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THE FINITE LARMOR RADIUS APPROXIMATION*

EMMANUEL FRÉNOD† AND ERIC SONNENDRÜCKER‡

Abstract. The presence of a large external magnetic field in a plasma introduces an additional time-scale which is very constraining for the numerical simulation. Hence it is very useful to introduce averaged models which remove this time-scale. However, depending on other parameters of the plasma, different starting models for the asymptotic analysis may be considered. We introduce here a generic framework for our analysis which fits many of the possible regimes and apply it in particular to justify the finite Larmor radius approximation both in the linear case and in the nonlinear case in the plane transverse to the magnetic field.

Key words. Vlasov–Poisson equations, kinetic equations, homogenization, two-scale convergence, multiple time scales

AMS subject classifications. 82D10, 35B27, 35Q99, 76X05

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1. Introduction.

The main goal of this paper is the investigation of an asymptotic regime taking place in the description of the behavior of charged particles under the action of a strong external magnetic field and called the finite Larmor radius approximation. This approximation has a natural field of application in tokamak physics.

This work was announced in Frénod and Sonnendrücker [9] and follows Frénod and Sonnendrücker [8, 10], where we exhibited global asymptotic behavior of plasmas. Those global behaviors have also been mathematically put in light by Golse and Saint-Raymond [12, 11] and Grenier [14]. The context of the finite Larmor radius approximation is more local. Its object is to describe the behavior of the considered plama's particles when the observation length scale is comparable with their Larmor radius.

We choose to lead our study in the framework of the Vlasov equation which writes, in this context

$$
\begin{aligned}
&\frac{\partial f^\epsilon}{\partial t} + \mathbf{v}_\parallel \cdot \nabla_x f^\epsilon + \frac{1}{\epsilon}\mathbf{v}_\perp \cdot \nabla_x f^\epsilon + \left(\mathbf{E} + \frac{1}{\epsilon}\mathbf{v} \times \mathbf{m}\right) \cdot \nabla_v f^\epsilon = 0, \\
&f^\epsilon_{|t=0} = f_0,
\end{aligned}
\tag{1.1}
$$

where ϵ is a small parameter which will tend to 0. In (1.1) the distribution function $f^\epsilon \equiv f^\epsilon(t, \mathbf{x}, \mathbf{v})$; $t \in [0, T)$ for some $T < \infty$ is the time, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_x$ is the position, and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_v$ is the velocity. We denote $\mathcal{O} = \mathbb{R}^3_x \times \mathbb{R}^3_v$, $\Omega = [0, T) \times \mathbb{R}^3_x$, and $\mathcal{Q} = [0, T) \times \mathcal{O}$. The magnetic field $\mathbf{m}$ is supposed to be $\mathbf{e}_1$, the first vector of the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$. For any vector $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v}_\parallel$ stands for $\mathbf{v}_\parallel = (\mathbf{v} \cdot \mathbf{m})\mathbf{m} = v_1\mathbf{e}_1$ and $\mathbf{v}_\perp$ for $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = v_2\mathbf{e}_2 + v_3\mathbf{e}_3$. The electric field $\mathbf{E} \equiv \mathbf{E}(t, \mathbf{x})$ is external and nonoscillating.

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†Laboratoire de Mathématiques et Applications des Mathématiques, Université de Bretagne Sud, BP573, Campus de Tohannic, F-56017 Vannes, France (emmanuel.frenod@univ-ubs.fr).

‡CNRS, Institut Elie Cartan, Université Henri Poincaré, Nancy and Institut de Recherche en Mathématiques Avancées, Université Louis Pasteur, 7 rue Rene Descartes, F-67084 Strasbourg Cedex, France (sonnen@math.u-strasbg.fr).

In order to make the process $\epsilon \to 0$ in (1.1), we assume

$$f_0 \geq 0, \quad f_0 \in L^1 \cap L^2(\mathcal{O}), \tag{1.2}$$

and for $\mathbf{E}$, we assume

$$\mathbf{E} \in \mathcal{C}^1(\Omega). \tag{1.3}$$

Then we have the following theorem.

THEOREM 1.1. *Under assumptions* (1.2) *and* (1.3), *for each* $\epsilon > 0$, *there exists a unique solution* f^ϵ *of* (1.1) *in* $L^\infty(0, T, L^1 \cap L^2(\mathcal{O}))$. *As* $\epsilon \to 0$,

$$f^\epsilon \rightharpoonup f \text{ in } L^\infty(0, T, L^2(\mathcal{O})) \text{ weak} - *, \tag{1.4}$$

where f *is the unique solution of*

(1.5)
$$\frac{\partial f}{\partial t} + \mathbf{v}_\parallel \cdot \nabla_x f + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(-\tau)\mathbf{E}(t, \mathbf{x} + \mathcal{R}(\tau)\mathbf{v})\, d\tau\right) \cdot \nabla_x f$$
$$+ \frac{1}{2\pi}\left(\int_0^{2\pi} R(-\tau)\mathbf{E}(t, \mathbf{x} + \mathcal{R}(\tau)\mathbf{v})\, d\tau\right) \cdot \nabla_v f = 0,$$

$$f_{|t=0} = \frac{1}{2\pi}\int_0^{2\pi} f_0(\mathbf{x} + \mathcal{R}(-\tau)\mathbf{v}, R(-\tau)\mathbf{v})\, d\tau,$$

where the matrices R *and* $\mathcal{R}$ *are given by*

$$R(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau \\ 0 & -\sin\tau & \cos\tau \end{pmatrix}, \quad \mathcal{R}(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin\tau & 1 - \cos\tau \\ 0 & \cos\tau - 1 & \sin\tau \end{pmatrix}. \tag{1.6}$$

The way to prove this theorem uses the 2-scale convergence defined as follows.

THEOREM 1.2 (see Nguetseng [18] and Allaire [2]). *If a sequence* f^ϵ *is bounded in* $L^\infty(0, T; W)$, *for a Banach spaces* W *being the dual of a separable space and being compactly embedded in* $\mathcal{D}'(\mathcal{O})$, *then for every period* θ *there exists a* θ*-periodic profile* $F_\theta(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^\infty_\theta(\mathbb{R}_\tau; W))$ *such that for all* $\psi_\theta(t, \tau, \mathbf{x}, \mathbf{v})$ *regular, with compact support with respect to* $(t, \mathbf{x}, \mathbf{v})$ *and* θ*-periodic with respect to* τ, *we have, up to a subsequence,*

$$\int_{\mathcal{Q}} f^\epsilon \psi_\theta^\epsilon \, dt\, d\mathbf{x}\, d\mathbf{v} \to \int_{\mathcal{Q}} \int_0^\theta F_\theta \psi_\theta \, d\tau\, dt\, d\mathbf{x}\, d\mathbf{v}. \tag{1.7}$$

We then say that f^ϵ *two scale converges to* F_θ. *Above,* $L^\infty_\theta(\mathbb{R}_\tau)$ *stands for the space of functions being* $L^\infty(\mathbb{R})$ *and being* θ*-periodic and* $\psi_\theta^\epsilon \equiv \psi_\theta(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v})$.

The profile F_θ *is called the* θ*-periodic two scale limit of* f^ϵ *and the link between* F_θ *and the weak*$-*$ *limit* f *of* (f^ϵ) *is given by*

$$\int_0^\theta F_\theta(t, \tau, \mathbf{x}, \mathbf{v})\, d\tau = f(t, \mathbf{x}, \mathbf{v}). \tag{1.8}$$

Moreover, if a sequence (g^ϵ) *strongly converges to* g *in a second Banach space* W' *(with the same assumption for* W' *as for* W*), such that the product makes sense in a third Banach space* W'', *then,*

$$f^\epsilon g^\epsilon \text{ 2-scale converges to } F_\theta g \in L^\infty(0, T; L^\infty_\theta(\mathbb{R}_\tau; W'')). \tag{1.9}$$

Remark. Our definition of the two scale convergence by (1.7) does not comply with the averaging rule usually used. Otherwise the right-hand side of (1.7) would be divided by θ.

The proof of Theorem 1.1 consists in finding a constraint equation for the two scale limit F of f^ϵ, using a weak formulation with oscillating test function of (1.1). This constraint imposes a form to F. Then using oscillating test functions satisfying the constraint equation in the previously evoked weak formulation gives the equation satisfied by F. Integrating this last equation yields finally (1.5).

As the proof of Theorem 1.1 in this paper and of Theorems 1.1 and 3.2 of Frénod and Sonnendrücker [8] are very close, we develop here a generic framework inside which all those proofs may be included. This generic framework consists in considering a conservation law linearly perturbed:

$$\begin{aligned} &\frac{\partial u^\epsilon}{\partial t} + \mathbf{A}\cdot\nabla_x u^\epsilon + \frac{1}{\epsilon}\mathbf{L}\cdot\nabla_x u^\epsilon = 0, \\ &u^\epsilon_{t=0} = u_0. \end{aligned} \tag{1.10}$$

In this system, $u^\epsilon \equiv u^\epsilon(t,\mathbf{x})$, $t\in[0,T)$ for some $T<\infty$ and $\mathbf{x}\in\mathbb{R}^n=\mathcal{O}$. Let us mention that $\mathbf{x}$ here is an abstract variable which is not connected to the position which is also denoted by $\mathbf{x}$ in the Vlasov equation. We denote $\mathcal{Q}=[0,T)\times\mathcal{O}$, and we assume $\mathbf{A}\equiv\mathbf{A}(t,\mathbf{x})\in L^\infty(0,T;L^2_{loc}(\mathcal{O}))$, with $\nabla_x\cdot\mathbf{A}=0$ and $\mathbf{L}\equiv M\mathbf{x}+N$, where M is a real $n\times n$ matrix with constant entries, satisfying $\mathrm{tr}M=0$ and where $N\in\mathrm{Im}M$. We moreover assume that $e^{\tau M}$ is θ-periodic for a given $\theta\in\mathbb{R}$. The generic theorem writes as follows.

