

# Existence result for a model of *Proteus mirabilis* swarm\*

E. Frénod<sup>†</sup>

March 8, 2005

**Abstract:** In this paper we present a modification of the usual *Proteus mirabilis* Swarm model. For the obtained model (which is a two phase model with a non-linear diffusion term containing memory) we set up a collection of a priori estimates. Those estimates allow to get an existence and uniqueness result.

## 1 Introduction and results

*Proteus mirabilis* is a bacterium that can be either a short cell we call “swimmer” or an elongated cell capable of translocation we call “swarmer”. A model of behaviour of *Proteus mirabilis* colony has been proposed by Esipov and Shapiro [8], based on ideas of Gurtin [10].

In this paper, we prove an existence result to a model which is, in a way, a generalization but also a regularization of the Esipov and Shapiro [8] model.

The model under consideration here is a two phase model with non-linear diffusion term containing memory for one of the two phases. It involves two functions  $\rho$  and  $Q$ . The function  $\rho = \rho(t, a, x)$  refers, at time  $t \in [0, T)$ ,  $0 < T < +\infty$ , to the density of swarmer of age  $a \in [0, A)$ ,  $0 < A \leq +\infty$  at position  $x \in \Omega$ , where  $\Omega$  is a regular sub-domain of  $\mathbb{R}^2$ , with boundary  $\partial\Omega$ . In each point  $x$  of  $\partial\Omega$ ,  $\vec{\nu} = \vec{\nu}(x)$  stands for the unit normal vector pointing outside  $\Omega$ . The function  $Q = Q(t, x)$  stands, at time  $t$ , for the biomass density of swimmers on  $\Omega$ .

For a constant  $\tau$ , those two functions are supposed to satisfy the following system:

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau} Q + \int_0^A \rho(\cdot, a, \cdot) e^{a/\tau} \mu(\cdot, a, \cdot) da + \chi(A) \rho(\cdot, A, \cdot) e^{A/\tau}, \quad \text{on } [0, T) \times \Omega, \quad (1.1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = -\mu \rho + \nabla \cdot [(D(\mathcal{M}, Q, P) + d) \nabla \rho], \quad \text{on } [0, T) \times [0, A) \times \Omega, \quad (1.2)$$

$$\rho(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q, \quad \text{on } [0, T) \times \Omega, \quad (1.3)$$

$$\rho(0, \cdot, \cdot) = \rho_0, \quad \text{on } [0, A) \times \Omega, \quad (1.4)$$

$$Q(0, \cdot) = Q_0, \quad \text{on } \Omega, \quad (1.5)$$

$$\frac{\partial \rho}{\partial \vec{\nu}} = 0, \quad \text{on } [0, T) \times [0, A) \times \partial\Omega. \quad (1.6)$$

Above,  $\chi(A)$  is an artifice allowing to take into account a possible maximum age  $A$  beyond which swarmer cannot exist. It has the following definition:

$$\chi(A) = 1, \quad \forall A \in \mathbb{R}, \quad \chi(+\infty) = 0. \quad (1.7)$$

---

\*This work has been partially supported by “La Région Bretagne”, F-35031 Rennes, Program 1042: “Renouvellement des compétences dans les laboratoires de recherche”, Operation A1C872.

<sup>†</sup>LMAM et Lemel, Université de Bretagne Sud, Centre Yves Coppens, Campus de Tohannic, F-56000, Vannes.

Denoting by  $C_b^k$  the space of functions having continuous and bounded derivatives until the order  $k$ ,  $\mu = \mu(t, a, x)$  is a function such that

$$\mu \in C_b^2([0, T) \times [0, A) \times \Omega), \quad \mu \geq 0, \quad \lim_{a \rightarrow A} \mu(t, a, x) = \bar{\mu} \quad \text{uniformly in } x \text{ and } t, \quad (1.8)$$

with  $\bar{\mu} \geq c(1 - \chi(A))$ , for a constant  $c > 0$ . The function  $\xi = \xi(t, Q)$  satisfies

$$\xi \in C_b^2([0, T) \times \mathbb{R}), \quad 0 \leq \xi \leq 1. \quad (1.9)$$

In the second equation,  $\nabla$  stands for the gradient with respect to the  $x$ -variable, and  $\nabla \cdot$  for the divergence. The diffusion coefficient is the sum of a constant, a priori small,

$$d > 0, \quad \text{and of } D = D(\mathcal{M}, P, Q), \text{ a non negative } C_b^1 \text{ function of its arguments.} \quad (1.10)$$

In (1.10),  $P$  is defined for  $0 \leq a_{min} < A$  by

$$P(t, x) = \int_{a_{min}}^A \rho(t, a, x) e^{a/\tau} da, \quad (1.11)$$

and  $Q$  is given by the first equation of the system.

The memory (or hysteresis) term  $\mathcal{M} = \mathcal{M}_{[P]}(t, x)$  keeps information on the value of  $P$  in the past. For four thresholds,  $P_{min} < p_{min} < p_{max} < P_{max}$ , with  $P_{min}$  close to  $p_{min}$  and  $P_{max}$  close to  $p_{max}$ ,  $\mathcal{M}$  is defined as the solution to:

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial t} &= \frac{1}{P_{max} - p_{max}} H_r \left( \frac{P - p_{max}}{P_{max} - p_{max}} \right) H_r(1 - \mathcal{M}) - \frac{1}{p_{min} - P_{min}} H_r \left( \frac{p_{min} - P}{p_{min} - P_{min}} \right) H_r(\mathcal{M}), \\ \mathcal{M}(0, \cdot) &= \mathcal{M}_0, \end{aligned} \quad (1.12)$$

with, denoting  $P_0 = P(0, \cdot)$ ,

$$\mathcal{M}_0 \in C_b^1(\Omega), \quad 0 \leq \mathcal{M}_0 \leq 1, \quad \mathcal{M}_0 = 0 \text{ where } P_0 < P_{min} \text{ and } \mathcal{M}_0 = 1 \text{ where } P_0 > P_{max}, \quad (1.13)$$

and with

$$H_r(p) = 0 \text{ if } p \leq 0, \quad H_r(p) = p \text{ if } 0 \leq p \leq 1 \quad \text{and } H_r(p) = 1 \text{ if } p \geq 1. \quad (1.14)$$

We now turn to the statement of the main result of this paper.

**THEOREM 1.1** *Under assumptions (1.8), (1.9), (1.10) and (1.13), if  $\rho_0 \geq 0 \in L^1 \cap W^{2,2} \cap W^{1,4}([0, A) \times \Omega, e^{a/\tau} dadx)$  satisfies*

$$\begin{aligned} \int_{\Omega} (\rho_0)^2 e^{2a/\tau} dx &\leq b \int_{\Omega} (\rho_0)^2 e^{a/\tau} dx, \quad \forall a \in [0, A), \\ \int_0^A \int_{\Omega} |\nabla \rho_0|^4 e^{4a/\tau} dadx &\leq b \int_0^A \int_{\Omega} |\nabla \rho_0|^4 e^{a/\tau} dadx, \\ \int_0^A \int_{\Omega} |\nabla \rho_0|^2 e^{2a/\tau} dx &\leq b \int_0^A \int_{\Omega} |\nabla \rho_0|^2 e^{a/\tau} dx, \end{aligned} \quad (1.15)$$

for a constant  $b$ , and if  $Q_0 \geq 0 \in L^1 \cap W^{2,2} \cap W^{1,4}(\Omega)$ ; then, there exists a unique solution  $(Q, \rho) \in L^\infty(0, T; ((L^1 \cap W^{1,2}(\Omega)) \times (L^1 \cap W^{1,2}([0, A) \times \Omega, e^{a/\tau} dadx)))$  to system (1.1)-(1.6) coupled with (1.11) and (1.12). Moreover,  $Q \geq 0$  and  $\rho \geq 0$ .

The precise definitions of the spaces at work in the Theorem are given in the beginning of section 4.

We now give references where modelling and mathematical methods are developed on age-structured population problem: Gurtin and Mac Camy [11], Marcati [22], Andreasen [1, 2, 3]; possibly with diffusion: Gurtin [10], Di Blasio and Lamberti [7], Di Blasio [6], Mac Camy [21], Gurtin and Mac Camy [12], Busenberg and Iannelli [5], Langlais [17, 18], Kubo and Langlais [15], Huang [13] and Esipov and Shapiro [8]. For simulation methods we refer to Lopez and Trigiante [20], Milner [24] Kim [14], Esipov and Shapiro [8], Medvedev, Kapper and Koppel [23], Ayati and Dupont [4]. Concerning the biological description of *Proteus mirabilis* colony behaviour, we refer for instance to Rauprich *et al* [25], Gué, Dupont, Dufour and Sire [9] and theirs references.

The paper is organized as follows: in section 2 we present the way to go from the Esipov and Shapiro model to system (1.1)-(1.6). Then section 3 is devoted to a priori estimates for the solution to (1.1)-(1.6). By a usual procedure consisting in linearizing and passing to limit, we prove the Theorem in section 4.

**Acknowledgements:** I would like to thank O. Sire for having introduced to me the swarm model of Esipov and Shapiro [8] and for stimulating discussions.

I would also like to thank F. Granger who, despite stopping its PhD thesis for personal reasons, made the first steps towards the result.

## 2 Model

*Proteus mirabilis* is a pathogenic bacterium of urinary track that when standing in liquid medium, consists in a usual short “swimmer cell” or “swimmer”. When placed on agar medium, if the bacterium density is large enough, it may undergo a differentiation process producing an elongated cell with several nucleus called “swarmer cell” or “swarmer”. Those swarmers are capable of translocation allowing the bacterial colony to colonise the medium.

