Modélisation long terme de l'océan en zone côtière

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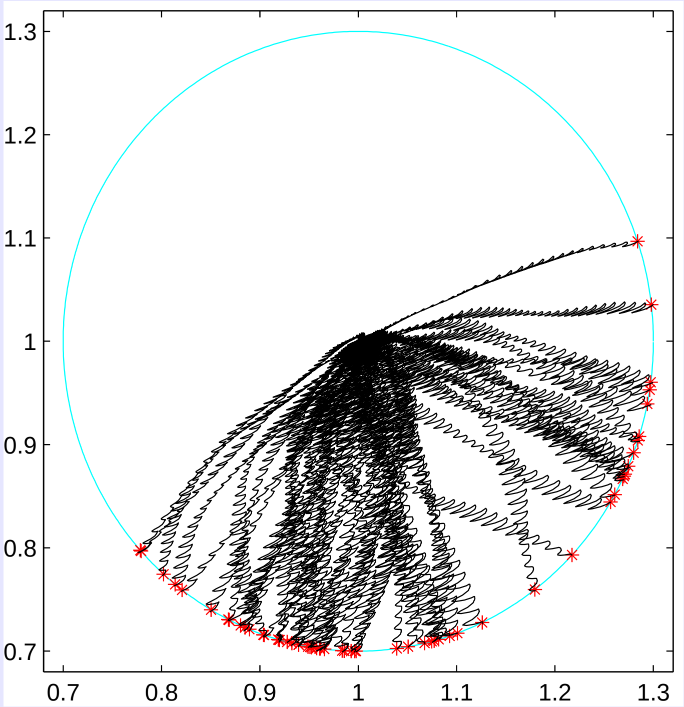
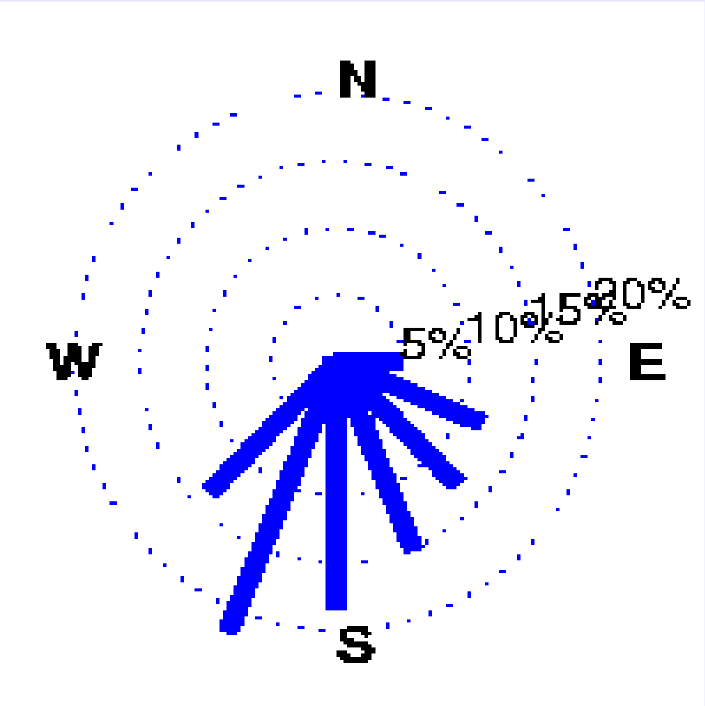
Motivation

Dérive à long terme d'objets dans l'océan côtier



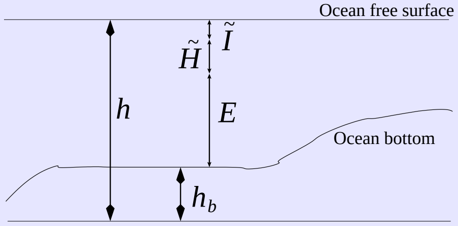


Extrait de **P. Ailliot, E. Frénod & V. Monbet** (2006). Long term drift forecast in the ocean with tide and wind. *Multiscale Modeling and Simulation, Vol 5, No 2, p 514-531*.

Construction du modèle de référence

Point de départ : Saint-Venant

$h \equiv h(t, \mathbf{x})$: altitude de la surface libre de l'océan,
$\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2) \equiv \mathbf{m}(t, \mathbf{x})$: vitesse de l'océan



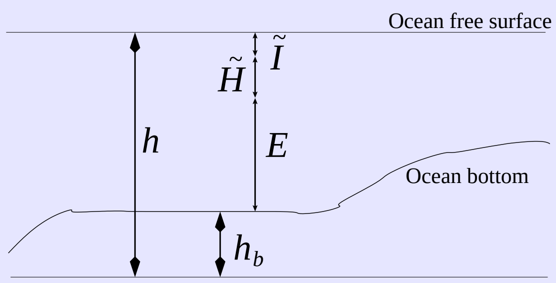
$$\frac{\partial h}{\partial t} + \nabla(h - h_b) \cdot \mathbf{m} + (h - h_b)\nabla \cdot \mathbf{m} = 0,$$

$$\frac{\partial \mathbf{m}}{\partial t} + (\nabla \mathbf{m})\mathbf{m} + f\mathbf{m}^{\perp} + g\nabla h - c\Delta \mathbf{m} - c\frac{(\nabla \mathbf{m})\nabla(h - h_b)}{h - h_b}$$

$$+ \frac{\frac{\kappa}{h-h_b}}{1 + \frac{\kappa}{c}(h - h_b)}\mathbf{m} = \frac{\frac{\mu}{h-h_b}}{1 + \frac{\mu}{c}(h - h_b)}(\tilde{\mathbf{W}} - \mathbf{m}).$$

$(\mathbf{m}^{\perp} = (-\mathbf{m}_2, \mathbf{m}_1))$.