THEOREM 1.3. *Under the assumptions above, if, moreover, the sequence* (u^ϵ) *of solution of* (1.10) *satisfies*

$$\|u^\epsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))}\le C, \tag{1.11}$$

for some constants C independent on ϵ, then, extracting a subsequence,

$$u^\epsilon \text{ 2-scale converges to a } \theta\text{-periodic profile } U\in L^\infty(0,T;L^\infty_\theta(\mathbb{R}_\tau;L^2(\mathcal{O}))) \tag{1.12}$$

and

$$u^\epsilon \rightharpoonup u \text{ in } L^\infty(0,T;L^2(\mathcal{O})) \text{ weak}-*. \tag{1.13}$$

We have

$$U(t,\tau,\mathbf{x}) = U_0(t, e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}), \tag{1.14}$$

where $\overline{N}$ is such that $-M\overline{N}=N$ and where $U_0\equiv U_0(t,\mathbf{y})$ is solution of

$$\begin{aligned} &\frac{\partial U_0}{\partial t} + \frac{1}{\theta}\int_0^\theta e^{-\sigma M}\mathbf{A}(t, e^{\sigma M}(\mathbf{y}-\overline{N})+\overline{N})\,d\sigma\cdot\nabla_y U_0 = 0, \\ &U_{0|t=0} = \tfrac{1}{\theta}u_0. \end{aligned} \tag{1.15}$$

Moreover, u is solution of

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{\theta}\int_0^\theta e^{-\sigma M}\mathbf{A}(t, e^{\sigma M}(\mathbf{x}-\overline{N})+\overline{N})d\sigma\cdot\nabla_x u = 0, \\ &u_{|t=0}(\mathbf{x}) = \frac{1}{\theta}\int_0^\theta u_0(e^{-\sigma M}(\mathbf{x}-\overline{N})+\overline{N})\,d\sigma. \end{aligned} \tag{1.16}$$

When restricting to the plane perpendicular to $\mathbf{m}$, we may extend the previous result to the Vlasov–Poisson system.

We suppose now that f^ϵ does not depend on x_1 and v_1, and we use the following notations: $t \in [0,T)$, $T < \infty$, still denotes the time, the position- and velocity-variables become $\mathbf{x} = (x_2, x_3) \in \mathbb{R}^2_x$ and $\mathbf{v} = (v_2, v_3) \in \mathbb{R}^2_v$. We set $\mathcal{O} = \mathbb{R}^2_x \times \mathbb{R}^2_v$, $\Omega = [0,T) \times \mathbb{R}^2_x$, and $\mathcal{Q} = [0,T) \times \mathcal{O}$. For clarity, we denote $\mathcal{O}' = \mathbb{R}^2_y \times \mathbb{R}^2_u$ and $\mathcal{Q} = [0,T) \times \mathcal{O}'$. The electric field $\mathbf{E}^\epsilon \equiv \mathbf{E}^\epsilon(t, \mathbf{x})$ standing in the Vlasov equation is now given by the Poisson equation, and then the system we work with writes

$$\begin{aligned} &\frac{\partial f^\epsilon}{\partial t} + \frac{1}{\epsilon}\mathbf{v} \cdot \nabla_x f^\epsilon + \left(\mathbf{E}^\epsilon + \frac{1}{\epsilon}\mathbf{v} \times \mathbf{m}\right) \cdot \nabla_v f^\epsilon = 0, \\ &f^\epsilon_{|t=0} = f_0, \\ &\mathbf{E}^\epsilon = -\nabla \phi^\epsilon, \ \ -\Delta \phi^\epsilon = \rho^\epsilon, \\ &\rho^\epsilon = \int_{\mathbb{R}^2_v} f^\epsilon \, d\mathbf{v}. \end{aligned} \tag{1.17}$$

As the first equation in (1.17) is bidimensional, we precise the sense to give to $\mathbf{v} \times \mathbf{m}$

$$\mathbf{v} \times \mathbf{m} = \begin{pmatrix} v_3 \\ -v_2 \end{pmatrix}. \tag{1.18}$$

We assume

$$f_0 \geq 0, \ \ f_0 \in L^1 \cap L^p(\mathcal{O}), \ \ 0 < \int_{\mathcal{O}} f_0(1 + |\mathbf{v}|^2) \, d\mathbf{v} < +\infty, \tag{1.19}$$

for some $p \geq 2$, and we have the following theorem.

THEOREM 1.4. *Under assumption* (1.19)*, for each* ϵ*, there exists a solution* $(f^\epsilon, \mathbf{E}^\epsilon)$ *of* (1.17) *in* $L^\infty(0,T; L^1 \cap L^p(\mathcal{O})) \times L^\infty(0,T; W^{1,\frac{3}{2}}(\mathbb{R}^2_x))$ *for any* $T \in \mathbb{R}^+$*. Moreover, this solution is bounded in* $L^\infty(0,T; L^1 \cap L^p(\mathcal{O})) \times L^\infty(0,T; W^{1,\frac{3}{2}}(\mathbb{R}^2_x))$ *independently on* ϵ*.*

If we consider a sequence $(f^\epsilon, \mathbf{E}^\epsilon)$ *of such solutions, extracting a subsequence, we have*

$$\begin{aligned} &f^\epsilon \text{ 2-scale converges to } F \in L^\infty(0,T; L^\infty_{2\pi}(\mathbb{R}_\tau; L^p(\mathcal{O}))), \\ &\mathbf{E}^\epsilon \text{ 2-scale converges to } \mathcal{E} \in L^\infty(0,T; L^\infty_{2\pi}(\mathbb{R}_\tau; W^{1,\frac{3}{2}}(\mathbb{R}^2_x))), \end{aligned} \tag{1.20}$$

where $F \equiv F(t, \tau, \mathbf{x}, \mathbf{v})$ *and* $\mathcal{E} \equiv \mathcal{E}(t, \tau, \mathbf{x})$*.*

Moreover, there exists a function $G \equiv G(t, \mathbf{y}, \mathbf{u}) \in L^\infty(0,T; L^1 \cap L^p(\mathcal{O}'))$ *such that*

$$F(t, \tau, \mathbf{x}, \mathbf{v}) = G(t, \mathbf{x} + \mathcal{R}(-\tau)\mathbf{v}, R(-\tau)\mathbf{v}), \tag{1.21}$$

and $(G, \mathcal{E})$ *is solution of*

(1.22)
$$\frac{\partial G}{\partial t} + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(-\tau)\mathcal{E}(t,\tau,\mathbf{y}+\mathcal{R}(\tau)\mathbf{u})\,d\tau\right)\cdot\nabla_y G + \frac{1}{2\pi}\left(\int_0^{2\pi} R(-\tau)\mathcal{E}(t,\tau,\mathbf{y}+\mathcal{R}(\tau)\mathbf{u})\,d\tau\right)\cdot\nabla_u G = 0,$$

$$G_{|t=0} = \frac{1}{2\pi} f_0,$$

$$\mathcal{E} \equiv \mathcal{E}(t,\tau,\mathbf{x}), \ \textit{with}\ \mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v}, R(-\tau)\mathbf{v})\,d\mathbf{v},$$

with R *and* $\mathcal{R}$ *given by*

$$R(\tau) = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}, \quad \mathcal{R}(\tau) = \begin{pmatrix} \sin\tau & 1-\cos\tau \\ \cos\tau - 1 & \sin\tau \end{pmatrix}. \tag{1.23}$$

In order to prove this theorem, we modify the generic framework previously introduced. We consider here

$$\frac{\partial u^\epsilon}{\partial t} + \mathbf{A}^\epsilon\cdot\nabla_x u^\epsilon + \frac{1}{\epsilon}\mathbf{L}\cdot\nabla_x u^\epsilon = 0, \qquad u^\epsilon_{t=0} = u_0, \tag{1.24}$$

where the notations are similar as for (1.10): $u^\epsilon \equiv u^\epsilon(t,\mathbf{x})$, $t \in [0,T)$, $T < \infty$; $\mathbf{x} \in \mathbb{R}^n = \mathcal{O}$, $\mathcal{Q} = [0,T)\times\mathcal{O}$. We suppose, as previously, that $\mathbf{L} \equiv M\mathbf{x} + N$, where M is a constant entry matrix satisfying $\mathrm{tr}M = 0$ and $e^{\tau M}$ is θ-periodic and where $N \in \mathrm{Im}M$. The assumptions we make on $\mathbf{A}^\epsilon$ are the following: we suppose that, for all $\epsilon > 0$, $\nabla_x\cdot\mathbf{A}^\epsilon = 0$ and that, for some $q > 1$,

$$\mathbf{A}^\epsilon \text{ 2-scale converges to } \mathcal{A} \in L^\infty(0,T; L^\infty_\theta(\mathbb{R}_\tau; W^{1,q}(K))) \tag{1.25}$$

for all compact sets $K \subset \mathbb{R}^n$ and where $\mathcal{A} \equiv \mathcal{A}(t,\tau,\mathbf{x})$ is θ-periodic in τ.

We have the following.

THEOREM 1.5. *Under the assumptions above, if, moreover, the sequence* (u^ϵ) *of solutions of* (1.24) *satisfies*

$$\|u^\epsilon\|_{L^\infty(0,T;L^p(\mathcal{O}))} \le C, \tag{1.26}$$

for some $p > 1$ *such that* $\frac{1}{p} + \frac{1}{q'} < 1$, *where* $\frac{1}{q'} = \mathrm{Max}\{\frac{1}{q} - \frac{1}{n}, 0\}$; *then, extracting a subsequence,*

$$u^\epsilon \text{ 2-scale converges to a profile } U \in L^\infty(0,T; L^\infty_\theta(\mathbb{R}_\tau; L^p(\mathcal{O}))). \tag{1.27}$$

Moreover, we have

$$U(t,\tau,\mathbf{x}) = U_0(t, e^{-\tau M}(\mathbf{x}-\overline{N}) + \overline{N}), \tag{1.28}$$

where $\overline{N}$ *is such that* $-M\overline{N} = N$ *and where* $U_0 \equiv U_0(t,\mathbf{y})$ *is solution of*

$$\frac{\partial U_0}{\partial t} + \int_0^\theta e^{-\sigma M}\mathcal{A}(t,\sigma,e^{\sigma M}(\mathbf{y}-\overline{N})+\overline{N})\,d\sigma\cdot\nabla_y U_0 = 0, \qquad U_{0|t=0} = \frac{1}{\theta}u_0. \tag{1.29}$$

The proof of this theorem consists in finding the constraint equation imposed on U by the $\frac{1}{\epsilon}L$ operator. This yields (1.28). Then we remove the essential oscillation of u^ϵ by defining $w^\epsilon(t,\mathbf{y}) = u^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N}) + \overline{N}))$. Using the equation w^ϵ satisfies, denoting $(W_0^{1,r}(K))^*$ the dual of $W_0^{1,r}(K)$, we prove that $\frac{\partial w^\epsilon}{\partial t}$ is bounded in $L^\infty(0,T;(W_0^{1,r}(K))^*)$, for some $r>1$ ($\frac{1}{r^*} = \frac{1}{p} + \frac{1}{q} - \frac{1}{n}$, $\frac{1}{r} + \frac{1}{r^*} = 1$), which, applying the Aubin–Lions lemma, gives that $w^\epsilon \to \theta U_0$ strongly in $L^\infty(0,T;(W_0^{1,q}(K))^*)$ for any compact set $K \subset \mathbb{R}^n$. This fact, coupled with (1.25), enables us to pass to the limit in the equation satisfied by w^ϵ and find (1.29).