The macroscopic model built by Esipov and Shapiro [8] describes this swarm phenomenon at the colony scale. We shall explain this model now. The swarmer behaviour depends on their own age. This dependence is taken into account by introducing the age dependent density of swarmers  $\rho(t, a, x)$ . The link between this density  $\rho$  and the biomass density is a consequence of the fact that the mass of each cell is in direct proportion with its length and that the length growth of a swarmer is also in direct proportion with its length. Then the biomass density, at time  $t$ , of swarmers of age  $a$  at position  $x$  is  $\rho(t, a, x)e^{a/\tau}$ , where  $\tau$  is the growth rate of the biomass. The first age-depending behaviour of the swarmers is that they actively participate in group migration only after an age  $a_{min}$ . Then the definition of the biomass density  $P$  of swarmers capable of active translocation is given by (1.11). The second age-depending behaviour is that the swarmers dedifferentiate themselves and give swimmers. On this topic, Esipov and Shapiro [8] consider two situations. In the first one (Model A), the swarmers dedifferentiate themselves at a given age  $a_{max}$ . The second situation consider that swarmers may dedifferentiate at each time with a probability  $1/\bar{a}$  (Model B).

Now, we are able to write the swimmer evolution equation for the biomass density  $Q$ . Its evolution results from the classical cellular division, with a characteristic time which is the biomass growth rate  $\tau$ , subtracting the proportion of bacteria undergoing differentiation and adding the dedifferentiation product. In the case of Model A, the evolution equation for  $Q$  is then

$$\frac{\partial Q}{\partial t} = \frac{1 - \xi}{\tau} Q + \rho(\cdot, a_{max}, \cdot) e^{a_{max}/\tau}, \quad Q(0, \cdot) = Q_0. \quad (2.1)$$

Here,  $Q_0$  stands for the initial swimmer density and  $\xi/\tau$  is the fraction of swimmer population to

produce swimmers. In the case of Model B, the evolution equation is

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau}Q + \int_0^t \rho(\cdot, a, \cdot) \frac{e^{a/\tau}}{a} da, \quad Q(0, \cdot) = Q_0. \quad (2.2)$$

We turn to the evolution of the swarmer density  $\rho$ . Its evolution is linked with ageing and dedifferentiation process, but also to swarm. This last phenomenon is modelled by a non linear diffusion term with memory. The evolution equation for  $\rho$  is then

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = \nabla \cdot [D(\mathcal{M}, Q, P)\nabla \rho], \quad (2.3)$$

in the case of Model A; and, in the case of Model B, it is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = -\frac{1}{a}\rho + \nabla \cdot [D(\mathcal{M}, Q, P)\nabla \rho]. \quad (2.4)$$

Both of those equations are equipped with the following initial and boundary conditions:

$$(a): \rho(\cdot, 0, \cdot) = \frac{\xi}{\tau}Q, \quad (b): \rho(0, \cdot, \cdot) = 0, \quad \text{and} \quad (c): \frac{\partial \rho}{\partial \nu} \Big|_{\partial \Omega} = 0. \quad (2.5)$$

The first of those three conditions means the fraction  $\xi/\tau$  of swimmers undergoing the differentiation process produces swimmers of age 0. The initial condition on  $\rho$  means that, at the beginning of the process, there is no swarmer. The boundary condition means that swarmer cannot leave the domain  $\Omega$ . In the diffusion term  $\nabla \cdot [D(\mathcal{M}, Q, P)\nabla \rho]$  appearing in (2.3) and (2.4), and modelling the swarm, the diffusion factor  $D(\mathcal{M}, Q, P)$  depends on the present value of  $Q$  and  $P$  but also on the history of  $P$ .

The term  $\mathcal{M}$  then keeps in memory informations concerning the history of the swarmer density. Esipov and Shapiro [8] defines  $\mathcal{M}$  as being set to 1 in a given point  $x$  if the value of  $P$  in  $x$  reaches a threshold  $P_{max}$ . Then it remains at the value 1 until the value of  $P$  in  $x$  reaches another value  $P_{min} < P_{max}$ . Then they suggest to take

$$D(\mathcal{M}, Q, P) = \overline{D}_0 \mathcal{M} \gamma\left(\frac{P}{P_{max}}\right) \exp\left(\frac{-Q}{Q_{sat}}\right), \quad (2.6)$$

for given values of  $\overline{D}_0$  and  $Q_{sat}$  and with,  $H$  being such that  $H(p) = 0$  if  $p < 0$  and  $H(p) = 1$  if  $p > 0$ ,

$$\gamma(p) = \left(p - \frac{P_{min}}{P_{max}}\right)H\left(p - \frac{P_{min}}{P_{max}}\right) \text{ or } \gamma(p) = p - \frac{P_{min}}{P_{max}} \text{ or } \gamma(p) = p^2 \text{ or } \gamma(p) = 1. \quad (2.7)$$

We also mention that Medvedev, Kapper and Koppel [23] studied this model taking, for a given value of  $k$ ,

$$D(\mathcal{M}, Q, P) = D(Q, P) = \frac{\overline{D}_0 P}{P + kQ}. \quad (2.8)$$

Now, we explain in what sense the model (1.1)-(1.6) is a generalization and a regularization of the Esipov and Shapiro [8] model.

First, because of the initial condition (2.5.b), it is an easy game to see that, at least formally, the solution  $\rho$  to (2.3) or (2.4) satisfies

$$\rho(t, a, x) = 0 \text{ when } a > t. \quad (2.9)$$

Hence we can replace (2.2) by

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau}Q + \int_0^{+\infty} \rho(\cdot, a, \cdot) \frac{e^{a/\tau}}{a} da, \quad Q(0, \cdot) = Q_0. \quad (2.10)$$

Making this allows to take under considerations, with no lost of consistency, initial data  $\rho(0, \cdot, \cdot)$  that are not 0 coming to the more general initial and boundary conditions (1.3)-(1.6).

Secondly, equations (2.1) and (2.10) are particular cases of the general equation (1.1) with assumption (1.8). The case of equation (2.1) is recovered setting  $\mu = 0$  and  $A = a_{max}$  and the case of (2.10) is recovered setting  $A = +\infty$  and  $\mu = 1/\bar{a}$ . We also see that (1.2) with  $d = 0$  is a general framework inside which (2.3) and (2.4) may enter directly.

Concerning the fraction  $\xi$  of swimmers to produce swarmers, it seems to depend on experimental conditions and to be 0 when the swimmer density is high. It seems then to be reasonable to set that  $\xi$  satisfies (1.9).

The first regularization effect we consider in our model (1.1)-(1.6) consists in adding  $d > 0$  in (1.2). This may be justified by experimental arguments saying that swarmers always experience a small random motion even before and after swarming.

The second regularization effect which constitutes the most visible modification of the model concerns the memory term  $\mathcal{M}$ . In order to explain the way to go from the definition of  $\mathcal{M}$  by Esipov and Shapiro [8] to the definition (1.12), we first notice that the term  $\mathcal{M}$  of Esipov and Shapiro could be defined formally, in any  $x$ , as the solution to the following equation:

$$\frac{\partial \mathcal{M}}{\partial t}(\cdot, x) = (1 - \mathcal{M}(\cdot, x))\delta_{\{t/P(t,x)=P_{max}\}} - \mathcal{M}(\cdot, x)\delta_{\{t/P(t,x)=P_{min}\}}, \quad \mathcal{M}(0, x) = 0, \quad (2.11)$$

where  $\delta_{\{t/P(t,x)=P_{max}\}}$  stand for the Dirac measure in instant  $t$  where  $P(t, x) = P_{max}$ . Of course this equation has no real mathematical meaning. But formally, if  $\mathcal{M} = 0$  and if the value  $P_{max}$  is reached at a given time  $\tilde{t}$ , the solution of (2.11) experiences a jump  $+1$ . When  $P = P_{max}$ , and  $\mathcal{M} = 1$ , nothing happens and this is what is needed. In the same way, if at a given time  $\tilde{t}$ ,  $P = P_{min}$ , nothing happens if  $\mathcal{M} = 0$  and  $\mathcal{M}$  experiences a jump  $-1$  if  $\mathcal{M} = 1$ . Now, it is easy to see that the right hand side of (1.12) is nothing but a regularization of the right hand value of (2.11). This is the reason why we make this choice to define  $\mathcal{M}$ .

### 3 A priori estimates

The key point to get the existence result is a collection of a priori estimates satisfied by  $(Q, \rho)$ . They are mathematical translations of biological properties. In order to set those estimates comfortably, we assume that the assumptions and the conclusions of Theorem 1.1 are satisfied.

#### 3.1 $L^1$ and $L^2$ estimates

For  $p > 0$ , we denote

$$\|\rho(t)\|_p = \left( \int_0^A \int_{\Omega} |\rho(t, a, x)|^p e^{a/\tau} dx da \right)^{1/p}, \quad \|Q(t)\|_p = \left( \int_{\Omega} |Q(t, x)|^p dx \right)^{1/p}, \quad (3.1)$$

and we have the following estimates saying that the total biomass grows exponentially with a growth rate  $\tau$ .

