of ε, we have $\big|\mathbf{Z}(t,\theta;\mathbf{Y}^0_\varepsilon)-\mathbf{Z}(t,\theta;\mathbf{Y}^0)\big|\leq c|\mathbf{Y}^0_\varepsilon-\mathbf{Y}^0|$, for any $t\in[s,s+T]$ and any $\theta\in\mathbb{R}$, the proof of the following proposition is straightforward.

PROPOSITION 3.4 *The sequence of functions $(\mathbf{Y}^1_\varepsilon(\cdot;\mathbf{x},s))$ is bounded in $L^\infty([s,s+T])$.*

Concerning now the derivative of $\mathbf{Y}^1_\varepsilon$ we have

Proposition 3.5 *The function $\mathbf{Y}^1_\varepsilon(\cdot;\mathbf{x},s)$ is solution to:*

$$\frac{d\mathbf{Y}^1_\varepsilon}{dt} = [\tilde\alpha^1(\mathbf{Y}^0,\mathbf{Y}^1_\varepsilon)]_\varepsilon + \mathcal{O}_1(\varepsilon),\quad \mathbf{Y}^1_\varepsilon(s;\mathbf{x},s)=0. \tag{3.15}$$

where $\tilde\alpha^1$ is defined by (1.18) and where $\|\mathcal{O}_1(\varepsilon)\|_{L^\infty([s,s+T])}\le c\varepsilon$ for a constant c and ε small enough.

Proof. From the definition (1.29) of $\mathbf{Y}^1_\varepsilon$, we get: $\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\big\} = \frac{1}{\varepsilon}\big([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon - [\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big)$. Derivating this equation, using $\frac{d\mathbf{Y}^0_\varepsilon}{dt}=[\tilde\alpha^0(\mathbf{Y}^0_\varepsilon)]_\varepsilon$, $\frac{\partial}{\partial\theta}\big(\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big)=\tilde\alpha^0(\mathbf{Y}^0)-\tilde{\mathbf{a}}^0(\mathbf{Y}^0)$, the expressions of $\{\nabla_z\mathbf{Z}(\mathbf{y}^0)\}\{\tilde\alpha^0(\mathbf{y}^0)\}$ obtained from (1.13), of $\frac{\partial\mathbf{Z}}{\partial\theta}$ given by (1.4) and of $\frac{\partial\nabla_z\mathbf{Z}}{\partial\theta}$ given by (2.9), a direct computation leads

$$\begin{aligned}\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\Big\{\frac{d\mathbf{Y}^1_\varepsilon}{dt}\Big\} &= \frac{1}{\varepsilon}\Big(\big[\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon - \big[\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\big]_\varepsilon\Big)+\\ &\frac{1}{\varepsilon^2}\Big(\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon)-\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big) - \frac{1}{\varepsilon}\Big\{\nabla_x\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\Big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\} -\\ &\Big\{\frac{\partial\nabla_z\mathbf{Z}}{\partial t}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\} - \Big\{\nabla_z^2\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\tilde{\mathbf{a}}^0(\mathbf{Y}^0),\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\} -\\ &\Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\frac{\partial}{\partial t}\big(\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big)\Big\}_\varepsilon.\end{aligned} \tag{3.16}$$

Using the Taylor formula with integral form of the remainder, we may write

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(\big[\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0_\varepsilon)\big]_\varepsilon - \big[\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\big]_\varepsilon\Big)\\ &= \Big\{\int_0^1\nabla_x\mathbf{a}\big(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)+\xi(\mathbf{Z}(\mathbf{Y}^0_\varepsilon)-\mathbf{Z}(\mathbf{Y}^0)\big)\,d\xi\Big\}_\varepsilon\Big\{\frac{1}{\varepsilon}\big([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big)\Big\}\\ &= \Big\{\nabla_x\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\frac{1}{\varepsilon}\big([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big)\Big\}+\frac{1}{\varepsilon}\mathcal{O}\Big(\big|[\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon-[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\big|^2\Big)\\ &= \Big\{\nabla_x\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\}+\mathcal{O}(\varepsilon).\end{aligned} \tag{3.17}$$

In the same way, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^2}\Big(\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0_\varepsilon)]_\varepsilon)-\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big) - \frac{1}{\varepsilon}\Big\{\nabla_x\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\Big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\}\\ &= \frac{1}{2}\Big\{\nabla_x^2\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\Big\{\mathbf{Y}^1_\varepsilon+\big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\}\Big\}^2+\mathcal{O}(\varepsilon).\end{aligned} \tag{3.18}$$

Now computing $\frac{\partial}{\partial t}(\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0))$ and injecting (3.17) and (3.18) in (3.16) we get (3.15), ending the proof of the Proposition. ∎

Now, the proof of the next Proposition can be led easily.

Proposition 3.6 *We have*

$$\mathbf{Y}^1_\varepsilon \longrightarrow \mathbf{Y}^1 \text{ in } L^\infty([s,s+T]), \tag{3.19}$$

where $\mathbf{Y}^1$ is the unique solution to

$$\frac{d\mathbf{Y}^1}{dt} = \frac{1}{2\pi}\int_0^{2\pi}\tilde\alpha^1(t,\theta,\mathbf{Y}^0,\mathbf{Y}^1)\,d\theta = \tilde{\mathbf{a}}^1(t,\mathbf{Y}^0,\mathbf{Y}^1),\quad \mathbf{Y}^1(s,\mathbf{x},s)=0. \tag{3.20}$$

Proof. From (3.15) and because of the regularity of $\tilde\alpha^1$ and the boundness of $\mathbf{Y}^1_\varepsilon$, we deduce that $\frac{d\mathbf{Y}^1}{dt}$ is bounded in $L^\infty([s,s+T])$. Then we have that (3.19) holds true for a subsequence. Then passing to the limit as $\varepsilon\to0$ in (3.15), we obtain (3.20). As the solution to (3.20) is unique we deduce that the convergence (3.19) holds true for the whole sequence $(\mathbf{Y}^1_\varepsilon)$, ending the proof. ∎

Now, defining $\mathbf{X}^1_\varepsilon$ from $\mathbf{Y}^1_\varepsilon$ by (1.31) and using the definition (1.24) of $\mathbf{X}^1$ in terms of $\mathbf{Y}^1$, we deduce from (3.19) that $\mathbf{X}^1_\varepsilon - [\mathbf{X}^1]_\varepsilon = \big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon \big\{\mathbf{Y}^1_\varepsilon - \mathbf{Y}^1\big\} \longrightarrow 0$, as $\varepsilon \to 0$. Then, Theorem 1.4 is true for k=1.

Now in order to manage the induction procedure to prove the Theorem for any k, we need to express the function $\mathcal{O}_1(\varepsilon)$ taking place in Proposition 3.4. In view of (3.17), and (3.18) the following Proposition is straightforward.

PROPOSITION 3.7 *The function $\mathcal{O}_1(\varepsilon)$ of equation (3.15) has the following explicit expression:*

$$\begin{aligned}\mathcal{O}_1(\varepsilon) = &\Big\{ \int_0^1 \nabla_x \mathbf{a}\big(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0) + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}\{\mathbf{Y}^1_\varepsilon + [\theta \tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\}\big)\, d\xi - \nabla_x \mathbf{a}\big(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon \\ &\qquad \Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon \Big\{\mathbf{Y}^1_\varepsilon + \big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\} + \\ &\Big\{ \int_0^1 (1-\xi)\, \nabla_x^2 \mathbf{b}\big([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}\{\mathbf{Y}^1_\varepsilon + [\theta \tilde{\mathbf{A}}^0(\mathbf{Y}^0))]_\varepsilon\}\big)\, d\xi - \frac{1}{2}\nabla_x^2 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\} \\ &\qquad \Big\{\Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon \Big\{\mathbf{Y}^1_\varepsilon + \big[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)\big]_\varepsilon\Big\}\Big\}^2 .\end{aligned} \tag{3.21}$$

3.3 Higher order approximation, proof of Theorem 1.4

All along this subsection, we fix $k \geq 2$, we make the assumptions of Theorem 1.4 and we suppose that this Theorem is true until $k-1$. In other words we have: $\mathbf{a} \in (C_b^{k+1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$, $\mathbf{b} \in (C_b^{k+2}(\mathbb{R}\times\mathbb{R}^d))^d$, $\mathbf{Z} \in (C^{k+2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$, $\nabla_z \mathbf{Z} \in (C_b^{k+1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^{d^2}$ and $\frac{\partial \mathbf{Z}}{\partial t} \in (C_b^{k+1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$. From the definition of $\tilde{\alpha}^i$ and $\tilde{\mathbf{a}}^i$, we deduce $\tilde{\alpha}^i \in (C_b^{k+1-i}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d))^d$ and $\tilde{\mathbf{a}}^i \in (C_b^{k+1-i}(\mathbb{R}\times\mathbb{R}^d))^d$. Since $\tilde{\mathbf{a}}^i(t,\mathbf{y}^0,\dots,\mathbf{y}^i)$ is the mean value of the function $\tilde{\alpha}^i(t,\cdot,\mathbf{y}^0,\dots,\mathbf{y}^i)$, we deduce $(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^i) \mapsto \theta\tilde{\mathbf{A}}^i(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^i)$ is $2\pi-$periodic in θ and $(C_b^{k+1-i}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\dots\times\mathbb{R}^d))^d$. Moreover we assume that for $i=1,\dots,k-1$, $\mathbf{Y}^i_\varepsilon$ defined by (1.30) is solution to

$$\frac{d\mathbf{Y}^i_\varepsilon}{dt} = \Big[\tilde{\alpha}^i(\mathbf{Y}^0,\dots \mathbf{Y}^{i-1},\mathbf{Y}^i_\varepsilon)\Big]_\varepsilon + \mathcal{O}_i(\varepsilon), \quad \mathbf{Y}^i_\varepsilon(s;\mathbf{x},s) = 0, \tag{3.22}$$

where $\tilde{\alpha}^i$ is defined by (1.18) and where $\|\mathcal{O}_i(\varepsilon)\|_{L^\infty([s,s+T])} \leq c\varepsilon$ for a constant c and ε small enough and given by (3.21) or (B.1) or (B.2) of appendix B.

We consider now $\mathbf{Y}^k_\varepsilon$ defined by (1.30), and we have

PROPOSITION 3.8 *The sequence $(\mathbf{Y}^k_\varepsilon(\cdot;\mathbf{x},s))$ is bounded in $L^\infty([s,s+T])$ and there exits a constant c independent of ε such that*

$$\|\mathbf{Y}^{k-1}_\varepsilon - \mathbf{Y}^{k-1}\|_{L^\infty([s,s+T])} \leq c\varepsilon. \tag{3.23}$$

Proof. Derivating (1.30) we get

$$\begin{aligned}\frac{d\mathbf{Y}^k_\varepsilon}{dt} = &\frac{1}{\varepsilon}\big([\tilde{\alpha}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-2},\mathbf{Y}^{k-1}_\varepsilon)]_\varepsilon - [\tilde{\alpha}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon\big) + \frac{1}{\varepsilon}\mathcal{O}_{k-1}(\varepsilon) - \\ &\Big[\theta\big(\sum_{j=0}^{k-1}\{\nabla_{y^j}\tilde{\mathbf{A}}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})\}\{\tilde{\mathbf{a}}^j(\mathbf{Y}^0,\dots,\mathbf{Y}^j)\} + \frac{\partial\tilde{\mathbf{A}}^{k-1}}{\partial t}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})\big)\Big]_\varepsilon.\end{aligned} \tag{3.24}$$

Using the regularity of $\tilde{\alpha}^{k-1}$ we obtain $\frac{1}{\varepsilon}\big|[\tilde{\alpha}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-2},\mathbf{Y}^{k-1}_\varepsilon)]_\varepsilon - [\tilde{\alpha}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon\big| \leq \frac{c}{\varepsilon}|\mathbf{Y}^{k-1}_\varepsilon - \mathbf{Y}^{k-1}| \leq c\big(|\mathbf{Y}^k_\varepsilon| + \big|[\theta\tilde{\mathbf{A}}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon\big|\big)$. Then, multiplying (3.24) by $\mathbf{Y}^k_\varepsilon$ we obtain

$$\left|\frac{d\big(|\mathbf{Y}^k_\varepsilon|^2\big)}{dt}\right| \leq c\big(|\mathbf{Y}^k_\varepsilon|^2 + 1\big), \tag{3.25}$$

for a constant c which does not depend on ε. This yields that $(\mathbf{Y}^k_\varepsilon)$ remains in a bounded set which does not depend on ε, proving the first part of the proposition. As (3.23) is an obvious consequence of (1.30) and of the boundness of $(\mathbf{Y}^k_\varepsilon)$, the proposition is proved. ■

Using the definition of $\tilde\alpha^{k-1}$ (see (1.18)) we have:

$$\begin{aligned}
&\big[\tilde\alpha^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-2},\mathbf{Y}^{k-1}_\varepsilon)\big]_\varepsilon - \big[\tilde\alpha^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})\big]_\varepsilon = \Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}^{-1}_\varepsilon \\
&\quad\Big\{\Big\{\nabla_x\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\big\{\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}\big\} + \\
&\quad\Big\{\nabla^2_x\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}\big\}\,,\,\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^1+[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\big\}\Big\} - \\
&\quad\Big\{\frac{\partial\nabla_z\mathbf{Z}}{\partial t}(\mathbf{Y}^0)\Big\}_\varepsilon\big\{\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}\big\} - \Big\{\nabla^2_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\tilde{\mathbf{a}}^0(\mathbf{Y}^0),\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}\Big\}\Big\},
\end{aligned} \tag{3.26}$$

for $k\geq 3$ and

$$\begin{aligned}
&\big[\tilde\alpha^{1}(\mathbf{Y}^0,\mathbf{Y}^{1}_\varepsilon)\big]_\varepsilon - \big[\tilde\alpha^{1}(\mathbf{Y}^0,\mathbf{Y}^{1})\big]_\varepsilon = \Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}^{-1}_\varepsilon\Big\{\Big\{\nabla_x\mathbf{a}(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\big\{\mathbf{Y}^{1}_\varepsilon-\mathbf{Y}^{1}\big\} + \\
&\quad\Big\{\nabla^2_x\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^{1}_\varepsilon-\mathbf{Y}^{1}\big\}\,,\,\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^1+[\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\big\}\Big\} + \\
&\quad\frac{1}{2}\Big\{\nabla^2_x\mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\big\{\nabla_z\mathbf{Z}(\mathbf{Y}^0)\big\}_\varepsilon\big\{\mathbf{Y}^{1}_\varepsilon-\mathbf{Y}^{1}\big\}\Big\}^2 - \\
&\quad\Big\{\frac{\partial\nabla_z\mathbf{Z}}{\partial t}(\mathbf{Y}^0)\Big\}_\varepsilon\big\{\mathbf{Y}^{1}_\varepsilon-\mathbf{Y}^{1}\big\} - \Big\{\nabla^2_z\mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\tilde{\mathbf{a}}^0(\mathbf{Y}^0),\mathbf{Y}^{1}_\varepsilon-\mathbf{Y}^{1}\Big\}\Big\}.
\end{aligned} \tag{3.27}$$

Replacing in those expressions $\mathbf{Y}^{k-1}_\varepsilon-\mathbf{Y}^{k-1}$ by $\varepsilon(\mathbf{Y}^k_\varepsilon+[\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon)$, integrating by parts the integrals in the expression of $\mathcal{O}_{k-1}$ (3.21), (B.1) or (B.2) and using $\mathbf{Y}^i_\varepsilon=\mathbf{Y}^i+\varepsilon\big(\mathbf{Y}^{i+1}+[\theta\tilde{\mathbf{A}}^i(\mathbf{Y}^0,\dots,\mathbf{Y}^i)]_\varepsilon\big)+\dots+\varepsilon^{k-i-1}\big(\mathbf{Y}^{k-1}+[\theta\tilde{\mathbf{A}}^{k-2}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-2})]_\varepsilon\big)+\varepsilon^{k-i}\big(\mathbf{Y}^k_\varepsilon+[\theta\tilde{\mathbf{A}}^{k-1}(\mathbf{Y}^0,\dots,\mathbf{Y}^{k-1})]_\varepsilon\big)$, which can be easily deduced from (1.30), we obtain from (3.24):

$$\frac{d\mathbf{Y}^k_\varepsilon}{dt} = \Big[\tilde\alpha^k(\mathbf{Y}^0,\dots\mathbf{Y}^{k-1},\mathbf{Y}^k_\varepsilon)\Big]_\varepsilon + \mathcal{O}_k(\varepsilon),\quad \mathbf{Y}^k_\varepsilon(s;\mathbf{x},s)=0. \tag{3.28}$$

Then $(\frac{d\mathbf{Y}^k_\varepsilon}{dt})$ is bounded in $L^\infty([s,s+T])$ and we have the following Proposition.

