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INTEGRAL OPERATOR APPROACH OVER OCTONIONS TO SOLUTION OF NONLINEAR PDE

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Abstract

Integration of nonlinear partial differential equations with the help of the non-commutative integration over octonions is studied. An apparatus permitting to take into account symmetry properties of PDOs is developed. For this purpose formulas for calculations of commutators of integral and partial differential operators are deduced. Transformations of partial differential operators and solutions of partial differential equations are investigated. Theorems providing solutions of nonlinear PDEs are proved. Examples are given. Applications to PDEs of hydrodynamics and other types PDEs are described.

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1. Introduction

Analysis over hypercomplex numbers develops fast and has important applications in geometry and partial differential equations including that of nonlinear (see [3-9, 23-28] and references therein). As a consequence it gives new opportunities for integration of different types of partial differential equations (PDEs). It is worth to mention that the quaternion skew field $\mathbf{H} = \mathcal{A}_2$, the octonion algebra $\mathbf{O} = \mathcal{A}_3$ and Cayley-Dickson algebras $\mathcal{A}_r$ have found a lot of applications not only in mathematics, but also in theoretical physics (see [3-8] and references therein).

This article is devoted to analytic approaches to solution of PDEs and taking into account their symmetry properties. For this purpose the octonion algebra is used. This is actual especially in recent period because of increasing interest to non-commutative analysis and its applications. It is worth to mention that each problem of PDE can be reformulated using the octonion algebra. The approach over octonions enlarges a class of PDEs which can be analytically integrated in comparison with approaches over the real field and the complex field.

We exploit a new approach based on the non-commutative integration over non-associative Cayley-Dickson algebras that to integrate definite types of nonlinear PDEs. This work develops further results of the previous article [23]. The obtained below results open new perspectives and permit to integrate nonlinear PDEs with variable coefficients and analyze symmetries of solutions as well.

In the following sections integration of nonlinear PDEs with the help of the non-commutative integration over quaternions, octonions and Cayley-Dickson algebras is studied. For this purpose formulas for calculations of commutators of integral and partial differential operators are deduced. Transformations of partial differential operators and solutions of partial differential equations are investigated. An apparatus permitting to take into account symmetry properties of PDOs is developed. Theorems providing solutions of nonlinear PDEs are proved. Examples are given. Applications to PDEs used in hydrodynamics and other types PDEs are described. The results of this paper can be applied to integration of some kinds of nonlinear Sobolev type PDEs as well.

All main results of this paper are obtained for the first time. They can be used for further investigations of PDEs and properties of their solutions. For example, generalized PDEs including terms such as Δ^p or ∇^p for $p > 0$ or even complex p can be investigated.

2. Integral Operators over Octonions

To avoid misunderstandings we first present our definitions and notations.

1. Notations and definitions

By $\mathcal{A}_r$ we denote the Cayley-Dickson algebra over the real field $\boldsymbol{R}$ with generators $i_0, \ldots, i_{2^r-1}$ so that $i_0 = 1$, $i_j^2 = -1$ for each $j \geq 1$, $i_j i_k = -i_k i_j$ for each $j \neq k \geq 1$, $2 \leq r \in \mathbf{N}$.

Henceforward PDEs are considered on a domain U in $\mathcal{A}_r^m$ satisfying conditions 2.1(D1) and (D2) [23].

2. Operators

Let X and Y be two $\boldsymbol{R}$ linear normed spaces which are also left and right $\mathcal{A}_r$ modules, where $2 \leq r$, such that

(1) $0 \leq \| ax \|_X \leq | a | \| x \|_X$ and $\| xa \|_X \leq | a | \| x \|_X$ for all $x \in X$ and $a \in \mathcal{A}_r$ and

(2) $\| x + y \|_X \leq \| x \|_X + \| Y \|_X$ for all $x, y \in X$ and

(3) $\| bx \|_X = | b | \| x \|_X = \| xb \|_X$ for each $b \in \mathbf{R}$ and $x \in X$, where for $r = 2$ and $r = 3$. Condition (1) takes the form

(1’) $0 \le \| ax \|_X = |a| \| x \|_X = \| xa \|_X$ for all $x \in X$ and $a \in \mathcal{A}_r$.