**LEMMA 3.1** *If the assumptions of Theorem 1.1 are valid, and if the solution  $(Q, \rho)$  given by this same Theorem exists, then it satisfies*

$$\|\rho(t)\|_1 + \|Q(t)\|_1 = (\|\rho_0\|_1 + \|Q_0\|_1)e^{t/\tau}. \quad (3.2)$$

*Proof.* Multiplying equation (1.2) by  $e^{a/\tau}$ , integrating then with respect to  $x$  and  $a$  yields:

$$\frac{d\|\rho\|_1}{dt} + \lim_{a \rightarrow A} \left( \int_{\Omega} \rho e^{a/\tau} dx \right) - \int_{\Omega} \rho(\cdot, 0, \cdot) dx = \frac{1}{\tau} \|\rho\|_1 - \int_0^A \int_{\Omega} \mu \rho e^{a/\tau} dx da, \quad (3.3)$$

which also reads

$$\frac{d\|\rho\|_1}{dt} + \chi(A) e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) dx - \int_{\Omega} \frac{\xi}{\tau} Q dx = \frac{1}{\tau} \|\rho\|_1 - \int_0^A \int_{\Omega} \mu \rho e^{a/\tau} dx da. \quad (3.4)$$

Integrating now (1.1) with respect to  $x$ , we obtain

$$\frac{d\|Q\|_1}{dt} = \int_{\Omega} \frac{1-\xi}{\tau} Q dx + \int_0^A \int_{\Omega} \mu \rho e^{a/\tau} dx da + \chi(A) e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) dx. \quad (3.5)$$

Summing up (3.4) and (3.5) finally gives

$$\frac{d(\|\rho\|_1 + \|Q\|_1)}{dt} = \frac{1}{\tau} (\|\rho\|_1 + \|Q\|_1), \quad (3.6)$$

proving the Lemma. ■

The second Lemma concerns the  $L^2$ -norms of  $\rho$  and  $Q$ . It mathematically translates that the biomass cannot be so gathered that a null area set contains a positive biomass quantity.

**LEMMA 3.2** *If the assumptions of Theorem 1.1 are realized and if the solution  $(Q, \rho)$  exists, then for any  $t \in [0, T)$ , it satisfies*

$$\|\rho(t)\|_2 + \|Q(t)\|_2 \leq c(\|\rho_0\|_2 + \|Q_0\|_2), \quad (3.7)$$

for a constant  $c$  ( $c$  only depends on  $A, \tau, T$  and  $\tilde{\mu} = \sup\{\mu(t, a, x), t \in [0, T), a \in [0, A), x \in \Omega\}$ ).

*Proof.* First, integrating (1.2) with respect to  $x$ , we get:

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \left( \int_{\Omega} \rho dx \right) = - \int_{\Omega} \mu \rho dx \leq 0. \quad (3.8)$$

Making the same, after multiplying (1.2) by  $\rho$ , gives:

$$\frac{1}{2} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \left( \int_{\Omega} \rho^2 dx \right) = - \int_{\Omega} \mu \rho^2 dx - \int_{\Omega} (D + d) |\nabla \rho|^2 dx \leq 0. \quad (3.9)$$

Defining, for a fixed  $\alpha$  and for  $p = 1$  or  $2$ ,

$$r_p : s \mapsto \int_{\Omega} \rho^p(s, \alpha + s, x) dx, \quad (3.10)$$

we have

$$r_p'(s) = \left( \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \left( \int_{\Omega} \rho^p dx \right) \right) (s, \alpha + s) \leq 0. \quad (3.11)$$

Now, since for fixed  $t$  and  $a$  and with  $\alpha = a - t$  we have  $\int_{\Omega} \rho^p(t, a, x) dx = r_p(t)$ , if  $t < a$  the relation  $r_p(t) \leq r_p(0) = \int_{\Omega} \rho^p(0, a - t, x) dx$  reads

$$\int_{\Omega} \rho(t, a, x) dx \leq \int_{\Omega} \rho_0(a - t, x) dx, \quad \int_{\Omega} \rho^2(t, a, x) dx \leq \int_{\Omega} \rho_0^2(a - t, x) dx. \quad (3.12)$$

In the case when  $t > a$ , the relation  $r_p(t) \leq r_p(t-a) = \int_{\Omega} \rho^p(t-a, 0, x) dx$  gives

$$\int_{\Omega} \rho(t, a, x) dx \leq \int_{\Omega} \frac{\xi}{\tau} Q(t-a, x) dx da, \quad \int_{\Omega} \rho^2(t, a, x) dx \leq \int_{\Omega} \frac{\xi^2}{\tau^2} Q^2(t-a, x) dx da, \quad (3.13)$$

Secondly, we multiply equation (1.2) by  $e^{a/\tau} \rho$  and we integrate in  $x$  and  $a$ . Since

$$\int_0^A \frac{\partial \rho}{\partial a} \rho e^{a/\tau} da = - \int_0^A \frac{\partial \rho}{\partial a} \rho e^{a/\tau} da - \frac{1}{\tau} \int_0^A \rho^2 e^{a/\tau} da + \left[ \rho^2 e^{a/\tau} \right]_{a=0}^{a=A}, \quad (3.14)$$

and since

$$\int_{\Omega} \rho^2(t, 0, x) dx = \int_{\Omega} \frac{\xi^2}{\tau^2} Q^2 dx, \quad (3.15)$$

we obtain

$$\begin{aligned} \frac{d\|\rho\|_2^2}{dt} + \chi(A)e^{A/\tau} \int_{\Omega} \rho^2(\cdot, A, \cdot) dx + 2 \int_0^A \int_{\Omega} (D+d)|\nabla \rho|^2 e^{a/\tau} dx da \\ + 2 \int_0^A \int_{\Omega} \mu \rho^2 e^{a/\tau} dx da = \int_{\Omega} \frac{\xi^2}{\tau^2} Q^2 dx + \frac{1}{\tau} \|\rho\|_2^2. \end{aligned} \quad (3.16)$$

As the second, third and fourth terms of the left hand side of equation (3.16) are non negative, we may deduce

$$\frac{d\|\rho\|_2^2}{dt} \leq \frac{1}{\tau^2} \|Q\|_2^2 + \frac{1}{\tau} \|\rho\|_2^2. \quad (3.17)$$

In the same way, multiplying now (1.1) by  $Q$  and integrating yields

$$\frac{d\|Q\|_2^2}{dt} = 2 \int_{\Omega} \frac{1-\xi}{\tau} Q^2 dx + 2 \int_0^A \int_{\Omega} \mu \rho Q e^{a/\tau} dx da + 2\chi(A)e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) Q dx. \quad (3.18)$$

Concerning the second term of the right hand side of the last equality, using Young's inequality and formula (3.12) and (3.13), we get

$$\begin{aligned} \int_0^A \int_{\Omega} \mu \rho Q e^{a/\tau} dx da &\leq \int_0^A \left( \int_{\Omega} \mu^2 \rho^2 dx \right)^{1/2} \left( \int_{\Omega} Q^2 dx \right)^{1/2} e^{a/\tau} da \\ &\leq \tilde{\mu} \left[ \int_0^t \left( \int_{\Omega} \frac{\xi^2}{\tau^2} Q^2(t-a, \cdot) dx \right)^{1/2} e^{a/\tau} da \right. \\ &\quad \left. + H(A-t)e^{t/\tau} \int_t^A \left( \int_{\Omega} \rho_0^2(a-t, \cdot) e^{2(a-t)/\tau} dx \right)^{1/2} da \right] \|Q\|_2 \quad (3.19) \\ &\leq \tilde{\mu} \max(e^{T/\tau}, \frac{e^{T/\tau}}{\tau^2}) \left[ \int_0^t \left( \int_{\Omega} Q^2(t-a, \cdot) dx \right)^{1/2} da \right. \\ &\quad \left. + H(A-t) \int_t^A \left( \int_{\Omega} \rho_0^2(a-t, \cdot) e^{2(a-t)/\tau} dx \right)^{1/2} da \right] \|Q\|_2, \end{aligned}$$

where  $H(a) = 0$  if  $a < 0$  and  $H(a) = 1$  if  $a \in [0, +\infty]$  and where  $\tilde{\mu} = \sup\{\mu(t, a, x), t \in [0, T], a \in [0, A], x \in \Omega\}$ .

The third term of the right hand side of (3.18) may also be estimated:

$$\chi(A)e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) Q dx \leq \chi(A)e^{A/\tau} \left( \int_{\Omega} \rho^2(\cdot, A, \cdot) dx \right)^{1/2} \left( \int_{\Omega} Q^2 dx \right)^{1/2}. \quad (3.20)$$

Hence, applying again (3.12) and (3.13),

$$\begin{aligned} \chi(A)e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) Q \, dx &\leq \chi(A)e^{A/\tau} \left( \int_{\Omega} Q^2(t-A, \cdot) \, dx \right)^{1/2} \|Q\|_2, \quad \text{if } t > A, \\ &\leq \chi(A)e^{2t/\tau} \left( \int_{\Omega} \rho_0^2(A-t, \cdot) e^{2(A-t)/\tau} \, dx \right)^{1/2} \|Q\|_2, \quad \text{if } t < A. \end{aligned} \quad (3.21)$$

Using (3.19), (3.21) and (1.15) in (3.18), for four non negative constant  $c_1, c_2, c_3$  and  $c_4$  we have

$$\frac{d\|Q\|_2^2}{dt} \leq \left( c_1\|Q\|_2 + \int_0^t \left( \int_{\Omega} Q^2(t-a, \cdot) \, dx \right)^{1/2} da + c_3\|\rho_0\|_2 + c_4 \left( H(t-A) \int_{\Omega} Q^2(t-A, \cdot) \, dx \right)^{1/2} \right) \|Q\|_2. \quad (3.22)$$

Setting

$$F(t) = \sup_{s \in [0, T]} \|Q(s)\|_2^2 + \sup_{s \in [0, T]} \|\rho(s)\|_2^2, \quad (3.23)$$

we have

$$F'(t) \leq \max(0, \frac{d\|Q(t)\|_2^2}{dt}) + \max(0, \frac{d\|\rho(t)\|_2^2}{dt}), \quad (3.24)$$

and from (3.17) and (3.22) we deduce

$$F'(t) \leq ((c_1 + c_2T + c_3 + c_4)\sqrt{F(t)})\sqrt{F(t)} + \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)F(t) \leq c_5F(t), \quad (3.25)$$

for a constant  $c_5$ , from which we deduce that

$$F(t) \leq e^{c_5T} F(0), \quad (3.26)$$

and, as a consequence, that for a constant  $c_6$ ,

$$\|Q(t)\|_2^2 + \|\rho(t)\|_2^2 \leq c_6(\|Q_0\|_2^2 + \|\rho_0\|_2^2). \quad (3.27)$$

Finally we get (3.7) as a consequence of (3.27), this ends the proof.  $\blacksquare$

As a consequence of the Lemma 3.2 we have the following Corollary whose biological meaning is: Biomass cannot be created from nothing.