Theorem 1.4 is a direct application of Theorem 1.5 once the wanted regularity of $\mathbf{E}^\epsilon$ is proved. This is done with the help of classical kinetic energy estimates and the regularization property of the Laplace operator.

The paper is organized as follows. In section 2 we present the scaling leading to the finite Larmor radius approximation. We show how to obtain (1.1) and system (1.17). The next section is devoted to the deduction of the asymptotic behavior of the linear Vlasov equation. Finally, in section 4 we prove Theorems 1.5 and 1.4 concerning the nonlinear case.

2. Scaling: The finite Larmor radius regime. Approximate models in the case of a large external magnetic field have been used by physicists for a long time and the corresponding gyrokinetic ordering is due to Taylor and Hastie [24] and Rutherford and Frieman [19]. We also refer to [6] for a further discussion. And for a physical introduction of the finite Larmor radius model, we refer to [15, 17]. Our scaling assumptions follow from those works.

We present here the scaling leading to (1.1) and system (1.17). We exhibit the important parameters playing a role when charged particles are submitted to a strong magnetic field. For this purpose we consider the following Vlasov–Poisson system

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_x f + \frac{q}{m}(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x}))\cdot\nabla_v f = 0 \\ &f_{|t=0} = f_0, \\ &\mathbf{E} = -\nabla\phi, \qquad -\Delta\phi = \frac{q}{\epsilon_0}\rho, \\ &\rho(t,\mathbf{x}) = \int_{\mathbb{R}^3_v} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}, \end{aligned} \tag{2.1}$$

before any scaling, which can be considered as a natural model to describe the behavior of charged particles under the action of an external magnetic field $\mathbf{B}(t,\mathbf{x})$.

We define some characteristic scales: $\bar{t}$ stands for a characteristic time, $\overline{L_\parallel}$ for a characteristic length in the direction of the magnetic field, $\overline{L_\perp}$ for a characteristic length in the direction orthogonal to the magnetic field, $\bar{v}$ for a characteristic velocity. Denoting, for any vector $\mathbf{x}$, $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ its components parallel and perpendicular to the magnetic field, we now define new variables t', $\mathbf{x}'$, and $\mathbf{v}'$, by $t = \bar{t}t'$, $\mathbf{x}_\parallel = \overline{L_\parallel}\mathbf{x}'_\parallel$, $\mathbf{x}_\perp = \overline{L_\perp}\mathbf{x}'_\perp$, and $\mathbf{v} = \bar{v}\mathbf{v}'$, making the characteristic scales the unities. In the same way, we define the scaling factors for the fields: $\overline{E}$ for the electric field and $\overline{B}$ for the magnetic field and the new fields $\mathcal{E}$ and $\mathcal{B}$ are given by $\overline{E}\mathcal{E}(t',\mathbf{x}') = \mathbf{E}(\bar{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp)$ and $\overline{B}\mathcal{B}(t',\mathbf{x}') = \mathbf{B}(\bar{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp)$. Lastly, defining a scaling factor $\overline{f}$ for the repartition function, noticing that f is a repartition function on the phase-space, it is natural to define the new repartition function by

$$\overline{f} f'(t', \mathbf{x}', \mathbf{v}') = \overline{L_\parallel}\, \overline{L_\perp}^2 \overline{v}^3 f(\overline{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp, \overline{v}\mathbf{v}'). \tag{2.2}$$

With those new variables and fields we deduce the scaling equations.

2.1. Scaling of the Vlasov equation. Let us begin with the Vlasov equation; we obtain that f' is solution of

$$\frac{\partial f'}{\partial t'} + \frac{\overline{v}\overline{t}}{\overline{L_\parallel}}\mathbf{v}'_\parallel \cdot \nabla_{x'} f' + \frac{\overline{v}\overline{t}}{\overline{L_\perp}}\mathbf{v}'_\perp \cdot \nabla_{x'} f' + \left(\frac{q\overline{E}\overline{t}}{m\overline{v}} \mathcal{E}(t', \mathbf{x}') + \frac{q\overline{B}\overline{t}}{m} \mathbf{v}' \times \mathcal{B}(t', \mathbf{x}') \right) \cdot \nabla_{v'} f' = 0. \tag{2.3}$$

Now, we introduce the characteristic cyclotron frequency: $\overline{\omega}_c = \frac{q\overline{B}}{m}$ and the characteristic Larmor radius: $\overline{a}_L = \frac{\overline{v}}{\overline{\omega}_c}$. Using those physical quantities, (2.3) becomes

$$\frac{\partial f'}{\partial t'} + \overline{t}\overline{\omega}_c \frac{\overline{a}_L}{\overline{L_\parallel}}\mathbf{v}'_\parallel \cdot \nabla_{x'} f' + \overline{t}\overline{\omega}_c \frac{\overline{a}_L}{\overline{L_\perp}}\mathbf{v}'_\perp \cdot \nabla_{x'} f' + \left(\overline{t}\overline{\omega}_c \frac{\overline{E}}{\overline{v}\overline{B}} \mathcal{E}(t', \mathbf{x}') + \overline{t}\overline{\omega}_c \mathbf{v}' \times \mathcal{B}(t', \mathbf{x}') \right) \cdot \nabla_{v'} f' = 0. \tag{2.4}$$

Assuming the magnetic field is strong consists essentially in setting

$$\overline{t}\overline{\omega_c} = \frac{1}{\epsilon} \text{ and } \frac{\overline{E}}{\overline{v}\overline{B}} = \epsilon \tag{2.5}$$

for a small parameter ϵ, and the finite Larmor radius regime consists in choosing

$$\frac{\overline{a}_L}{\overline{L_\parallel}} = \epsilon \text{ and } \frac{\overline{a}_L}{\overline{L_\perp}} = 1. \tag{2.6}$$

Hence the rescaled Vlasov equation writes

$$\frac{\partial f'}{\partial t'} + \mathbf{v}'_\parallel \cdot \nabla_{x'} f' + \frac{1}{\epsilon}\mathbf{v}'_\perp \cdot \nabla_{x'} f' + \left(\mathcal{E}(t', \mathbf{x}') + \frac{1}{\epsilon}\mathbf{v} \times \mathcal{B}(t', \mathbf{x}')\right) \cdot \nabla_{v'} f' = 0. \tag{2.7}$$

Concerning the initial data, under the scaling (2.2), the second equation of (2.1) directly gives

$$f'_{|t'=0} = \frac{\overline{L_\parallel}\, \overline{L_\perp}^2 \overline{v}^3}{\overline{f}} f_0(\overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp, \overline{v}\mathbf{v}'). \tag{2.8}$$

Hence, if we assume that the scales of variations of the initial data f_0 (before scaling) are of the same order as the characteristic lengths used, and that $\overline{f} = \overline{L_\parallel}\, \overline{L_\perp}^2 \overline{v}^3$, it is natural to consider (1.1) as a relevant model to understand local behavior of charged particles under the action of a strong external constant magnetic field.

This is the reason why we study (1.1) in section 3.

2.2. Scaling of the Poisson equation. We now turn to the Poisson equation given by the third and fourth equations of (2.1). For this purpose, we define the new electric potential by

$$\overline{E}\,\overline{L_\parallel}\phi'(t', \mathbf{x}') = \phi(\overline{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp), \tag{2.9}$$

and the new particle density by

$$\rho'(t', \mathbf{x}') = \int f'(t', \mathbf{x}', \mathbf{v}')\, d\mathbf{v}'. \tag{2.10}$$

Direct computations give

$$\rho'(t', \mathbf{x}') = \frac{\overline{L_\parallel}\,\overline{L_\perp}^2}{\overline{f}} \rho(\bar{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp), \tag{2.11}$$

$$\nabla\phi(\bar{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp) = \overline{E} \begin{pmatrix} \nabla_{\mathbf{x}'_\parallel}\phi'(t', \mathbf{x}') \\ \frac{\overline{L_\parallel}}{\overline{L_\perp}} \nabla_{\mathbf{x}'_\perp}\phi'(t', \mathbf{x}') \end{pmatrix}, \tag{2.12}$$

and

$$\Delta\phi(\bar{t}t', \overline{L_\parallel}\mathbf{x}'_\parallel, \overline{L_\perp}\mathbf{x}'_\perp) = \frac{\overline{E}}{\overline{L_\parallel}} \Big(\Delta_{\mathbf{x}'_\parallel}\phi'(t', \mathbf{x}') + \frac{\overline{L_\parallel}^2}{\overline{L_\perp}^2} \Delta_{\mathbf{x}'_\perp}\phi'(t', \mathbf{x}') \Big). \tag{2.13}$$

Hence the Poisson equation $-\Delta\phi = \frac{q}{\epsilon_0}\rho$ becomes

$$-\Big(\Delta_{\mathbf{x}'_\parallel}\phi' + \frac{\overline{L_\parallel}^2}{\overline{L_\perp}^2} \Delta_{\mathbf{x}'_\perp}\phi' \Big) = \frac{q}{\epsilon_0} \frac{\overline{f}}{\overline{E}\,\overline{L_\perp}^2} \rho' \tag{2.14}$$

and the definition of the electric field $\mathbf{E} = -\nabla\phi$ yields

$$\mathcal{E} = -\begin{pmatrix} \nabla_{\mathbf{x}'_\parallel}\phi' \\ \frac{\overline{L_\parallel}}{\overline{L_\perp}} \nabla_{\mathbf{x}'_\perp}\phi' \end{pmatrix}. \tag{2.15}$$