Retirer l'onde de marée



. $h = h_b + E + \tilde{\mathcal{H}} + \tilde{\mathcal{I}}$,
. E : altitude moyenne de l'eau,
. $\tilde{\mathcal{H}}$: variation de hauteur due à la marée.
. $\mathbf{m} = \tilde{\mathbf{M}} + \tilde{\mathbf{N}}$,
. $\tilde{\mathbf{M}}$: vitesse de l'eau due à la marée.

$$\frac{\partial \tilde{\mathcal{H}}}{\partial t} + \nabla(E + \tilde{\mathcal{H}}) \cdot \tilde{\mathbf{M}} + (E + \tilde{\mathcal{H}}) \nabla \cdot \tilde{\mathbf{M}} = 0,$$

$$\frac{\partial \tilde{\mathbf{M}}}{\partial t} + \left(\nabla \tilde{\mathbf{M}}\right) \tilde{\mathbf{M}} + f \tilde{\mathbf{M}}^{\perp} + g \nabla (E + \tilde{\mathcal{H}} + h_b) = 0,$$

Equations pour $\tilde{\mathcal{I}}$ et $\tilde{\mathbf{N}}$ - 1

$h = h_b + E + \tilde{\mathcal{H}} + \tilde{\mathcal{I}}$ et $\mathbf{m} = \tilde{\mathbf{M}} + \tilde{\mathbf{N}}$ dans

$$\frac{\partial h}{\partial t} + \nabla(h - h_b)\cdot \mathbf{m} + (h - h_b)\nabla \cdot \mathbf{m} = 0,$$

$$\frac{\partial \mathbf{m}}{\partial t} + \big(\nabla \mathbf{m}\big)\mathbf{m} + f\mathbf{m}^{\perp} + g\nabla(h - h_b) - c\Delta \mathbf{m} - c\frac{\big(\nabla \mathbf{m}\big)\nabla(h - h_b)}{h - h_b}$$

$$+\frac{\frac{\kappa}{h-h_b}}{1 + \frac{\kappa}{c}(h - h_b)}\mathbf{m} = \frac{\frac{\mu}{h-h_b}}{1 + \frac{\mu}{c}(h - h_b)}(\tilde{\mathbf{W}} - \mathbf{m}).$$

On utilise :

$$\frac{\partial \tilde{\mathcal{H}}}{\partial t} + \nabla(E + \tilde{\mathcal{H}})\cdot \tilde{\mathbf{M}} + (E + \tilde{\mathcal{H}})\nabla \cdot \tilde{\mathbf{M}} = 0,$$

$$\frac{\partial \tilde{\mathbf{M}}}{\partial t} + \big(\nabla \tilde{\mathbf{M}}\big)\tilde{\mathbf{M}} + f\tilde{\mathbf{M}}^{\perp} + g\nabla(E + \tilde{\mathcal{H}}) = 0,$$

Equations pour $\tilde{\mathcal{I}}$ et $\tilde{\mathbf{N}}$ - 2

$$\frac{\partial \tilde{\mathcal{I}}}{\partial t} + \nabla(E+\tilde{\mathcal{H}})\cdot\tilde{\mathbf{N}} + (E+\tilde{\mathcal{H}})(\nabla\cdot\tilde{\mathbf{N}}) + (\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{M}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{M}}) + (\nabla\tilde{\mathcal{I}})\cdot\tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla\cdot\tilde{\mathbf{N}}) = 0,$$

$$\frac{\partial \tilde{\mathbf{N}}}{\partial t} + (\nabla\tilde{\mathbf{N}})\tilde{\mathbf{M}} + (\nabla\tilde{\mathbf{M}})\tilde{\mathbf{N}} + (\nabla\tilde{\mathbf{N}})\tilde{\mathbf{N}} + f\tilde{\mathbf{N}}^{\perp} + g\nabla\tilde{\mathcal{I}} - c\Delta\tilde{\mathbf{M}} - c\Delta\tilde{\mathbf{N}}$$
$$- c\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+\tilde{\mathcal{H}})}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} - c\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} - c\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+\tilde{\mathcal{H}})}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}$$
$$- c\frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}} + \frac{\frac{\kappa}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\kappa}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{M}} + \frac{\frac{\kappa}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\kappa}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{N}}$$
$$= \frac{\frac{\mu}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\mu}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}(\tilde{\mathbf{W}}-\tilde{\mathbf{M}}) - \frac{\frac{\mu}{E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}}}}{1+\frac{\mu}{c}(E+\tilde{\mathcal{H}}+\tilde{\mathcal{I}})}\tilde{\mathbf{N}}.$$

Ces équations sont considérées comme les équations de référence.

adimensionnement

variables et champs adimensionnés

temps de ref : $\overline{t}$ long. de ref : $\overline{L}$

Variables adimensionnées : $t = \overline{t}t', \mathbf{x} = \overline{L}\mathbf{x}'.$

Vitesse caract. de la marée : $\overline{M}$ Taille caract. de la perturb. : $\overline{N}$