**COROLLARY 3.3** *If the assumptions of Theorem 1.1 are realized and if  $\rho_0 = 0$  and  $Q_0 = 0$  then the solution  $(Q, \rho)$  given by the Theorem satisfies*

$$\rho = 0 \quad \text{and} \quad Q = 0. \quad (3.28)$$

In order to establish the previous estimates, we have assumed that  $Q \geq 0$  and  $\rho \geq 0$ . We can show that this is a consequence of the non negativity of  $Q_0$  and  $\rho_0$ .

**LEMMA 3.4** *If the assumptions of Theorem 1.1 are true and if there exists a solution  $(Q, \rho)$  to system (1.1)-(1.6), then  $Q \geq 0$  and  $\rho \geq 0$ .*

The proof of this Lemma is close to the one of Lemma 3.2.

*Proof.* We define

$$\rho^- = \min(\rho, 0), \quad Q^- = \min(Q, 0). \quad (3.29)$$

Of course,

$$\rho^-(0, \cdot, \cdot) = 0, \quad Q^-(0, \cdot) = 0, \quad \rho^-(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q^-. \quad (3.30)$$



Now, multiplying (3.30) by  $e^{a/\tau}\rho^-$  and integrating, since

$$\frac{\partial \rho}{\partial t} \rho^- = \frac{\partial \rho^-}{\partial t} \rho^-, \quad (3.31)$$

proceeding as while establishing (3.17), we get

$$\frac{d\|\rho^-\|_2^2}{dt} \leq \frac{1}{\tau^2}\|Q^-\|_2^2 + \frac{1}{\tau}\|\rho^-\|_2^2. \quad (3.32)$$

Multiplying now (1.1) by  $Q^-$  and integrating gives

$$\begin{aligned} \frac{d\|Q^-\|_2^2}{dt} &= 2 \int_{\Omega} \frac{1-\xi}{\tau} Q^{-2} dx + 2 \int_0^A \int_{\Omega} \mu \rho Q^- e^{a/\tau} dx da + 2\chi(A) e^{A/\tau} \int_{\Omega} \rho(\cdot, A, \cdot) Q^- dx \\ &\leq 2 \int_{\Omega} \frac{1-\xi}{\tau} Q^{-2} dx + 2 \int_0^A \int_{\Omega} \mu \rho^- Q^- e^{a/\tau} dx da + 2\chi(A) e^{A/\tau} \int_{\Omega} \rho^-(\cdot, A, \cdot) Q^- dx. \end{aligned} \quad (3.33)$$

Since, as we had (3.12) and (3.13), we have here:

$$\int_{\Omega} \rho^{-2} dx = \int_{\Omega} \rho_0^{-2} dx \text{ if } t < a \quad \text{and} \quad \int_{\Omega} \rho^{-2} dx = \int_{\Omega} \frac{\xi^2}{\tau^2} Q^{-2}(t-a, \cdot) dx \text{ if } t > a, \quad (3.34)$$

we can finish the proof as in the proof of Lemma 3.2 and get

$$\|Q^-(t)\| + \|\rho^-(t)\| \leq c'(\|Q_0^-\| + \|\rho_0^-\|) = 0, \quad (3.35)$$

giving the Lemma. ■

### 3.2 Estimates on the derivatives

As a by product of the proof of Lemma 3.2, we can deduce from (3.16) the following result insuring a first control on the regularity of  $\rho$ .

**COROLLARY 3.5** *If the assumptions and the conclusion of Theorem 1.1 are valid then, for any  $0 \leq s \leq T$ ,*

$$\int_0^s \|\nabla \rho\|_2^2 dt = \int_0^s \int_0^A \int_{\Omega} |\nabla \rho|_2^2 dx da dt \leq \frac{c}{d}, \quad (3.36)$$

for a constant  $c$  (depending only on  $A, \tau, T, \tilde{\mu}, \|\rho_0\|_2$  and  $\|Q_0\|_2$ ).

*Proof.* Integrating (3.16) from 0 to  $s$  yields:

$$\begin{aligned} &\chi(A) e^{A/\tau} \int_0^s \int_{\Omega} \rho^2(\cdot, A, \cdot) dx dt + 2 \int_0^s \int_0^A \int_{\Omega} (D+d) |\nabla \rho|^2 e^{a/\tau} dx da dt \\ &+ 2 \int_0^s \int_0^A \int_{\Omega} \mu \rho^2 e^{a/\tau} dx da dt = \int_0^s \int_{\Omega} \frac{\xi^2}{\tau^2} Q^2 dx dt + \frac{1}{\tau} \int_0^s \|\rho\|_2^2 dt + \|\rho(0)\|_2^2 - \|\rho(s)\|_2^2. \end{aligned} \quad (3.37)$$

Using the previous estimate concerning  $\rho$  and  $Q$ , we may deduce

$$d \int_0^s \int_0^A \int_{\Omega} |\nabla \rho|^2 e^{a/\tau} dx da dt \leq c, \quad (3.38)$$

and (3.36) follows. ■

Because of the form of the non linearity in (1.1)-(1.6), we need a supplementary estimate concerning

$$\int_0^T \|\Delta \rho\|_2^2 dt = \int_0^T \int_0^A \int_{\Omega} |\Delta \rho|_2^2 dx da dt. \quad (3.39)$$

This estimate is a consequence of an estimate on  $\|\nabla \rho\|_4$  and on

$$\|\rho\|_{\infty} = \sup \{|\rho(t, a, x)|, t \in [0, T], a \in [0, A], x \in \Omega\}, \quad \|Q\|_{\infty} = \sup \{|Q(t, x)|, t \in [0, T], x \in \Omega\}, \quad (3.40)$$

that we now set.

LEMMA 3.6 *The solution  $(Q, \rho)$  given by Theorem 1.1 satisfies*

$$\|\rho\|_{\infty} + \|Q\|_{\infty} \leq k, \quad (3.41)$$

where  $k$  is a constant depending only on  $T, A, \sup_{t \in [0, T]} \|\rho(t)\|_2$  and  $\sup_{t \in [0, T]} \|Q(t)\|_2$  (which are estimated by Lemma 3.7)

*Proof.* As we already see that  $\|\rho(t)\|_2$  is bounded, using a method similar to Ladyzenskaja, Solonnikov and Ural'ceva [16], (paragraph III-8) we deduce that

$$|\chi(A)\rho(\cdot, A, \cdot)e^{A/\tau}| \leq k_1, \quad (3.42)$$

where the constant  $k_1$  only depends only on  $A$  and  $\sup_{t \in [0, T]} \|\rho(t)\|_2$ . Then defining

$$\tilde{P} = \int_0^A \rho e^{a/\tau} da, \quad (3.43)$$

we deduce from (1.2) that  $\tilde{P}$  is solution to the following parabolic equation

$$\frac{\partial \tilde{P}}{\partial t} + \chi(A)\rho(\cdot, A, \cdot)e^{A/\tau} = - \int_0^A \mu \rho e^{a/\tau} da + \nabla \cdot ((D + d)\nabla \tilde{P}) + \frac{\xi}{\tau} Q, \quad (3.44)$$

from which we get that

$$\|\tilde{P}\|_{\infty} = \sup \{|\tilde{P}(t, x)|, t \in [0, T], x \in \Omega\} \leq k_2, \quad (3.45)$$

where  $k_2$  only depends on  $\sup_{t \in [0, T]} \|\rho(t)\|_2, \sup_{t \in [0, T]} \|Q(t)\|_2$  and  $\sup_{x \in \Omega} |\tilde{P}_0| = \sup_{x \in \Omega} \left| \int_0^A \rho_0 e^{a/\tau} da \right|$  which is finite by assumption.

Then, (3.45) and (3.42) give that  $\|Q\|_{\infty}$  is finite, and as a consequence, (1.2) and (1.3) finally give the bound on  $\|\rho\|_{\infty}$ , ending the proof.  $\blacksquare$

LEMMA 3.7 *The solution  $(Q, \rho)$  given by Theorem 1.1 satisfies*

$$\|\nabla Q\|_4 + \|\nabla \rho\|_4 = \left( \int_{\Omega} |\nabla Q|^4 dx \right)^{1/4} + \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{a/\tau} dx da \right)^{1/4} \leq \frac{C}{d}, \quad (3.46)$$

for a constant  $C$  ( $C$  does not depend on  $d$ ).