PROPOSITION 3.9 *The following convergence holds true, $\mathbf{Y}^k$ being the unique solution to (1.25),*

$$\mathbf{Y}^k_\varepsilon \to \mathbf{Y}^k \ \textit{in}\ L^\infty([s,s+T]), \tag{3.29}$$

This means that Theorem 1.4 is valid for k.

In view of Proposition 3.9 and equation (3.28), we have enough to make the induction procedure proving Theorem 1.4 for any $k\geq 1$. ■

4 Application to gyrokinetic

The dynamical systems associated with equations (1.7), (1.8) and (1.9) are in the form of (1.5), i.e.:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{X}_\varepsilon\\ \mathbf{V}_\varepsilon\end{pmatrix} = \mathbf{a}(t,\frac{t-s}{\varepsilon},\mathbf{X}_\varepsilon,\mathbf{V}_\varepsilon)+\frac{1}{\varepsilon}\mathbf{b}(t,\mathbf{X}_\varepsilon,\mathbf{V}_\varepsilon)\,,\quad \begin{pmatrix}\mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s)\\ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s)\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} \tag{4.1}$$

with variable $(\mathbf{x},\mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ in place of $\mathbf{x}$ and with ad-hoc fields $\mathbf{a}$ and $\mathbf{b}$. Then we can apply our result saying that the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s), \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ can be expanded as

$$\begin{aligned}\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s) &= \mathbf{X}^0(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon\mathbf{X}^1(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon^2\mathbf{X}^2(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \dots,\\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s) &= \mathbf{V}^0(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon\mathbf{V}^1(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon^2\mathbf{V}^2(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \dots,\end{aligned} \tag{4.2}$$

and that this expansion may be justified until any order if the regularity of the fields is enough.

In this section, we first lead the computations in the case of Isotope Resonant Separation Regime with the restriction that $\mathcal{M}$ is constant and $\mathbf{B}(t,\theta,\mathbf{x}) = 0$ until the order 1. Then, with the same restrictions, we treat the case of the Guiding Centre Regime until the order 2 and we give the result for the Finite Larmor Radius Regime until the order 0. The forth example concerns the Guiding Centre Regime with a variable strong magnetic field and a constant $\mathbf{B}$ until the order 1.

In the three first examples, the singular perturbation is linear: $\mathbf{b}(t,\mathbf{Z}) = \beta\mathbf{Z}$, with a skew-symmetric matrix β. Hence, $\mathbf{Z}(t,\theta;\mathbf{z}) = e^{\theta\beta}\mathbf{z}$ and $\nabla_z\mathbf{Z}(t,\theta;\mathbf{y}^0) = e^{\theta\beta}$. Then the deduction of the asymptotic expansion of $\mathbf{X}_\varepsilon$ does not really need the heavy framework presented in this paper. In particular, the definitions of $\mathbf{Y}_\varepsilon^0$, $\mathbf{Y}_\varepsilon^1$ and $\mathbf{Y}_\varepsilon^k$ reduce to only one formulation.
From the physical point of view, the last example is relevant to understand the behaviour of a plasma in a tokamak. This case involves a singular perturbation depending on the solution and then we need the framework presented in this paper to treat it. This example shows that the generic computations made before, coupled with the use of Maple, enable to deduce the result relatively comfortably.

4.1 Isotope Resonant Separation Regime with constant strong magnetic field

In the case of Isotope Resonance Separation Regime, i.e. of equation (1.9), $\mathbf{a}$ and $\mathbf{b}$ are:

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \begin{pmatrix}\mathbf{v}\\ \mathbf{E}(t,\theta,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\theta,\mathbf{x})\end{pmatrix}, \quad \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix}0\\ \mathbf{v}\times\mathcal{M}\end{pmatrix}. \tag{4.3}$$

For simplicity, we restrict to the case when $\mathbf{B}(t,\theta,\mathbf{x}) = 0$ and when $\mathcal{M} = \mathbf{e}_1$ is a constant vector, ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$) being the frame of $\mathbb{R}^3$. Then $\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})$ and $\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\}^{-1}$ are given by

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w}) = \begin{pmatrix}\mathbf{z}\\ R(\theta)\mathbf{w}\end{pmatrix}, \quad \left\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\right\}^{-1} = \begin{pmatrix}I & 0\\ 0 & R(-\theta)\end{pmatrix}, \tag{4.4}$$

where $R(\theta)$ is the matrix of the rotation of angle $-\theta$ around $\mathcal{M}$,

$$R(\theta) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}. \tag{4.5}$$

We have the following result.

THEOREM 4.1 *If $\mathbf{E}(t,\theta,\mathbf{x})$ is $C_b^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3)$ and $2\pi-$periodic in θ, the first term of the expansion (4.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s), \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon\,, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}\,, \quad \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,, \quad \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{4.6}$$

is given by

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,,\quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s)=R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{4.7}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$\frac{d\mathbf{Y}^0}{dt}=\mathbf{U}^0_{\|}\,,\quad \frac{d\mathbf{U}^0}{dt}=\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\mathbf{E}(t,\theta,\mathbf{Y}^0)\,d\theta\,,\quad \mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{x}\,,\quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}. \tag{4.8}$$

Proof. In view of (1.13) and (1.12), here we have

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}R(\theta)\mathbf{u}^0\\ R(-\theta)\mathbf{E}(t,\theta,\mathbf{y}^0)\end{pmatrix}\,,\quad \tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}\mathbf{u}^0_{\|}\\ \displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\mathbf{E}(t,\theta,\mathbf{y}^0)\,d\theta\end{pmatrix}. \tag{4.9}$$

Then the proof of the Theorem is straightforward. ■

In order to obtain the system satisfied by the second term $(\mathbf{Y}^1,\mathbf{U}^1)$ of the expansion we notice that $\mathcal{R}(\theta)=-R(\frac{\pi}{2}+\theta)+R(\frac{\pi}{2})$ is such that $\int_0^\theta R(\sigma)\,d\sigma=\theta P+\mathcal{R}(\theta)$, with P the matrix of the orthogonal projection onto $\mathcal{M}$. We have

$$\mathcal{R}(\theta)=\begin{pmatrix}0&0&0\\0&\sin\theta&1-\cos\theta\\0&\cos\theta-1&\sin\theta\end{pmatrix}\ \text{and}\ P=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}. \tag{4.10}$$

Then, from (1.17), we get:

$$\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}\mathcal{R}(\theta)\mathbf{u}^0\\ \Big(\displaystyle\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\big)\end{pmatrix}, \tag{4.11}$$

where we denote $\int_0^\theta f(\sigma)\,d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}f(\sigma)\,d\sigma$ by $\big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\big)(f(\sigma))$. We also get

$$\begin{aligned}\tilde{\alpha}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)=&\begin{pmatrix}R(\theta)\big(\mathbf{u}^1+\Big(\displaystyle\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)\big)\\ R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)(\mathbf{y}^1+\mathcal{R}(\theta)\mathbf{u}^0)\end{pmatrix}-\\ &\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big[\begin{pmatrix}R(\sigma)\big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{y}^0)\,d\varsigma\big)\\ R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{y}^0)\mathbf{u}^0_{\|}\end{pmatrix}+\begin{pmatrix}0\\ R(-\sigma)\displaystyle\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{y}^0)\end{pmatrix}\Big]\end{aligned} \tag{4.12}$$

Then using $\big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\big)\big(R(\sigma)\big)=\mathcal{R}(\theta)$ and $\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\theta)\,d\theta=R(\frac{\pi}{2})-P$, we obtain

$$\begin{aligned}&\tilde{\mathbf{a}}^1(t,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)=\\ &\begin{pmatrix}\mathbf{u}^1_{\|}+\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(\theta-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)d\theta\\ \big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)d\theta\big)\mathbf{y}^1+\big(\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)\mathcal{R}(\theta)d\theta\big)\mathbf{u}^0\end{pmatrix}-\end{aligned}$$

$$\begin{pmatrix} (R(\frac{\pi}{2})-P)\big(\frac{1}{2\pi}\int_0^{2\pi} R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{y}^0)\,d\varsigma\big) \\ \big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)\,d\theta\big)\mathbf{u}_{\|}^0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{y}^0)\Big)\,d\theta \end{pmatrix}. \qquad (4.13)$$

Hence we can state the following Theorem.

THEOREM 4.2 *If* $\mathbf{E}(t,\theta,\mathbf{x})$ *is* $C_b^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3)$ *and* $2\pi-$*periodic in* θ*, the second term of the expansion (4.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to (4.6) is given by*

$$\begin{aligned} &\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \\ &\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s) + R(\theta)\big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\big)\big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big), \end{aligned} \qquad (4.14)$$

where $(\mathbf{Y}^1,\mathbf{U}^1)$ *is solution to*

$$\begin{aligned} \frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}_{\|}^1 + \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(\theta-\sigma)\mathbf{E}(t,\sigma,\mathbf{Y}^0)\Big)\,d\theta - \\ &\qquad (R(\frac{\pi}{2})-P)\big(\frac{1}{2\pi}\int_0^{2\pi} R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{Y}^0)\,d\varsigma\big), \\ \frac{d\mathbf{U}^1}{dt} &= \big(\frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{Y}^0)d\theta\big)\mathbf{Y}^1 + \big(\frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{Y}^0)\mathcal{R}(\theta)d\theta\big)\mathbf{U}^0 - \\ &\quad \big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{Y}^0)\Big)\,d\theta\big)\mathbf{U}_{\|}^0 - \\ &\qquad \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{Y}^0)\Big)\,d\theta, \\ \mathbf{Y}^1(s;\mathbf{x},\mathbf{v},s) &= 0, \quad \mathbf{U}^1(s;\mathbf{x},\mathbf{v},s) = 0. \end{aligned} \qquad (4.15)$$

4.2 Guiding Centre Regime with constant strong magnetic field

In the case of the Guiding Centre Regime, i.e. of equation (1.7), we have

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \mathbf{a}(t,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x}) \end{pmatrix}, \text{ and } \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} 0 \\ \mathcal{N} + \mathbf{v}\times\mathcal{M} \end{pmatrix}, \qquad (4.16)$$

As previously, we make the restriction $\mathbf{B}(t,\theta,\mathbf{x}) = \mathcal{N} = 0$ and $\mathcal{M} = \mathbf{e}_1$. Since this situation is similar to the previous one with the only difference that $\mathbf{E}(t,\mathbf{x})$ does not depend on θ we can directly deduce the following Theorem.

THEOREM 4.3 *If* $\mathbf{E}(t,\mathbf{x})$ *is* $C_b^2(\mathbb{R}\times\mathbb{R}^3)$*, the first and second terms of the expansion (4.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon\,,\ \ \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}\,,\ \ \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,,\ \ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \qquad (4.17)$$

are given by

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,,\quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{4.18}$$

and

$$\begin{aligned} &\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s),\\ &\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)), \end{aligned} \tag{4.19}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}^0_\parallel\,,\quad \frac{d\mathbf{U}^0}{dt} = \mathbf{E}_\parallel(t,\mathbf{Y}^0)\,,\quad \mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,,\quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{4.20}$$

and where $(\mathbf{Y}^1,\mathbf{U}^1)$ *is solution to*

$$\begin{aligned} \frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}^1_\parallel + (R(\frac{\pi}{2}) - P)\mathbf{E}(t,\mathbf{Y}^0) = \mathbf{U}^1_\parallel + \mathbf{E}(t,\mathbf{Y}^0)\times\mathcal{M},\\ \frac{d\mathbf{U}^1}{dt} &= P\,\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^1 + \frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2}) - P\big)\mathbf{U}^0 +\\ &\quad \frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2}) - P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)\mathbf{U}^0 - (R(-\frac{\pi}{2}) - P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}^0_\parallel -\\ &\qquad (R(-\frac{\pi}{2}) - P)\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0),\\ \mathbf{Y}^1&(s;\mathbf{x},\mathbf{v},s) = 0,\quad \mathbf{U}^1(s;\mathbf{x},\mathbf{v},s) = 0. \end{aligned} \tag{4.21}$$

In the computations leading to this Theorem, we use, among other formula,

$$\Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\Big) = -\mathcal{R}(-\theta),\quad \frac{1}{2\pi}\int_0^{2\pi} -\mathcal{R}(-\theta)\,d\theta = R(-\frac{\pi}{2}) - P,\quad -R(\theta)\mathcal{R}(-\theta) = \mathcal{R}(\theta), \tag{4.22}$$

and

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathcal{R}(\theta)d\theta = \frac{1}{2}&\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2}) - P\big) +\\ &\frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2}) - P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P). \end{aligned} \tag{4.23}$$

Now we turn to the characterisation of $(\mathbf{X}^2,\mathbf{V}^2)$. For this purpose we notice that here

$$\tilde{\alpha}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1) = \begin{pmatrix} R(\theta)\mathbf{u}^1 + \mathcal{R}(\theta)\mathbf{E}(t,\mathbf{y}^0)\\ R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1 + R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\theta)\mathbf{u}^0 + \mathcal{R}(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_\parallel + \mathcal{R}(-\theta)\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{y}^0) \end{pmatrix}. \tag{4.24}$$

In order to get now $\theta\tilde{\mathbf{A}}^1$ we need first to compute

$$\Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\big(\mathcal{R}(\sigma)\big) = I - R(\theta)\,,\quad \Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\big(\mathcal{R}(-\sigma)\big) = I - R(-\theta). \tag{4.25}$$

Secondly,

$$\Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\big(R(-\sigma)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\sigma)\big) =$$

$$P\nabla_x \mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) + (I-P)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sin^2\theta}{2} & \sin\theta - \frac{\sin 2\theta}{4} \\ 0 & \frac{\sin 2\theta}{4} - \sin\theta & \frac{\sin^2\theta}{2} \end{pmatrix} +$$

$$\big(R(-\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\sin 2\theta}{4} & 1-\cos\theta - \frac{\sin^2\theta}{2} \\ 0 & \frac{\sin^2\theta}{2} + \cos\theta - 1 & \frac{-\sin 2\theta}{4} \end{pmatrix}, \tag{4.26}$$

which also reads

$$\Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\big(R(-\sigma)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\sigma)\big) = P\nabla_x \mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) +$$
$$\frac{1}{2}\big(-\mathcal{R}(-\theta) + R(\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P\big) + \frac{1}{2}\big(R(-\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(R(\frac{\pi}{2}) - P\big). \tag{4.27}$$

Hence integrating (4.24) we have

$$\theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1) = \begin{pmatrix} \mathcal{R}(\theta)\mathbf{u}^1 + (I-R(\theta))\mathbf{E}(t,\mathbf{y}^0) \\ -\mathcal{R}(-\theta)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1 - (I-R(-\theta))\Big(\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_{\|} + \frac{\partial \mathbf{E}}{\partial t}(t,\mathbf{y}^0)\Big) \end{pmatrix} +$$
$$\begin{pmatrix} 0 \\ \Big(P\nabla_x \mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) + \frac{1}{2}\big(-\mathcal{R}(-\theta) + R(\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P\big)\Big)\mathbf{u}^0 \end{pmatrix} +$$
$$\begin{pmatrix} 0 \\ \Big(\frac{1}{2}\big(R(-\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(R(\frac{\pi}{2}) - P\big)\mathbf{u}^0 \end{pmatrix}. \tag{4.28}$$

In order to obtain the expression of $\tilde{\alpha}^2$, we need to compute

$$\big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0)\big\}^{-1}\big\{\nabla_{x,v}\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0))\big\}\big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0)\big\}$$
$$\left\{\begin{pmatrix}\mathbf{y}^2 \\ \mathbf{u}^2\end{pmatrix} + \theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\right\} =$$
$$\begin{pmatrix} 0 & R(\theta) \\ R(-\theta)\nabla_x \mathbf{E}(t,\mathbf{y}^0) & 0 \end{pmatrix}\left(\begin{pmatrix}\mathbf{y}^2 \\ \mathbf{u}^2\end{pmatrix} + \theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\right) =$$

$$\begin{pmatrix} R(\theta)\mathbf{u}^2 + \mathcal{R}(\theta)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1 - (R(\theta)-I)\Big(\nabla_x \mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_{\|} + \frac{\partial \mathbf{E}}{\partial t}(t,\mathbf{y}^0)\Big) + \\ \Big(P\nabla_x \mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) + \frac{1}{2}\big(R(\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta) + R(\frac{\pi}{2}) - P\big) + \\ \frac{1}{2}\big(R(\theta-\frac{\pi}{2}) - P\big)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\big(R(\frac{\pi}{2}) - P\big)\Big)\mathbf{u}^0 \\ \\ R(-\theta)\nabla_x \mathbf{E}(t,\mathbf{y}^0)\Big(\mathbf{y}^2 + \mathcal{R}(\theta)\mathbf{u}^1 + (I-R(\theta))\mathbf{E}(t,\mathbf{y}^0)\Big) \end{pmatrix}, \tag{4.29}$$

$$\{\nabla^2_{x,v}\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0))\}\left\{\begin{pmatrix}\mathbf{y}^0\\ \mathbf{u}^0\end{pmatrix}+\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)\right\}^2 = \begin{pmatrix} 0 \\ \left\{\nabla^2_x\mathbf{E}(t,\mathbf{y}^0)\right\}^2\Big(\{\mathbf{y}^0,\mathbf{y}^0\}+2\{\mathbf{y}^0,\mathcal{R}(\theta)\mathbf{u}^0\}+\{\mathcal{R}(\theta)\mathbf{u}^0,\mathcal{R}(\theta)\mathbf{u}^0\}\Big)\end{pmatrix}. \tag{4.30}$$

A direct but heavy computation also leads the derivatives $\nabla_{\mathbf{y}^0,\mathbf{u}^0}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\,\tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0)$, $\nabla_{\mathbf{y}^1,\mathbf{u}^1}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\,\tilde{\mathbf{a}}^1(t,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)$, and $\dfrac{\partial\tilde{\mathbf{A}}^1}{\partial t}(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)$.