Such spaces X and Y will be called $\mathcal{A}_r$ normed spaces.

An $\mathcal{A}_r$ normed space complete relative to its norm will be called an $\mathcal{A}_r$ Banach space.

An ***R*** linear $\mathcal{A}_r$ additive operator A is called *invertible* if it is densely defined and one-to-one and has a dense range $\mathcal{R}(A)$.

Henceforward, if an expression of the form

(4) $\sum_k [(I - A_x)_k f(x, y)]_k g(y) = u(x, y)$

will appear on a domain U, which need to be inverted we consider the case when

(RS) $(I - A_x)$ is either right strongly $\mathcal{A}_r$ linear, or right $\mathcal{A}_r$ linear (see their definitions in [23]) and ${}_k f \in X_0$ for each k, or ***R*** linear and ${}_k g(y) \in \mathbf{R}$ for each k and every $y \in U$, at each point $x \in U$, since ***R*** is the center of the Cayley-Dickson algebra $\mathcal{A}_r$, where $2 \le r$.

3. First order PDOs

We consider an arbitrary first order partial differential operator σ given by the formula

(1) $\sigma f = \sum_{j=0}^{2^r - 1} i_j^* (\partial f / \partial z_{\xi(j)}) \psi_j$,

where f is a differentiable $\mathcal{A}_r$ -valued function on the domain U satisfying Conditions 1(D1, D2), $2 \le r$, $i_0, ..., i_{2^r - 1}$ are the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$, $a^* = \tilde{a} := a_0 i_0 - a_1 i_1 - \cdots - a_{2^r - 1} i_{2^r - 1}$ for each $a = a_0 i_0 + a_1 i_1 + \cdots + a_{2^r - 1} i_{2^r - 1}$ in $\mathcal{A}_r$ with $a_0, ..., a_{2^r - 1} \in \mathbf{R}$; ψ_j are real constants so that $\sum_j \psi_j^2 > 0$, $\xi : \{0, 1, ..., 2^r - 1\} \to \{0, 1, ..., 2^r - 1\}$ is a surjective bijective mapping, i.e. ξ belongs to the symmetric group S_{2^r} (see also Section 2 in [22]).

For an ordered product $\{_1 f \cdots _k f\}_{q(k)}$ of differentiable functions $_s f$ we put

(2) $${}^s\sigma\{_1 f \cdots _k f\}_{q(k)} = \sum_{j=0}^{2^r-1} i_j^* \{_1 f \cdots (\partial_s f / \partial z_{\xi(j)}) \cdots _k f\}_{q(k)} \psi_j ,$$

where a vector $q(k)$ indicates on an order of the multiplication in the curled brackets (see also Section 2 [17, 16]), so that

(3) $$\sigma\{_1 f \cdots _k f\}_{q(k)} = \sum_{s=1}^{k} {}^s\sigma\{_1 f \cdots _k f\}_{q(k)}.$$

4. Integral operators

We consider integral operators of the form:

(1) $$K(x, y) = F(x, y) + p_\sigma \int_x^\infty F(z, y) N(x, z, y) dz,$$

where σ is an $\boldsymbol{R}$-linear partial differential operator as in Section 3 and $_\sigma\int$ is the non-commutative line integral (anti-derivative operator) over the Cayley-Dickson algebra $\mathcal{A}_r$ from [22] or Subsection 4.2.5 [18], where F and K are continuous functions with values in the Cayley-Dickson algebra $\mathcal{A}_r$ or more generally in the real algebra $Mat_{n\times n}(\mathcal{A}_r)$ of $n \times n$ matrices with entries in $\mathcal{A}_r$, p is a nonzero real parameter. For definiteness we take the right $\mathcal{A}_r$ linear anti-derivative operator $_\sigma\int g(z) dz$.