*Proof.* Multiplying equation (1.2) by  $-\nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau}$ , and integrating in  $a$  and  $x$  gives:

$$\begin{aligned} & - \int_0^A \int_{\Omega} \frac{\partial \rho}{\partial t} \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da - \int_0^A \int_{\Omega} \frac{\partial \rho}{\partial a} \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da = \\ & \int_0^A \int_{\Omega} \mu \rho \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da - \int_0^A \int_{\Omega} \nabla \cdot [(D + d) \nabla \rho] \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da. \end{aligned} \quad (3.47)$$

Making a double integration by part, and following a straightforward computation procedure, we get

$$\begin{aligned} & \int_0^A \int_{\Omega} \nabla \cdot [(D + d) \nabla \rho] \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da = \\ & \int_0^A \int_{\Omega} (D + d) (|\nabla \rho|^2 |\nabla^2 \rho|^2 + 2\mathbf{H}(\nabla \rho, \nabla^2 \rho)) e^{a/\tau} dx da + \mathbf{E}, \end{aligned} \quad (3.48)$$

where

$$|\nabla^2 \rho|^2 = \left( \frac{\partial^2 \rho}{\partial x_1^2} \right)^2 + 2 \left( \frac{\partial^2 \rho}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial^2 \rho}{\partial x_2^2} \right)^2, \quad (3.49)$$

$$\mathbf{H}(\nabla \rho, \nabla^2 \rho) = \left( \frac{\partial \rho}{\partial x_1} \frac{\partial^2 \rho}{\partial x_1^2} + \frac{\partial \rho}{\partial x_2} \frac{\partial^2 \rho}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial \rho}{\partial x_1} \frac{\partial^2 \rho}{\partial x_1 \partial x_2} + \frac{\partial \rho}{\partial x_2} \frac{\partial^2 \rho}{\partial x_2^2} \right)^2, \quad (3.50)$$

and

$$\begin{aligned} \mathbf{E} &= \int_0^A \int_{\Omega} \sum_{i,j=1}^2 \left( \frac{\partial D(\mathcal{M}, Q, P)}{\partial x_i} \frac{\partial \rho}{\partial x_j} \right) \\ & \quad \left( |\nabla \rho|^2 \frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2 \left( \frac{\partial \rho}{\partial x_1} \frac{\partial^2 \rho}{\partial x_1 \partial x_j} + \frac{\partial \rho}{\partial x_2} \frac{\partial^2 \rho}{\partial x_j \partial x_2} \right) \frac{\partial \rho}{\partial x_i} \right) e^{a/\tau} dx da. \end{aligned} \quad (3.51)$$

Since

$$\frac{\partial D(\mathcal{M}, Q, P)}{\partial x_i} = \frac{\partial D}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial x_i} + \frac{\partial D}{\partial Q} \frac{\partial Q}{\partial x_i} + \frac{\partial D}{\partial P} \frac{\partial P}{\partial x_i}, \quad (3.52)$$

in view of the regularity of  $D$ , of equation (1.12) that gives a control on  $\partial \mathcal{M} / \partial x_i$  in terms of  $\partial P / \partial x_i$ , we get for a constant  $C_1$

$$\begin{aligned} |\mathbf{E}| &\leq C_1 \int_0^A \int_{\Omega} (|\nabla P| + |\nabla Q|) |\nabla \rho|^3 |\nabla^2 \rho| e^{a/\tau} dx da \\ &\leq \frac{d}{4} \int_0^A \int_{\Omega} |\nabla \rho|^2 |\nabla^2 \rho|^2 e^{a/\tau} dx da + \frac{C_1^2}{d} \int_0^A \int_{\Omega} (|\nabla P| + |\nabla Q|)^2 |\nabla \rho|^2 e^{a/\tau} dx da; \end{aligned} \quad (3.53)$$

in order to get the last expression in (3.53), we used  $UV \leq \frac{d}{4}U^2 + \frac{1}{d}V^2$  with  $V = C_1(|\nabla P| + |\nabla Q|)|\nabla \rho|$ .

Concerning the other terms of (3.47), since  $\frac{\partial |\nabla \rho|^4}{\partial t} = 2 \frac{\partial |\nabla \rho|^2}{\partial t} |\nabla \rho|^2 = 4 \frac{\partial \nabla \rho}{\partial t} \cdot \nabla \rho |\nabla \rho|^2$ , making an integration by parts, we get

$$- \int_0^A \int_{\Omega} \frac{\partial \rho}{\partial t} \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da = \frac{1}{4} \frac{d \|\nabla \rho\|_4^4}{dt}. \quad (3.54)$$

In a similar way,

$$\begin{aligned}
& - \int_0^A \int_{\Omega} \frac{\partial \rho}{\partial a} \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da = \frac{1}{4} \int_0^A \int_{\Omega} \frac{\partial (|\nabla \rho|^4 e^{a/\tau})}{\partial a} dx da - \frac{1}{4\tau} \|\nabla \rho\|_4^4 \\
& = \frac{1}{4} \chi(A) \int_{\Omega} |\nabla \rho(\cdot, A, \cdot)|^4 e^{A/\tau} dx - \frac{1}{4\tau^4} \int_{\Omega} (\xi^4 + (\frac{\partial \xi}{\partial Q} Q)^4) |\nabla Q|^4 dx - \frac{1}{4\tau} \|\nabla \rho\|_4^4, \quad (3.55)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_0^A \int_{\Omega} \mu \rho \nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{a/\tau} dx da \right| \\
& = \left| - \int_0^A \int_{\Omega} \mu |\nabla \rho|^4 e^{a/\tau} dx da - \int_0^A \int_{\Omega} \nabla \mu \rho |\nabla \rho|^2 e^{a/\tau} dx da \right| \leq C_2 (\|\nabla \rho\|_4^4 + 1), \quad (3.56)
\end{aligned}$$

using the regularity of  $\mu$  and the estimate on  $\sup(\rho)$  given by Lemma 3.6.

The regularity of  $\xi$  and the estimate on  $\sup(Q)$  give  $|\xi^4 + (\frac{\partial \xi}{\partial Q} Q)^4| \leq C_3$  for a constant  $C_3$ . Hence (3.47) yields

$$\begin{aligned}
& \frac{1}{4} \frac{d\|\nabla \rho\|_4^4}{dt} + \frac{1}{4} \chi(A) \int_{\Omega} |\nabla \rho(\cdot, A, \cdot)|^4 e^{A/\tau} dx + \int_0^A \int_{\Omega} (D+d) (|\nabla \rho|^2 |\nabla^2 \rho|^2) e^{a/\tau} dx da \\
& + 2 \int_0^A \int_{\Omega} (D+d) \mathbf{H}(\nabla \rho, \nabla^2 \rho) e^{a/\tau} dx \leq \frac{C_3}{4\tau^4} \int_{\Omega} |\nabla Q|^4 dx + \frac{1}{4\tau} \|\nabla \rho\|_4^4 + C_2 (\|\nabla \rho\|_4^4 + 1) \\
& + \frac{d}{4} \int_0^A \int_{\Omega} |\nabla \rho|^2 |\nabla^2 \rho|^2 e^{a/\tau} dx da + \frac{C_1^2}{d} \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{a/\tau} dx da + C_4 (\|\nabla Q\|_4^4 + \|\nabla \rho\|_4^4) \right), \quad (3.57)
\end{aligned}$$

and passing the fourth term of the right hand side in the left hand side we can deduce

$$\frac{d\|\nabla \rho\|_4^4}{dt} \leq \frac{C_5}{d} (\|\nabla \rho\|_4^4 + \|\nabla Q\|_4^4 + 1), \quad (3.58)$$

for a constant  $C_5$ .

Multiplying equation (1.2) by  $-\nabla \cdot (|\nabla \rho|^2 \nabla \rho) e^{4a/\tau}$ , and making the same operations as previously, we obtain an inequality which is (3.57) with  $e^{a/\tau}$  replaced by  $e^{4a/\tau}$  and  $\|\nabla Q\|_4^4$  replaced by  $\int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da$ . From this, we can deduce

$$\begin{aligned}
& \frac{d(\int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da)}{dt} + \chi(A) \int_{\Omega} |\nabla \rho(\cdot, A, \cdot)|^4 e^{4A/\tau} dx \\
& \leq \frac{C_6}{d} \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da + \|\nabla Q\|_4^4 + 1 \right). \quad (3.59)
\end{aligned}$$

On another hand, computing the gradient of (1.1), and multiplying by  $\nabla Q |\nabla Q|^2$  yields

$$\begin{aligned}
& \frac{1}{4} \frac{d\|\nabla Q\|_4^4}{dt} = \frac{1}{\tau} \int_{\Omega} \frac{\partial \xi}{\partial Q} Q |\nabla Q|^4 dx + \int_{\Omega} \frac{1-\xi}{\tau} |\nabla Q|^4 dx + \int_{\Omega} \left( \int_0^A \nabla \mu \rho e^{a/\tau} da \right) \cdot \nabla Q |\nabla Q|^2 dx \\
& + \int_{\Omega} \left( \int_0^A \mu \nabla \rho e^{a/\tau} da \right) \cdot \nabla Q |\nabla Q|^2 dx + \chi(A) \int_{\Omega} (\nabla \rho(\cdot, A, \cdot) e^{A/\tau}) \cdot \nabla Q |\nabla Q|^2 dx. \quad (3.60)
\end{aligned}$$