Setting now the same ratio as in (2.5) and (2.6) and considering that the scales of variations of the initial data are of the same order as the characteristic lengths, the rescaled Vlasov–Poisson system writes

$$\begin{gathered} \frac{\partial f'}{\partial t'} + \mathbf{v}'_\parallel \cdot \nabla_{x'} f' + \frac{1}{\epsilon}\mathbf{v}'_\perp \cdot \nabla_{x'} f' + \Big(\mathcal{E}(t', \mathbf{x}') + \frac{1}{\epsilon}\mathbf{v} \times \mathcal{B}(t', \mathbf{x}') \Big) \cdot \nabla_{v'} f' = 0, \\ f_{|t=0} = f'_0, \\ \mathcal{E} = -\begin{pmatrix} \nabla_{\mathbf{x}'_\parallel}\phi' \\ \frac{1}{\epsilon}\nabla_{\mathbf{x}'_\perp}\phi' \end{pmatrix}, \quad -\Big(\Delta_{\mathbf{x}'_\parallel}\phi' + \frac{1}{\epsilon^2}\Delta_{\mathbf{x}'_\perp}\phi' \Big) = \gamma\rho', \\ \rho'(t', \mathbf{x}') = \int_{\mathbb{R}^3_v} f'(t', \mathbf{x}', \mathbf{v}')\, d\mathbf{v}', \end{gathered} \tag{2.16}$$

with $\gamma = \frac{q}{\epsilon_0} \frac{\overline{f}}{\overline{E}\overline{L_\perp}^2}$.

For the study we lead in section 4 we consider the previous system with $\gamma = \frac{1}{\epsilon}$ and with $\mathcal{B} = \mathbf{m} = \mathbf{e}_1$. We moreover assume that none of the fields depend on the component parallel to the magnetic fields $\mathbf{x}_\parallel$ and $\mathbf{v}_\parallel$. In this case the Poisson equation from which we remove the $\mathbf{x}_\parallel-$dependency

$$\mathcal{E} = -\begin{pmatrix} 0 \\ \frac{1}{\epsilon}\nabla_{\mathbf{x}'_\perp}\phi' \end{pmatrix}, \quad -\frac{1}{\epsilon^2}\Delta_{\mathbf{x}'_\perp}\phi' = \frac{1}{\epsilon}\rho', \tag{2.17}$$

is equivalent to, removing the magnetic field direction,

$$\mathcal{E} = -\nabla\phi^*, \quad -\Delta\phi^* = \rho^*, \tag{2.18}$$

where ϕ^* is nothing but $\frac{1}{\epsilon}\phi'$ and with

$$\rho^* = \int_{\mathbb{R}^2_v} f' \, d\mathbf{v}, \tag{2.19}$$

explaining the interest of studying system (1.17).

3. Homogenization of the Vlasov equation. In this section, we provide the homogenization of the Vlasov equation (1.1) and prove Theorem 1.1. Since the contexts of (1.1) and the equation studied in Frénod and Sonnendrücker [8] are similar, we develop a generic framework and apply it to prove Theorem 1.1. We then show that this generic framework applies also to prove Theorems 1.1 and 3.2 of Frénod and Sonnendrücker [8].

3.1. Generic framework—proof of Theorem 1.3. The framework inside which the problem we want to homogenize enters is the following conservation law singularly linearly perturbed:

$$\begin{aligned} &\frac{\partial u^\epsilon}{\partial t} + \mathbf{A}\cdot\nabla_x u^\epsilon + \frac{1}{\epsilon}\mathbf{L}\cdot\nabla_x u^\epsilon = 0, \\ &u^\epsilon_{|t=0} = u_0, \end{aligned} \tag{3.1}$$

where $u^\epsilon \equiv u^\epsilon(t,\mathbf{x})$, $t \in [0,T)$ for some $T < \infty$, and $\mathbf{x} \in \mathbb{R}^n = \mathcal{O}$. We denote $\mathcal{Q} = [0,T)\times\mathcal{O}$, and we assume $\mathbf{A} \equiv \mathbf{A}(t,\mathbf{x}) \in L^\infty(0,T;L^2_{loc}(\mathcal{O}))$, with $\nabla_x\cdot\mathbf{A} = 0$ and $\mathbf{L} \equiv M\mathbf{x} + N$, where M is a real $n\times n$ matrix with constant entries satisfying $\mathrm{tr}M = 0$ and where $N \in \mathrm{Im}M$, which implies that $\nabla_x\cdot\mathbf{L} = 0$.We moreover assume that $e^{\tau M}$ is θ-periodic for a given $\theta\in\mathbb{R}$.

The proof of Theorem 1.3, characterizing the limit of (3.1), is led in three steps. First, we look for the constraint imposed by the operator $(\frac{1}{\epsilon}\mathbf{L}\cdot\nabla_x)$ on the profile U, 2-scale limit of (u^ϵ). Studying the characteristics associated with this constraint, we obtain the form (1.14) it gives to U. In the second step, using test functions satisfying the constraint in the weak formulation of (3.1), we get the equation satisfied by U_0. In view of formula (1.8) linking the 2-scale limit to the weak$-*$ limit, in the last step, we integrate the equation satisfied by U_0 to deduce (1.16).

Under the assumption (1.11), we may apply the result of Nguetseng [18] and Allaire [2] (see Theorem 1.2). Then, for any period $\tilde\theta$ there exists a $\tilde\theta$-periodic profile $U_{\tilde\theta}(t,\tau,\mathbf{x}) \in L^\infty(0,T;L^\infty_{\tilde\theta}(\mathbb{R}_\tau;L^2(\mathcal{O})))$ such that, for any regular function $\psi_{\tilde\theta}(t,\tau,\mathbf{x})$ compactly supported in $(t,\mathbf{x})$ and $\tilde\theta$-periodic in τ, we have

$$\int_{\mathcal{Q}} u^\epsilon(t,\mathbf{x})\psi_{\tilde\theta}\left(t,\frac{t}{\epsilon},\mathbf{x}\right)\,dt\,d\mathbf{x} \to \int_{\mathcal{Q}}\int_0^{\tilde\theta} U(t,\tau,\mathbf{x})\psi_{\tilde\theta}(t,\tau,\mathbf{x})\,d\tau\,dt\,d\mathbf{x}. \tag{3.2}$$

Now, we write a weak formulation of (3.1) with oscillating test functions $(\psi_{\tilde\theta})^\epsilon = \psi_{\tilde\theta}(t,\frac{t}{\epsilon},\mathbf{x})$, with $\psi_{\tilde\theta}(t,\tau,\mathbf{x})$ previously defined. Since $\nabla_x\cdot\mathbf{A} = \nabla_x\cdot\mathbf{L} = 0$, it writes

(3.3)
$$\int_{\mathcal{Q}} u^\epsilon\left(\left(\frac{\partial\psi_{\tilde\theta}}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial\psi_{\tilde\theta}}{\partial\tau}\right)^\epsilon + \mathbf{A}\cdot(\nabla_x\psi_{\tilde\theta})^\epsilon + \frac{1}{\epsilon}\mathbf{L}\cdot(\nabla_x\psi_{\tilde\theta})^\epsilon\right)dtd\mathbf{x} = -\int_{\mathcal{O}} u_0\psi_{\tilde\theta}(0,0,\mathbf{x})\,d\mathbf{x}.$$

Multiplying (3.3) by ϵ and passing to the limit gives the following constraint equation for the $\tilde\theta$-periodic profile $U_{\tilde\theta}$:

$$\frac{\partial U_{\tilde\theta}}{\partial \tau} + \mathbf{L}\cdot\nabla_x U_{\tilde\theta} = 0 \text{ in } \mathcal{D}'(\mathbb{R}_\tau \times \mathcal{O}). \tag{3.4}$$

This equation says that $U_{\tilde\theta}$ is constant along the characteristics of the following dynamical system:

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{L}(\mathbf{X}(\tau)) = M\mathbf{X}(\tau) + N. \tag{3.5}$$

Using the assumptions made on $\mathbf{L}$, writing $\mathbf{X}(\tau;\mathbf{x},s)$ for the solution of (3.5) satisfying $\mathbf{X}(s;\mathbf{x},s)=\mathbf{x}$, we obtain

$$\mathbf{X}(\tau;\mathbf{x},s) = e^{(\tau-s)M}(\mathbf{x}-\overline{N})+\overline{N}. \tag{3.6}$$

Hence from (3.4), we deduce, on the one hand, that for any $\tilde\theta$ the $\tilde\theta$-periodic profile writes

$$U_{\tilde\theta}(t,\tau,\mathbf{x}) = U_0(t, e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}) \tag{3.7}$$

for a function $U_0 \equiv U_0(t,\mathbf{y}) \in L^\infty(0,T;L^2(\mathcal{O}'))$. On the other hand, we take the θ-periodicity of $e^{-\tau M}$ under consideration. In view of (3.7), we deduce that if $\tilde\theta$ and θ are incommensurable, $U_{\tilde\theta}$ cannot depend on τ, and then contains no information concerning the oscillations of (u^ϵ). Yet if $\tilde\theta$ equals (or is multiple of) θ, the profile $U_{\tilde\theta}$ naturally satisfies its $\tilde\theta$-periodicity condition once (3.7) is satisfied.

Hence, among every possible periodic profile, we are incited to work now with the θ-periodic one

$$U := U_\theta, \tag{3.8}$$

which writes

$$U(t,\tau,\mathbf{x}) = U_0(t, e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}) \tag{3.9}$$

for $U_0 \equiv U_0(t,\mathbf{y}) \in L^\infty(0,T;L^2(\mathcal{O}'))$, which is the equality (1.14) of Theorem 1.3.

Now, we seek the equation U_0 satisfies. For this purpose, we build oscillating test functions satisfying the constraint and use them in the weak formulation (3.3).