Then in view of (1.18), we get $\tilde{\alpha}^2$, which, integrating with respect to θ leads to $\tilde{\mathbf{a}}^2$ and to the equation satisfied by $(\mathbf{X}^2,\mathbf{V}^2)$. Then we have the following Theorem.

THEOREM 4.4 *If $\mathbf{E}(t,\mathbf{x})$ is $C^3_b(\mathbb{R}\times\mathbb{R}^3)$, the third term of the expansion (4.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to (4.17) is given by*

$$\begin{aligned}
&\mathbf{X}^2(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^2(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\mathbf{U}^1+(I-R(\theta))\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)),\\
&\mathbf{V}^2(t,\theta;\mathbf{x},\mathbf{v},s)=R(\theta)\mathbf{U}^2(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s)-\\
&\quad(R(\theta)-I)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\Big(\mathbf{U}^0_{\parallel}(t;\mathbf{x},\mathbf{v},s)+\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\Big)+\\
&\quad\Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))(I-R(\theta))+\frac{1}{2}\big(R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big(\mathcal{R}(\theta)+R(\frac{\pi}{2})-P\big)+\\
&\qquad\frac{1}{2}\big(R(\theta-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s).
\end{aligned}\tag{4.31}$$

Moreover $(\mathbf{Y}^2,\mathbf{U}^2)$ is solution to

$$\begin{aligned}
\frac{d\mathbf{Y}^2}{dt}&=\mathbf{U}^2_{\parallel}+(R(\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^1+(I-P)\Big(\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}^0_{\parallel}+\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\Big)+\\
&\Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P)+\big(R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0-\\
&\frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(I-P\big)\mathbf{U}^0-\frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(R(\frac{\pi}{2})-P)\mathbf{U}^0,
\end{aligned}\tag{4.32}$$

$$\begin{aligned}
\frac{d\mathbf{U}^2}{dt}&=P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^2+\\
&\frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2})-P\big)\mathbf{U}^1+\frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)\mathbf{U}^1+\\
&P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{E}(t,\mathbf{Y}^0)-\Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)P+\frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)+\\
&\frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(R(-\frac{\pi}{2})-P)\Big)\mathbf{E}(t,\mathbf{Y}^0)+\\
&\left\{\nabla^2_x\mathbf{E}(t,\mathbf{Y}^0)\right\}^2\Big(\{\mathbf{Y}^0,\mathbf{Y}^0\}+2\{\mathbf{Y}^0,\mathbf{U}^0_{\parallel}\}+\{(I-P)\mathbf{U}^0,(I-P)\mathbf{U}^0\}+\\
&\qquad\{(R(-\frac{\pi}{2})-P)\mathbf{U}^0,(R(-\frac{\pi}{2})-P)\mathbf{U}^0\}\Big)+\\
&(P-R(-\frac{\pi}{2}))\{\nabla^2_x\mathbf{E}(t,\mathbf{Y}^0)\}\{\mathbf{Y}^1,\mathbf{U}^0_{\parallel}\}+(I-P)\Big(\{\nabla^2_x\mathbf{E}(t,\mathbf{Y}^0)\}\{\mathbf{U}^0_{\parallel},\mathbf{U}^0_{\parallel}\}+\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{U}^0_{\parallel}\Big)-\\
&P\{\nabla^2_x\mathbf{E}(t,\mathbf{Y}^0)\}\{(I-P)\mathbf{U}^0,\mathbf{U}^0_{\parallel}\}-
\end{aligned}$$

$$\frac{1}{4}(I-P)\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{(I-P)\mathbf{U}^0,\mathbf{U}_\|^0\}-\frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{\big(R(\frac{\pi}{2})-P\big)\mathbf{U}^0,\mathbf{U}_\|^0\}-$$
$$\Big(-(I-P)\Big(\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\Big)+P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P)+$$
$$\frac{1}{4}(I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P)+\frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{E}_\|(t,\mathbf{Y}^0)+$$
$$(P-R(-\frac{\pi}{2}))\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}_\|^1+(P-R(-\frac{\pi}{2}))\nabla_x\mathbf{E}(t,\mathbf{y}^0)(R(\frac{\pi}{2})-P)\mathbf{E}(t,\mathbf{Y}^0)+$$
$$(P-R(-\frac{\pi}{2}))\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{Y}^1-(I-P)\Big(\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{U}_\|^0+\frac{\partial^2\mathbf{E}}{\partial t^2}(t,\mathbf{Y}^0)\Big)-P\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)(I-P)-$$
$$\Big(\frac{1}{4}(I-P)\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)(I-P)+\frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0. \quad (4.33)$$

4.3 Finite Larmor Radius Regime with constant strong magnetic field

In the case of Finite Larmor Radius Regime (1.8) with $\mathcal{M}=\mathbf{e}_1$, we have

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v})=\begin{pmatrix}\mathbf{v}_\|\\ \mathbf{E}(t,\mathbf{x})\end{pmatrix} \text{ and } \mathbf{b}(t,\mathbf{x},\mathbf{v})=\mathbf{b}(\mathbf{v})=\begin{pmatrix}\mathbf{v}_\perp\\ \mathbf{v}\times\mathcal{M}\end{pmatrix}, \quad (4.34)$$

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})=\begin{pmatrix}\mathbf{z}+\mathcal{R}(\theta)\mathbf{w}\\ R(\theta)\mathbf{w}\end{pmatrix}, \quad \{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{x},\mathbf{v})\}^{-1}=\begin{pmatrix}I & \mathcal{R}(-\theta)\\ 0 & R(-\theta)\end{pmatrix}. \quad (4.35)$$

with R and $\mathcal{R}$ defined by (4.5) and (4.10).

Here we give the result for this case only for the order 0. We have

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}\mathbf{u}_\|^0+\mathcal{R}(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\\ R(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\end{pmatrix}, \quad (4.36)$$

and

$$\tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}\mathbf{u}_\|^0+\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\,d\theta\\ \frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\,d\theta\end{pmatrix}. \quad (4.37)$$

Hence we have the following Theorem.

THEOREM 4.5 *If we assume that $\mathbf{E}(t,\mathbf{x})$ is $C_b^1(\mathbb{R}\times\mathbb{R}^3)$, the first term of the expansion (4.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to*

$$\frac{d\mathbf{X}_\varepsilon}{dt}=\mathbf{V}_{\varepsilon\|}+\frac{1}{\varepsilon}\mathbf{V}_{\varepsilon\perp}, \quad \frac{d\mathbf{V}_\varepsilon}{dt}=\mathbf{E}(t,\mathbf{X}_\varepsilon)+\frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}, \ \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s)=\mathbf{x}, \ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}, \quad (4.38)$$

is given by

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s)=R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \quad (4.39)$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ is solution to

$$\frac{d\mathbf{Y}^0}{dt}=\mathbf{U}_\|^0+\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\theta)\mathbf{E}(\mathbf{Y}^0+\mathcal{R}(\theta)\mathbf{U}^0,t)\,d\theta, \quad \frac{d\mathbf{U}^0}{dt}=\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\mathbf{E}(\mathbf{Y}^0+\mathcal{R}(\theta)\mathbf{U}^0,t)\,d\theta, \quad (4.40)$$

with the initial conditions

$$\mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{x}, \quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}. \quad (4.41)$$

4.4 Guiding Centre Regime with variable strong magnetic field

Here we study a situation representative of what happens inside a tokamak, i.e., the Guiding Centre Regime with a variable $\mathcal{M}$, with $\mathcal{N}=0$ and $\mathbf{B}=\mathbf{e}_3$. In other words we consider the following system:

$$\frac{d\mathbf{X}_\varepsilon}{dt}=\mathbf{V}_\varepsilon\,,\quad \frac{d\mathbf{V}_\varepsilon}{dt}=\mathbf{E}(t,\mathbf{X}_\varepsilon)+\mathbf{V}_\varepsilon\times\mathbf{e}_3+\frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}(\mathbf{X}_\varepsilon)\,,\quad \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s)=\mathbf{x}\,,\quad \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}, \tag{4.42}$$

where

$$\mathcal{M}(\mathbf{x})=\frac{1}{\sqrt{{x_1}^2+{x_2}^2}}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}=\rho^T(\mathbf{x})\mathbf{e}_1,\ \text{with}\ \rho(\mathbf{x})=\frac{1}{\sqrt{{x_1}^2+{x_2}^2}}\begin{pmatrix}-x_2 & x_1 & 0\\ -x_1 & x_2 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{4.43}$$

where $\mathbf{x}=(x_1,x_2,x_3)$ in the frame $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ of $\mathbb{R}^3$. In this case

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v})=\mathbf{a}(t,\mathbf{x},\mathbf{v})=\begin{pmatrix}\mathbf{v}\\ \mathbf{E}(t,\mathbf{x})+\mathbf{v}\times\mathbf{e}_3\end{pmatrix},\ \text{and}\ \mathbf{b}(t,\mathbf{x},\mathbf{v})=\mathbf{b}(\mathbf{x},\mathbf{v})=\begin{pmatrix}0\\ \mathbf{v}\times\mathcal{M}(\mathbf{x})\end{pmatrix}, \tag{4.44}$$

and thus, $R(\theta)$ being defined by (4.5),

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})=\begin{pmatrix}\mathbf{z}\\ \rho^T(\mathbf{z})R(\theta)\rho(\mathbf{z})\mathbf{w}\end{pmatrix},\ \text{where}\ \rho^TR\rho=\begin{pmatrix}\frac{{z_2}^2+{z_1}^2\cos(\theta)}{{z_1}^2+{z_2}^2} & \frac{z_1z_2(\cos(\theta)-1)}{{z_1}^2+{z_2}^2} & -\frac{z_1\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}}\\ \frac{z_1z_2(\cos(\theta)-1)}{{z_1}^2+{z_2}^2} & \frac{{z_1}^2+{z_2}^2\cos(\theta)}{{z_1}^2+{z_2}^2} & -\frac{z_2\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}}\\ \frac{z_1\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} & \frac{z_2\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} & \cos(\theta)\end{pmatrix}, \tag{4.45}$$

and we have the following Theorem.

THEOREM 4.6 *If* $\mathbf{E}(t,\mathbf{x})$ *is* $C_b^2(\mathbb{R}\times\mathbb{R}^3)$, *the first term of the expansion (4.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to (4.42) is given by*

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,,\quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\rho^T(\mathbf{x})R(\theta)\rho(\mathbf{x})\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{4.46}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$\frac{d\mathbf{Y}^0}{dt}=\overline{A}(\mathbf{Y}^0)\mathbf{U}^0,\qquad \frac{d\mathbf{U}^0}{dt}=\overline{\beta}(\mathbf{Y}^0,\mathbf{U}^0)+\overline{A}(\mathbf{Y}^0)\mathbf{E}_{\|}(t,\mathbf{Y}^0)+\mathbf{U}^0\times\overline{A}(\mathbf{Y}^0)\mathbf{e}_3,$$

$$\textit{with}\ \overline{A}(\mathbf{y})=\frac{1}{{y_1}^2+{y_2}^2}\begin{pmatrix}{y_2}^2 & -y_1y_2 & 0\\ -y_1y_2 & {y_1}^2 & 0\\ 0 & 0 & 0\end{pmatrix},\ \textit{and}\ \overline{\beta}(\mathbf{y},\mathbf{u})=\begin{pmatrix}\frac{(y_2u_1-y_1u_2)u_2}{{y_1}^2+{y_2}^2}\\ \frac{(y_1u_2-y_2u_1)u_1}{{y_1}^2+{y_2}^2}\\ 0\end{pmatrix}, \tag{4.47}$$

$$\mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{x},\,,\quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}.$$

The term $\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s)$ *is given by*

$$\begin{aligned}
{\mathbf{X}^1}_1(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega^2}\Big({\mathbf{Y}^0}_1\Omega(\cos(\theta)-1){\mathbf{U}^0}_3+{\mathbf{Y}^0}_1({\mathbf{Y}^0}_1{\mathbf{U}^0}_1+{\mathbf{Y}^0}_2{\mathbf{U}^0}_2)\sin(\theta)+{\mathbf{Y}^1}_1\Omega^2\Big),\\
{\mathbf{X}^1}_2(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega^2}\Big({\mathbf{Y}^0}_2\Omega(\cos(\theta)-1){\mathbf{U}^0}_3+{\mathbf{Y}^0}_2({\mathbf{Y}^0}_1{\mathbf{U}^0}_1+{\mathbf{Y}^0}_2{\mathbf{U}^0}_2)\sin(\theta)+{\mathbf{Y}^1}_2\Omega^2\Big),\\
{\mathbf{X}^1}_3(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega}\Big((-{\mathbf{Y}^0}_1{\mathbf{U}^0}_1-{\mathbf{Y}^0}_2{\mathbf{U}^0}_2)(\cos(\theta)-1)+{\mathbf{Y}^1}_3\Omega+\sin(\theta){\mathbf{U}^0}_3\Omega\Big),
\end{aligned} \tag{4.48}$$

where $\Omega = \sqrt{{\mathbf{Y}^0}_1^2 + {\mathbf{Y}^0}_2^2}$ *and where* $\mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s)$ *is solution to*

$$\frac{d{\mathbf{Y}^1}_1}{dt} = \bigg(\Big(- \Omega^2 {\mathbf{Y}^0}_2{\mathbf{U}^0}_2 + 2{\mathbf{Y}^0}_1{\mathbf{Y}^0}_2(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Big){\mathbf{Y}^1}_1 + \Big((2{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 - {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Omega^2 + 2{{\mathbf{Y}^0}_2}^2(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Big){\mathbf{Y}^1}_2 + {{\mathbf{Y}^0}_2}^2{\mathbf{U}^1}_1\Omega^2 - {\mathbf{Y}^0}_1{\mathbf{Y}^0}_2\Omega^2{\mathbf{U}^1}_2 - \Omega^3{\mathbf{Y}^0}_1\mathbf{E}_3 - {\mathbf{Y}^0}_2{\mathbf{U}^0}_3\Omega^3 - 2{\mathbf{Y}^0}_2{\mathbf{U}^0}_3(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Omega\bigg)\Big/\Omega^4,$$

$$\frac{d{\mathbf{Y}^1}_2}{dt} = \bigg(\Big((2{\mathbf{U}^0}_2{\mathbf{Y}^0}_1 - {\mathbf{U}^0}_1{\mathbf{Y}^0}_2)\Omega^2 - 2{{\mathbf{Y}^0}_1}^2(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Big){\mathbf{Y}^1}_1 + \Big(- \Omega^2{\mathbf{Y}^0}_1{\mathbf{U}^0}_1 - 2{\mathbf{Y}^0}_1{\mathbf{Y}^0}_2(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Big){\mathbf{Y}^1}_2 - {\mathbf{Y}^0}_1{\mathbf{Y}^0}_2\Omega^2{\mathbf{U}^1}_1 + {{\mathbf{Y}^0}_1}^2{\mathbf{U}^1}_2\Omega^2 - \Omega^3{\mathbf{Y}^0}_2\mathbf{E}_3 + \Omega^3{\mathbf{Y}^0}_1{\mathbf{U}^0}_3 + 2{\mathbf{Y}^0}_1{\mathbf{U}^0}_3(-{\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {\mathbf{U}^0}_2{\mathbf{Y}^0}_1)\Omega\bigg)\Big/\Omega^4,$$

$$\frac{d{\mathbf{Y}^1}_3}{dt} = \bigg(\Omega^2{\mathbf{Y}^0}_1\mathbf{E}_1 + \Omega^2{\mathbf{Y}^0}_2\mathbf{E}_2 + \Big({\mathbf{U}^0}_2{\mathbf{Y}^0}_1 + {{\mathbf{U}^0}_1}^2 - {\mathbf{U}^0}_1{\mathbf{Y}^0}_2 + {{\mathbf{U}^0}_2}^2\Big)\Omega^2 - \Big({\mathbf{Y}^0}_1{\mathbf{U}^0}_1 + {\mathbf{Y}^0}_2{\mathbf{U}^0}_2\Big)^2\bigg)\Big/\Omega^3,$$

$$\mathbf{Y}^1(s;\mathbf{x},\mathbf{v},s) = 0, \tag{4.49}$$

with $\mathbf{E}$ *being evaluated in* $\mathbf{Y}^0$. *The result concerning* $\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s)$ *is given in the appendix A.*