Because of the regularity of  $\xi$  and  $\mu$  and of Lemma 3.6, since, applying Young's inequality,

$$\begin{aligned} & \int_{\Omega} \left( \int_0^A \mu \nabla \rho e^{a/\tau} da \right) \cdot \nabla Q |\nabla Q|^2 dx \\ & \leq \left( \int_{\Omega} \left( \int_0^A \mu |\nabla \rho| e^{a/\tau} da \right)^4 dx \right)^{1/4} \left( \int_{\Omega} (|\nabla Q|^3)^{4/3} dx \right)^{3/4} \\ & \leq \tilde{\mu} \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da \right)^{1/4} \|\nabla Q\|_4^3, \end{aligned} \quad (3.61)$$

and

$$\begin{aligned} \chi(A) \int_{\Omega} (\nabla \rho(\cdot, A, \cdot) e^{A/\tau}) \cdot \nabla Q |\nabla Q|^2 dx & \leq \chi(A) \left( \int_{\Omega} |\nabla \rho(\cdot, A, \cdot)|^4 e^{4A/\tau} dx \right)^{1/4} \|\nabla Q\|_4^3 \\ & \leq \chi(A) \frac{C'_6}{d} \left( \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da \right)^{1/4} + \|\nabla Q\|_4 + 1 \right) \|\nabla Q\|_4^3, \end{aligned} \quad (3.62)$$

we deduce from (3.60)

$$\frac{d\|\nabla Q\|_4^4}{dt} \leq C_4 (\|\nabla Q\|_4^4 + 1) + \frac{C'_5}{d} \left( \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da \right)^{1/4} + \|\nabla Q\|_4 + 1 \right) \|\nabla Q\|_4^3. \quad (3.63)$$

Inequalities (3.58), (3.59), (3.63) and the assumptions on  $Q_0$  and  $\rho_0$  give

$$\|\nabla Q\|_4 + \|\nabla \rho\|_4 + \left( \int_0^A \int_{\Omega} |\nabla \rho|^4 e^{4a/\tau} dx da \right)^{1/4} \leq \frac{C}{d}, \quad (3.64)$$

and finally the Lemma. ■

LEMMA 3.8 *The solution  $(Q, \rho)$  given by Theorem 1.1 satisfies*

$$\|\nabla Q(t)\|_2^2 \leq \frac{c}{d}, \quad \|\nabla \rho(t)\|_2^2 \leq \frac{c}{d} \quad \text{for any } 0 \leq t \leq T, \quad (3.65)$$

$$\int_0^s \|\Delta \rho(t)\|_2^2 dt = \int_0^s \int_0^A \int_{\Omega} |\Delta \rho(t)|^2 e^{a/\tau} dx da dt \leq \frac{c}{d^2}, \quad \text{for any } 0 \leq s \leq T, \quad (3.66)$$

$$\int_0^s \left\| \frac{\partial Q}{\partial t}(t) \right\|_2^2 dt \leq c, \quad \int_0^s \left\| \frac{\partial \rho}{\partial t}(t) \right\|_2^2 dt \leq \frac{c}{d^2} \quad \text{and} \quad \int_0^s \left\| \frac{\partial \rho}{\partial a}(t) \right\|_2^2 dt \leq \frac{c}{d^2} \quad \text{for any } 0 \leq s \leq T, \quad (3.67)$$

for a constant  $c$  (which does not depend on  $d$ ).

*Proof.* Multiplying (1.2) by  $(-\Delta\rho)e^{a/\tau}$  and integrating in  $a$  and  $x$  gives

$$\begin{aligned}
& \frac{d\|\nabla\rho\|_2^2}{dt} + \chi(A)e^{A/\tau} \int_{\Omega} |\nabla\rho(\cdot, A, \cdot)|^2 dx + 2 \int_0^A \int_{\Omega} \mu |\nabla\rho|^2 e^{a/\tau} dx da \\
& \quad + 2 \int_0^A \int_{\Omega} (D+d) |\Delta\rho|^2 e^{a/\tau} dx da = \int_0^A \int_{\Omega} \nabla\mu \cdot \nabla\rho \rho e^{a/\tau} dx da \\
& \quad - \int_0^A \int_{\Omega} \nabla[D(\mathcal{M}, Q, P)] \cdot \nabla\rho \Delta\rho e^{a/\tau} dx da + \frac{1}{\tau} \|\nabla\rho\|_2^2 + \int_{\Omega} (\xi^2 + (\frac{\partial\xi}{\partial Q}Q)^2) |\nabla Q|^2 dx \\
& \leq c_1 (\|\nabla\rho\|_2 + \|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2) + \int_0^A \int_{\Omega} \left( \frac{\partial D}{\partial \mathcal{M}} \nabla\mathcal{M} + \frac{\partial D}{\partial P} \nabla P + \frac{\partial D}{\partial Q} \nabla Q \right) \nabla\rho \Delta\rho e^{a/\tau} dx da \\
& \leq c_1 (\|\nabla\rho\|_2 + \|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2) + \frac{1}{d} \int_0^A \int_{\Omega} \left( \frac{\partial D}{\partial \mathcal{M}} \nabla\mathcal{M} + \frac{\partial D}{\partial P} \nabla P + \frac{\partial D}{\partial Q} \nabla Q \right)^2 |\nabla\rho|^2 e^{a/\tau} dx da \\
& \quad + \frac{d}{4} \int_0^A \int_{\Omega} |\Delta\rho|^2 e^{a/\tau} dx da, \quad (3.68)
\end{aligned}$$

for a constant  $c_1$ . Now transferring the last term of the right hand side in the left hand side and using the estimate on  $\|\nabla\rho\|_4$  and  $\|\nabla Q\|_4$  which also give an estimate on  $\|\nabla\mathcal{M}\|_4$  and  $\|\nabla P\|_4$ , we get, for a constant  $c_2$

$$\frac{d\|\nabla\rho\|_2^2}{dt} + \frac{5d}{4} \|\Delta\rho\|_2^2 \leq c_2 (\|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2 + \frac{1}{d}), \quad (3.69)$$

In a similar way, we can also get

$$\frac{d(\int_0^A \int_{\Omega} |\nabla\rho|^2 e^{2a/\tau} dx da)}{dt} \leq c_3 \left( \int_0^A \int_{\Omega} |\nabla\rho|^2 e^{2a/\tau} dx da + \|\nabla Q\|_2^2 + \frac{1}{d} \right), \quad (3.70)$$

and

$$\frac{d\|\nabla Q\|_2^2}{dt} \leq c_4 \left( \int_0^A \int_{\Omega} |\nabla\rho|^2 e^{2a/\tau} dx da + \|\nabla Q\|_2^2 \right). \quad (3.71)$$

From the three last inequalities we get (3.65). Integrating (3.69) from 0 to  $s$  gives (3.66). The estimate on  $\|\nabla Q(t)\|_2$  is then obtained as the estimate on  $\|\nabla Q(t)\|_4$ . Estimate (3.67) is finally a direct consequence of equations (1.1) and (1.2).  $\blacksquare$

REMARK 3.1 We could also prove that  $\|\nabla Q\|_{\infty}$  and  $\|\nabla\rho\|_{\infty}$  are bounded.

## 4 Existence and uniqueness of the solution

Once the a priori estimates are set, the proof of existence is classical and in the spirit of Ladyzenskaja, Solonnikov and Ural'ceva [16]. It essentially consists in linearization and passing to the limit using the estimates.

In the following  $L^p(\Omega)$  and  $L^p([0, A] \times \Omega, e^{a/\tau} dadx)$  are the functional spaces associated with the norms (3.1),  $W^{k,p}(\Omega)$  and  $W^{k,p}([0, A] \times \Omega, e^{a/\tau} dadx)$  are the Sobolev spaces composed of functions whose derivatives until order  $k$  are in  $L^p(\Omega)$  or  $L^p([0, A] \times \Omega, e^{a/\tau} dadx)$ .  $L^p([0, T] \times \Omega)$  and  $L^p([0, T] \times [0, A] \times \Omega, e^{a/\tau} dadxdt)$  are the spaces of functions having finite norm

$$\left( \int_0^T \int_0^A \int_{\Omega} |\rho(t, a, x)|^p e^{a/\tau} dx dadt \right)^{1/p} \quad \text{or} \quad \left( \int_0^T \int_{\Omega} |Q(t, x)|^p dx dt \right)^{1/p}, \quad (4.1)$$

and  $W^{k,p}([0, T] \times \Omega)$  and  $W^{k,p}([0, T] \times [0, A] \times \Omega, e^{a/\tau} dadxdt)$  are their associated Sobolev spaces. Finally, for a functional space  $W$ ,  $L^{\infty}(0, T; W)$  stands for the functions whose norm in  $W$  is finite for any  $t \in [0, T]$ .

## 4.1 Linearization

We linearize the system (1.1)-(1.6) Then using classical results on pde and ode, we give an existence and uniqueness result for the solution to this linearized system.