For any $\varphi(t,\mathbf{y})$, regular and compactly supported, we define $\psi(t,\tau,\mathbf{x}) = \varphi(t, e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N})$ and we inject in (3.3) test function $(\psi)^\epsilon = \psi(t,\frac{t}{\epsilon},\mathbf{x})$. Acting in such a way, the terms containing the constraint vanishes. We have then

$$\int_{\mathcal{Q}} u^\epsilon\left(\left(\frac{\partial\psi}{\partial t}\right)^\epsilon + \mathbf{A}\cdot(\nabla_x\psi)^\epsilon\right)dtd\mathbf{x} = -\int_{\mathcal{O}} u_0\psi(0,0,\mathbf{x})\,d\mathbf{x}, \tag{3.10}$$

which passing to the limit yields

$$\int_{\mathcal{Q}}\int_0^\theta U\left(\frac{\partial\psi}{\partial t}+\mathbf{A}\cdot\nabla\psi\right)dtd\mathbf{x}d\tau = -\int_{\mathcal{O}} u_0\psi(0,0,\mathbf{x})\,d\mathbf{x}. \tag{3.11}$$

In (3.11) we use the expression of U in terms of U_0, the expression of ψ in term of φ, without forgetting

$$\nabla_x\psi(t,\tau,\mathbf{x}) = (e^{-\tau M})^T\nabla_y\varphi(t, e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}), \tag{3.12}$$

denoting $(e^{-\tau M})^T$ the transpose of $e^{-\tau M}$; and we make the change of variable $\mathbf{x} \mapsto \mathbf{y} = e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N}$. This gives

$$\int_{\mathcal{Q}'} \int_0^\theta U_0 \left(\frac{\partial \varphi}{\partial t} + e^{-\tau M} \mathbf{A}(t, e^{\tau M}(\mathbf{y} - \overline{N}) + \overline{N}) \cdot \nabla_y \varphi \right) dt d\mathbf{y} d\tau = - \int_{\mathcal{O}} u_0 \varphi(0, \mathbf{y})\, d\mathbf{y}. \tag{3.13}$$

An easy computation coupled with the fact that $\nabla_x \cdot \mathbf{A} = 0$ gives

$$\nabla_y \cdot (e^{-\sigma M} \mathbf{A}(t, e^{\sigma M}(\mathbf{y} - \overline{N}) + \overline{N})) = (\nabla_x \cdot \mathbf{A})(t, e^{\sigma M}(\mathbf{y} - \overline{N}) + \overline{N}) = 0. \tag{3.14}$$

Hence, knowing that neither U_0 nor φ depend on τ, we deduce that (3.13) is the weak formulation of

$$\begin{aligned} &\frac{\partial U_0}{\partial t} + \frac{1}{\theta} \int_0^\theta e^{-\sigma M} \mathbf{A}(t, e^{\sigma M}(\mathbf{y} - \overline{N}) + \overline{N})\, d\sigma \cdot \nabla_y U_0 = 0, \\ &U_{0|t=0} = \frac{1}{\theta} u_0, \end{aligned} \tag{3.15}$$

proving the second part of Theorem 1.3.

In order to get (1.16) we use the fact that

$$u(t, \mathbf{x}) = \int_0^\theta U(t, \tau, \mathbf{x})\, d\tau = \int_0^\theta U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\tau. \tag{3.16}$$

Replacing in (3.15) $\mathbf{y}$ by $e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N}$ and integrating in τ we get

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\int_0^\theta U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\tau \right) \\ &\quad + \frac{1}{\theta} \int_0^\theta \int_0^\theta e^{-\sigma M} \mathbf{A}(t, e^{(\sigma - \tau)M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\sigma \cdot \nabla_y U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\tau = 0, \\ &\int_0^\theta U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N}))_{|t=0}\, d\tau = \frac{1}{\theta} \int_0^\theta u_0(e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\tau. \end{aligned} \tag{3.17}$$

An easy computation yields

$$\nabla_x (U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})) = (e^{-\tau M})^T (\nabla_y U_0)(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N}), \tag{3.18}$$

and then replacing in the second term of the first equation in (3.17) gives

$$\begin{aligned} &\int_0^\theta \int_0^\theta e^{-\sigma M} \mathbf{A}(t, e^{(\sigma - \tau)M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\sigma \cdot (e^{\tau M})^T \nabla_x (U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\tau \\ &\quad = \int_0^\theta \int_0^\theta e^{(\tau - \sigma)M} \mathbf{A}(t, e^{(\sigma - \tau)M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\sigma \cdot \nabla_x (U_0(t, e^{-\tau M}(\mathbf{x} - \overline{N}) + \overline{N}))\, d\tau. \end{aligned} \tag{3.19}$$

Yet by the periodicity of $\tau \mapsto e^{\tau M}$ we deduce that

$$\int_0^\theta e^{(\tau - \sigma)M} \mathbf{A}(t, e^{(\sigma - \tau)M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\sigma = \int_0^\theta e^{-\sigma M} \mathbf{A}(t, e^{\sigma M}(\mathbf{x} - \overline{N}) + \overline{N})\, d\sigma \tag{3.20}$$

does not depend on τ.

We may finally conclude that (3.17) reads

$$(3.21)\quad \begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{\theta}\int_0^\theta e^{-\sigma M}\mathbf{A}(t, e^{\sigma M}(\mathbf{x}-\overline{N})+\overline{N})d\sigma \cdot \nabla\, u = 0,\\ &u_{|t=0}(\mathbf{x}) = \frac{1}{\theta}\int_0^\theta u_0(e^{-\sigma M}(\mathbf{x}-\overline{N})+\overline{N})\, d\sigma. \end{aligned}$$

achieving the proof of Theorem 1.3.

3.2. Application to the Vlasov equation—proof of Theorem 1.1. Using assumption (1.2) made on f_0 and the following property of f^ϵ solution of (1.1)

$$(3.22)\quad \frac{d}{dt}\|f^\epsilon(t,\cdot,\cdot)\|_{L^2(\mathcal{O})} = 0,$$

obtained by a direct integration in $\mathbf{x}$ and $\mathbf{v}$ of the first equation in (1.1), after multiplication by f^ϵ, we deduce that

$$(3.23)\quad \|f^\epsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))} \le C$$

for some constants C.

Hence, the Vlasov equation (1.1) enters the generic framework previously built with

$$(3.24)\quad \mathbf{A}(t,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v}_\parallel \\ \mathbf{E}(t,\mathbf{x})\end{pmatrix} (\in \mathbb{R}^6) \text{ and } \mathbf{L}(t,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v}_\perp \\ \mathbf{v}\times\mathbf{m}\end{pmatrix} (\in \mathbb{R}^6).$$

Then the differential system defining the characteristics $(\dot{X},\dot{V}) = \mathbf{L}(X,V)$ becomes

$$\frac{dX_\perp}{d\tau} = V_\perp, \qquad \frac{dV}{d\tau} = V\times\mathbf{m}.$$

An easy computation then yields $V(\tau;\mathbf{v},s) = R(\tau-s)\mathbf{v}$ and $X(\tau;(\mathbf{x},\mathbf{v}),s) = \mathbf{x} + \mathcal{R}(\tau-s)\mathbf{v}$, with $\mathcal{R}(\tau)$ and $R(\tau)$ given by (1.6). Hence $e^{\tau M}$ reads in this case

$$(3.25)\quad e^{\tau M} = \begin{pmatrix} I & \mathcal{R}(\tau) \\ 0 & R(\tau)\end{pmatrix}.$$

We can then deduce

$$(3.26)\quad f^\epsilon \text{ 2-scale converges to } F \in L^\infty(0,T;L^\infty_{2\pi}(\mathbb{R}_\tau;L^2(\mathcal{O}))),$$

and applying Theorem 1.3, we can deduce that there exists a function $G \equiv G(t,\mathbf{y},\mathbf{u}) \in L^\infty(0,T;L^2(\mathcal{O}'))$ such that

$$(3.27)\quad F(t,\tau,\mathbf{x},\mathbf{v}) = G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v}, R(-\tau)\mathbf{v}),$$

where G is the unique solution of

$$(3.28)\quad \begin{aligned} &\frac{\partial G}{\partial t} + \mathbf{u}_\parallel\cdot\nabla_y G + \frac{1}{2\pi}\left(\int_0^{2\pi}\mathcal{R}(-\tau)\mathbf{E}(t,\mathbf{y}+\mathcal{R}(\tau)\mathbf{u})\, d\tau\right)\cdot\nabla_y G\\ &\qquad + \frac{1}{2\pi}\left(\int_0^{2\pi} R(-\tau)\mathbf{E}(t,\mathbf{y}+\mathcal{R}(\tau)\mathbf{u})\, d\tau\right)\cdot\nabla_v G = 0,\\ &G_{|t=0} = \frac{1}{2\pi}f_0. \end{aligned}$$

Always applying Theorem 1.3, we also deduce that the weak$-*$ limit f of (f^ϵ) is the unique solution of (1.5).

The fact that the whole sequence (f^ϵ) 2-scale converges to F and weak$-*$ converges to f is a direct consequence of the uniqueness of the solution of (3.28) and (1.5). This ends the proof of Theorem 1.1.