This Theorem is the consequence of the following computations. Setting $A = \rho^T R\rho$ we have

$$\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\} = \begin{pmatrix} I & 0 \\ \nabla_z(A\mathbf{w}) & A \end{pmatrix} \text{ and } \{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\}^{-1} = \begin{pmatrix} I & 0 \\ A^T\nabla_z(A\mathbf{w}) & A^T \end{pmatrix}. \tag{4.50}$$

Then, since in this example $\frac{\partial \mathbf{Z}}{\partial t} = 0$, and

$$\mathbf{a}(t,\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})) = \begin{pmatrix} A\mathbf{w} \\ \mathbf{E} + A\mathbf{w}\times\mathbf{e}_3 \end{pmatrix}, \tag{4.51}$$

applying (1.12) and (1.13) we get

$$\tilde{\alpha}^0(t,\theta,\mathbf{z},\mathbf{w}) = \begin{pmatrix} A\mathbf{w} \\ -A^T\nabla_z(A\mathbf{w})(A\mathbf{w}) + A^T\mathbf{E} + \mathbf{w}\times A^T\mathbf{e}_3 \end{pmatrix}, \tag{4.52}$$

and since $\int_0^{2\pi} A d\theta = \int_0^{2\pi} A^T d\theta = 2\pi\overline{A}$, and $\int_0^{2\pi} -A^T\nabla_z(A\mathbf{w})(A\mathbf{w}) = \overline{\beta}$, we finally get the first part of the Theorem.

The second part of the Theorem is obtained following the computation program described in the previous sections using Maple.

A Appendix : $\mathbf{V}^1$ for the Guiding Centre Regime with variable strong magnetic field

The computations of this appendix have been realized using Maple5.5, on computers of Medicis Centre, Ecole Polytechnique, Palaiseau, France.

The electric field $\mathbf{E}$ and its derivatives being evaluated in $\mathbf{Y}^0$, and Ω standing for $\sqrt{(\mathbf{Y}^0{}_1)^2+(\mathbf{Y}^0{}_2)^2}$, the term $\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s)$ of the expansion (4.2) of the solution to (4.42) is given by :

$$
\begin{aligned}
\mathbf{V}^1{}_1(t;\mathbf{x},\mathbf{v},s) = \Bigg(\bigg(\Big(&(2\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Omega^2-2\mathbf{Y}^0{}_1{}^2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1+\\
&\Big(\mathbf{Y}^0{}_1\mathbf{U}^0{}_2\Omega^2-2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2+\mathbf{Y}^0{}_1{}^2\mathbf{U}^1{}_1\Omega^2+\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2+\Omega^3\mathbf{Y}^0{}_1\mathbf{E}_3+\\
&\Omega^3\mathbf{Y}^0{}_2\mathbf{U}^0{}_3+\mathbf{Y}^0{}_2\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\bigg)\cos(\theta)+\Big((-\Omega^3\mathbf{U}^0{}_3+\Omega\mathbf{U}^0{}_3\mathbf{Y}^0{}_1{}^2)\mathbf{Y}^1{}_1+\\
&\mathbf{Y}^0{}_1\mathbf{U}^0{}_3\mathbf{Y}^0{}_2\Omega\mathbf{Y}^1{}_2-\mathbf{Y}^0{}_1\Omega^3\mathbf{U}^1{}_3+\Omega^2\mathbf{Y}^0{}_1{}^2\mathbf{E}_1+\Omega^2\mathbf{Y}^0{}_1\mathbf{E}_2\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\Omega^4+\\
&\mathbf{U}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2\Big)\sin(\theta)+\Big(-\Omega^2\mathbf{Y}^0{}_2\mathbf{U}^0{}_2+2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1+\\
&\Big((2\mathbf{U}^0{}_1\mathbf{Y}^0{}_2-\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2+2\mathbf{Y}^0{}_2{}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2+\mathbf{Y}^0{}_2{}^2\mathbf{U}^1{}_1\Omega^2-\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2-\\
&\Omega^3\mathbf{Y}^0{}_1\mathbf{E}_3-\Omega^3\mathbf{Y}^0{}_2\mathbf{U}^0{}_3-\mathbf{Y}^0{}_2\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\Big/\Omega^4, \quad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}^1{}_2(t;\mathbf{x},\mathbf{v},s) = \Bigg(\bigg(&(\mathbf{U}^0{}_1\mathbf{Y}^0{}_2\Omega^2-2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_1+\Big((\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+2\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Omega^2-\\
&2\mathbf{Y}^0{}_2{}^2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2+\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_1+\mathbf{Y}^0{}_2{}^2\mathbf{U}^1{}_2\Omega^2+\Omega^3\mathbf{Y}^0{}_2\mathbf{E}_3-\Omega^3\mathbf{Y}^0{}_1\mathbf{U}^0{}_3-\\
&\mathbf{Y}^0{}_1\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\bigg)\cos(\theta)+\Big(\mathbf{Y}^0{}_1\mathbf{U}^0{}_3\mathbf{Y}^0{}_2\Omega\mathbf{Y}^1{}_1+(-\Omega^3\mathbf{U}^0{}_3+\Omega\mathbf{U}^0{}_3\mathbf{Y}^0{}_2{}^2)\mathbf{Y}^1{}_2-\\
&\mathbf{Y}^0{}_2\Omega^3\mathbf{U}^1{}_3+\Omega^2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\mathbf{E}_1+\Omega^2\mathbf{Y}^0{}_2{}^2\mathbf{E}_2-\mathbf{U}^0{}_1\Omega^4-\mathbf{U}^0{}_1(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2\Big)\sin(\theta)+\\
&\Big((2\mathbf{Y}^0{}_1\mathbf{U}^0{}_2-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2)\Omega^2-2\mathbf{Y}^0{}_1{}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1+\\
&(-\Omega^2\mathbf{Y}^0{}_1\mathbf{U}^0{}_1-2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2))\mathbf{Y}^1{}_2-\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_1+\mathbf{Y}^0{}_1{}^2\mathbf{U}^1{}_2\Omega^2-\\
&\Omega^3\mathbf{Y}^0{}_2\mathbf{E}_3+\Omega^3\mathbf{Y}^0{}_1\mathbf{U}^0{}_3+\mathbf{Y}^0{}_1\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\Big/\Omega^4, \quad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}^1{}_3(t;\mathbf{x},\mathbf{v},s) = \Bigg(\bigg(&\Omega^3\mathbf{U}^1{}_3-\Omega^2\mathbf{Y}^0{}_1\mathbf{E}_1-\Omega^2\mathbf{Y}^0{}_2\mathbf{E}_2+(\mathbf{U}^0{}_1\mathbf{Y}^0{}_2-\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2-\\
&(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2)^2\bigg)\cos(\theta)+\Big((\mathbf{U}^0{}_1\Omega^2-\mathbf{Y}^0{}_1(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_1+\\
&(\mathbf{U}^0{}_2\Omega^2-\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_2+\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2+\Omega^3\mathbf{E}_3+\mathbf{Y}^0{}_1\Omega^2\mathbf{U}^1{}_1\Big)\sin(\theta)+\\
&\Omega^2\mathbf{Y}^0{}_1\mathbf{E}_1+\Omega^2\mathbf{Y}^0{}_2\mathbf{E}_2+(\mathbf{U}^0{}_1{}^2+\mathbf{Y}^0{}_1\mathbf{U}^0{}_2-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2{}^2)\Omega^2-(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)^2\Bigg)\Big/\Omega^3, \quad \text{(A.3)}
\end{aligned}
$$

where $\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s))$ is solution to:

$$
\begin{aligned}
\frac{d\mathbf{U}^1{}_1}{dt} = 1/4\,\Bigg(\bigg(&-4\,\mathbf{Y}^0{}_2\,\Omega^8\,\mathbf{Y}^0{}_1\,\tfrac{\partial\mathbf{E}_2}{\partial x_3}+4\mathbf{Y}^0{}_2{}^2\,\Omega^8\,\tfrac{\partial\mathbf{E}_1}{\partial x_3}\bigg)\,\mathbf{Y}^1{}_3+\Big(-4\Omega^9\,\mathbf{Y}^0{}_2+(-\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_3+4\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_1-4\mathbf{U}^0{}_1\\
&\mathbf{Y}^0{}_2{}^2)\,\Omega^7+(-\mathbf{Y}^0{}_1{}^4\,\mathbf{U}^0{}_1-2\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_1+2\,\mathbf{Y}^0{}_2{}^4\,\mathbf{U}^0{}_1+2\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_2+\mathbf{Y}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_3-\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1{}^3-2\,\mathbf{Y}^0{}_1{}^2\\
&\mathbf{U}^0{}_1\,\mathbf{Y}^0{}_2{}^2+\mathbf{Y}^0{}_1{}^3\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_2-4\,\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_2{}^3\,\mathbf{Y}^0{}_1)\,\Omega^5+\mathbf{Y}^0{}_1\,(3\,\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2{}^2+\mathbf{Y}^0{}_2{}^4+\mathbf{Y}^0{}_1{}^4+\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1{}^3)\,(\mathbf{Y}^0{}_1\,\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\\
&\mathbf{U}^0{}_2)\,\Omega^3\Big)\,\mathbf{E}_3+\Big(4\,\Omega^8\,\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_2-\mathbf{Y}^0{}_1{}^3\,\Omega^6\,\mathbf{U}^0{}_3+\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1{}^2\,(-\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_2-\mathbf{U}^0{}_1\,\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1+\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_3+\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2)\\
&\Omega^4+\mathbf{Y}^0{}_1{}^4\,\mathbf{Y}^0{}_2\,(\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1)\,(\mathbf{Y}^0{}_1\,\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_2)\,\Omega^2\Big)\,\mathbf{U}^1{}_1+\Big(-\Omega^7\,\mathbf{Y}^0{}_1\,\mathbf{U}^0{}_3+\mathbf{Y}^0{}_1\,(8\,\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_3-\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_1-\mathbf{U}^0{}_1\\
&\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1+5\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2+2\,\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_3)\,\Omega^5-\mathbf{Y}^0{}_1{}^2\,(8\,\mathbf{Y}^0{}_1\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_2{}^2-\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_1+7\,\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_3-\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2{}^2\\
&\mathbf{U}^0{}_2+6\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_3+7\,\mathbf{Y}^0{}_1{}^3\,\mathbf{U}^0{}_3-\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_1-\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_2)\,\Omega^3\Big)\,\mathbf{Y}^1{}_1{}^2+\Big(\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2{}^5\,\mathbf{U}^0{}_2{}^3-8\,\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_2{}^7+7\\
&\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_1{}^7+\mathbf{Y}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^5\,\mathbf{U}^0{}_1{}^3+6\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_1{}^7\,\mathbf{U}^0{}_2{}^2+\mathbf{Y}^0{}_1{}^6\,\mathbf{U}^0{}_1{}^3\,\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_2{}^4\,\mathbf{Y}^0{}_1{}^3\,\mathbf{U}^0{}_2{}^3+5\,\mathbf{U}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^5\,\mathbf{Y}^0{}_2{}^2\\
&\mathbf{U}^0{}_3+\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1\,\mathbf{Y}^0{}_1{}^7-5\,\mathbf{Y}^0{}_1{}^3\,\mathbf{Y}^0{}_2{}^4\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2{}^2+3\,\mathbf{Y}^0{}_1{}^5\,\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1{}^2+19\,\mathbf{Y}^0{}_1{}^4\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2{}^2\,\mathbf{Y}^0{}_2{}^3+9\,\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\\
&\mathbf{Y}^0{}_1{}^6\,\mathbf{Y}^0{}_2-9\,\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_2{}^6\,\mathbf{Y}^0{}_1+\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_1{}^4\,\mathbf{Y}^0{}_2{}^3-16\,\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3\,\mathbf{Y}^0{}_2{}^4\,\mathbf{Y}^0{}_1{}^3+3\,\mathbf{Y}^0{}_2{}^4\,\mathbf{Y}^0{}_1{}^3\,\mathbf{U}^0{}_2{}^2\,\mathbf{U}^0{}_1+\mathbf{Y}^0{}_1{}^5\\
&\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_1{}^2+3\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_1\,\mathbf{Y}^0{}_1{}^4\,\mathbf{U}^0{}_2{}^2-15\,\mathbf{Y}^0{}_2{}^5\,\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_1{}^2\,\mathbf{U}^0{}_3+3\,\mathbf{Y}^0{}_2{}^3\,\mathbf{Y}^0{}_1{}^4\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1{}^2-5\,\mathbf{Y}^0{}_2{}^6\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2{}^2\\
&\mathbf{Y}^0{}_1+10\,\mathbf{Y}^0{}_2{}^5\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^2+8\,\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^6-6\,\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2{}^5\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1+19\,\mathbf{Y}^0{}_1{}^5\,\mathbf{Y}^0{}_2{}^2\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1+7\\
&\mathbf{Y}^0{}_1{}^6\,\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_2\,\mathbf{U}^0{}_1+3\,\mathbf{Y}^0{}_1{}^4\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_1\,\mathbf{U}^0{}_2+16\,\mathbf{Y}^0{}_1{}^3\,\mathbf{Y}^0{}_2{}^4\,\mathbf{U}^0{}_3\,\mathbf{U}^0{}_1\,\mathbf{U}^0{}_2\Big)\,\Omega+(-4\,\Omega^8\,\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_1-\mathbf{Y}^0{}_1{}^2\,\Omega^6\\
&\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_3+\mathbf{Y}^0{}_1\,\Big(\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_3-\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_2{}^4+\mathbf{U}^0{}_2\,\mathbf{Y}^0{}_2{}^3\,\mathbf{Y}^0{}_1-\mathbf{Y}^0{}_1\,\mathbf{Y}^0{}_2{}^3\,\mathbf{U}^0{}_1+\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_1\,\mathbf{Y}^0{}_2{}^2+\mathbf{Y}^0{}_2{}^2\,\mathbf{Y}^0{}_1{}^2\,\mathbf{U}^0{}_3+\mathbf{U}^0{}_2\\
&\mathbf{Y}^0{}_2\,\mathbf{Y}^0{}_1{}^3+\mathbf{Y}^0{}_1{}^4\,\mathbf{U}^0{}_1\Big)\,\Omega^4-\mathbf{Y}^0{}_1{}^5\,(\mathbf{Y}^0{}_2+\mathbf{Y}^0{}_1)\,(\mathbf{Y}^0{}_1\,\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\,\mathbf{U}^0{}_2)\,\Omega^2)\,\mathbf{U}^1{}_2+\Big(6\,\Omega^6\,\mathbf{Y}^0{}_1{}^2\,\mathbf{Y}^0{}_2\,\mathbf{E}_1\,-2\,\mathbf{Y}^0{}_1\,(-\mathbf{Y}^0{}_2{}^2+2
\end{aligned}
$$