Let a domain U be provided with a foliation by locally rectifiable paths $\{\gamma^\alpha : \alpha \in \Lambda\}$ (see also [22] or [18]).

5. Proposition

Let $F \in C^m(U^2, Mat_{n\times n}(\mathcal{A}_r))$ *and* $N \in C^m(U^3, Mat_{n\times n}(\mathcal{A}_r))$ *and let*

(1) $\lim_{z \to \infty} {}^1\sigma_z^k \, {}^2\sigma_x^s \, {}^2\sigma_z^l F(z, y) N(x, z, y) = 0$

for each x, y *in a domain* U *satisfying Conditions* 1 $(D1, D2)$ *with* $\infty \in U$ *and every non-negative integers* $0 \le k\ s, l \in \mathbf{Z}$ *such that* $k + s + l \le m$. *Suppose also that* ${}_\sigma\int_x^\infty \partial_x^\alpha \partial_y^\beta \partial_z^\omega [F(z, y) N(x, z, y)] dz$ *converges uniformly by parameters* x, y *on each compact subset* $W \subset U \subset \mathcal{A}_r^2$ *for each* $|\alpha| + |\beta| + |\omega| \le m$, *where* $\alpha = (\alpha_0, ..., \alpha_{2^r - 1})$, $|\alpha| = \alpha_0 + \dots + \alpha_{2^r - 1}$, $\partial_x^\alpha = \partial^{|\alpha|} / \partial x_0^{\alpha_0} \cdots \partial x_{2^r - 1}^{\alpha_{2^r - 1}}$. *Then the non-commutative line integral* ${}_\sigma\int_x^\infty F(z, y) N(x, z, y)$ *from* Section 4 *satisfies the identities*:

(2) $\sigma_x^m \, {}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz$

$= {}^2\sigma_x^m \, {}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz + A_m(F, N)(x, y),$

(3) ${}^1\sigma_z^m \, {}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz$

$= (-1)^m \, {}^2\sigma_z^m \, {}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz + B_m(F, N)(x, y),$

where

(4)

$A_m(F, N)(x, y) = - {}^2\sigma_x^{m-1} [F(x, y) N(x, z, y)]|_{z=x} + \sigma_x A_{m-1}(F, N)(x, y)$

for $m \ge 2$,

(5)

$B_m(F(z, y), N(x, z, y)) = (-1)^m \, {}^2\sigma_z^{m-1} F(x, y) N(x, z, y) + {}^1\sigma_z B_{m-1}(F(z, y), N(x, z, y))$

for $m \geq 2,\ B_m(F,\ N)(x,\ y) = B_m(F(z,\ y),\ N(x,\ z,\ y))|_{z=x}$;

(6) $A_1(F,\ N)(x,\ y) = -F(x,\ y)N(x,\ x,\ y)$,

(7) $B_1(F(z,\ y),\ N(x,\ z,\ y)) = -F(z,\ y)N(x,\ z,\ y)$,

σ_x *is an operator* σ *acting by the variable* $x \in U \subset \mathcal{A}_r$.

Proof. Using the conditions of this proposition and the theorem about differentiability of improper integrals by parameters (see, for example, Part IV, Chapter 2, Section 4 in [12]) we get the equality

$$_\sigma\int_x^\infty \partial_x^\alpha \partial_y^\beta \partial_z^\omega [F(z,\ y)N(x,\ z,\ y)]dz = \partial_x^\alpha \partial_y^\beta\ {}_\sigma\int_x^\infty \partial_z^\omega [F(z,\ y)N(x,\ z,\ y)]dz$$

for each $|\alpha| + |\beta| + |\omega| \leq m$.