3.3. Link with physical models. In order to compare the model we obtain with the finite Larmor radius approximation used by physicists, we restrain to the plane orthogonal to the magnetic field. Denoting here $R(\tau)$ and $\mathcal{R}(\tau)$ there restrictions to this plan, we introduce the Larmor radius variable $\mathbf{r} = \mathbf{v}^\perp$ and the guiding center variable $\mathbf{x}_C = \mathbf{x} - \mathbf{r}$, where for any vector $\mathbf{v} = (v_2, v_3)$, $\mathbf{v}^\perp$ stand for $\mathbf{v}^\perp = (-v_3, v_2)$. In this new variables, (3.28) reads:

$$\begin{aligned} \frac{\partial f}{\partial t} - \frac{1}{2\pi}\left(\int_0^{2\pi} \mathbf{E}^\perp(t, \mathbf{x}_C + R(\tau)\mathbf{r})\, d\tau\right) \cdot \nabla_{x_C} f \\ + \frac{1}{2\pi}\left(\int_0^{2\pi} R(-\tau)\mathbf{E}^\perp(t, \mathbf{x}_C + R(\tau)\mathbf{r})\, d\tau\right) \cdot \nabla_r f = 0. \end{aligned} \tag{3.29}$$

Then, assuming that the distribution function is a Maxwellian distribution, i.e., $f \equiv n(\mathbf{x}_c, t)e^{-\mathbf{r}^2/(2\sigma^2)}/(2\pi\sigma)$, we integrate (3.29) with respect to $\mathbf{r}$. This procedure cancels the third term. Indeed

$$\int_0^{2\pi} R(-\tau)\mathbf{E}^\perp(t, \mathbf{x}_C + R(\tau)\mathbf{r})\, d\tau \tag{3.30}$$

depends only on $|\mathbf{r}|$ and then the integrand is odd. Then we get

$$\frac{\partial n}{\partial t} - \int_{\mathbb{R}^2_r} \frac{1}{2\pi}\left(\int_0^{2\pi} \mathbf{E}^\perp(t, \mathbf{x}_C + R(\tau)\mathbf{r})\, d\tau\right) e^{-\mathbf{r}^2/(2\sigma^2)}/(2\pi\sigma)\, d\mathbf{r} \cdot \nabla_{x_C} n = 0, \tag{3.31}$$

which is the model introduced by Hansen et al. [15].

3.4. About previous results. Notice that Theorem 1.1 of Frénod and Sonnendrücker [8], proving that the asymptotic behavior of

$$\begin{aligned} &\frac{\partial f^\epsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\epsilon + \left(\mathbf{E} + \mathbf{v} \times \left(\mathbf{B} + \frac{\mathbf{m}}{\epsilon}\right)\right) \cdot \nabla_v f^\epsilon = 0, \\ &f^\epsilon_{|t=0} = f_0 \end{aligned} \tag{3.32}$$

is given by

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v}_\parallel \cdot \nabla_x f + (\mathbf{E}_\parallel + \mathbf{v} \times \mathbf{B}_\parallel) \cdot \nabla_v f = 0, \\ &f_{|t=0} = \frac{1}{2\pi}\int_0^{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v}, \tau))\, d\tau, \end{aligned} \tag{3.33}$$

is also a consequence of Theorem 1.3 by setting

$$\mathbf{A} = \begin{pmatrix} \mathbf{v} \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 0 \\ \mathbf{v} \times \mathbf{m} \end{pmatrix}. \tag{3.34}$$

This is the same for Theorem 3.2 of [8] with

$$\mathbf{A} = \begin{pmatrix} \mathbf{v} \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 0 \\ \mathbf{n} + \mathbf{v} \times \mathbf{m} \end{pmatrix}. \tag{3.35}$$

This theorem says that the weak$-*$ limit of the solution of

$$\begin{aligned} &\frac{\partial f^\epsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\epsilon + \left(\left(\mathbf{E} + \frac{\mathbf{n}}{\epsilon}\right) + \mathbf{v} \times \left(\mathbf{B} + \frac{\mathbf{m}}{\epsilon}\right)\right) \cdot \nabla_v f^\epsilon = 0, \\ &f^\epsilon_{|t=0} = f_0, \end{aligned} \tag{3.36}$$

with $\mathbf{n} = \mathbf{e}_2$, satisfies

(3.37)

$$\frac{\partial f}{\partial t} + \begin{pmatrix} v_1 \\ 0 \\ -1 \end{pmatrix} \cdot \nabla_x f + \left[\begin{pmatrix} E_1 - B_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 + 1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \nabla_v f = 0,$$

$$f_{|t=0} = \frac{1}{2\pi} \int_0^{2\pi} f_0(\mathbf{x}, \mathbf{u}(\mathbf{v}, \tau))\, d\tau.$$

4. 2-scale limit of the 2D Vlasov–Poisson system.

The aim of this section is to characterize the equation satisfied by the 2-scale limit of the sequence $(f^\epsilon, \mathbf{E}^\epsilon)$ of the Vlasov–Poisson system (1.17) For this purpose, we generalize the generic framework to the case when the operator $\mathbf{A}^\epsilon$ is oscillating. Then we apply the results obtained in this new generic framework to prove Theorem 1.4.

4.1. Generalized generic framework–proof of Theorem 1.5.

We consider here

$$\begin{aligned} &\frac{\partial u^\epsilon}{\partial t} + \mathbf{A}^\epsilon \cdot \nabla_x u^\epsilon + \frac{1}{\epsilon}\mathbf{L} \cdot \nabla_x u^\epsilon = 0, \\ &u^\epsilon_{t=0} = u_0, \end{aligned} \tag{4.1}$$

where the notations are the same as for (1.10): $u^\epsilon \equiv u^\epsilon(t, \mathbf{x})$, $t \in [0, T)$, $T < \infty$; $\mathbf{x} \in \mathbb{R}^n = \mathcal{O}$, $\mathcal{Q} = [0, T) \times \mathcal{O}$. We suppose, as previously, that $\mathbf{L} \equiv M\mathbf{x} + N$, where M is a constant entry matrix satisfying tr$M = 0$ and $e^{\tau M}$ is θ-periodic. The assumptions we make on $\mathbf{A}^\epsilon$ are the following: we suppose that, for all $\epsilon > 0$, $\nabla_x \cdot \mathbf{A}^\epsilon = 0$ and that, for some $q > 1$,

$$\mathbf{A}^\epsilon \text{ 2-scale converges to } \mathcal{A} \in L^\infty(0, T; L^\infty_\theta(\mathbb{R}_\tau; W^{1,q}(K))) \tag{4.2}$$

for all compact sets $K \subset \mathbb{R}^n$ and where $\mathcal{A} \equiv \mathcal{A}(t, \tau, \mathbf{x})$ is θ-periodic in τ.

The proof of Theorem 1.5 begins as the proof of Theorem 1.3 in the sense that the constraint equation and its consequences are similar. Hence relation (1.28) is obvious. In order to get the equation U_0 satisfies, we proceed as follows: we define $w^\epsilon(t, \mathbf{x}) = u^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{x} - \overline{N}) + \overline{N})$, which is the function u^ϵ from which the essential oscillation is removed. This idea has also been used in Frénod and Sonnendrücker [10], Grenier [13, 14], Schochet [20], Joly, Metivier, and Rauch [16], and Colin [5]. Using the equation satisfied by w^ϵ, we show that

$$w^\epsilon \to \theta U_0 \text{ in } L^\infty(0, T; (W^{1,q}_0(K))^*), \tag{4.3}$$

where $(W_0^{1,q}(K))^*$ is the dual of $(W_0^{1,q}(K))$. This fact coupled with the assumption on $\mathbf{A}^\epsilon$ enables us to pass to the limit and find (1.29).

Under assumption (1.26) we may deduce, up to subsequences,

$$u^\epsilon \text{ 2-scale converges to } U \in L^\infty(0,T;L^\infty_\theta(\mathbb{R}_\tau;L^p(\mathcal{O}))). \tag{4.4}$$

The weak formulation of (4.1) with θ-periodic oscillating functions $(\psi)^\epsilon \equiv \psi(t,\frac{t}{\epsilon},\mathbf{x})$ writes

$$\int_{\mathcal{Q}} u^\epsilon\left(\left(\frac{\partial\psi}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial\psi}{\partial\tau}\right)^\epsilon + \mathbf{A}^\epsilon\cdot(\nabla_x\psi)^\epsilon + \frac{1}{\epsilon}\mathbf{L}\cdot(\nabla_x\psi)^\epsilon\right)dtd\mathbf{x} = -\int_{\mathcal{O}} u_0\psi(0,0,\mathbf{x})\,d\mathbf{x}. \tag{4.5}$$

Proceeding as in subsection 3.1 we obtain that U satisfies

$$\frac{\partial U}{\partial\tau} + \mathbf{L}\cdot\nabla_x U = 0 \text{ in } \mathcal{D}'(\mathbb{R}_\tau\times\mathcal{O}) \tag{4.6}$$

and then

$$U(t,\tau,\mathbf{x}) = U_0(t,e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}) \tag{4.7}$$

for $U_0 \equiv U_0(t,\mathbf{y}) \in L^\infty(0,T;L^2(\mathcal{O}'))$, which is (1.28) of Theorem 1.5.

Now we look for the equation U_0 satisfies. For this purpose, we define

$$w^\epsilon(t,\mathbf{y}) = u^\epsilon(t,e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}), \tag{4.8}$$

and we have the following lemma which characterizes the asymptotic limit of w^ϵ.

LEMMA 4.1. *The sequence* (w^ϵ) *satisfies*

$$w^\epsilon \to \theta U_0 \text{ in } L^\infty(0,T;(W_0^{1,q}(K))^*), \tag{4.9}$$

where U_0 *is linked with the profile* U *by* (1.28).

Proof. First, we prove that w^ϵ 2-scale converges to U_0 and w^ϵ weakly$-*$ converges to θU_0. Second, we show that $\frac{\partial w^\epsilon}{\partial t}$ is bounded in $L^\infty(0,T;(W_0^{1,r}(K))^*)$ for some $r>1$. Since w^ϵ is bounded in $L^\infty(0,T;L^p(\mathcal{O}))$, the Aubin–Lions lemma leads to the conclusion.

We take any function $\phi(t,\tau,\mathbf{y})$ regular, with compact support in t and $\mathbf{y}$ and θ-periodic with respect to τ. We have

$$\begin{aligned}\int_{\mathcal{O}'} w^\epsilon(t,\mathbf{y})\phi\left(t,\frac{t}{\epsilon},\mathbf{y}\right)dtd\mathbf{y} &= \int_{\mathcal{O}'} u^\epsilon(t,e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\phi\left(t,\frac{t}{\epsilon},\mathbf{y}\right)dtd\mathbf{y}\\ &= \int_{\mathcal{O}} u^\epsilon(t,\mathbf{x})\phi\left(t,\frac{t}{\epsilon},e^{-\frac{t}{\epsilon}M}(\mathbf{x}-\overline{N})+\overline{N}\right)dtd\mathbf{x}.\end{aligned} \tag{4.10}$$

This last quantity converges to

$$\int_{\mathcal{O}}\int_0^\theta U(t,\tau,\mathbf{x})\phi(t,\tau,e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N})\,dtd\mathbf{x}d\tau = \int_{\mathcal{O}}\int_0^\theta U_0\phi\,dtd\mathbf{y}d\tau, \tag{4.11}$$

proving w^ϵ 2-scale converges to U_0. Since U_0 does not depend on τ, we immediately deduce $w^\epsilon \rightharpoonup \theta U_0$ weakly$-*$.