Hauteur d'eau moyenne : $\overline{E}$

Marnage caract. : $\overline{H}$ Taille caract. de la perturb. : $\overline{I}$

Champs adimensionnées :

$$\tilde{\mathbf{M}}'(t', \mathbf{x}') = \frac{1}{\overline{M}} \tilde{\mathbf{M}}(\overline{t}t', \overline{L}\mathbf{x}'), \qquad \tilde{\mathbf{N}}'(t', \mathbf{x}') = \frac{1}{\overline{N}} \tilde{\mathbf{N}}(\overline{t}t', \overline{L}\mathbf{x}'),$$

$$\tilde{\mathcal{H}}'(t', \mathbf{x}') = \frac{1}{\overline{H}} \tilde{\mathcal{H}}(\overline{t}t', \overline{L}\mathbf{x}'), \qquad \tilde{\mathcal{I}}'(t', \mathbf{x}') = \frac{1}{\overline{I}} \tilde{\mathcal{I}}(\overline{t}t', \overline{L}\mathbf{x}'),$$

$$E'(\mathbf{x}') = \frac{1}{\overline{E}} E(\overline{L}\mathbf{x}'), \qquad \tilde{\mathbf{W}}'(t', \mathbf{x}') = \frac{1}{\overline{W}} \mathbf{W}(\overline{t}t', \overline{L}\mathbf{x}').$$

Equations adimensionnées

$$\frac{\partial \tilde{\mathcal{I}}'}{\partial t'} + \frac{\overline{H}}{\overline{I}}\,\frac{\overline{N}\overline{t}}{\overline{L}}\left[\left(\frac{\overline{E}}{\overline{\overline{H}}}\nabla E' + \nabla \tilde{\mathcal{H}}'\right)\cdot \tilde{\mathbf{N}}' + \left(\frac{\overline{E}}{\overline{\overline{H}}}E' + \tilde{\mathcal{H}}'\right)\nabla \cdot \tilde{\mathbf{N}}' \right]$$

$$+ \frac{\overline{M}\overline{t}}{\overline{L}}\left[\nabla \tilde{\mathcal{I}}' \cdot \tilde{\mathbf{M}}' + \tilde{\mathcal{I}}'\,\nabla \cdot \tilde{\mathbf{M}}'\right] + \frac{\overline{N}\overline{t}}{\overline{L}}\left[\nabla \tilde{\mathcal{I}} \cdot \tilde{\mathbf{N}}' + \tilde{\mathcal{I}}'\,\nabla \cdot \tilde{\mathbf{N}}'\right] = 0,$$

$$
\begin{aligned}
&\frac{\partial \tilde{\mathbf{N}}'}{\partial t'} + \frac{\overline{M}\,\overline{t}}{\overline{L}}\left[\nabla\tilde{\mathbf{N}}'\tilde{\mathbf{M}}' + \nabla\tilde{\mathbf{M}}'\tilde{\mathbf{N}}'\right] + \frac{\overline{N}\,\overline{t}}{\overline{L}}\nabla\tilde{\mathbf{N}}'\tilde{\mathbf{N}}' + f\overline{t}\tilde{\mathbf{N}}'^{\perp} \\
&\quad + \frac{g\overline{t}}{\overline{N}}\frac{\overline{I}}{\overline{L}}\nabla\tilde{\mathcal{I}}' - \frac{c\overline{t}}{\overline{L}^2}\frac{\overline{M}}{\overline{N}}\nabla\cdot\tilde{\mathbf{M}}' - \frac{c\overline{t}}{\overline{L}^2}\nabla\cdot\tilde{\mathbf{N}}' \\
&\quad - \frac{c\overline{t}}{\overline{L}^2}\frac{\overline{M}}{\overline{N}}\frac{\nabla\tilde{\mathbf{M}}'\left(\nabla E' + \frac{\overline{H}}{\overline{E}}\nabla\tilde{\mathcal{H}}'\right)}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'} - \frac{c\overline{t}}{\overline{L}^2}\frac{\overline{M}}{\overline{N}}\frac{\overline{I}}{\overline{E}}\frac{\nabla\tilde{\mathbf{M}}'\nabla\tilde{\mathcal{I}}'}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'} \\
&\quad - \frac{c\overline{t}}{\overline{L}^2}\frac{\nabla\tilde{\mathbf{N}}'\left(\nabla E' + \frac{\overline{H}}{\overline{E}}\nabla\tilde{\mathcal{H}}'\right)}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'} - \frac{c\overline{t}}{\overline{L}^2}\frac{\overline{I}}{\overline{E}}\frac{\nabla\tilde{\mathbf{N}}'\nabla\tilde{\mathcal{I}}'}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'} \\
&\quad + \frac{\kappa\overline{t}}{\overline{E}}\frac{\overline{M}}{\overline{N}}\frac{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'}}{1 + \frac{\kappa\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\tilde{\mathbf{M}}' \\
&\quad + \frac{\kappa\overline{t}}{\overline{E}}\frac{\frac{1}{E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'}}{1 + \frac{\kappa\overline{E}}{c}\left(E' + \frac{\overline{H}}{\overline{E}}\tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}}\tilde{\mathcal{I}}'\right)}\tilde{\mathbf{N}}' = \quad \ldots/\ldots
\end{aligned}
$$