We set  $Q^0 = Q_0$  and  $\rho^0 = \rho_0$  and for  $n \in \mathbb{N}^*$ , we consider  $(Q^n, \rho^n)$  solution to:

$$\frac{\partial Q^n}{\partial t} = \frac{1-\xi}{\tau} Q^n + \int_0^A \rho^n(\cdot, a, \cdot) e^{a/\tau} \mu(\cdot, a, \cdot) da + \chi(A) \rho^n(\cdot, A, \cdot) e^{A/\tau}, \quad \text{on } [0, T] \times \Omega, \quad (4.2)$$

$$\frac{\partial \rho^n}{\partial t} + \frac{\partial \rho^n}{\partial a} = -\mu \rho^n + \nabla \cdot [(D(\mathcal{M}^{n-1}, Q^{n-1}, P^{n-1}) + d) \nabla \rho^n], \quad \text{on } [0, T] \times [0, A] \times \Omega, \quad (4.3)$$

$$\rho^n(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q^n, \quad \text{on } [0, T] \times \Omega, \quad (4.4)$$

$$\rho^n(0, \cdot, \cdot) = \rho_0, \quad \text{on } [0, A] \times \Omega, \quad (4.5)$$

$$Q^n(0, \cdot) = Q_0, \quad \text{on } \Omega, \quad (4.6)$$

$$\frac{\partial \rho^n}{\partial \vec{\nu}} = 0, \quad \text{on } [0, T] \times [0, A] \times \partial \Omega, \quad (4.7)$$

where for  $n \in \mathbb{N}$ ,

$$P^n(t, x) = \int_{a_{\min}}^A \rho^n(t, a, x) e^{a/\tau} da, \quad (4.8)$$

and  $\mathcal{M}^n$  is solution to

$$\begin{aligned} \frac{\partial \mathcal{M}^n}{\partial t} &= \frac{1}{P_{\max} - p_{\max}} H_r \left( \frac{P^n - p_{\max}}{P_{\max} - p_{\max}} \right) H_r(1 - \mathcal{M}^n) \\ &\quad - \frac{1}{p_{\min} - P_{\min}} H_r \left( \frac{p_{\min} - P^n}{p_{\min} - P_{\min}} \right) H_r(\mathcal{M}^n), \\ \mathcal{M}^n(0, \cdot) &= \mathcal{M}_0. \end{aligned} \quad (4.9)$$

**THEOREM 4.1** *Under assumptions (1.8), (1.9), (1.10) and (1.13), if  $\rho_0 \geq 0 \in L^1 \cap W^{2,2} \cap W^{1,4}([0, A] \times \Omega, e^{a/\tau} dadx)$  satisfies (1.15) and if  $Q_0 \geq 0 \in L^1 \cap W^{2,2} \cap W^{1,4}(\Omega)$  then for any  $n \in \mathbb{N}$ , there exists a unique solution  $(Q^n, \rho^n) \in C_b^0(0, T; (L^1 \cap W^{1,2}(\Omega)) \times (L^1 \cap W^{1,2}([0, A] \times \Omega, e^{a/\tau} dadx)))$  to system (4.2) - (4.7) coupled with (4.8) and (4.9). Moreover,  $Q^n \geq 0$ ,  $\rho^n \geq 0$  and  $(Q^n, \rho^n)$  satisfies estimates (3.2), (3.7), (3.41), (3.46), (3.65), (3.66) and (3.67) with constants independent of  $n$ .*

The proof of this Theorem uses only classical pde and ode arguments. Hence we only sketch it.

*Proof.* The proof consists in an induction procedure. Because of the assumptions on  $(Q_0, \rho_0)$  and the definition of  $(Q^0, \rho^0)$ , the Theorem is true for  $n = 0$ .

Then, if the Theorem is true for  $n - 1$ , by regularization arguments, we can get that that  $\mathcal{M}^{n-1}$  exists and is unique on  $[0, T] \times \Omega$  and that  $\mathcal{M}^{n-1} \in C_b^0(0, T; W^{1,2}(\Omega)) \cap C_b^1(0, T; L^\infty(\Omega))$ . Hence we deduce that, for each  $l \in \mathbb{N}^*$ , there exists a unique solution  $(Q^{n,l}, \rho^{n,l}) \in C_b^0(0, T; (L^1 \cap W^{1,2}(\Omega)) \times (L^1 \cap W^{1,2}([0, A] \times \Omega, e^{a/\tau} dadx)))$  to

$$\frac{\partial Q^{n,l}}{\partial t} = \frac{1}{\tau} Q^{n,l} - \frac{\xi}{\tau} Q^{n,l-1} + \int_0^A \rho^{n,l}(\cdot, a, \cdot) e^{a/\tau} \mu(\cdot, a, \cdot) da + \chi(A) \rho^{n,l}(\cdot, A, \cdot) e^{A/\tau}, \quad (4.10)$$

$$\frac{\partial \rho^{n,l}}{\partial t} + \frac{\partial \rho^{n,l}}{\partial a} = -\mu \rho^{n,l} + \nabla \cdot [(D(\mathcal{M}^{n-1}, Q^{n-1}, P^{n-1}) + d) \nabla \rho^{n,l}], \quad (4.11)$$

$$\rho^{n,l}(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q^{n,l-1}, \quad \rho^{n,l}(0, \cdot, \cdot) = \rho_0, \quad \frac{\partial \rho^{n,l}}{\partial \vec{\nu}} \Big|_{\partial \Omega} = 0, \quad (4.12)$$

$$Q^{n,l}(0, \cdot) = Q_0, \quad (4.13)$$

where  $Q^{n,0}$  is defined as  $Q^{n,0} = Q^{n-1}$ . This deduction involves first a classical semi-group or Galerkin routine in order to deduce that there exists a unique solution  $\rho^{n,l}$  to (4.11) - (4.12) as soon as the existence of  $(Q^{n,l-1}, \rho^{n,l-1})$  is achieved. These routines are explained in Lions and Magenes [19], Ladyzenskaja, Solonnikov and Ural'ceva [16], or in a context close to our in Langlais [17]. Once the existence of  $\rho^{n,l}$  is established, the existence and uniqueness of  $Q^{n,l}$  follows. Now, following the way leading to (3.17) and (3.22) we deduce that  $(Q^{n,l}, \rho^{n,l})$  satisfies

$$\frac{d\|\rho^{n,l}\|_2^2}{dt} \leq \frac{1}{\tau^2}\|Q^{n,l-1}\|_2^2 + \frac{1}{\tau}\|\rho^{n,l}\|_2^2, \quad (4.14)$$

$$\begin{aligned} \frac{d\|Q^{n,l}\|_2^2}{dt} \leq & \left( c_1\|Q^{n,l}\|_2 + c'_1\|Q^{n,l-1}\|_2 \int_0^t \left( \int_\Omega (Q^{n,l})^2(t-a, \cdot) dx \right)^{1/2} da + c_3\|\rho_0\|_2 \right. \\ & \left. + c_4 \left( H(t-A) \int_\Omega (Q^{n,l})^2(t-A, \cdot) dx \right)^{1/2} \right) \|Q^{n,l}\|_2, \end{aligned} \quad (4.15)$$

which is enough to deduce that  $(\|\rho^{n,l}\|_2 + \|Q^{n,l}\|_2)$  is bounded. As a consequence of this bound, we get that, for a subsequence still denoted  $l$ ,  $(Q^{n,l}, \rho^{n,l}) \rightarrow (Q^n, \rho^n)$  in  $L^\infty(0, T; (L^2(\Omega) \times (L^2([0, A] \times \Omega, e^{a/\tau} dadx)))$  weakly-\*, where  $(Q^n, \rho^n)$  is solution to (4.2) - (4.7). Finally, the estimates are led in the same way as in section 3. The uniqueness follows directly from the linear character of (4.2) - (4.7). Hence, the Theorem is true for  $n$ .

The induction procedure is then straightforward to end the proof of the Theorem.  $\blacksquare$

## 4.2 Existence

From estimates (3.65), (3.66) and (3.67) we deduce that the sequence  $(Q^n, \rho^n)$  is bounded in  $W^{1,2}([0, T] \times \Omega) \times W^{1,2}([0, T] \times [0, A] \times \Omega)$ . Hence, up to a subsequence still denoted  $n$ , we have  $(Q^n, \rho^n) \rightarrow (Q, \rho)$  in  $W^{1,2}([0, T] \times \Omega) \times W^{1,2}([0, T] \times [0, A] \times \Omega, e^{a/\tau} dadxdt)$  weakly, and then, in  $L^2([0, T] \times \Omega) \times L^2([0, T] \times [0, A] \times \Omega, e^{a/\tau} dadxdt)$  strongly.

From this we can also deduce that  $P^n \rightarrow P$  in  $L^2([0, T] \times \Omega)$  strongly, with  $P$  defined from  $\rho$  by (1.11). In view of (4.9), we can deduce that  $(\mathcal{M}^n)$ ,  $(\partial\mathcal{M}^n/\partial t)$  and, taking the gradient of (4.9),  $(\nabla\mathcal{M}^n)$  are bounded in  $L^2([0, T] \times \Omega)$ . Then extracting again a subsequence still denoted  $n$ , we deduce  $\mathcal{M}^n \rightarrow \mathcal{M}$  strongly, where  $\mathcal{M}$  is solution to (1.12).

Using now the regularity of  $D$ , we obtain  $D(\mathcal{M}^{n-1}, Q^{n-1}, P^{n-1}) \rightarrow D(\mathcal{M}, Q, P)$  in  $L^2([0, T] \times \Omega)$  strongly.

The regularity of trace operators gives  $\chi(A)\rho^n(\cdot, A, \cdot) \rightarrow \chi(A)\rho(\cdot, A, \cdot)$ ,  $\rho^n(\cdot, 0, \cdot) \rightarrow \rho(\cdot, 0, \cdot)$ ,  $\rho^n(0, \cdot, \cdot) \rightarrow \rho(0, \cdot, \cdot)$ ,  $Q^n(0, \cdot) \rightarrow Q^n(0, \cdot)$  weakly, and using (3.66),  $\partial\rho^n/\partial\vec{\nu}|_{\partial\Omega} \rightarrow \partial\rho/\partial\vec{\nu}|_{\partial\Omega}$  weakly.

Then passing to the limit in (4.2) - (4.9), we obtain that  $(Q, \rho)$  is solution to (1.1) - (1.6) coupled with (1.11) and (1.12).

Once this existence result is established, using regularizations and truncations, we can lead the computations of section 3 giving the additional regularity and the non negativity of  $Q$  and  $\rho$ .

It now remains to prove the uniqueness of the solution.