$$ {\mathbf{Y}^0_1}^2) \Omega^6 \mathbf{E}_2 +4 \Omega^8 {\mathbf{Y}^0_2}^2 \frac{\partial \mathbf{E}_1}{\partial x_2} -4 \Omega^8 \mathbf{Y}^0_1 \frac{\partial \mathbf{E}_2}{\partial x_2} \mathbf{Y}^0_2 - \Omega^8 \mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_1 (-\mathbf{Y}^0_1 \mathbf{U}^0_3 \mathbf{U}^0_2 + 3 \mathbf{Y}^0_2 {\mathbf{U}^0_3}^2 + \mathbf{U}^0_1 \mathbf{Y}^0_1 $$
$$ \mathbf{U}^0_2 + 12 \mathbf{Y}^0_2 {\mathbf{U}^0_2}^2 + 2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 - \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 4 \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 + 3 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 + \mathbf{Y}^0_1 {\mathbf{U}^0_3}^2) \Omega^6 + (-{\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + 2 $$
$$ {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 \mathbf{U}^0_2 + 10 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 \mathbf{U}^0_2 + \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3 - 3 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 - 8 \mathbf{Y}^0_2 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^3 - {\mathbf{Y}^0_1}^4 \mathbf{U}^0_2 $$
$$ \mathbf{U}^0_1 - 3 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^4 + \mathbf{U}^0_2 {\mathbf{Y}^0_1}^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 + 2 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_1}^2 - 9 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^2 - 2 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 2 {\mathbf{Y}^0_1}^3 $$
$$ \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 - 4 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 - 6 {\mathbf{U}^0_1}^2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 - 8 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^4 - {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_3}^2 \mathbf{Y}^0_1) \Omega^4 - \mathbf{Y}^0_2 (-8 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^5 - 2 \mathbf{Y}^0_1 $$
$$ {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_2}^2 - 7 \mathbf{Y}^0_2 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^4 - {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_3}^2 \mathbf{Y}^0_1 + 9 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 - 2 {\mathbf{Y}^0_1}^3 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^2 - 4 {\mathbf{Y}^0_1}^4 \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 - 9 $$
$$ {\mathbf{Y}^0_1}^4 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{U}^0_1 - 14 {\mathbf{Y}^0_2}^3 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 - 9 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^2 - 7 {\mathbf{Y}^0_1}^5 {\mathbf{U}^0_3}^2 + 11 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + \mathbf{Y}^0_2 \mathbf{U}^0_3 {\mathbf{Y}^0_1}^4 $$
$$ \mathbf{U}^0_1 + {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 + 7 \mathbf{U}^0_1 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^5 - 14 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_2 + \mathbf{U}^0_2 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^2 {\mathbf{Y}^0_1}^3 + 12 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 $$
$$ \mathbf{U}^0_2 + 2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 + \mathbf{U}^0_2 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 {\mathbf{Y}^0_1}^2 + 10 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^5 + 8 {\mathbf{Y}^0_1}^4 \mathbf{Y}^0_2 {\mathbf{U}^0_2}^2 - 6 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_2}^3 {\mathbf{Y}^0_1}^2) \Omega^2 + 2 $$
$$ {\mathbf{Y}^0_1}^4 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)\Big) \mathbf{Y}^1_2 + \Big(- 6 {\mathbf{Y}^0_2}^5 {\mathbf{U}^0_1}^3 + {\mathbf{Y}^0_1}^4 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^2 - 2 $$
$$ \mathbf{U}^0_3 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^5 + 5 {\mathbf{Y}^0_2}^2 {\mathbf{Y}^0_1}^3 {\mathbf{U}^0_2}^3 - 5 \mathbf{U}^0_3 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_1}^5 - \mathbf{U}^0_1 {\mathbf{Y}^0_1}^6 \mathbf{U}^0_3 + 8 {\mathbf{Y}^0_2}^5 {\mathbf{U}^0_1}^2 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + {\mathbf{Y}^0_1}^3 $$
$$ {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_1}^2 - {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 - 4 \mathbf{Y}^0_2 {\mathbf{U}^0_2}^3 {\mathbf{Y}^0_1}^4 + 6 {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_1}^3 \mathbf{Y}^0_1 - \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^4 {\mathbf{U}^0_1}^3 - 6 \mathbf{Y}^0_1 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_2}^2 + 12 $$
$$ {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_1}^2 - 5 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_3 \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 - {\mathbf{Y}^0_1}^2 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 - {\mathbf{Y}^0_1}^4 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 - 19 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^3 $$
$$ \mathbf{U}^0_2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_2 + 4 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^5 + 14 \mathbf{U}^0_1 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 - {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 {\mathbf{U}^0_1}^2 + {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^4 $$
$$ \mathbf{U}^0_2 - {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{U}^0_1 - 6 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 {\mathbf{U}^0_2}^2 + 9 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 + 12 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_3 \mathbf{Y}^0_1 - 18 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_2}^2 $$
$$ \mathbf{U}^0_1 {\mathbf{Y}^0_2}^3 + 18 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 {\mathbf{U}^0_1}^2 + {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{U}^0_3 - 5 {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 \mathbf{U}^0_1 \mathbf{U}^0_3 - 6 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 \mathbf{U}^0_3 - 2 {\mathbf{Y}^0_2}^5 \mathbf{Y}^0_1 $$
$$ \mathbf{U}^0_1 \mathbf{U}^0_3 - 2 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_1 \mathbf{U}^0_3 + 5 {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 \mathbf{U}^0_3 + 2 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 \mathbf{U}^0_3 - 2 {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1 \mathbf{U}^0_3 + 2 {\mathbf{Y}^0_1}^3 $$
$$ \mathbf{U}^0_2 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 - 3 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_2 {\mathbf{U}^0_1}^2 - \mathbf{Y}^0_2 \mathbf{U}^0_1 {\mathbf{Y}^0_1}^4 {\mathbf{U}^0_2}^2 - 17 \mathbf{U}^0_1 \mathbf{U}^0_3 {\mathbf{Y}^0_1}^4 \mathbf{Y}^0_2 \mathbf{U}^0_2 - 4 \mathbf{U}^0_1 \mathbf{U}^0_2 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_3 $$
$$ \mathbf{Y}^0_1 - 17 \mathbf{U}^0_1 \mathbf{U}^0_2 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^2 {\mathbf{Y}^0_1}^3 + 4 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^6 \mathbf{U}^0_3\Big) \Omega^3 + \Big(6 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^4 + 6 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + {\mathbf{Y}^0_1}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 - {\mathbf{Y}^0_1}^3 $$
$$ \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 - 6 {\mathbf{U}^0_1}^2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 + {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 \mathbf{U}^0_2 - 12 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 \mathbf{U}^0_2 + 6 \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3 - 4 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^3 \mathbf{Y}^0_2 $$
$$ \mathbf{U}^0_3 + {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 5 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 \mathbf{U}^0_1 {\mathbf{Y}^0_1}^2 + 11 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 2 \mathbf{U}^0_1 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 - 3 \mathbf{U}^0_3 {\mathbf{Y}^0_1}^2 $$
$$ \mathbf{Y}^0_2 {\mathbf{U}^0_2}^2 - \mathbf{Y}^0_2 {\mathbf{U}^0_2}^3 {\mathbf{Y}^0_1}^2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 {\mathbf{U}^0_2}^2 - \mathbf{U}^0_3 {\mathbf{Y}^0_1}^3 {\mathbf{U}^0_2}^2 - 2 \mathbf{Y}^0_1 {\mathbf{U}^0_3}^3 {\mathbf{Y}^0_2}^2 - 4 \mathbf{U}^0_3 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^4 - 5 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^3 $$
$$ \mathbf{Y}^0_2 + 2 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_3}^3 \mathbf{Y}^0_2 - 2 {\mathbf{Y}^0_1}^3 {\mathbf{U}^0_1}^2 \mathbf{U}^0_3 + 2 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 + 6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 {\mathbf{U}^0_2}^2 - 4 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 \mathbf{U}^0_3 $$
$$ \mathbf{Y}^0_2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 \mathbf{U}^0_3 \mathbf{U}^0_2\Big) \Omega^5 + \Big(4 \Omega^7 \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{Y}^0_1 + \mathbf{Y}^0_2 (6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 - 5 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 - 6 {\mathbf{Y}^0_1}^2 $$
$$ \mathbf{Y}^0_2 \mathbf{U}^0_3 + 6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1 - 6 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 - 5 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3) \Omega^5 + \mathbf{Y}^0_1 \mathbf{Y}^0_2 (3 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3 \mathbf{Y}^0_2 - {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 9 {\mathbf{Y}^0_2}^2 $$
$$ {\mathbf{Y}^0_1}^2 \mathbf{U}^0_3 - \mathbf{U}^0_2 {\mathbf{Y}^0_2}^3 \mathbf{Y}^0_1 + 5 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3 + 3 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^4 + 2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3) \Omega^3\Big) \mathbf{E}_2 $$
$$ +\Big(\Omega^8 \mathbf{Y}^0_1 \mathbf{U}^0_3 + (2 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 + 4 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 - 4 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 4 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3 - 4 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1) $$
$$ \Omega^6 - \mathbf{Y}^0_1 (\mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3 + 2 {\mathbf{Y}^0_2}^4 + 5 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 + 2 {\mathbf{Y}^0_1}^4) \mathbf{U}^0_3 \Omega^4\Big) \mathbf{U}^1_3 + \Big(- 4 \Omega^7 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 + \mathbf{Y}^0_1 (6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 - 5 $$
$$ {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_2 + {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 - 6 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 + 6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1 - 6 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 - 5 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3) \Omega^5 + {\mathbf{Y}^0_1}^2 (3 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3 $$
$$ \mathbf{Y}^0_2 - {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 9 {\mathbf{Y}^0_2}^2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_3 - \mathbf{U}^0_2 {\mathbf{Y}^0_2}^3 \mathbf{Y}^0_1 + 5 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3 + 3 \mathbf{U}^0_3 $$
$$ {\mathbf{Y}^0_2}^4 + 2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3) \Omega^3\Big) \mathbf{E}_1 + \Big((\mathbf{Y}^0_1 (-2 \mathbf{U}^0_3 + \mathbf{U}^0_2) \Omega^7 + \mathbf{Y}^0_1 (-\mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 8 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 + 4 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_3 - \mathbf{U}^0_1 $$
$$ \mathbf{Y}^0_2 \mathbf{Y}^0_1 - 2 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 9 \mathbf{U}^0_3 \mathbf{Y}^0_1 \mathbf{Y}^0_2) \Omega^5 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 (7 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_3 - \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 - {\mathbf{Y}^0_2}^3 \mathbf{U}^0_2 - {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 $$
$$ \mathbf{U}^0_1 + 6 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 + 8 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_3 + 9 \mathbf{Y}^0_1 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^2 - \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1) \Omega^3\Big) \mathbf{Y}^1_2 + ({\mathbf{Y}^0_1}^2 \Omega^7 - {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) $$
$$ \Omega^5) \mathbf{U}^1_1 + (\mathbf{Y}^0_1 \mathbf{Y}^0_2 \Omega^7 - \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \Omega^5) \mathbf{U}^1_2 + (-\mathbf{Y}^0_1 (\mathbf{Y}^0_2 + 2 \mathbf{Y}^0_1) \Omega^7 + {\mathbf{Y}^0_1}^2 (\mathbf{Y}^0_1 \mathbf{Y}^0_2 + 2 {\mathbf{Y}^0_1}^2 + 3 $$
$$ {\mathbf{Y}^0_2}^2) \Omega^5) \mathbf{U}^1_3 + (\mathbf{Y}^0_1 (-8 {\mathbf{Y}^0_2}^2 + 2 {\mathbf{Y}^0_1}^2 + 3 \mathbf{Y}^0_1 \mathbf{Y}^0_2) \Omega^6 - {\mathbf{Y}^0_1}^3 (\mathbf{Y}^0_1 \mathbf{Y}^0_2 + 2 {\mathbf{Y}^0_1}^2 + 3 {\mathbf{Y}^0_2}^2) \Omega^4) \mathbf{E}_1 + \big(\mathbf{Y}^0_2 (3 \mathbf{Y}^0_1 $$
$$ \mathbf{Y}^0_2 - 4 {\mathbf{Y}^0_2}^2 + 6 {\mathbf{Y}^0_1}^2) \Omega^6 - \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^2 (\mathbf{Y}^0_1 \mathbf{Y}^0_2 + 2 {\mathbf{Y}^0_1}^2 + 3 {\mathbf{Y}^0_2}^2) \Omega^4\big) \mathbf{E}_2 + (\Omega^8 \mathbf{Y}^0_1 - \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \Omega^6) $$
$$ \mathbf{E}_3 +4 \Omega^8 {\mathbf{Y}^0_2}^2 \frac{\partial \mathbf{E}_1}{\partial x_1} -4 \Omega^8 \mathbf{Y}^0_1 \mathbf{Y}^0_2 \frac{\partial \mathbf{E}_2}{\partial x_1} -\mathbf{Y}^0_1 (-\mathbf{U}^0_2 + 2 \mathbf{U}^0_1) \Omega^8 + \big(5 \mathbf{Y}^0_1 \mathbf{Y}^0_2 {\mathbf{U}^0_2}^2 - 2 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 2 {\mathbf{Y}^0_1}^2 $$
$$ \mathbf{Y}^0_2 \mathbf{U}^0_2 - {\mathbf{Y}^0_1}^2 \mathbf{U}^0_2 \mathbf{U}^0_1 - \mathbf{U}^0_2 {\mathbf{Y}^0_1}^3 - \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_1 - 2 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_3}^2 - 6 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + 2 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 - 8 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^2 - 9 $$
$$ \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 4 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^2\big) \Omega^6 + \big(- 6 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^2 + {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_3}^2 - 6 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_3}^2 \mathbf{Y}^0_1 + 14 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 $$
$$ \mathbf{U}^0_2 + 4 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{U}^0_2 + 2 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^4 - 13 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_1}^4 - 2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 3 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_2 \mathbf{U}^0_1 - 13 \mathbf{Y}^0_2 {\mathbf{U}^0_3}^2 $$
$$ {\mathbf{Y}^0_1}^3 - 8 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 + \mathbf{U}^0_2 {\mathbf{Y}^0_1}^3 \mathbf{Y}^0_2 \mathbf{U}^0_3 - 6 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 + 4 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 + \mathbf{U}^0_3 \mathbf{U}^0_1 \mathbf{Y}^0_2 {\mathbf{Y}^0_1}^3\big) \Omega^4 + \big(3 $$
$$ {\mathbf{Y}^0_1}^4 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^2 + 4 {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_2}^2 - 3 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_1}^2 - 5 {\mathbf{Y}^0_1}^3 {\mathbf{Y}^0_2}^3 {\mathbf{U}^0_2}^2 + 7 {\mathbf{Y}^0_1}^4 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 - 3 {\mathbf{Y}^0_1}^3 \mathbf{U}^0_1 $$
$$ {\mathbf{Y}^0_2}^3 \mathbf{U}^0_2 + 4 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^4 \mathbf{U}^0_2 + 6 {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{U}^0_1 - {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 \mathbf{U}^0_1 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^4 {\mathbf{Y}^0_2}^2 $$
$$ \mathbf{U}^0_1 \mathbf{U}^0_3 - {\mathbf{Y}^0_1}^3 \mathbf{U}^0_2 {\mathbf{Y}^0_2}^3 \mathbf{U}^0_3 + 2 {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2 + 18 {\mathbf{Y}^0_1}^3 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_2}^3 + 6 \mathbf{Y}^0_1 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_2}^5 - 6 {\mathbf{Y}^0_1}^6 \mathbf{U}^0_1 \mathbf{U}^0_2 - 5 \mathbf{Y}^0_1 $$
$$ {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^5 + 5 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^5 {\mathbf{U}^0_2}^2 + 7 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^4 - 11 {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 {\mathbf{U}^0_2}^2 - 8 \mathbf{U}^0_1 \mathbf{U}^0_2 {\mathbf{Y}^0_2}^6 + 7 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_3}^2 + 21 $$
$$ {\mathbf{Y}^0_1}^4 {\mathbf{U}^0_3}^2 {\mathbf{Y}^0_2}^2 + 13 {\mathbf{Y}^0_1}^5 \mathbf{Y}^0_2 {\mathbf{U}^0_3}^2 + 16 {\mathbf{Y}^0_1}^2 {\mathbf{Y}^0_2}^4 {\mathbf{U}^0_2}^2 - 18 \mathbf{Y}^0_1 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^5 \mathbf{U}^0_2 + {\mathbf{Y}^0_2}^6 {\mathbf{U}^0_2}^2 - 6 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^6 + 8 $$
$$ {\mathbf{U}^0_1}^2 {\mathbf{Y}^0_2}^6 + 13 {\mathbf{Y}^0_1}^6 {\mathbf{U}^0_3}^2) \Omega^2 + 2 {\mathbf{Y}^0_1}^5 (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) (-\mathbf{U}^0_1 \mathbf{Y}^0_2 + \mathbf{U}^0_2 \mathbf{Y}^0_1) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2)\Big) \mathbf{Y}^1_1 - 4 \Omega^9 \mathbf{Y}^0_1 $$
$$ \frac{\partial \mathbf{E}_3}{\partial t} -2 \mathbf{U}^0_3 (\mathbf{Y}^0_1 + 2 \mathbf{U}^0_2) \Omega^9 + \mathbf{U}^0_3 \Big(- 5 \mathbf{Y}^0_1 {\mathbf{U}^0_2}^2 - 3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 + 4 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{U}^0_1 + {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1\Big) \Omega^7 - 2 \mathbf{Y}^0_1 \Big(- 3 $$
$$ \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 - {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + 2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_2\Big) \Omega^7 \frac{\partial \mathbf{E}_3}{\partial x_2} +6 \Omega^7 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_3 \mathbf{Y}^0_2 \frac{\partial \mathbf{E}_2}{\partial x_1} +6 \Omega^7 \mathbf{Y}^0_1 \mathbf{U}^0_3 {\mathbf{Y}^0_2}^2 \frac{\partial \mathbf{E}_2}{\partial x_2} +2 \mathbf{Y}^0_1 (-2 $$
$$ {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2) \mathbf{U}^0_3 \Omega^7 \frac{\partial \mathbf{E}_1}{\partial x_1} +2 \mathbf{Y}^0_1 \Big(- 2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 + 3 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1\Big) \Omega^7 \frac{\partial \mathbf{E}_3}{\partial x_1} +2 \Omega^9 \mathbf{Y}^0_1 \frac{\partial \mathbf{E}_3}{\partial x_3} \mathbf{U}^0_3 - 2 $$
$$ (-2 {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^7 \frac{\partial \mathbf{E}_1}{\partial x_3} -6 \mathbf{Y}^0_1 \mathbf{Y}^0_2 (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^7 \frac{\partial \mathbf{E}_2}{\partial x_3} +2 \mathbf{Y}^0_2 (-2 {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2) $$
$$ \mathbf{U}^0_3 \Omega^7 \frac{\partial \mathbf{E}_1}{\partial x_2} +({\mathbf{Y}^0_2}^2 \mathbf{U}^0_3 \Omega^5 \mathbf{Y}^0_1 - \Omega^7 \mathbf{Y}^0_1 \mathbf{U}^0_3) {\mathbf{Y}^1_2}^2\Big) \Big/ \Omega^{10}, $$