$$= \frac{\mu \overline{t}}{\overline{E}} \frac{\frac{1}{E' + \frac{\overline{H}}{\overline{E}} \tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}} \tilde{\mathcal{I}}'}}{1 + \frac{\mu \overline{E}}{c} (E' + \frac{\overline{H}}{\overline{E}} \tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}} \tilde{\mathcal{I}}')} (\frac{\overline{W}}{\overline{N}} \tilde{\mathbf{W}}' - \frac{\overline{M}}{\overline{N}} \tilde{\mathbf{M}}')$$

$$- \frac{\mu \overline{t}}{\overline{E}} \frac{\frac{1}{E' + \frac{\overline{H}}{\overline{E}} \tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}} \tilde{\mathcal{I}}'}}{1 + \frac{\mu \overline{E}}{c} (E' + \frac{\overline{H}}{\overline{E}} \tilde{\mathcal{H}}' + \frac{\overline{I}}{\overline{E}} \tilde{\mathcal{I}}')} \tilde{\mathbf{N}}'.$$

Le petit paramètre ε

Temps d'observation : envir. trois mois : $\overline{t} \sim 100$ jours $\sim 2400h$

Fréquence de marée : $\overline{\omega}$. $\quad \dfrac{1}{\overline{\omega}} \sim 13h$

$$\text{Petit paramètre : } \varepsilon = \frac{1}{\overline{t}\,\overline{\omega}} \sim \frac{1}{200}$$

Hypothèse sur $\tilde{\mathbf{M}}'$ et $\tilde{\mathcal{H}}'$:

$$\tilde{\mathbf{M}}'(t',\mathbf{x}') = \mathbf{M}'(t',\overline{\omega}\,\overline{t}t',\mathbf{x}'), \qquad \tilde{\mathcal{H}}'(t',\mathbf{x}') = \mathcal{H}'(t',\overline{\omega}\,\overline{t}t',\mathbf{x}'),$$

avec $\theta \mapsto (\mathbf{M}'(t',\theta,\mathbf{x}'),\mathcal{H}'(t',\theta,\mathbf{x}'))$ $1-$périodique.

On suppose que $\tilde{\mathbf{N}}'$ et $\tilde{\mathcal{I}}'$ sont vraiment des perturbations :

$$\frac{\overline{N}}{\overline{M}} \sim \frac{\overline{I}}{\overline{H}} \sim \varepsilon$$

Domaine côtier au dessus d'un plateau continental

$\overline{L} \sim 500km, \quad \overline{E} \sim 300m,$

$\overline{H} \sim 3m,$

$\overline{M} \sim 0.5km/h \quad$ DONC $\quad \boxed{\dfrac{\overline{M}}{\overline{\omega}} \sim 5km}$

$$\boxed{\dfrac{\frac{\overline{M}}{\overline{\omega}}}{\overline{L}} \sim 2\varepsilon} \qquad \boxed{\dfrac{\overline{H}}{\overline{E}} \sim 2\varepsilon}$$

Les autres rapports de grandeurs caractéristiques

PAR EXEMPLE

- Latitude de 45^o, $f \sim \pi/jour \sim 4\ 10^{-5}/s$, DONC $\boxed{f\bar{t} \sim \pi/2\varepsilon}$.
- Si c est la viscosité de l'eau, $c \sim 10^{-2}cm^2/s \sim 10^{-7}km^2/jour$, donc $c\bar{t} \sim 10^{-5}km^2$, DONC $\boxed{\frac{c\bar{t}}{\overline{L}^2} \sim 13\varepsilon^5}$.
- μ coeff. de frott. eau-air, $\mu \sim 10^{-6}m/s \sim 10^{-4}km/jour$, DONC $\boxed{\frac{\mu\overline{E}}{c} \sim 3\ 10^2 \sim \frac{1.5}{\varepsilon}}$.

Pour le vent : condition de tempête : $\boxed{\frac{\overline{M}}{\overline{W}} \sim \frac{0.5}{100} \sim \varepsilon}$.

Le modèle

$$\tilde{\mathbf{M}}(t,\mathbf{x}) = \mathbf{M}(t,\tfrac{t}{\varepsilon},\mathbf{x}) \text{ et } \tilde{\mathcal{H}}(t,\mathbf{x}) = \mathcal{H}(t,\tfrac{t}{\varepsilon},\mathbf{x})$$

$$\frac{\partial \tilde{\mathcal{I}}}{\partial t} + \nabla(\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}}) \cdot \tilde{\mathbf{N}} + (\frac{1}{\varepsilon}E + 2\tilde{\mathcal{H}})\nabla \cdot \tilde{\mathbf{N}} + 2(\nabla\tilde{\mathcal{I}}) \cdot \tilde{\mathbf{M}} + 2\tilde{\mathcal{I}}(\nabla \cdot \tilde{\mathbf{M}})$$
$$+ 2\varepsilon\Big((\nabla\tilde{\mathcal{I}}) \cdot \tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla \cdot \tilde{\mathbf{N}})\Big) = 0,$$