In virtue of Theorems 2.4.1 and 2.5.2 [22] or 4.2.5 and 4.2.23 and Corollary 4.2.6 [18] there are satisfied the equalities

(8) $\sigma_x\ {}_\sigma\int_x^\infty g(z)dz = -g(x)$ and

(9) $_\sigma\int_{{}_0x}^x [\sigma_z f(z)]dz = f(x) - f({}_0x)$

for each continuous function g and a continuously differentiable function f, where ${}_0x$ is a marked point in U,

(10) ${}^1\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz$

$$:= \sum_{j=0}^{2^r-1} {}_\sigma\int_x^\infty \{i_j^*[(\partial F(z,\ y)/\partial z_{\xi(j)})N(x,\ z,\ y)]\psi_j\}dz \text{ and}$$

(11) ${}^2\sigma_z\ {}_\sigma\int_{{}_0x}^x F(z,\ y)N(x,\ z,\ y)dz$

$$:= \sum_{j=0}^{2^r-1} {}_\sigma\int_{0^x}^{\infty} \{i_j^*[F(z, y)(\partial N(x, z, y)/\partial z_{\xi(j)})]\psi_j\}dz \text{ and}$$

$$(12)\quad {}^2\sigma_x \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz$$

$$:= \sum_{j=0}^{2^r-1} {}_\sigma\int_x^\infty \{i_j^*[F(z, y)(\partial N(x, z, y)/\partial z_{\xi(j)})]\psi_j\}dz.$$

Therefore, from equalities (8, 9), 3(3) and 4(5) and Condition (1) we infer that:

$$(13)\quad \sigma_x \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz$$

$$= {}^2\sigma_x \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz - F(x, y)N(x, x, y),$$

since $F(z, y)N(x, z, y)|_x^\infty = -F(x, y)N(x, x, y)$, that demonstrates Formula (2) for $m = 1$ and $A_1 = -F(x, y)N(x, x, y)$. Proceeding by induction for $p = 2, ..., m$ leads to the identities:

$$(14)\quad \sigma_x^p \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz$$

$$= \sigma_x\left[{}^2\sigma_x^{p-1} \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz\right] + \sigma_x A_{p-1}(F, N)(x, z, y)$$

$$= {}^2\sigma_x^p \, {}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz$$

$$- [2\sigma_x^{p-1}F(z, y)N(x, z, y)]|_{z=x} + \sigma_x A_{p-1}(F, N)(x, y).$$

Thus (14) implies Formulas (2, 4, 6). Then with the help of Formulas (8, 9) and Condition (1) we infer also that

$$(15)\ {}^1\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz$$

$$= -{}^2\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz + F(z,\ y)N(x,\ z,\ y)|_x^\infty$$

$$= -F(x,\ y)N(x,\ x,\ y) - {}^2\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz.$$

Thus, formulas (3) for $m = 1$ and (7) are valid. Then we deduce Formulas (3, 5) by induction on $p = 2, ..., m$:

$$(16)\ {}^1\sigma_z^p\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz$$

$${}^1\sigma_z^{p-1}\left[{}^1\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz\right]$$

$$= {}^1\sigma_z^{p-1}\left[-\ {}^2\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz\right]$$

$$-\ [{}^1\sigma_z^{p-1}F(z,\ y)N(x,\ z,\ y)]|_{z=x}$$

$$= {}^1\sigma_z^{p-2}\left\{{}^1\sigma_z\left[-\ {}^2\sigma_z\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz\right]\right\}$$

$$-\ [{}^1\sigma_z^{p-1}F(z,\ y)N(x,\ z,\ y)]|_{z=x}$$

$$= {}^1\sigma_z^{p-2}\left\{(-{}^2\ \sigma_z)_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz\right\}$$

$$+\ [{}^1\sigma_z^{p-2}(\ {}^2\sigma_z F(z,\ y)N(x,\ z,\ y))]|_{z=x}$$

$$-\ [{}^1\sigma_z^{p-1}F(z,\ y)N(x,\ z,\ y)]|_{z=x}$$

$$= \cdots = (-{}^2\sigma_z)^p{}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz + B_p(F(z,\ y),\ N(x,\ z,\ y))|_{z=x}$$