$$ \frac{d\mathbf{U}^1_2}{dt} = 1/4 \Big(- 4 \Omega^9 \mathbf{Y}^0_2 \frac{\partial \mathbf{E}_3}{\partial t} -2 \mathbf{U}^0_3 (\mathbf{Y}^0_2 - 4 \mathbf{U}^0_1) \Omega^9 - \mathbf{U}^0_3 \Big(3 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 + 4 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 - 4 \mathbf{U}^0_1 \mathbf{Y}^0_1 \mathbf{U}^0_2 + \mathbf{Y}^0_2 $$
$$ {\mathbf{U}^0_2}^2 - 2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_2 - \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 + 4 \mathbf{Y}^0_2 {\mathbf{U}^0_1}^2\Big) \Omega^7 + 2 (-{\mathbf{Y}^0_2}^2 + 2 {\mathbf{Y}^0_1}^2) (\mathbf{Y}^0_1 \mathbf{U}^0_1 + \mathbf{Y}^0_2 \mathbf{U}^0_2) \Omega^7 \frac{\partial \mathbf{E}_2}{\partial x_3} +2 \Omega^9 $$
$$ \mathbf{Y}^0_2 \frac{\partial \mathbf{E}_3}{\partial x_3} \mathbf{U}^0_3 + 2 \mathbf{Y}^0_2 \big(- 2 \mathbf{U}^0_1 {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2 \mathbf{U}^0_1 + 3 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1\big) \Omega^7 \frac{\partial \mathbf{E}_3}{\partial x_1} -2 \mathbf{Y}^0_2 \big(- 3 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{Y}^0_1 - {\mathbf{Y}^0_2}^2 $$
$$ \mathbf{U}^0_2 + 2 {\mathbf{Y}^0_1}^2 \mathbf{U}^0_2\big) \Omega^7 \frac{\partial \mathbf{E}_3}{\partial x_2} +\Big(- 2 \mathbf{Y}^0_1 (-2 {\mathbf{Y}^0_2}^2 + {\mathbf{Y}^0_1}^2) \Omega^6 \mathbf{E}_1 -6 \Omega^6 {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{E}_2 -4 \Omega^8 \mathbf{Y}^0_1 \mathbf{Y}^0_2 \frac{\partial \mathbf{E}_1}{\partial x_2} +4 \Omega^8 $$
$$ {\mathbf{Y}^0_1}^2 \frac{\partial \mathbf{E}_2}{\partial x_2} +(-2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + \mathbf{U}^0_1 \mathbf{Y}^0_2) \Omega^8 + (-2 {\mathbf{U}^0_2}^2 {\mathbf{Y}^0_1}^2 - 6 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_1}^2 + 4 \mathbf{U}^0_2 {\mathbf{Y}^0_1}^3 - 2 {\mathbf{Y}^0_1}^2 {\mathbf{U}^0_3}^2 - {\mathbf{Y}^0_2}^3 \mathbf{U}^0_1 - 2 $$
$$ {\mathbf{Y}^0_1}^2 \mathbf{Y}^0_2 \mathbf{U}^0_1 - \mathbf{Y}^0_2 \mathbf{U}^0_3 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 3 \mathbf{Y}^0_1 {\mathbf{Y}^0_2}^2 \mathbf{U}^0_2 + {\mathbf{Y}^0_2}^2 {\mathbf{U}^0_3}^2 + {\mathbf{U}^0_3}^2 \mathbf{Y}^0_1 \mathbf{Y}^0_2 - 11 \mathbf{U}^0_1 \mathbf{Y}^0_2 \mathbf{U}^0_2 \mathbf{Y}^0_1 + 3 {\mathbf{U}^0_1}^2 $$

$$\mathbf{Y}^0{}_2{}^2)\ \Omega^6 + (-4\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^4 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 8\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2 + 10\ \mathbf{U}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3 - 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 + \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 + 5\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - 7\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3{}^2$$

$$\mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^2 - \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^3 + 2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + 7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 - 11\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3$$

$$\mathbf{Y}^0{}_2 + \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2 + 2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2)\ \Omega^4 + \mathbf{Y}^0{}_2\ (7\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 - 18\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_1 - 14$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2 - 6\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 + 18\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^5 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1 - 20\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_2{}^2 - 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2 + \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^5 - 16\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 31\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 6\ \mathbf{Y}^0{}_2{}^4$$

$$\mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1 - 2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 + 9\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^2 + 7\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 8\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5 + 7\ \mathbf{Y}^0{}_1{}^3$$

$$\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1 + 7\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2 + 26\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2)\ \Omega^2 + 2$$

$$\mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ (-\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)\Big)\ \mathbf{Y}^1{}_2 + \Big(\mathbf{Y}^0{}_2\ \Omega^8\ \mathbf{U}^0{}_3 - (-\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_2$$

$$\mathbf{Y}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^3 - \mathbf{Y}^0{}_2{}^3)\ \mathbf{U}^0{}_3\ \Omega^6 - \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 2\ \mathbf{Y}^0{}_2{}^4 + 5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^4)\ \mathbf{U}^0{}_3\ \Omega^4\Big)\ \mathbf{U}^1{}_3 + \Big(\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5 + 2$$

$$\mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^3 - \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1{}^2 - \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - 5\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 8\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2$$

$$\mathbf{U}^0{}_1{}^2 - 15\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 - 11\ \mathbf{U}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - 2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1 - 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 4\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^5$$

$$\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 6\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 + 2\ \mathbf{Y}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - 5\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 + 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - 6\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 + 10\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3$$

$$\mathbf{Y}^0{}_1 + 9\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_2{}^5 - 2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^3 - \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^3 + \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - 31\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3$$

$$\mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2\Big)\ \Omega^3 + (-\mathbf{Y}^0{}_2\ \Omega^7\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \Omega^5)\ \mathbf{Y}^1{}_2{}^2 + \Big(4\ \Omega^8\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_2\ (4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 4\ \mathbf{U}^0{}_1$$

$$\mathbf{Y}^0{}_2{}^2 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^6 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3)\ \Omega^4 - \mathbf{Y}^0{}_1{}^4$$

$$\mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^2\Big)\ \mathbf{U}^1{}_2 + \Big(-4\ \mathbf{Y}^0{}_2\ \Omega^8\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^6 + \mathbf{Y}^0{}_2$$

$$\mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3)\ \Omega^4 + \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^2\Big)$$

$$\mathbf{U}^1{}_1 + \Big(\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2{}^3 - 3\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^7 - 2\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^7 + \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5\ \mathbf{U}^0{}_1{}^3 + 10\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3 + 17$$

$$\mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_2{}^3 + 17\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^6\ \mathbf{Y}^0{}_2 + 17\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^6\ \mathbf{Y}^0{}_1 + 21\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^4$$

$$\mathbf{Y}^0{}_2{}^3 + 33\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3 + 3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^2\ \mathbf{U}^0{}_1 + 15\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 6\ \mathbf{Y}^0{}_2{}^6$$

$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1 + 3\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2 + 12\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 11\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 3\ \mathbf{Y}^0{}_1{}^6\ \mathbf{Y}^0{}_2$$

$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 11\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 + 11\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_2{}^7\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_2{}^3 + \mathbf{Y}^0{}_2{}^3$$

$$\mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^3 + 3\ \mathbf{Y}^0{}_2{}^5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1\Big)\ \Omega + \Big(2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^2 - 3$$

$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 + 11\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2$$

$$\mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 - 13\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - 4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 + \mathbf{U}^0{}_3$$

$$\mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^4 - 7\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3{}^3\ \mathbf{Y}^0{}_2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3{}^3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^3 + 3$$

$$\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\Big)\ \Omega^5 + \Big(4\ \mathbf{Y}^0{}_1{}^2\ \Omega^8\ \frac{\partial \mathbf{E}_2}{\partial x_3}\ -4\ \mathbf{Y}^0{}_1\ \Omega^8\ \frac{\partial \mathbf{E}_1}{\partial x_3}\ \mathbf{Y}^0{}_2\Big)\ \mathbf{Y}^1{}_3 + \Big(4\ \Omega^9\ \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_1\ (\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + 4\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)$$

$$\Omega^7 + (-3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 5\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4 + 3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1 - 6\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1 - 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2)\ \Omega^5 + \mathbf{Y}^0{}_2\ (3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_2{}^4 + \mathbf{Y}^0{}_1{}^4 + \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3)\ (\mathbf{Y}^0{}_1$$

$$\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^3\Big)\ \mathbf{E}_3 + \Big(4\ \Omega^7\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_1\ (-8\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_1{}^3$$

$$\mathbf{U}^0{}_3 - 3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1)\ \Omega^5 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 9\ \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2$$

$$\mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1 + 5\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3)\ \Omega^3\Big)\ \mathbf{E}_1 + \Big(-\mathbf{Y}^0{}_2\ \Omega^7$$

$$\mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\ (8\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 + 5\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3)\ \Omega^5 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (8\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3$$

$$\mathbf{Y}^0{}_2{}^2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1 + 7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 6\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + 7\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2)$$

$$\Omega^3\Big)\ \mathbf{Y}^1{}_1{}^2 + \Big((\mathbf{Y}^0{}_2\ (-2\ \mathbf{U}^0{}_3 + \mathbf{U}^0{}_2)\ \Omega^7 + \mathbf{Y}^0{}_2\ (-\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 8\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 - 2\ \mathbf{Y}^0{}_2{}^2$$

$$\mathbf{U}^0{}_2 + 9\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^5 - \mathbf{Y}^0{}_2{}^2\ (7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1 + 6\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + 8$$

$$\mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3 + 9\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1)\ \Omega^3\Big)\ \mathbf{Y}^1{}_2 + (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \Omega^7 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ \Omega^5)\ \mathbf{U}^1{}_1 + (\mathbf{Y}^0{}_2{}^2$$

$$\Omega^7 - \mathbf{Y}^0{}_2{}^3\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ \Omega^5)\ \mathbf{U}^1{}_2 + (-\mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1)\ \Omega^7 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^2)\ \Omega^5)\ \mathbf{U}^1{}_3 + ((6$$

$$\mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^2 - 2\ \mathbf{Y}^0{}_1{}^3 - 4\ \mathbf{Y}^0{}_2{}^3 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2)\ \Omega^6 - \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^2)\ \Omega^4)\ \mathbf{E}_1 + (-\mathbf{Y}^0{}_2\ (-10\ \mathbf{Y}^0{}_1$$

$$\mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 - \mathbf{Y}^0{}_2{}^2)\ \Omega^6 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^2)\ \Omega^4)\ \mathbf{E}_2 + (\Omega^8\ \mathbf{Y}^0{}_2 - \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ \Omega^6)\ \mathbf{E}_3\ -4\ \Omega^8$$

$$\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E}_1}{\partial x_1}\ +4\ \Omega^8\ \mathbf{Y}^0{}_1{}^2\ \frac{\partial \mathbf{E}_2}{\partial x_1}\ +(\mathbf{U}^0{}_1{}^2 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 - 2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^8 + \Big(\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1 - 2\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2 - 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2 - \mathbf{U}^0{}_1$$

$$\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - 2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 15\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 3$$

$$\mathbf{U}^0{}_2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1 + 2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^6 + \big(\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - 10\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1 - 7\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1$$

$$\mathbf{U}^0{}_2 + 6\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 + 11\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2 - 8\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 6\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 - 19\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2$$

$$\mathbf{U}^0{}_2 + 2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2 + 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2 - 11\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^2 - 16$$

$$\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^3\big)\ \Omega^4 + \big(-3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 10\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1{}^2 - 16\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 + 6\ \mathbf{Y}^0{}_1{}^4$$

$$\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 + 11\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 + 8\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - 12\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 21\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^3 + 7\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3{}^2$$

$$\mathbf{Y}^0{}_2{}^5 + 8\ \mathbf{Y}^0{}_1{}^6\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5 - 5\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2{}^2 - 3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - 6\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 4\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2$$

$$\mathbf{Y}^0{}_2{}^6 + 18\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 + 13\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^2 + 13\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 4\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2 + 3$$

$$\mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^6 + 6\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_3{}^2 + 3\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_1{}^6\big)\ \Omega^2 + 2\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (-\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2$$

$$\mathbf{U}^0{}_2)\Big)\ \mathbf{Y}^1{}_1 + \Big(-4\ \Omega^7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\ (-8\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3 - 3\ \mathbf{Y}^0{}_2{}^3$$

$$\mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1)\ \Omega^5 + \mathbf{Y}^0{}_2{}^2\ (3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 9\ \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1 + 5\ \mathbf{Y}^0{}_1{}^4$$

$$\mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3)\ \Omega^3\Big)\ \mathbf{E}_2 + 6\ \Omega^7\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \frac{\partial \mathbf{E}_1}{\partial x_2}\ \mathbf{U}^0{}_3 + 6\ \Omega^7\ \mathbf{Y}^0{}_1{}^2$$

$$\mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E}_1}{\partial x_1}\ \mathbf{U}^0{}_3 - 6\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^7\ \frac{\partial \mathbf{E}_1}{\partial x_3}\ -2\ \mathbf{Y}^0{}_1\ (-\mathbf{Y}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^2)\ \mathbf{U}^0{}_3\ \Omega^7\ \frac{\partial \mathbf{E}_2}{\partial x_1}\ -2\ \mathbf{Y}^0{}_2\ (-\mathbf{Y}^0{}_2{}^2 + 2$$