$$\frac{\partial \tilde{\mathbf{N}}}{\partial t} + 2(\nabla \tilde{\mathbf{N}})\tilde{\mathbf{M}} + 2(\nabla \tilde{\mathbf{M}})\tilde{\mathbf{N}} + 2\varepsilon(\nabla \tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{\pi}{2\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{1}{4\varepsilon}\nabla\tilde{\mathcal{I}}$$
$$- 13\varepsilon^4\Delta\tilde{\mathbf{M}} - 13\varepsilon^5\Delta\tilde{\mathbf{N}} - 13\varepsilon^4\frac{(\nabla\tilde{\mathbf{M}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}$$
$$- 26\varepsilon^6\frac{(\nabla\tilde{\mathbf{M}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} - 13\varepsilon^5\frac{(\nabla\tilde{\mathbf{N}})\nabla(E+2\varepsilon\tilde{\mathcal{H}})}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}$$
$$- 26\varepsilon^7\frac{(\nabla\tilde{\mathbf{N}})\nabla\tilde{\mathcal{I}}}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}} + \frac{3}{\varepsilon}\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{M}}$$
$$+ 3\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{0.8}{\varepsilon^2}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}}$$
$$= 6\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}(\frac{1}{\varepsilon}\tilde{\mathbf{W}} - \tilde{\mathbf{M}})$$
$$- 6\varepsilon\frac{\frac{1}{E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}}}}{1+\frac{1.5}{\varepsilon}(E+2\varepsilon\tilde{\mathcal{H}}+2\varepsilon^2\tilde{\mathcal{I}})}\tilde{\mathbf{N}}.$$

Un résultat d’existence

Système simplifié

$$\frac{\partial \tilde{\mathcal{I}}}{\partial t} + (\nabla \tilde{\mathcal{H}}) \cdot \tilde{\mathbf{N}} + (\frac{1}{\varepsilon} + \tilde{\mathcal{H}})(\nabla \cdot \tilde{\mathbf{N}})$$
$$+ (\nabla \tilde{\mathcal{I}}) \cdot \tilde{\mathbf{M}} + \tilde{\mathcal{I}}(\nabla \cdot \tilde{\mathbf{M}}) + \varepsilon\big((\nabla \tilde{\mathcal{I}}) \cdot \tilde{\mathbf{N}} + \tilde{\mathcal{I}}(\nabla \cdot \tilde{\mathbf{N}})\big) = 0,$$

$$\frac{\partial \tilde{\mathbf{N}}}{\partial t} + (\nabla \tilde{\mathbf{N}})\tilde{\mathbf{M}} + (\nabla \tilde{\mathbf{M}})\tilde{\mathbf{N}} + \varepsilon(\nabla \tilde{\mathbf{N}})\tilde{\mathbf{N}} + \frac{1}{\varepsilon}\tilde{\mathbf{N}}^{\perp} + \frac{1}{\varepsilon}\nabla \tilde{\mathcal{I}} = \tilde{\mathbf{W}}.$$

$$\tilde{\mathbf{M}}(t, \mathbf{x}) = \mathbf{M}(t, \tfrac{t}{\varepsilon}, \mathbf{x}), \tilde{\mathcal{H}}(t, \mathbf{x}) = \mathcal{H}(t, \tfrac{t}{\varepsilon}, \mathbf{x}), \qquad \tilde{\mathbf{W}}(t, \mathbf{x}) = \mathbf{W}(t, \tfrac{t}{\varepsilon}, \mathbf{x})$$

Système hyperbolique symétrisable - 1

$$\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}_1, \tilde{\mathbf{N}}_2),\ \mathbf{u}^\perp = (0, \tilde{\mathbf{N}}^\perp)$$

$$B^1(t, \frac{t}{\varepsilon}, \mathbf{x}, \varepsilon\mathbf{u}) = \begin{pmatrix} \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 & \frac{1}{\varepsilon} + \tilde{\mathcal{H}} + \varepsilon\tilde{\mathcal{I}} & 0 \\ \frac{1}{\varepsilon} & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 & 0 \\ 0 & 0 & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 \end{pmatrix}$$

$$F(t, \frac{t}{\varepsilon}, \mathbf{x}, \mathbf{u}) = \begin{pmatrix} -(\frac{\partial\tilde{\mathcal{H}}}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathcal{H}}}{\partial x_2}\tilde{\mathbf{N}}_2) - (\frac{\partial\tilde{\mathbf{M}}_1}{\partial x_1} + \frac{\partial\tilde{\mathbf{M}}_2}{\partial x_2})\tilde{\mathcal{I}} \\ \tilde{\mathbf{W}}_1 - (\frac{\partial\tilde{\mathbf{M}}_1}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathbf{M}}_1}{\partial x_2}\tilde{\mathbf{N}}_2) \\ \tilde{\mathbf{W}}_2 - (\frac{\partial\tilde{\mathbf{M}}_2}{\partial x_1}\tilde{\mathbf{N}}_1 + \frac{\partial\tilde{\mathbf{M}}_2}{\partial x_2}\tilde{\mathbf{N}}_2) \end{pmatrix},$$

$$\boxed{\frac{\partial\mathbf{u}}{\partial t} + B^1\frac{\partial\mathbf{u}}{\partial x_1} + B^2\frac{\partial\mathbf{u}}{\partial x_2} + \frac{1}{\varepsilon}\mathbf{u}^\perp = F.}$$