and

$$B_p(F(z, y), N(x, z, y)) = -(-{}^2\sigma_z)^{p-1} F(z, y) N(x, z, y)$$

$$+{}^1\sigma_z B_{p-1}(F(z, y), N(x, z, y)).$$

6. Corollary

If suppositions of Proposition 5 *are satisfied, then*

(1)

$$A_2(F, N)(x, y) = -\sigma_x[F(x, y)N(x, x, y)] - {}^2\sigma_x[F(z, y)N(x, z, y)]|_{z=x},$$

(2) $A_3(F, N)(x, y) = -\sigma_x^2[F(x, y)N(x, x, y)]$

$- \sigma_x({}^2\sigma_x[F(x, y)N(x, z, y)]|_{z=x}) - {}^2\sigma_x^2[F(x, y)N(x, z, y)]|_{z=x},$

(3) $A_m(F, N)(x, y) = -\sum_{j=0}^{m-1} \sigma_x^j \{[{}^2\sigma_x^{m-1-j} F(z, y)N(x, z, y)]|_{z=x}\},$

(4) $B_2(F(z, y), N(x, z, y)) = -{}^1\sigma_z[F(z, y)N(x, z, y)]$

$$+ {}^2\sigma_z[F(z, y)N(x, z, y)],$$

(5) $B_3(F(z, y), N(x, z, y)) = -{}^1\sigma_z^2[F(z, y)N(x, z, y)]$

$+{}^1\sigma_z({}^2\sigma_z[F(z, y)N(x, z, y)]) - {}^2\sigma_z^2[F(z, y)N(x, z, y)],$

(6) $B_m(F(z, y), N(x, z, y))$

$= \left[\sum_{k=0}^{m-1} (-1)^{k+1}\, {}^1\sigma_z^{m-1-k}\, {}^2\sigma_z^k\right] F(z, y)N(x, z, y),$

(7) $A_2(F, N)(x, y) - B_2(F, N)(x, y) = -2\, {}^2\sigma_x[F(x, y)N(x, x, y)],$

where $\sigma_x N(x, x, y) = [\sigma_x N(x, z, y) + \sigma_z N(x, z, y)]|_{z=x},$

(8) $A_3(F, N)(x, y) - B_3(F, N)(x, y) = -(3\, {}^2\sigma_x^2 + {}^2\sigma_x\, {}^2\sigma_z +$

$2\ {}^2\sigma_z\ {}^2\sigma_x)[F(x,\ y)N(x,\ z,\ y)]|_{z=x}$

$-(2\ {}^1\sigma_x\ {}^2\sigma_x + {}^2\sigma_x\ {}^1\sigma_x)[F(x,\ y)N(x,\ x,\ y)].$

Particularly, if either p is even and $\psi_0 = 0$, *or* $F \in Mat_{n\times n}(\mathbf{R})$ *and* $N \in Mat_{n\times n}(\mathcal{A}_r)$, *then*

(8) ${}^2\sigma_x^p[F(z,\ y)N(x,\ z,\ y)] = F(z,\ y)\sigma_x^p N(x,\ z,\ y)$ *and*

$${}^2\sigma_z^p[F(z,\ y)N(x,\ z,\ y)] = F(z,\ y)\sigma_z^p N(x,\ z,\ y).$$

Proof. From formulas 5(13-16) identities (1-8) follow by induction, since

$A_m(F, N)(x, y) = -[{}^2\sigma_x^{m-1}F(z, y)N(x, z, y)]|_{z=x} + \sigma_x A_{m-1}(F, N)(x, y)$

$= \cdots = -[{}^2\sigma_x^{m-1}F(z, y)N(x, z, y)]|_{z=x} - \sigma_x\{[{}^2\sigma_x^{m-2}F(z, y)N(x, z, y)]|_{z=x}\}$