$$
\mathbf{Y}^0_1{}^2)\ \mathbf{U}^0_3\ \Omega^7\ \frac{\partial \mathbf{E}_2}{\partial x_2} \Big) \Big/ \Omega^{10},
$$

$$
\begin{aligned}
&\frac{d\mathbf{U}^1_3}{dt} = 1/4 \Big(4\ \Omega^7\ \mathbf{Y}^0_2\ \frac{\partial \mathbf{E}_2}{\partial t} + 4\ \Omega^7\ \frac{\partial \mathbf{E}_1}{\partial t}\ \mathbf{Y}^0_1 + 2\ \mathbf{Y}^0_1 \Big(-3\ \mathbf{U}^0_1\ \mathbf{Y}^0_2\ \mathbf{Y}^0_1 - \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_2\Big)\ \Omega^5\ \frac{\partial \mathbf{E}_1}{\partial x_2} + 2\ \Omega^7 \\
&\mathbf{U}^0_3\ \mathbf{Y}^0_2\ \frac{\partial \mathbf{E}_3}{\partial x_2} - 2\ \Omega^7\ \mathbf{U}^0_3\ \mathbf{Y}^0_1\ \frac{\partial \mathbf{E}_1}{\partial x_3} + 2\ \Omega^7\ \mathbf{U}^0_3\ \frac{\partial \mathbf{E}_3}{\partial x_1}\ \mathbf{Y}^0_1 - 2\ \Omega^7\ \mathbf{U}^0_3\ \mathbf{Y}^0_2\ \frac{\partial \mathbf{E}_2}{\partial x_3} - 2\ \mathbf{Y}^0_2 \Big(-2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1 + 3 \\
&\mathbf{Y}^0_2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1\Big)\ \Omega^5\ \frac{\partial \mathbf{E}_2}{\partial x_1} + 2\ \mathbf{Y}^0_2 \Big(-3\ \mathbf{U}^0_1\ \mathbf{Y}^0_2\ \mathbf{Y}^0_1 - \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_2\Big)\ \Omega^5\ \frac{\partial \mathbf{E}_2}{\partial x_2} - 2\ \mathbf{Y}^0_1 \Big(-2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_1{}^2 \\
&\mathbf{U}^0_1 + 3\ \mathbf{Y}^0_2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1\Big)\ \Omega^5\ \frac{\partial \mathbf{E}_1}{\partial x_1} + \Big(-2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^5\ \mathbf{U}^0_3{}^2 - 2\ \mathbf{Y}^0_2{}^5\ \mathbf{U}^0_1{}^3 - 2\ \mathbf{Y}^0_2{}^2\ \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_2{}^3 + 2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2{}^3\ \mathbf{Y}^0_1{}^4 + 8\ \mathbf{Y}^0_2{}^4 \\
&\mathbf{U}^0_1{}^3\ \mathbf{Y}^0_1 + \mathbf{Y}^0_1\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_2{}^2 + \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1{}^2 + \mathbf{Y}^0_1{}^4\ \mathbf{U}^0_3\ \mathbf{Y}^0_2\ \mathbf{U}^0_1{}^2 + \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2{}^2 - 8\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1{}^2 \\
&\mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2{}^2 + 4\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2\ \mathbf{U}^0_1{}^2 - \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2{}^2 - \mathbf{U}^0_1{}^2\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3 - \mathbf{U}^0_1{}^2 \\
&\mathbf{Y}^0_2{}^4\ \mathbf{U}^0_3\ \mathbf{Y}^0_1 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_2{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^3 + 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_2\ \mathbf{U}^0_1{}^2 - 2\ \mathbf{Y}^0_2\ \mathbf{U}^0_1\ \mathbf{Y}^0_1{}^4\ \mathbf{U}^0_2{}^2 + 4\ \mathbf{U}^0_1\ \mathbf{U}^0_2\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^2 \\
&\mathbf{Y}^0_1{}^3 + 4\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2{}^3 + 4\ \mathbf{Y}^0_2{}^2\ \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_1{}^3 - \mathbf{Y}^0_1{}^4\ \mathbf{Y}^0_2\ \mathbf{U}^0_3\ \mathbf{U}^0_2{}^2 - 3\ \mathbf{U}^0_2\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2 - 5\ \mathbf{U}^0_2\ \mathbf{Y}^0_2\ \mathbf{U}^0_3{}^2 \\
&\mathbf{Y}^0_1{}^4 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_3{}^2 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^4\ \mathbf{U}^0_3{}^2 - 3\ \mathbf{U}^0_2\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1 - 3\ \mathbf{U}^0_2\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_2{}^3\ \mathbf{Y}^0_1{}^2 - 6\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_1 \\
&\mathbf{Y}^0_1\ \mathbf{U}^0_2{}^2 + 4\ \mathbf{U}^0_1\ \mathbf{U}^0_2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3\ \mathbf{Y}^0_1{}^2 - 2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1{}^3\ \mathbf{Y}^0_1{}^2 + 2\ \mathbf{Y}^0_2{}^5\ \mathbf{U}^0_2\ \mathbf{U}^0_1{}^2\Big)\ \Omega + \Big(4\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_2\ \mathbf{U}^0_1 - 4\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2 \\
&\mathbf{U}^0_1{}^2 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3{}^2 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2 - 2\ \mathbf{Y}^0_1\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_2 + \mathbf{Y}^0_1\ \mathbf{U}^0_1{}^2\ \mathbf{U}^0_3 + \mathbf{U}^0_3\ \mathbf{Y}^0_1\ \mathbf{U}^0_2{}^2 + 4\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ \mathbf{U}^0_2{}^2 - 4 \\
&\mathbf{U}^0_2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2\Big)\ \Omega^5 + \Big(\mathbf{U}^0_3\ (-3\ \mathbf{U}^0_2 + \mathbf{Y}^0_1)\ \Omega^6 - \mathbf{Y}^0_1\ \mathbf{U}^0_3\ (2\ \mathbf{Y}^0_1\ \mathbf{U}^0_1 + 3\ \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega^4 + \mathbf{U}^0_3\ (-\mathbf{U}^0_2\ \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^3 + 7 \\
&\mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2 + 4\ \mathbf{Y}^0_1{}^4\ \mathbf{U}^0_1 - 3\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1 + 3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^4 + 10\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + 2\ \mathbf{U}^0_2\ \mathbf{Y}^0_2{}^3\ \mathbf{Y}^0_1 + 4\ \mathbf{U}^0_2\ \mathbf{Y}^0_2{}^4) \\
&\Omega^2 + \mathbf{Y}^0_2\ \mathbf{U}^0_3\ (-\mathbf{Y}^0_2{}^5\ \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^3 + 2\ \mathbf{Y}^0_1{}^5\ \mathbf{U}^0_2 + 2\ \mathbf{Y}^0_1{}^4\ \mathbf{Y}^0_2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_2 - 3 \\
&\mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_1 - 3\ \mathbf{Y}^0_1{}^4\ \mathbf{Y}^0_2\ \mathbf{U}^0_1 - 5\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1)\Big)\ \mathbf{Y}^1_2 + \Big(4\ \Omega^7\ \mathbf{Y}^0_1 + 2\ \mathbf{Y}^0_1\ (6\ \mathbf{U}^0_2\ \mathbf{Y}^0_1 + \mathbf{Y}^0_2 \\
&\mathbf{U}^0_3 - 6\ \mathbf{U}^0_1\ \mathbf{Y}^0_2)\ \Omega^5 - 2\ \mathbf{Y}^0_2{}^2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ (\mathbf{Y}^0_1\ \mathbf{U}^0_3 + \mathbf{U}^0_2\ \mathbf{Y}^0_1 - \mathbf{U}^0_1\ \mathbf{Y}^0_2)\ \Omega^3 + 2\ \mathbf{Y}^0_2{}^2\ (\mathbf{Y}^0_1{}^3 + 2\ \mathbf{Y}^0_2{}^3 + \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^2) \\
&(-\mathbf{U}^0_1\ \mathbf{Y}^0_2 + \mathbf{U}^0_2\ \mathbf{Y}^0_1)\ \Omega\Big)\ \mathbf{E}_2 + \Big(((-2\ \mathbf{Y}^0_1\ \mathbf{U}^0_1 + 3\ \mathbf{Y}^0_2\ \mathbf{U}^0_2 + 3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1)\ \Omega^5 + (-2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2 - 6\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 - 5 \\
&\mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2 - 2\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_1 + 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1 - 3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^3)\ \Omega^3 + \mathbf{Y}^0_2\ (\mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^3 - \mathbf{Y}^0_2{}^3 + 3\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2) \\
&(\mathbf{Y}^0_1\ \mathbf{U}^0_1 + \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega)\ \mathbf{Y}^1_2 + (\mathbf{Y}^0_2\ \Omega^5\ \mathbf{Y}^0_1 - \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \Omega^3)\ \mathbf{U}^1_1 + (\Omega^5\ \mathbf{Y}^0_2{}^2 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \\
&\Omega^3)\ \mathbf{U}^1_2 + (\Omega^7 - \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \Omega^5)\ \mathbf{U}^1_3 + (-\mathbf{Y}^0_1\ \Omega^6 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \Omega^4)\ \mathbf{E}_1 + (-\mathbf{Y}^0_2\ \Omega^6 + \mathbf{Y}^0_2{}^2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \\
&\Omega^4)\ \mathbf{E}_2 + (\mathbf{Y}^0_2\ \Omega^6 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \Omega^4)\ \mathbf{E}_3 + \big(\mathbf{U}^0_1\ \mathbf{Y}^0_2 - 4\ \mathbf{U}^0_1\ \mathbf{U}^0_3 - \mathbf{Y}^0_2\ \mathbf{U}^0_3 - \mathbf{U}^0_2\ \mathbf{Y}^0_1 - \mathbf{Y}^0_1\ \mathbf{U}^0_3\big)\ \Omega^6 + \big(3 \\
&\mathbf{U}^0_3\ \mathbf{U}^0_2\ \mathbf{Y}^0_2{}^2 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1 + \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1 + \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_3 - \mathbf{U}^0_1{}^2\ \mathbf{Y}^0_2{}^2 - \mathbf{U}^0_2{}^2\ \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^2 \\
&\mathbf{Y}^0_2\ \mathbf{U}^0_3 - \mathbf{Y}^0_2\ \mathbf{U}^0_3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1 - \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2 - 2\ \mathbf{Y}^0_1\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_2 + 2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_1\big) \\
&\Omega^4 + (\mathbf{U}^0_1{}^2\ \mathbf{Y}^0_2{}^4 + 5\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^3 - 5\ \mathbf{U}^0_3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^4 + \mathbf{U}^0_2{}^2\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2 + \mathbf{U}^0_1{}^2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3 - 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2 \\
&\mathbf{U}^0_2 + 8\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2{}^2 + 2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3 + 2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2\ \mathbf{U}^0_3 - 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1 \\
&\mathbf{U}^0_2 - 2\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2 - \mathbf{U}^0_1\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3 - 3\ \mathbf{U}^0_2\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^4 + 4\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_3\ \mathbf{U}^0_1 + 4\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_1{}^4)\ \Omega^2 + \mathbf{Y}^0_1 \\
&\mathbf{U}^0_3\ (-\mathbf{Y}^0_2{}^5\ \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^3 + 2\ \mathbf{Y}^0_1{}^5\ \mathbf{U}^0_2 + 2\ \mathbf{Y}^0_1{}^4\ \mathbf{Y}^0_2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_2 - 3\ \mathbf{Y}^0_1{}^2 \\
&\mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^4\ \mathbf{U}^0_1 - 3\ \mathbf{Y}^0_1{}^4\ \mathbf{Y}^0_2\ \mathbf{U}^0_1 - 5\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1)\Big)\ \mathbf{Y}^1_1 + \Big(-2\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1{}^4 + 2\ \mathbf{U}^0_1{}^2\ \mathbf{Y}^0_2{}^4 - 2\ \mathbf{Y}^0_2 \\
&\mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1{}^3 - 2\ \mathbf{U}^0_1{}^2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3 - 2\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2 - 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1\ \mathbf{U}^0_2 - 2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^3\ \mathbf{Y}^0_2\ \mathbf{U}^0_3 - 4\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1{}^2 \\
&\mathbf{Y}^0_2\ \mathbf{U}^0_2 + 2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_1\ \mathbf{Y}^0_1{}^2 + 2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3 - \mathbf{U}^0_3\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2{}^2 + 2\ \mathbf{Y}^0_1{}^3 \\
&\mathbf{U}^0_1\ \mathbf{U}^0_2{}^2 + \mathbf{U}^0_3\ \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_1{}^2\ \mathbf{U}^0_3 - 4\ \mathbf{Y}^0_2\ \mathbf{U}^0_3\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1 - 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_3\ \mathbf{U}^0_2{}^2 - \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_1{}^2\ \mathbf{U}^0_3 \\
&\mathbf{Y}^0_2 - 2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2{}^2\ \mathbf{U}^0_1\ \mathbf{Y}^0_1 + \mathbf{Y}^0_2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3{}^2 + 2\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1 + 3\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1 - 4\ \mathbf{Y}^0_2\ \mathbf{U}^0_1\ \mathbf{U}^0_3{}^2 \\
&\mathbf{Y}^0_1{}^2 + 2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2\ \mathbf{U}^0_1{}^2 - 2\ \mathbf{Y}^0_1\ \mathbf{U}^0_1{}^3\ \mathbf{Y}^0_2{}^2 + 4\ \mathbf{U}^0_2\ \mathbf{U}^0_3{}^2\ \mathbf{Y}^0_1{}^3\Big)\ \Omega^3 + \Big(-\mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \mathbf{U}^0_3\ \Omega^5 + \mathbf{Y}^0_2\ (-2\ \mathbf{Y}^0_1{}^2 \\
&\mathbf{Y}^0_2\ \mathbf{U}^0_2 + \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_3 - 2\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_1 + 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1 + 3\ \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_3 - \mathbf{Y}^0_1\ \mathbf{U}^0_3\ \mathbf{Y}^0_2{}^2 + 2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1{}^3 + \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_3) \\
&\Omega^3\Big)\ \mathbf{E}_3 + \Big((\mathbf{Y}^0_1\ \mathbf{U}^0_1 + \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega^5 - \mathbf{Y}^0_2{}^2\ (\mathbf{Y}^0_1\ \mathbf{U}^0_1 + \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega^3\Big)\ \mathbf{Y}^1_2{}^2 + \Big(-\Omega^6\ \mathbf{Y}^0_1\ \mathbf{U}^0_3 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1) \\
&\mathbf{U}^0_3\ \Omega^4\Big)\ \mathbf{U}^1_3 + (-2\ \mathbf{Y}^0_1\ \mathbf{U}^0_1 - 2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega^7\ \frac{\partial \mathbf{E}_3}{\partial x_3} + \Big(\mathbf{Y}^0_1{}^2\ (-\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \mathbf{U}^0_3\ \Omega^4 - \mathbf{Y}^0_1{}^2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ (-\mathbf{Y}^0_2 + \mathbf{Y}^0_1)^2 \\
&\mathbf{U}^0_3\ \Omega^2\Big)\ \mathbf{U}^1_1 + \Big(-4\ \Omega^7\ \mathbf{Y}^0_2 + (2\ \mathbf{Y}^0_1{}^2\ \mathbf{U}^0_3 + 12\ \mathbf{U}^0_1\ \mathbf{Y}^0_2{}^2 - 12\ \mathbf{Y}^0_2\ \mathbf{U}^0_2\ \mathbf{Y}^0_1)\ \Omega^5 - 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ (\mathbf{Y}^0_1 \\
&\mathbf{U}^0_3 + \mathbf{U}^0_2\ \mathbf{Y}^0_1 - \mathbf{U}^0_1\ \mathbf{Y}^0_2)\ \Omega^3 + 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_1{}^3 + 2\ \mathbf{Y}^0_2{}^3 + \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^2)\ (-\mathbf{U}^0_1\ \mathbf{Y}^0_2 + \mathbf{U}^0_2\ \mathbf{Y}^0_1)\ \Omega\Big)\ \mathbf{E}_1 + (2\ \mathbf{Y}^0_2 \\
&\mathbf{U}^0_2 + 2\ \mathbf{Y}^0_1\ \mathbf{U}^0_1)\ \Omega^7 + \Big(\mathbf{Y}^0_1{}^2\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ \mathbf{U}^0_3\ \Omega^4 - \mathbf{Y}^0_1\ (\mathbf{Y}^0_2 + \mathbf{Y}^0_1)\ (\mathbf{Y}^0_2{}^3 - \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2 + \mathbf{Y}^0_2\ \mathbf{Y}^0_1{}^2 + \mathbf{Y}^0_1{}^3)\ \mathbf{U}^0_3\ \Omega^2\Big) \\
&\mathbf{U}^1_2 + \Big((\mathbf{Y}^0_2\ \mathbf{U}^0_2 + 4\ \mathbf{Y}^0_1\ \mathbf{U}^0_1 + 2\ \mathbf{U}^0_1\ \mathbf{Y}^0_2)\ \Omega^5 + (-4\ \mathbf{Y}^0_1{}^3\ \mathbf{U}^0_1 - \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_1 - 2\ \mathbf{Y}^0_2{}^3\ \mathbf{U}^0_2 - 6\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2{}^2\ \mathbf{U}^0_1 + \mathbf{Y}^0_1 \\
&\mathbf{Y}^0_2{}^2\ \mathbf{U}^0_2 - \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_1 - \mathbf{Y}^0_1{}^2\ \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega^3 + \mathbf{Y}^0_1\ \mathbf{Y}^0_2\ (\mathbf{Y}^0_1{}^2 + 2\ \mathbf{Y}^0_1\ \mathbf{Y}^0_2 - \mathbf{Y}^0_2{}^2)\ (\mathbf{Y}^0_1\ \mathbf{U}^0_1 + \mathbf{Y}^0_2\ \mathbf{U}^0_2)\ \Omega\Big) \\
&\mathbf{Y}^1_1{}^2\Big) \Big/ \Omega^8,
\end{aligned}
$$