Système hyperbolique symétrisable - 2

$$A^0(t, \tfrac{t}{\varepsilon}, \mathbf{x}, \varepsilon^2\tilde{\mathcal{I}}) = \begin{pmatrix} \frac{1}{1+\varepsilon\tilde{\mathcal{H}}+\varepsilon^2\tilde{\mathcal{I}}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^1(t, \tfrac{t}{\varepsilon}, \mathbf{x}, \varepsilon\mathbf{u}) = A^0 B^1 = \begin{pmatrix} \frac{\tilde{\mathbf{M}}_1+\varepsilon\tilde{\mathbf{N}}_1}{1+\varepsilon\tilde{\mathcal{H}}+\varepsilon^2\tilde{\mathcal{I}}} & 0 & 0 \\ 0 & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 & 0 \\ 0 & 0 & \tilde{\mathbf{M}}_1 + \varepsilon\tilde{\mathbf{N}}_1 \end{pmatrix},$$

$$S^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A^0\frac{\partial \mathbf{u}}{\partial t} + A^1\frac{\partial \mathbf{u}}{\partial x_1} + A^2\frac{\partial \mathbf{u}}{\partial x_2} + \frac{1}{\varepsilon}S^1\frac{\partial \mathbf{u}}{\partial x_1} + \frac{1}{\varepsilon}S^2\frac{\partial \mathbf{u}}{\partial x_2} + \frac{1}{\varepsilon}\mathbf{u}^\perp = F_0 = A^0 F.$$

Estimations a priori

$$\sum_{|\alpha|\le s}\left|\frac{d\Big(\displaystyle\int A^0 D^\alpha \mathbf{u}\cdot D^\alpha \mathbf{u}\,d\mathbf{x}\Big)}{dt}\right| \le g_1(\|\mathbf{u}\|_s, \varepsilon\|\frac{\partial \mathbf{u}}{\partial t}\|_{s-1})$$

$$\sum_{|\alpha|\le s-1}\left|\frac{d\Big(\displaystyle\int A^0 D^\alpha(\varepsilon\frac{\partial \mathbf{u}}{\partial t})\cdot D^\alpha(\varepsilon\frac{\partial \mathbf{u}}{\partial t})\,d\mathbf{x}\Big)}{dt}\right| \le g_2(\|\mathbf{u}\|_s, \varepsilon\|\frac{\partial \mathbf{u}}{\partial t}\|_{s-1})$$

$$\frac{d\Big(\|\mathbf{u}\|_s + \varepsilon\Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{s-1}\Big)}{dt} \le g(\|\mathbf{u}\|_s + \varepsilon\Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{s-1})$$

Théorème d'existence

Si on munit le modèle de conditions initiales

$$(\tilde{\mathcal{I}}_{|t=0}, \tilde{\mathbf{N}}_{|t=0}) = (\tilde{\mathbf{N}}_0, \tilde{\mathcal{I}}_0) \in (H^s(\mathbb{R}^2))^3 \text{ pour } s > 3,$$

Il y a existence et unicité de la solution régulière

$$(\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) \in (C([0,T], (H^s(\mathbb{R}^2))^3) \cap (C^1([0,T], (H^{s-1}(\mathbb{R}^2))^3))$$

pour T indépendant de ε. De plus

$$\sup_{t\in[0,T]} \|(\tilde{\mathcal{I}}, \tilde{\mathbf{N}})\|_s \leq c.$$

Comportement asymptotique lorsque $\varepsilon \to 0$

Convergence à 2 échelles et convergence faible

Le théorème précédent implique : lorsque $\varepsilon \to 0$,

$$\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) \xrightarrow{\text{2 éch}} \mathbf{U} = (\mathcal{I}, \mathbf{N}) \in L^\infty([0,T], L^\infty(0,1; H^s(\mathbb{R}^2)^3))$$

i.e : $\Psi(t,\theta,\mathbf{x})$ régulière et $1-$périodique en θ :

$$\int_0^T \int_{\mathbb{R}^2} \mathbf{u}(t,\mathbf{x})\Psi(t,\frac{t}{\varepsilon},\mathbf{x})dtd\mathbf{x} \to \int_0^T \int_{\mathbb{R}^2} \int_0^1 \mathbf{U}(t,\theta,\mathbf{x})\Psi(t,\theta,\mathbf{x})dtd\mathbf{x}d\theta$$

$$\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) \overset{*}{\rightharpoonup} \int_0^1 \mathbf{U}d\theta = (\int_0^1 \mathcal{I}d\theta, \int_0^1 \mathbf{N}d\theta) \text{ dans } L^\infty([0,T], H^s(\mathbb{R}^2)^3)$$

Formulation faible

Formulation faible de

$$A^0\frac{\partial \mathbf{u}}{\partial t}+A^1\frac{\partial \mathbf{u}}{\partial x_1}+A^2\frac{\partial \mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}S^1\frac{\partial \mathbf{u}}{\partial x_1}+\frac{1}{\varepsilon}S^2\frac{\partial \mathbf{u}}{\partial x_2}+\frac{1}{\varepsilon}\mathbf{u}^\perp=F_0=A^0F.$$

On multiplie par $\Psi(t,\frac{t}{\varepsilon},\mathbf{x})$ régulière et à supp. compact dans $[0,T)\times\mathbb{R}^2$

$$-\int_0^T\int_{\mathbb{R}^2}\mathbf{u}\cdot\Big(\frac{\partial A^0\Psi}{\partial t}+\frac{1}{\varepsilon}A^0\frac{\partial \Psi}{\partial \theta}+\frac{1}{\varepsilon}\frac{\partial A^0}{\partial \theta}\Psi+\frac{\partial A^1\Psi}{\partial x_1}+\frac{\partial A^2\Psi}{\partial x_2}$$
$$+\frac{1}{\varepsilon}S^1\frac{\partial \Psi}{\partial x_1}+\frac{1}{\varepsilon}S^2\frac{\partial \Psi}{\partial x_2}+\frac{1}{\varepsilon}\Psi^\perp\Big)dtd\mathbf{x}$$
$$=\int_0^T\int_{\mathbb{R}^2}A^0F\cdot\Psi dtd\mathbf{x}+\int_{\mathbb{R}^2}\mathbf{u}_0\cdot A^0\Psi(0,0,\cdot)d\mathbf{x}.$$