Now, we seek the equation w^ϵ satisfies. We have

$$\frac{\partial w^\epsilon}{\partial t}(t,\mathbf{y}) = \frac{\partial u^\epsilon}{\partial t}(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}) + \frac{M}{\epsilon}e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})\cdot\nabla_x u^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}); \tag{4.12}$$

writing this last equality in $\mathbf{y} = e^{-\frac{t}{\epsilon}M}(\mathbf{x}-\overline{N})+\overline{N}$ we obtain

$$\begin{aligned}\frac{\partial w^\epsilon}{\partial t}(t, e^{-\frac{t}{\epsilon}M}(\mathbf{x}-\overline{N})+\overline{N}) &= \frac{\partial u^\epsilon}{\partial t}(t,\mathbf{x}) + \frac{M}{\epsilon}(\mathbf{x}-\overline{N})\cdot\nabla_x u^\epsilon(t,\mathbf{x})\\ &= \frac{\partial u^\epsilon}{\partial t}(t,\mathbf{x}) + \frac{1}{\epsilon}\mathbf{L}\cdot\nabla_x u^\epsilon(t,\mathbf{x}).\end{aligned} \tag{4.13}$$

Hence in view of the equation satisfied by u^ϵ and of

$$\nabla_y w^\epsilon(t,\mathbf{y}) = (e^{\frac{t}{\epsilon}M})^T\nabla_x u^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}), \tag{4.14}$$

we obtain that w^ϵ is solution of

$$\frac{\partial w^\epsilon}{\partial t} + \mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\cdot(e^{-\frac{t}{\epsilon}M})^T\nabla_y w^\epsilon = 0, \tag{4.15}$$

i.e.,

$$\frac{\partial w^\epsilon}{\partial t} + e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\cdot\nabla_y w^\epsilon = 0. \tag{4.16}$$

Having (4.16) at hand we can prove that $\frac{\partial w^\epsilon}{\partial t}$ is bounded in $L^\infty(0,T;(W_0^{1,r}(K))^*)$ for some $r>1$ and any compact $K\subset\mathbb{R}^n$. It is an easy game to show

$$\nabla_y\cdot\left[e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\right] = 0. \tag{4.17}$$

Hence, from (4.16) we deduce

$$\frac{\partial w^\epsilon}{\partial t} = -\nabla_y\cdot\left[e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\,w^\epsilon\right], \tag{4.18}$$

and since, due to its two scale convergence, $\mathbf{A}^\epsilon$ is bounded in $W_0^{1,q}(K)$ with a bound independent on t, a Sobolev embedding theorem implies that $\mathbf{A}^\epsilon$ is bounded in $L^{q'}(K)$, where q' is defined by $\frac{1}{q'} = \mathrm{Max}(\frac{1}{q}-\frac{1}{n},0)$, and since $\tau\mapsto e^{-\tau M}$ is periodic, we deduce that $\left(e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\right)$ is also bounded in $L^{q'}(K)$. Then as w^ϵ is bounded in $L^p(\mathcal{O})$, we get that $\left(e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\,w^\epsilon\right)$ is bounded in $L^{r^*}(K)$ with $\frac{1}{r^*} = \frac{1}{p}+\frac{1}{q'}$. We may then conclude that

$$\frac{\partial w^\epsilon}{\partial t} \text{ is bounded in } L^\infty(0,T;(W_0^{1,r}(K))^*), \text{ with } \frac{1}{r^*}+\frac{1}{r} = 1 \tag{4.19}$$

for any compact $K\subset\mathbb{R}^n$.

In order to conclude, we treat first the case when $r^* < q^*$, where q^* is such that $\frac{1}{q^*}+\frac{1}{q} = 1$. As K is compact, we have $L^{q^*}(K)\subset L^{r^*}(K)$. Since, by considering separately the functions and their derivatives, $(W_0^{1,q}(K))^*$ and $(W_0^{1,r}(K))^*$ are, respectively, isomorphic to $(L^{q^*}(K))^{n+1}$ and $(L^{r^*}(K))^{n+1}$ we deduce, on the one hand,

$$(W_0^{1,q}(K))^* \subset (W_0^{1,r}(K))^* \tag{4.20}$$

with continuous injection. On the other hand, as a consequence of the Rellich–Kondrachov theorem, we have

$$L^p(K) \subset (W_0^{1,q}(K))^* \quad \text{compactly.} \tag{4.21}$$

Hence, applying an analogue of the Aubin–Lions lemma proved by Simon [21], we deduce

$$\mathcal{U} = \left\{ u \in L^\infty(0,T;L^p(K)), \frac{\partial u}{\partial t} \in L^\infty(0,T;(W_0^{1,r}(K))^*) \right\} \tag{4.22}$$

is compactly embedded in $L^\infty(0,T;(W_0^{1,q}(K))^*)$. Then we deduce that w^ϵ converges strongly in $L^\infty(0,T;(W_0^{1,r}(K))^*)$, giving the conclusion of the Lemma.

The case $r^* \geq q^*$ is simpler. Indeed, in this case, we directly have from (4.19)

$$\frac{\partial w^\epsilon}{\partial t} \text{ is bounded in } L^\infty(0,T;(W_0^{1,q}(K))^*), \tag{4.23}$$

yielding directly the conclusion of the lemma. ∎

Having Lemma 4.1 at hand, we want to pass to the limit in (4.16). This will give the equation for U_0. In order to realize this, we have first to prove the following.

LEMMA 4.2. *We have*

$$\begin{aligned} &\mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}) \textit{ 2-scale converges to } \mathcal{A}(t,\tau,e^{\tau M}(\mathbf{y}-\overline{N})+\overline{N}) \\ &\qquad\qquad \textit{in } L^\infty(0,T;L^\infty_\theta(\mathbb{R}_\tau;W_0^{1,q}(K))). \end{aligned} \tag{4.24}$$

Proof. A direct computation gives

$$\begin{aligned} &\int_{\mathcal{Q}'} \mathbf{A}^\epsilon(t, e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\psi\left(t,\frac{t}{\epsilon},\mathbf{y}\right) dtd\mathbf{y} \\ &= \int_{\mathcal{Q}} \mathbf{A}^\epsilon(t,\mathbf{x})\psi\left(t,\frac{t}{\epsilon},e^{-\frac{t}{\epsilon}M}(\mathbf{x}-\overline{N})+\overline{N}\right) dtd\mathbf{x} \\ &\to \int_{\mathcal{Q}}\int_0^\theta \mathcal{A}(t,\tau,\mathbf{x})\psi(t,\tau,e^{-\tau M}(\mathbf{x}-\overline{N})+\overline{N}))\, dtd\mathbf{y}d\tau \\ &= \int_{\mathcal{Q}'}\int_0^\theta \mathcal{A}(t,\tau,e^{\tau M}(\mathbf{y}-\overline{N})+\overline{N}))\psi(t,\tau,\mathbf{y})\, dtd\mathbf{y}d\tau \end{aligned} \tag{4.25}$$

for any θ-periodic test function. This proves the lemma. ∎

Now, writing a weak formulation of (4.16), we have

$$\int_{\mathcal{Q}'} w^\epsilon\left[\frac{\partial\varphi}{\partial t} + e^{-\frac{t}{\epsilon}M}\mathbf{A}^\epsilon(t,e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N})\cdot\nabla_y\varphi\right] dtd\mathbf{y} = \int_{\mathcal{O}'} u_0\varphi(0,\cdot)\, d\mathbf{y} \tag{4.26}$$

for any $\varphi(t,\mathbf{y})$ regular and compactly supported in $\mathcal{Q}$. Let K be a compact containing the support of φ, since

$$\begin{aligned} &w^\epsilon \to \theta U_0 \text{ in } L^\infty(0,T;(W_0^{1,q}(K))^*), \\ &\mathbf{A}^\epsilon(t,e^{\frac{t}{\epsilon}M}(\mathbf{y}-\overline{N})+\overline{N}) \text{ 2-scale converges to } \mathcal{A}(t,e^{\tau M}(\mathbf{y}-\overline{N})+\overline{N}) \\ &\qquad\qquad \text{in } L^\infty(0,T;L^\infty_\theta(\mathbb{R}_\tau;W_0^{1,q}(K))), \end{aligned} \tag{4.27}$$

we can pass to the limit in (4.26) and find

(4.28)
$$\int_{\mathcal{Q}'} \theta U_0 \left[\frac{\partial \varphi}{\partial t} + \int_0^\theta e^{-\tau M} \mathcal{A}(t, \tau, e^{\tau M}(\mathbf{y} - \overline{N}) + \overline{N}) d\tau \cdot \nabla_y \varphi \right] dt d\mathbf{y} = \int_{\mathcal{O}'} u_0 \varphi(0, \cdot)\, d\mathbf{y}.$$

Noticing at last that neither U_0 nor φ depend on τ, (4.28) is nothing but a weak formulation of (1.29), proving Theorem 1.5.

Remark. In the case when $\mathbf{A}$ does not depend on ϵ, its 2-scale limit is $\frac{\mathbf{A}}{\theta}$, so that we indeed get the same result as in Theorem 1.3.

4.2. Application to the 2D Vlasov–Poisson system—proof of Theorem 1.4. In order to deduce Theorem 1.4 from Theorem 1.5 we essentially have to show (1.20) and to pass to the limit in the Poisson equation. Indeed, once those two things are proved, the theorem follows noticing that the Vlasov equation which is the first equation of (1.17) enters the generalized generic framework with

$$\mathbf{A}^\epsilon(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{E}^\epsilon(t, \mathbf{x}) \end{pmatrix} (\in \mathbb{R}^4) \text{ and } \mathbf{L}(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{v} \times \mathbf{m} \end{pmatrix} (\in \mathbb{R}^4). \tag{4.29}$$

In this case $e^{\tau M}$ becomes

$$e^{\tau M} = \begin{pmatrix} I & \mathcal{R}(\tau) \\ 0 & R(\tau) \end{pmatrix}, \tag{4.30}$$

with $\mathcal{R}(\tau)$ and $R(\tau)$ given by (1.23).