$-\sigma_x^2\{[{}^2\sigma_x^{m-3}F(z, y)N(x, z, y)]|_{z=x}\} - \cdots - \sigma_x^{m-2}\{[{}^2\sigma_x F(z, y)N(x, z, y)]|_{z=x}\}$

$-\sigma_x^{m-1}F(x,\ y)N(x,\ x,\ y)$ and

$B_m(F(z,\ y),\ N(x,\ z,\ y)) = -(-{}^2\sigma_z)^{m-1}F(z,\ y)N(x,\ z,\ y)$

$+{}^1\sigma_z B_{m-1}(F(z,\ y),\ N(x,\ z,\ y)) = \cdots =$

$-{}^1\sigma_z^{m-1}[F(z,\ y)N(x,\ z,\ y)] + {}^1\sigma_z^{m-2}({}^2\sigma_z[F(z,\ y)N(x,\ z,\ y)])$

$-{}^1\sigma_z^{m-3}({}^2\sigma_z^2[F(z, y)N(x, z, y)]) + \cdots + (-1)^m({}^2\sigma_z)^{m-1}[F(z, y)N(x, z, y)].$

Particularly when p is even and $\psi_0 = 0$, $p = 2k$, $k \in \mathbf{N}$ we get that

$$\sigma_x^p f(x) = A^k(x)$$

for p times differentiable function $F : U \to \mathcal{A}_r$, where

$Af = \sum_j b_j \partial^2 f(x)/\partial x_j^2$, $b_j = i^2_{\xi^{-1}(j)} \in \mathbf{R}$ according to Subsection 2.2 [22] or Formulas 4.2.4(7 -9) [18].

On the other hand the operators ${}^2\sigma_x^p$ and $2\sigma_z^p$ commute with the left multiplication on $F(z, y) \in Mat_{n\times n}(\mathbf{R})$, that is ${}^2\sigma_x^p[F(z, y)K(x, z)] = F(z, y)\sigma_x^p K(x, z)$ and ${}^2\sigma_z^p[F(z, y)K(x, z)] = F(z, y)\sigma_z^p K(x, z)$ for $p = 2k$, since R is the center of the Cayley-Dickson algebra $\mathcal{A}_r$.

3. Some Types of Integrable Nonlinear PDE

1. PDE

Partial differential operators L_j are considered on domains $\mathcal{D}(L_j)$ contained in suitable spaces of differentiable functions, for example, in the space $C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$ of infinitely differentiable by real variables functions on an open domain U in $\mathcal{A}_r$ and with values in $Mat_{n\times n}(\mathcal{A}_r)$, because U has the real shadow $U_\mathbf{R}$, where $n \in \mathbf{N}$.

Henceforth, if something other will not be specified, we shall take a function N may be depending on F, K and satisfying the following conditions:

(1) $N(x, y) = EK(x, y)$ with an operator E in the form

(2) $E = BST_g$,

(3) $[L_j, E] = 0$ for each j,

where B is a nonzero bounded right $\mathcal{A}_r$ linear (or strongly right $\mathcal{A}_r$ linear) operator, $S = S(x, y) \in Aut(Mat_{n\times n}(\mathcal{A}_r))$, so that B is independent of $x, y \in U$, $g \in Diff^\infty(U^2_\mathbf{R})$, $g = (g_1, g_2)$, $g_l(U^2_\mathbf{R}) = U_\mathbf{R}$ for $l = 1$ and $l = 2$, $U_\mathbf{R}$ denotes the real shadow of the domain U, $Aut(Mat_{n\times n}(\mathcal{A}_r))$ notates the automorphism group of the algebra $Mat_{n\times n}(\mathcal{A}_r)$,

(4) $T_g K(x, y) := K(g_1(x, y), g_2(x, y))$.