Equation de contrainte

On fait $\varepsilon \to 0$, sachant $A^0 \xrightarrow{\text{2 éch}} I$ et $\partial A^0/\partial\theta \xrightarrow{\text{2 éch}} 0$ (fonctions régulières)

$$-\int_0^T \int_{\mathbb{R}^2} \int_0^1 \mathbf{U} \cdot \left(\frac{\partial \Psi}{\partial \theta} + S^1 \frac{\partial \Psi}{\partial x_1} + S^2 \frac{\partial \Psi}{\partial x_2} + \Psi^\perp d\theta \right) dt d\mathbf{x} = 0,$$

i.e. $$\frac{\partial \mathbf{U}}{\partial \theta} + S^1 \frac{\partial \mathbf{U}}{\partial x_1} + S^2 \frac{\partial \mathbf{U}}{\partial x_2} + \mathbf{U}^\perp = 0,$$

$$\frac{\partial \mathcal{I}}{\partial \theta} + \nabla \cdot \mathbf{N} = 0, \qquad \frac{\partial \mathbf{N}}{\partial \theta} + \mathbf{N}^\perp + \nabla \mathcal{I} = 0.$$

Conséquences de l'équation de contrainte

Fourier sur équ. de contrainte : ($k \in 2\pi\mathbb{Z}$ duale de θ , $l \in \mathbb{R}^2$ duale de $\mathbf{x}$)

$$k\hat{\mathcal{I}} + l_1\hat{\mathbf{N}}_1 + l_2\hat{\mathbf{N}}_2 = 0,$$

$$k\hat{\mathbf{N}}_1 - \hat{\mathbf{N}}_2 + l_1\hat{\mathcal{I}} = 0, \qquad Det = -k(l_1^2 + l_2^2 - k^2 - 1)$$

$$k\hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_1 + l_2\hat{\mathcal{I}} = 0.$$

$(\mathcal{I}, \mathbf{N}) =$ (Une fonction indépendante de θ) +
(Une fonction de $L^\infty(0,1;(H^s(\mathbb{R}^2))^3)$ dont le spectre est portée par
$\{(k, l_1, l_2) \in \mathbb{R}^3,\ k \in 2\pi\mathbb{Z},\ l_1^2 + l_2^2 = k^2 + 1\}$)

$\Rightarrow$ $(\mathcal{I}, \mathbf{N})$ indépendante de θ, $\mathbf{u} = (\tilde{\mathcal{I}}, \tilde{\mathbf{N}}) \overset{*}{\rightharpoonup} \mathbf{U} = (\mathcal{I}, \mathbf{N})$

$$\nabla \cdot \mathbf{N} = 0, \quad \mathbf{N}^\perp + \nabla\mathcal{I} = 0.$$

DONC : $$\boxed{\mathbf{N}_1(t,\mathbf{x}) = -\frac{\partial\mathcal{I}}{\partial x_2}(t,\mathbf{x}), \quad \mathbf{N}_2(t,\mathbf{x}) = \frac{\partial\mathcal{I}}{\partial x_1}(t,\mathbf{x})}$$

Equation pour $\mathcal{I}$ - 1

Fonctions test Ψ telle que

$$\Psi_1(t,\mathbf{x}) = \varphi(t,\mathbf{x}) \quad \Psi_2(t,\mathbf{x}) = -\frac{\partial\varphi}{\partial x_2}(t,\mathbf{x}) \quad \Psi_3(t,\mathbf{x}) = \frac{\partial\varphi}{\partial x_1}(t,\mathbf{x}).$$

(vérifiant la contrainte) dans la formulation faible.

On fait $\varepsilon \to 0$,

sachant $A^0 \to I$; $1/\varepsilon\, \partial A^0/\partial\theta \xrightarrow{\text{2 éch}} 0$ et $A^1 \xrightarrow{\text{2 éch}} \int_0^1 \mathbf{M}_1 d\theta\, I$ (fonctions régulières)

$$-\int_0^T\int_{\mathbb{R}^2} \mathbf{U}\cdot\left(\frac{\partial\Psi}{\partial t} + \frac{\partial(\int_0^1 \mathbf{M}_1 d\theta)\Psi}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2 d\theta)\Psi}{\partial x_2}\right) dt d\mathbf{x}$$

$$= \int_0^T\int_{\mathbb{R}^2}\int_0^1 F d\theta\cdot\Psi dt d\mathbf{x} + \int_{\mathbb{R}^2} \mathbf{u}_0\cdot\Psi(0,\cdot)d\mathbf{x}.$$

Equation pour $\mathcal{I}$ - 2

On utilise la forme de $\mathbf{U}$ et Ψ.