Multiplying the Vlasov equation which is the first equation of (1.17) by $(f^\epsilon)^{p-1}$ and integrating in $\mathbf{x}$ and $\mathbf{v}$ we obtain

$$\|f^\epsilon\|_{L^\infty(0,T;L^p(\mathcal{O}))} \leq C \tag{4.31}$$

for some constants C. From this estimate, we deduce the following.

LEMMA 4.3. *Under assumption* (1.19)

$$f^\epsilon \text{ 2-scale converges to } F \in L^\infty(0, T; L^\infty_{2\pi}(\mathbb{R}_\tau; L^p(\mathcal{O}))). \tag{4.32}$$

The fact that $\mathbf{E}^\epsilon$ 2-scale converges takes a bit longer to obtain. We need first to show the following lemma.

LEMMA 4.4. *Under assumption* (1.19)*, we have*

$$\|(|v|^2 f^\epsilon)\|_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C \text{ and } \|\rho^\epsilon(\mathbf{x}, t)\|_{L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{R}^3_x))} \leq C \tag{4.33}$$

for some constant C.

Proof. Multiplying the Vlasov equation by $|\mathbf{v}|^2$, and integrating with respect to $\mathbf{x}$ and $\mathbf{v}$, we get

$$\frac{d}{dt} \int_{\mathcal{O}} f^\epsilon |\mathbf{v}|^2 \, d\mathbf{v}\, d\mathbf{x} - 2 \int_{\mathbb{R}^2_x} \mathbf{J}^\epsilon \cdot \mathbf{E}^\epsilon \, d\mathbf{x} = 0, \tag{4.34}$$

where

$$\mathbf{J}^\epsilon(\mathbf{x}, t) = \int_{\mathbb{R}^2_v} \mathbf{v} f^\epsilon \, d\, \mathbf{v}. \tag{4.35}$$

Now, integrating the Vlasov equation in $\mathbf{v}$ gives the continuity equation

$$\frac{\partial \rho^\epsilon}{\partial t} + \frac{1}{\epsilon}\nabla \cdot \mathbf{J}^\epsilon = 0. \tag{4.36}$$

Using this, we obtain

$$\int_{\mathbb{R}^2_x} \mathbf{J}^\epsilon \cdot \mathbf{E}^\epsilon \, d\mathbf{x} = -\int_{\mathbb{R}^2_x} \mathbf{J}^\epsilon \cdot \nabla\phi^\epsilon \, d\mathbf{x} = \int_{\mathbb{R}^2_x} \nabla \cdot \mathbf{J}^\epsilon \, \phi^\epsilon \, d\mathbf{x} = -\epsilon \int_{\mathbb{R}^2_x} \frac{\partial \rho_\epsilon}{\partial t} \, \phi^\epsilon \, d\mathbf{x}. \tag{4.37}$$

Using now the Poisson equation, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_x} (\nabla\phi^\epsilon)^2 \, d\mathbf{x} = -\int_{\mathbb{R}^2_x} \frac{\partial}{\partial t}\Delta\phi^\epsilon \, \phi^\epsilon \, d\mathbf{x} = \int_{\mathbb{R}^2_x} \frac{\partial \rho_\epsilon}{\partial t} \, \phi^\epsilon \, d\mathbf{x}. \tag{4.38}$$

Coupling (4.37) and (4.38) yields

$$-2\int_{\mathbb{R}^2_x} \mathbf{J}^\epsilon \cdot \mathbf{E}^\epsilon \, d\mathbf{x} = \epsilon \frac{d}{dt}\int_{\mathbb{R}^2_x} (\nabla\phi^\epsilon)^2 \, d\mathbf{x}, \tag{4.39}$$

and then (4.34) reads

$$\frac{d}{dt}\left[\int_{\mathcal{O}} f^\epsilon |\mathbf{v}|^2 \, d\mathbf{v} \, d\mathbf{x} + \epsilon \int_{\mathbb{R}^2_x} (\nabla\phi^\epsilon)^2 \, d\mathbf{x}, \right] = 0, \tag{4.40}$$

and as an immediate consequence we have

$$\|(|v|^2 \, f^\epsilon)\|_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C \tag{4.41}$$

for some constant C. The first part of the lemma is then proved.

Concerning ρ^ϵ we have

$$\rho^\epsilon(\mathbf{x}, t) = \int_{\mathbb{R}^2_v} f^\epsilon \, d\mathbf{v} = \int_{|v|<R} f^\epsilon \, d\mathbf{v} + \int_{|v|>R} f^\epsilon \, d\mathbf{v} \tag{4.42}$$

for any $R > 0$. Using the Cauchy–Schwartz inequality, we have

$$\int_{|v|<R} f^\epsilon \, d\mathbf{v} \leq \left(\int_{|v|<R} (f^\epsilon)^2 \, d\mathbf{v}\right)^{\frac{1}{2}} \left(\int_{|v|<R} d\mathbf{v}\right)^{\frac{1}{2}} \leq C_1 R \left(\int_{\mathbb{R}^2_v} (f^\epsilon)^2 \, d\mathbf{v}\right)^{\frac{1}{2}} \tag{4.43}$$

and

$$\int_{|v|>R} f^\epsilon \, d\mathbf{v} \leq \int_{|v|>R} \frac{|v|^2}{R^2} f^\epsilon \, d\mathbf{v} \leq \frac{1}{R^2}\int_{\mathbb{R}^2_v} |v|^2 f^\epsilon \, d\mathbf{v}. \tag{4.44}$$

Hence, we have for any $R > 0$

$$|\rho^\epsilon(\mathbf{x}, t)| \leq C_1 R \left(\int_{\mathbb{R}^2_v} (f^\epsilon)^2 \, d\mathbf{v}\right)^{\frac{1}{2}} + \frac{1}{R^2}\int_{\mathbb{R}^2_v} |v|^2 f^\epsilon \, d\mathbf{v}. \tag{4.45}$$

Taking the R which minimizes the right-hand side we obtain

$$|\rho^\epsilon(\mathbf{x}, t)| \leq C_2 \left(\int_{\mathbb{R}^2_v} (f^\epsilon)^2 \, d\mathbf{v}\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^2_v} |v|^2 f^\epsilon \, d\mathbf{v}\right)^{\frac{1}{3}} \tag{4.46}$$

and finally

$$\int_{\mathbb{R}^2_x} |\rho^\epsilon(\mathbf{x},t)|^{\frac{3}{2}}\,d\mathbf{x} \le C_3 \int_{\mathbb{R}^2_x} \left(\int_{\mathbb{R}^2_v} (f^\epsilon)^2\,d\mathbf{v}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2_v} |v|^2 f^\epsilon\,d\mathbf{v}\right)^{\frac{1}{2}} d\mathbf{x},$$
$$\le C_3 \left(\int_{\mathbb{R}^2_x\times\mathbb{R}^2_v} (f^\epsilon)^2\,d\mathbf{x}\,d\mathbf{v}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2_x\times\mathbb{R}^2_v} |v|^2 f^\epsilon\,d\mathbf{x}\,d\mathbf{v}\right)^{\frac{1}{2}},$$

thanks to the Hölder inequality. Now, knowing that the terms on the right-hand side are bounded, we have our estimate on ρ^ϵ. Hence the proof of the lemma is ended. □

As a direct consequence of Lemma 4.4, and of the regularization properties of the Laplace operator, we deduce that $\mathbf{E}^\epsilon$ is bounded in $L^\infty(0,T;W^{1,\frac{3}{2}}(\mathbb{R}^2_x))$ and the following lemma holds true.

Lemma 4.5. *Extracting a subsequence, we have*

$$E^\epsilon \text{ 2-scale converges to } \mathcal{E} \in L^\infty(0,T;L^\infty_{2\pi}(\mathbb{R}_\tau;W^{1,\frac{3}{2}}(\mathbb{R}^2_x))). \tag{4.47}$$

Hence we proved the two facts yielding the first two equations of (1.22) with the help of Theorem 1.5.

It now remains to pass to the 2-scale limit in the Poisson equation (1.17). This can be done easily writing a weak formulation of the Poisson equation with oscillating test functions,

$$\int_{\mathbb{R}^2_x} \nabla\phi^\epsilon(t,\mathbf{x})\cdot\nabla\psi\left(t,\frac{t}{\epsilon},\mathbf{x}\right)\,dtd\mathbf{x} = \int_{\mathcal{O}} f^\epsilon(t,\mathbf{x},\mathbf{v})\psi\left(t,\frac{t}{\epsilon},\mathbf{x}\right)\,dtd\mathbf{x}d\mathbf{v}, \tag{4.48}$$

in which case we can pass to the limit and obtain, denoting Φ the 2-scale limit of ϕ^ϵ

$$\int_{\mathbb{R}^2_x}\int_0^{2\pi} \nabla\Phi\cdot\nabla\psi\,dtd\mathbf{x}d\tau = \int_{\mathcal{O}}\int_0^{2\pi} F\psi\,dtd\mathbf{x}d\mathbf{v}d\tau$$
$$= \int_{\mathcal{O}}\int_0^{2\pi} G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v},R(-\tau)\mathbf{v})\psi\,dtd\mathbf{x}d\mathbf{v}d\tau,$$

which is the weak formulation of the third equation of (1.22), achieving, in view of what is said in the beginning of the subsection, the proof of Theorem 1.4.

Remark. The deduction of the equation satisfied by the weak−∗ limit f from (1.22) is an open problem. Indeed, writing an equation for $[G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v},R(-\tau)\mathbf{v})]$ from (1.22) introduces the τ-variable in the coefficients of $\nabla_x[G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v},R(-\tau)\mathbf{v})]$ and $\nabla_v[G(t,\mathbf{x}+\mathcal{R}(-\tau)\mathbf{v},R(-\tau)\mathbf{v})]$. Hence we cannot proceed as in the linear case. Moreover, since those coefficients also depend on $\mathbf{x}$ and $\mathbf{v}$, the nonlocal homogenization methods (see Tartar [22, 23], Amirat, Hamdache and Ziani [3, 4] Frénod and Hamdache [7], Alexandre [1] ...) do not work.

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