$$
\mathbf{U}^1(s; \mathbf{x}, \mathbf{v}, s) = 0.
$$

B Functions $\mathcal{O}_i(\varepsilon)$ for $i \geq 1$

The function $\mathcal{O}_2(\varepsilon)$ is given by

$$
\begin{aligned}
\mathcal{O}_2(\varepsilon) = \varepsilon \Big\{\nabla_x^2 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon \Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon \{\mathbf{Y}^1 + [\theta \tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon)\}\ ,\ \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon \{\mathbf{Y}^2_\varepsilon + [\theta \tilde{\mathbf{A}}^1(\mathbf{Y}^0, \mathbf{Y}^1)]_\varepsilon)\}\Big\} + \\
\frac{\varepsilon^2}{2} \Big\{\nabla_x^2 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon \Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon \{\mathbf{Y}^2_\varepsilon + [\theta \tilde{\mathbf{A}}^1(\mathbf{Y}^0, \mathbf{Y}^1)]_\varepsilon)\}^2\Big\} +
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{ \int_0^1 (1-\xi)\nabla_x^2 \mathbf{a}\big(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0) + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]\})\big)\, d\xi - \frac{1}{2}\nabla_x^2 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon \\
&\qquad \Big\{ \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon \{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\}\Big\}^2 + \\
&\frac{\varepsilon}{2}\Big\{\nabla_x^2 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^2 + [\theta\tilde{\mathbf{A}}^1(\mathbf{Y}^0,\mathbf{Y}^1)]_\varepsilon)\}\Big\}^2 + \frac{\varepsilon}{3}\Big\{\nabla_x^3 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\} \\
&\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon)\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon)\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^2 + [\theta\tilde{\mathbf{A}}^1(\mathbf{Y}^0,\mathbf{Y}^1)]_\varepsilon)\}\Big\} + \\
&\frac{\varepsilon^2}{3}\Big\{\nabla_x^3 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\ \Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon)\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^2 + [\theta\tilde{\mathbf{A}}^1(\mathbf{Y}^0,\mathbf{Y}^1)]_\varepsilon)\}, \\
&\qquad \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^2 + [\theta\tilde{\mathbf{A}}^1(\mathbf{Y}^0,\mathbf{Y}^1)]_\varepsilon)\}\Big\} + \\
&\frac{\varepsilon^3}{6}\Big\{\nabla_x^3 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\}\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^2 + [\theta\tilde{\mathbf{A}}^1(\mathbf{Y}^0,\mathbf{Y}^1)]_\varepsilon)\}\Big\}^3 + \\
&\Big\{ \int_0^1 \frac{(1-\xi)^2}{2}\nabla_x^3 \mathbf{b}\big([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0))]_\varepsilon\}\big)\, d\xi - \frac{1}{6}\nabla_x^3 \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\} \\
&\qquad \Big\{\Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon \Big\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0(\mathbf{Y}^0)]_\varepsilon\Big\}\Big\}^3. \qquad \text{(B.1)}
\end{aligned}
$$

For $i \geq 3$,

$$
\begin{aligned}
\mathcal{O}_i(\varepsilon) = \varepsilon\Bigg(&\frac{1}{2}\Big\{\nabla_x^2 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon \Big(\Big\{2\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon)\}\,,\, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^i + [\theta\tilde{\mathbf{A}}^{i-1}]_\varepsilon)\}\Big\} + \\
&\sum_{j=2}^{i-1}\Big\{\Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^{i+1-j} + [\theta\tilde{\mathbf{A}}^{i-j}]_\varepsilon\}\Big\}\Big) + \frac{1}{2}\{\nabla_x^3 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\}_\varepsilon \\
&\sum_{\substack{l+m+n\\=i+1}}\ \Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^l + [\theta\tilde{\mathbf{A}}^{l-1}]_\varepsilon\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^m + [\theta\tilde{\mathbf{A}}^{m-1}]_\varepsilon\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^n + [\theta\tilde{\mathbf{A}}^{n-1}]_\varepsilon\}\Big\} + \\
&\quad \cdots + \frac{i}{i!}\{\nabla_x^i \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\}_\varepsilon\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}, \ldots, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}, \\
&\qquad \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon\}\Big\}\Bigg) + \\
&\varepsilon^2\Bigg(\frac{1}{2}\Big\{\nabla_x^2 \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon\Big(2\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon)\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^i + [\theta\tilde{\mathbf{A}}^{i-1}]_\varepsilon)\}\Big\} + \\
&\sum_{j=3}^{i-1}\Big\{\Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^{i+2-j} + [\theta\tilde{\mathbf{A}}^{i+1-j}]_\varepsilon\}\Big\}\Big) + \ldots\Bigg) + \cdots + \\
&\quad + \cdots + \frac{\varepsilon^{i(i-1)}}{i!}\{\nabla_x^i \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\}_\varepsilon\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^i + [\theta\tilde{\mathbf{A}}^{i-1}]_\varepsilon\}\Big\}^i + \\
&\Big\{\int_0^1 \frac{(1-\xi)^{i-1}}{(i-1)!}\nabla_x^i \mathbf{a}\big(\cdot,\cdot,\mathbf{Z}(\mathbf{Y}^0) + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0]\})\big)\, d\xi - \frac{1}{i!}\nabla_x^i \mathbf{a}(\mathbf{Z}(\mathbf{Y}^0))\Big\}_\varepsilon \\
&\qquad \Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}\Big\}^i + \\
&\varepsilon\Bigg(\frac{1}{2}\{\nabla_x^2 \mathbf{b}[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\}\Big\{2\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon)\}\,,\, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^i + [\theta\tilde{\mathbf{A}}^{i-1}]_\varepsilon)\}\Big\} + \\
&\sum_{j=3}^{i-1}\Big\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon\}, \{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^{i+2-j} + [\theta\tilde{\mathbf{A}}^{i-j}]_\varepsilon\}\Big\} + \ldots\Bigg) + \cdots + \\
&\quad + \cdots + \frac{\varepsilon^{(i+1)(i-1)}}{(i+1)!}\{\nabla_x^{i+1} \mathbf{b}[\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon\}\{\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^i + [\theta\tilde{\mathbf{A}}^{i-1}]_\varepsilon\}\}^{i+1} \\
&\Big\{\int_0^1 \frac{(1-\xi)^i}{i!}\nabla_x^{i+1} \mathbf{b}\big([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon + \xi(\varepsilon\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0)]_\varepsilon\}\big)\, d\xi - \frac{1}{(i+1)!}\nabla_x^{i+1} \mathbf{b}([\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon)\Big\} \\
&\qquad \Big\{\Big\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\Big\}_\varepsilon\Big\{\mathbf{Y}_\varepsilon^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\Big\}\Big\}^{i+1}, \qquad \text{(B.2)}
\end{aligned}
$$

The expression of $\mathcal{O}_i(\varepsilon)$ is a bit heavy. Nevertheless, we notice that a part of it consists in terms writing ε at a given power, multiplied by $\nabla_x^q \mathbf{a}$ for a given q, acting on elements reading $\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}_\varepsilon^p + [\theta\tilde{\mathbf{A}}^{p-1}]_\varepsilon\}$ or $\{\nabla_z \mathbf{Z}(\mathbf{Y}^0)\}_\varepsilon\{\mathbf{Y}^p + [\theta\tilde{\mathbf{A}}^{p-1}]_\varepsilon\}$ for some exponents p. In those terms, if we make the sum of the exponents of the $\mathbf{Y}_\varepsilon^p$ or $\mathbf{Y}^p$ we obtain the power of ε plus i. Others terms have the same form with $\nabla_x^q \mathbf{a}$ replaced by $\nabla_x^q \mathbf{b}$. In those terms, if we make the sum of the exponents of the $\mathbf{Y}_\varepsilon^p$ or $\mathbf{Y}^p$ we obtain the power of ε plus $i+1$.

References

[1] G. Allaire. Homogenization and Two-scale Convergence. *SIAM J. Math. Anal.*, XXIII(6):1482–1518, 1992.

[2] A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures.* St. in Maths. and its Appl., Vol. 5. North Holland, 1978.

[3] N. N. Bogoliubov and J. A. Mitropolsky. *Asymptotical Methods in the Theory of Nonlinear Oscillations.* State Press for Phys. and Math. Literature,Moscow (In Russian). English translation: Hindustan Publishing Co., Delhi, India, 1963.

[4] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754, 2000.

[5] B. Cohen. Orbit averaging and subcycling in particle simulation of plasmas. *in Multiple Time Scales (Academic Press)*, 1985.

[6] D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee. Nonlinear gyrokinetic equations. *Phys. Fluids*, XXVI(12):3524–3535, 1983.

[7] E. Frénod. *Homogénéisation d'équations cinétiques avec potentiels oscillants.* PhD thesis, Université Paris Nord, Av J. B. Clément, F-93400 Villetaneuse, 12 1994.

[8] E. Frénod and K. Hamdache. Homogenisation of kinetic equations with oscillating potentials. *Proc. Royal Soc. Edinburgh*, 126A:1247–1275, 1996.

[9] E. Frénod, P. A. Raviart, and E. Sonnendrücker. Asymptotic expansion of the Vlasov equation in a large external magnetic field. *J. Math. Pures et Appl.*, 80(8):815–843, 2001.

[10] E. Frénod and E. Sonnendrücker. Asymptotic study of the Vlasov-Poisson equation with a large external fagnetic field. *Proceedings des Journées Elie Cartan, Nancy*, juin 1998.

[11] E. Frénod and E. Sonnendrücker. Homogenization of the Vlasov equation and of the Vlasov-Poisson system with a strong external magnetic field. *Asymp. Anal.*, 18(3,4):193–214, Dec. 1998.

[12] E. Frénod and E. Sonnendrücker. Approximation rayon de Larmor fini de l'équation de Vlasov. *C. R. Acad. Sci. Paris Sér.1 Math.*, 330(5):421–426, 2000.

[13] E. Frénod and E. Sonnendrücker. Long time behavior of the two dimensionnal Vlasov equation with a strong external magnetic field. *Math. Models Methods Appl. Sci.*, 10(4):539–553, 2000.

[14] E. Frénod and E. Sonnendrücker. The Finite Larmor Radius Approximation. *SIAM J. Math. Anal.*, 32(6):1227–1247, 2001.

[15] E. Frénod and F. Watbled. The Vlasov equation with strong magnetic field and oscillating electric field as a model for isotop resonant separation. *Elec. J. Diff. Eq.*, 2002(6):1–20, 2002.

[16] F. Golse and L. Saint Raymond. L'approximation centre guide pour l'équation de Vlasov-Poisson 2d. *C. R. Acad. Sci. Paris*, 328(10):865–870, 1998.

[17] F. Golse and L. Saint Raymond. The Vlasov-Poisson system with strong magnetic field. *J. Math. Pures. Appl.*, 78:791–817, 1999.

[18] H. Grad. Mathematical problems arising in plasma physics. In *Actes du Congrès International des Mathématiciens, Nice, 1970*, pages 105–113. Gauthier-Villars, Paris, 1971.

[19] E. Grenier. Oscillatory perturbation of the Navier-Stokes equations. *J. Maths. Pures Appl.*, 76:477–498, 1997.

[20] E. Grenier. Pseudo-differential energy estimates of singular perturbations. *Comm. Pure Appl. Maths.*, 50:821–865, 1997.

[21] P. E. Jabin. Large time concentrations for solutions to kinetic equations with energy dissipation. *Comm. in P.D.E.*, 25(3-4):541–557, 2000.

[22] J. L. Joly, G. Métivier, and J. Rauch. Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.*, 70:373–404, 1993.

[23] J. L. Joly, G. Métivier, and J. Rauch. Nonlinear oscillations beyond caustics. *Comm. Pure and Appl. Math.*, 48:443–529, 1996.

[24] J. L. Joly, G. Métivier, and J. Rauch. Global solutions to Maxwell equations in ferromagnetic medium. *Ann. Inst. H. Poincare*, 1(2):307–340, 2000.

[25] N. Krylov and N. N. Bogoliubov. *Introduction to Nonlinear Mechanics*. Princetown University Press, Princetown, NJ, 1943.

[26] W. W. Lee. Gyrokinetic approach in particle simulation. *Phys. Fluids*, 26(2):556–562, 1983.

[27] Robert G. Littlejohn. Hamiltonian formulation of guiding center motion. *Phys. Fluids*, 24, 1981.

[28] P. Lochak and C. Meunier. *Multiphase averaging for classical systems. With applications to adiabatic theorems*, volume 72 of *Applied Mathematical Sciences*. Springer-Verlag, 1988.

[29] R.E. Mickens. *Oscillations in Planar Dynamic Systems*, volume 37 of *Series on Adv. in Math. for Appl. Sci.* World Scientific, 1996.

[30] G. N'Guetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[31] H. Poincaré. *Méthodes Nouvelles de la Mecanique Céleste*. Gautier-Villars, Paris, 1889.

[32] L. Saint-Raymond. The gyrokinetic approximation for the Vlasov-Poisson system. *Math. Mod. Meth. Appl. Sci.*, 10(9):1305–1332, 2000.

[33] E. Sanchez-Palencia. Méthodes d'homogénéisation pour l'étude de matériaux hétérogènes. *Rend. Sem. Mat. Univ. Politec. Torino*, 36:15–25, 1978.

[34] J.A. Sanders and F. Verhulst. *Averaging methods in nonlinear dynamical systems*, volume 59 of *Applied Mathematical Sciences*. Springer-Verlag, 1985.

[35] S. Schochet. Fast singular limit of hyperbolic PDEs. *J. Diff. Equ.*, 114:476–512, 1994.

[36] D. Serre. Oscillations non linéaires des systèmes hyperboliques: méthodes et résultats qualitatifs. *Ann. Inst. Henri Poincaré*, 8:351–417, 1991.

[37] L. Tartar. Cours Peccot. *Collège de France*, 1977.

[38] A. N. Tikhonov. The dependence of the solutions of the differential equations on a small parameter. *Math. Sb.*, 22:193–204, 1948.

[39] A. N. Tikhonov. On systems of differential equations containing parameters. *Math. Sb.*, 27:147–156, 1950.

[40] A. N. Tikhonov. Systems of differential equations containing small parameters multiplying some of derivatives. *Math. Sb.*, 31:575–586, 1952.

[41] A.B. Vasilieva. On the development of singular perturbation theory at Moscow State University and elsewhere. *SIAM Review*, 36(3):440–452, 1994.

[42] F. Verhulst. *Nonlinear Differential Equations and Dynamical Systems*. Universitext. Springer, 1996.