### 4.3 Uniqueness

Consider  $(Q, \rho)$  with associated  $P$  and  $\mathcal{M}$  and  $(\tilde{Q}, \tilde{\rho})$  with associated  $\tilde{P}$  and  $\tilde{\mathcal{M}}$  two solutions of (1.1) - (1.6). They both satisfy the estimates and the difference  $(\tilde{Q}, \tilde{\rho}) = (Q - \tilde{Q}, \rho - \tilde{\rho})$  satisfies

$$\frac{\partial \tilde{Q}}{\partial t} = \frac{1-\xi}{\tau} \tilde{Q} + \int_0^A \tilde{\rho}(\cdot, a, \cdot) e^{a/\tau} \mu(\cdot, a, \cdot) da + \chi(A) \tilde{\rho}(\cdot, A, \cdot) e^{A/\tau}, \quad (4.16)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}}{\partial a} = -\mu \tilde{\rho} + \nabla \cdot [(D(\mathcal{M}, Q, P) + d) \nabla \tilde{\rho}] - \nabla \cdot [(D(\tilde{\mathcal{M}}, \tilde{Q}, \tilde{P}) - D(\mathcal{M}, Q, P)) \nabla \tilde{\rho}], \quad (4.17)$$

$$\tilde{\rho}(\cdot, 0, \cdot) = \frac{\xi}{\tau} \tilde{Q}, \quad \tilde{\rho}(0, \cdot, \cdot) = 0, \quad \tilde{Q}(0, \cdot) = 0, \quad \frac{\partial \tilde{\rho}}{\partial \mathcal{V}}|_{\partial \Omega} = 0. \quad (4.18)$$

Multiplying (4.17) by  $\tilde{\rho} e^{a/\tau}$  and integrating gives

$$\begin{aligned} \frac{d\|\tilde{\rho}\|_2^2}{dt} + \chi(A) e^{A/\tau} \int_{\Omega} \tilde{\rho}^2(\cdot, A, \cdot) dx + 2 \int_0^A \int_{\Omega} (D(\tilde{\mathcal{M}}, \tilde{Q}, \tilde{P}) + d) |\nabla \tilde{\rho}|^2 e^{a/\tau} dx da \\ = -2 \int_0^A \int_{\Omega} \mu \tilde{\rho}^2 e^{a/\tau} dx da + \int_{\Omega} \frac{\xi^2}{\tau^2} \tilde{Q}^2 dx + \frac{1}{\tau} \|\tilde{\rho}\|_2^2 \\ + \int_0^A \int_{\Omega} (D(\tilde{\mathcal{M}}, \tilde{Q}, \tilde{P}) - D(\mathcal{M}, Q, P)) \nabla \tilde{\rho} \cdot \nabla \tilde{\rho} e^{a/\tau} dx da \\ \leq k_1 (\|\tilde{\rho}\|_2^2 + \|\tilde{Q}\|_2^2) + k_2 \|\nabla \tilde{\rho}\|_{\infty} (\|\tilde{\rho}\|_2 + \|\tilde{Q}\|_2) \|\nabla \tilde{\rho}\|_2 \\ \leq k_1 (\|\tilde{\rho}\|_2^2 + \|\tilde{Q}\|_2^2) + \frac{k_2^2}{d} \|\nabla \tilde{\rho}\|_{\infty}^2 (\|\tilde{\rho}\|_2 + \|\tilde{Q}\|_2)^2 + \frac{d}{4} \|\nabla \tilde{\rho}\|_2^2, \end{aligned} \quad (4.19)$$

for constants  $k_1$  and  $k_2$ . Passing the term  $\frac{d}{4} \|\nabla \tilde{\rho}\|_2^2$  in the left hand side gives, for a constant  $k_3$

$$\frac{d\|\tilde{\rho}\|_2^2}{dt} \leq \frac{k_3}{d} (\|\tilde{\rho}\|_2^2 + \|\tilde{Q}\|_2^2). \quad (4.20)$$

Making the same, but multiplying (4.17) by  $\tilde{\rho} e^{2a/\tau}$  yields

$$\frac{d(\int_0^A \int_{\Omega} |\tilde{\rho}|^2 e^{2a/\tau} dx da)}{dt} + \chi(A) e^{2A/\tau} \int_{\Omega} \tilde{\rho}(\cdot, A, \cdot) dx \leq \frac{k_3}{d} \left( \int_0^A \int_{\Omega} |\tilde{\rho}|^2 e^{2a/\tau} dx da + \|\tilde{Q}\|_2^2 \right). \quad (4.21)$$

Multiplying (4.16) by  $\tilde{Q}$  gives

$$\frac{d\|\tilde{Q}\|_2^2}{dt} \leq \frac{k_4}{d} \left( \int_0^A \int_{\Omega} |\tilde{\rho}|^2 e^{2a/\tau} dx da + \|\tilde{Q}\|_2^2 \right), \quad (4.22)$$

for a constant  $k_4$ . From the last three inequalities we deduce, for a constant  $K$

$$\|\tilde{Q}\|_2^2 + \|\tilde{\rho}\|_2^2 + \int_0^A \int_{\Omega} |\tilde{\rho}|^2 e^{2a/\tau} dx da \leq K \left( \|\tilde{Q}|_{t=0}\|_2^2 + \|\tilde{\rho}|_{t=0}\|_2^2 + \int_0^A \int_{\Omega} |\tilde{\rho}|_{t=0}|^2 e^{2a/\tau} dx da \right) = 0, \quad (4.23)$$

giving  $\tilde{Q} = \tilde{\rho} = 0$  and then the uniqueness of the solution to (1.1) - (1.6).  $\blacksquare$

## References

- [1] V. Andreasen. Disease regulation of age-structured host populations. *Theo. Pop. Biol.*, 36(2):214–239, 1989.

- [2] V. Andreasen. The effect of age-dependent host mortality on the dynamics of an endemic disease. *Math. BioSciences*, 114:29–58, 1993.
- [3] V. Andreasen. Instability in SIR-model with age-dependent susceptibility. *Math. Pop. Dyn.*, 1:3–14, 1995.
- [4] B. P. Ayati and T. F. Dupont. Galerkin methods in age and space for a population model with nonlinear diffusion. *SIAM J. Numer. Anal.*, 40(3):1064–1076, 2002.
- [5] S. Busenberg and M. Iannelli. A class of nonlinear diffusion problems in age-dependent population dynamics. *Nonlinear Analysis, Theo., Meth., & Appl.*, 7(5):501–529, 1983.
- [6] G. Di Blasio. Non linear age-dependent population diffusion. *J. Math. Biol.*, 8:265–284, 1979.
- [7] G. Di Blasio and L. Lamberti. An initial-boundary value problem for age-dependent population diffusion. *SIAM J. Appl. Math.*, 35(3):593–616, November 1978.
- [8] S. E. Esipov and J. A. Shapiro. Kinetic model of *proteus mirabilis* swarm colony development. *J. Math. Biol.*, 36:249–268, 1998.
- [9] M. Gué, V. Dupont, A. Dufour, and O. Sire. Bacterial swarming: A biological time-resolved FTIR-ATR study of *proteus mirabilis* swarm-cell differentiation. *BioChemistry*, 40:11938–11945, 2001.
- [10] M. E. Gurtin. A system of equations for age-dependent population diffusion. *J. theor. Biol*, 40:389–392, 1973.
- [11] M. E. Gurtin and R. C. Mac Camy. Non-linear age-dependent population dynamics. 54:281–300, 1974.
- [12] M. E. Gurtin and R. C. Mac Camy. Diffusion model for age-structured population. *Math. BioSciences*, 54:49–59, 1981.
- [13] C. Huang. An age-dependent population model with nonlinear diffusion in  $R^n$ . *Quat. of Appl. Math.*, LII(2):377–398, june 1994.
- [14] M-Y. Kim. Galerkin methods for a model of population dynamics with nonlinear diffusion. *Num. Meth. for PDE*, 12:59–73, 1996.
- [15] M. Kubo and M. Langlais. Periodic solutions for population dynamics problem with age-dependent and spatial structure. *J. Math. Biol.*, 29:363–378, 1991.
- [16] O. A. Ladyzenskaja, Solonnikov V. A., and Ural'ceva N. N. *Linear and Quasi-linear Equation of Parabolic Type*. AMS, Translation of Mathematical Monographs, vol. 23.
- [17] M. Langlais. A nonlinear problem in age-dependent population diffusion. *Siam J. Math. Anal.*, 16(3):510–529, 1985.
- [18] M. Langlais. Large time behavior in a nonlinear age-dependent population dynamics problem with spatial diffusion. *J. Math. Biol.*, 26:319–346, 1988.
- [19] J. L. Lions and Magenes E. *Problèmes aux limites non homogènes et applications*, volume 17 , 18 of *Travaux et recherches mathématiques*. Dunod.
- [20] L. Lopez and D. Trigiante. A finite difference scheme for stiff problem arising in the numerical solution of a population dynamic model with spatial diffusion. *Nonlinear Analysis, Theo., Meth., & Appl.*, 9(1):1–12, 1985.

- [21] R. C. Mac Camy. A population model with nonlinear diffusion. *J. Diff. Equ.*, 39:52–72, 1981.
- [22] P. Marcati. Asymptotic behavior in age-dependent population dynamics with heredity renewal law. *SIAM J. Math. Anal.*, 12(6):904–916, November 1981.
- [23] G. E. Medvedev, T. J. Kapper, and Koppel N. A reaction-diffusion system with periodic front dynamics. *SIAM J. Appl. Math.*, 60(5):1601–1638, 2000.
- [24] F. A. Milner. A numerical method for a model of population dynamics with spatial diffusion. *Comp. Math. Applic.*, 19(4):31–43, 1990.
- [25] O. Rauprich, M. Matsuchita, C. J. Weijer, F. Siegert, S. E. Esipov, and J. A. Shapiro. Periodic phenomena in *proteus mirabilis* swarm colony development. *Jour. of Bacteriology.*, pages 6525–6538, Nov. 1996.