Condition (3) is implied by the following:

(5) $[L_j,\ B] = 0$ for each *j*,

(6) $L_{j,\,x,\,y}(T_g K(x,\ y)) = T_g(L_{j,\,x,\,y}K(x,\ y))$ and

(7) $L_{j,x,y}(S(x, y)K(x, y)) = S(x, y)(L_{j,x,y}K(x, y))$ for each *j* and each $x,\ y \in U$, where $L_{j,\,x,\,y}$ are PDOs considered below.

Evidently Conditions (6, 7) are fulfilled, when $L_{j,\,x,\,y}$ are polynomials of σ_x^k and σ_y^k, all coefficients of L_j are real and the following stronger conditions are imposed:

(8) $\sigma_x^k(T_g K(x,\ y)) = T_g(\sigma_x^k K(x,\ y)),\ \sigma_y^k(T_g K(x,\ y)) = T_g(\sigma_y^k K(x,\ y))$

and

(9) $\sigma_x^k(S(x,\ y)K(x,\ y)) = S(x,\ y)(\sigma_x^k K(x,\ y)),\ \sigma_y^k(S(x,\ y)K(x,\ y)) =$

$S(x,\ y)(\sigma_y^k K(x,\ y)),$

since $S|_{i_0\mathbf{R}} = I$.

2. General approach to solutions of nonlinear vector partial differential equations with the help of non-commutative integration over Cayley-Dickson algebras

We consider an equation over the Cayley-Dickson algebra $\mathcal{A}_r$ which is presented in the form:

(1) $K(x,\ y) = F(x,\ y) + p\ {}_\sigma\int_x^\infty F(z,\ y)N(x,\ z,\ y)dz,$

where *K*, *F* and *N* are continuous integrable functions of $\mathcal{A}_r$ variables $x, y, z \in U$ so that *F*, *K* and *N* have values in $Mat_{n\times n}(\mathcal{A}_r)$, where $n \geq 1$, $r \geq 2$, and *K* are related by $1(1,\ 2)$, $p \in \mathbf{R}\setminus\{0\}$ is a non-zero real constant. These functions *F*, *K* and *N* may depend on additional parameters $t,\ \tau,\ \cdots$. It is supposed that an operator

(2) $(I - \mathsf{A}_x E)K(x, y) = F(x, y)$ is invertible,

when $N(x, z, y) = E_y K(x, z)$ for each $x, y, z \in U$, so that $(I - \mathsf{A}_x E)^{-1}$ is continuous, where I denotes the unit operator,

(3) $\mathsf{A}_x K(x, y) := p\ {}_\sigma\int_x^\infty F(z, y)K(x, z)dz$

is an operator acting by variables x.

Then $\boldsymbol{R}$-linear partial differential operators L_k over the Cayley-Dickson algebra $\mathcal{A}_r$ are provided for $k = 1, ..., k_0$, where $k_0 \in \mathbf{N}$,

(4) $L_k f = \sum_j i_j^* (L_{k,j} f)$,

where f is a differentiable function in the domain of each operator L_k, $L_{k,j}$ are components of the operators L_k so that each $L_{k,j}$ is a PDO written in real variables with real coefficients. Next the conditions are imposed on the function F:

(5) $L_k F = 0$

for $k = 1, ..., k_0$, or sometimes stronger conditions:

(6) $\sum_{j \in \Psi_l} i_j^* [c_{k,j}(L_{k,0}F) + L_{k,j}F] = 0$

for each k and $1 \le l \le m$, where $c_{k,j}$ are constants $c_{k,j} \in \mathcal{A}_r$, $\Psi_l \subset \{0, 1, ..., 2^r - 1\}$ for each $\bigcup_l \Psi_l = \{0, 1, ..., 2^r - 1\}$, $\Psi_n \bigcap \Psi_l = \emptyset$ for each $n \ne l, 1 \le m \le 2^r$. Then with the help of Conditions either (5) or (6) we get the PDEs either

(7) $L_s[(I - \mathsf{A}_x E_y)K] = 0$ or

(8) $\sum_{j \in \Psi_k} i_j^* \{c_{k,j} L_{s,0}[(I - \mathsf{A}_x E_y)K] + L_{s,j}[(I - \mathsf{A}_x E_y)K]\} = 0$ for each

$k = 1, \ldots, m,$ respectively, for $s = 1, \ldots, k_1,$ where $k_1 \le k_0$. Hence

(9) $(I - \mathsf{A}_x E_y)(L_s K) = R_s(K)$ for $s = 1, \ldots, k_1,$ where

(10) $R_s(f) = (I - \mathsf{A}_x E_y)(L_s f) - L_s[(I - \mathsf{A}_x E_y) f]$.