$$
\begin{aligned}
-\int_0^T \int_{\mathbb{R}^2} \mathcal{I}\Big(\frac{\partial\varphi}{\partial t} + \frac{\partial(\int_0^1 \mathbf{M}_1 d\theta)\varphi}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2 d\theta)\varphi}{\partial x_2}\Big) \\
+ \frac{\partial\mathcal{I}}{\partial x_2}\Big(\frac{\partial\frac{\partial\varphi}{\partial x_2}}{\partial t} + \frac{\partial(\int_0^1 \mathbf{M}_1 d\theta)\frac{\partial\varphi}{\partial x_2}}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2 d\theta)\frac{\partial\varphi}{\partial x_2}}{\partial x_2}\Big) \\
+ \frac{\partial\mathcal{I}}{\partial x_1}\Big(\frac{\partial\frac{\partial\varphi}{\partial x_1}}{\partial t} + \frac{\partial(\int_0^1 \mathbf{M}_1 d\theta)\frac{\partial\varphi}{\partial x_1}}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2 d\theta)\frac{\partial\varphi}{\partial x_1}}{\partial x_2}\Big) dt d\mathbf{x} \\
= \int_0^T \int_{\mathbb{R}^2} -\Big(\frac{\partial(\int_0^1 \mathcal{H} d\theta)}{\partial x_1}(-\frac{\partial\mathcal{I}}{\partial x_2}) + \frac{\partial(\int_0^1 \mathcal{H} d\theta)}{\partial x_2}(\frac{\partial\mathcal{I}}{\partial x_1}) \\
+ (\frac{\partial(\int_0^1 \mathbf{M}_1 d\theta)}{\partial x_1} + \frac{\partial(\int_0^1 \mathbf{M}_2 d\theta)}{\partial x_2}) g\Big)\varphi + \quad \ldots/\ldots
\end{aligned}
$$

$$
\begin{aligned}
.... - \Big(\int_0^1 \mathbf{W}_1 d\theta - (\frac{\partial \int_0^1 \mathbf{M}_1 d\theta}{\partial x_1}(-\frac{\partial \mathcal{I}}{\partial x_2}) + \frac{\partial \int_0^1 \mathbf{M}_1 d\theta}{\partial x_2}(\frac{\partial \mathcal{I}}{\partial x_1})\Big) \frac{\partial \varphi}{\partial x_2} \\
+ \Big(\int_0^1 \mathbf{W}_2 d\theta - (\frac{\partial \int_0^1 \mathbf{M}_2 d\theta}{\partial x_1}(-\frac{\partial \mathcal{I}}{\partial x_2}) + \frac{\partial \int_0^1 \mathbf{M}_2 d\theta}{\partial x_2}(\frac{\partial \mathcal{I}}{\partial x_1})\Big) \frac{\partial \varphi}{\partial x_1} dt d\mathbf{x} \\
+ \int_{\mathbb{R}^2} \tilde{\mathcal{I}}_0 \varphi - (\tilde{\mathbf{N}}_0)_1 \frac{\partial \varphi}{\partial x_2} + (\tilde{\mathbf{N}}_0)_2 \frac{\partial \varphi}{\partial x_1} d\mathbf{x}.
\end{aligned}
$$

Equation pour $\mathcal{I}$ - 3

$$(\overline{\overline{\mathbf{M}}} = \int_0^1 \mathbf{M}\, d\theta),$$

$$\frac{\partial(\mathcal{I} - \Delta\mathcal{I})}{\partial t} + \overline{\overline{\mathbf{M}}} \cdot \nabla\mathcal{I} - \frac{\partial\Big(\overline{\overline{\mathbf{M}}}_1 \frac{\partial^2\mathcal{I}}{\partial x_1^2}\Big)}{\partial x_1} - \frac{\partial\Big(\overline{\overline{\mathbf{M}}}_2 \frac{\partial^2\mathcal{I}}{\partial x_1 \partial x_2}\Big)}{\partial x_1}$$

$$- \frac{\partial\Big(\overline{\overline{\mathbf{M}}}_1 \frac{\partial^2\mathcal{I}}{\partial x_1 \partial x_2}\Big)}{\partial x_2} - \frac{\partial\Big(\overline{\overline{\mathbf{M}}}_2 \frac{\partial^2\mathcal{I}}{\partial x_2^2}\Big)}{\partial x_2} - (\nabla\overline{\overline{\mathcal{H}}})^\perp \cdot \nabla\mathcal{I} + (\nabla \cdot \overline{\overline{\mathbf{M}}})\mathcal{I}$$

$$+ \frac{\partial\Big(\frac{\partial\overline{\overline{\mathbf{M}}}_2}{\partial x_1} \frac{\partial\mathcal{I}}{\partial x_2}\Big)}{\partial x_1} - \frac{\partial\Big(\frac{\partial\overline{\overline{\mathbf{M}}}_2}{\partial x_2} \frac{\partial\mathcal{I}}{\partial x_1}\Big)}{\partial x_1} - \frac{\partial\Big(\frac{\partial\overline{\overline{\mathbf{M}}}_1}{\partial x_1} \frac{\partial\mathcal{I}}{\partial x_2}\Big)}{\partial x_2} + \frac{\partial\Big(\frac{\partial\overline{\overline{\mathbf{M}}}_1}{\partial x_2} \frac{\partial\mathcal{I}}{\partial x_1}\Big)}{\partial x_2}$$

$$= \frac{\partial\overline{\overline{\mathbf{W}}}_1}{\partial x_2} - \frac{\partial\overline{\overline{\mathbf{W}}}_2}{\partial x_1},$$

$$\big(\mathcal{I} - \Delta\mathcal{I}\big)_{|t=0} = \tilde{\mathcal{I}}_0 + \frac{\partial(\tilde{\mathbf{N}}_0)_1}{\partial x_1} - \frac{\partial(\tilde{\mathbf{N}}_0)_2}{\partial x_2}.$$