The latter can be realized when

(11) $R_s(K) = (I - \mathsf{A}_x E_y) M_s(K)$ for $s = 1, \ldots, k_1,$

where $M_s(K)$ are operators or functionals acting on K. Therefore due to Condition (2) the function K must satisfy the PDEs or the partial integro-differential equations (PIDEs)

(12) $L_s K - M_s(K) = 0$ for $s = 1, \ldots, k_1$

which generally may be non-***R***-linear.

Henceforward, if something other will not be outlined, we consider the variants:

(13) $F, K, N \in Mat_{n\times n}(\mathcal{A}_r)$ with $2 \le r \le 3$ and B is the strongly right $\mathcal{A}_r$ -linear operator; or

(14) $F \in Mat_{n\times n}(\mathbf{R})$ and $K, N \in Mat_{n\times n}(\mathcal{A}_r)$ with $2 \le r$ and B is the right $\mathcal{A}_r$ -linear operator (see also Section 1), where $1 \le n \in \mathbf{N}$.

3. Theorem

Suppose that conditions of Proposition 2.5 *and* 2.2(RS) *are fulfilled over the Cayley-Dickson algebra* $\mathcal{A}_r$ *with* $2 \le r$ *and on a domain U satisfying Conditions* $2.1(D1, D2)$ *for the corresponding terms of operators* L_s *for all* $s = 1, \ldots, k_0$ *so that*

(1) *the appearing in the terms* $M_s(K)$ *integrals uniformly converge by parameters on compact sub-domains in U and*

(2) $\lim_{z\to\infty} \partial_x^{\alpha}\partial_y^{\beta}\partial_z^{\omega}[F(z, y)N(x, z, y)] = 0$

the limit converges uniformly by $x, y \in U \setminus V$ *for some compact subset V in U and for each* $|\alpha|+|\beta|+|\omega| \le m,$ *where* $1 \le m = \max\{\deg(L_s) : s = 1, ..., k_0\}$ *and*

(3) *the operator* $(I - \mathsf{A}_x E_y)$ *is invertible, where F is in the domain of PDOs* $L_1, ..., L_{k_0}, F(x, y) \in Mat_{n\times n}(\mathcal{A}_r)$ *and* $K(x, y) \in Mat_{n\times n}(\mathcal{A}_r), n \in \mathbf{N}$.

Then there exists a solution K of PDEs or PIDEs 2(12) *such that K is given by Formulas* 1(1, 2), 2(1) *and either* 2(5) *or* 2(6).

Proof. The anti-derivative operator $g \mapsto {}_\sigma\int_{{}_0x}^{x} g(z)dz$ is compact from $C^0(V, \mathcal{A}_r)$ into $C^0(V, \mathcal{A}_r)$ for a compact domain V in $\mathcal{A}_r$ where $C^0(V, \mathcal{A}_r)$ is the Banach space over $\mathcal{A}_r$ of all continuous functions $g : V \to \mathcal{A}_r$ supplied with the supremum norm $\| g \| := \sup_{x\in V} | g(x)|,$ ${}_0x$; is a marked point in $V,$ ${}_0x \in V.$ A function F satisfying the system of R linear PDEs 2(5) or 2(6) is continuous.

Therefore, due to conditions (1-3) the anti-derivative operator ${}_\sigma\int_x^\infty F(z, y)N(x, z, y)dz$ is compact. Hence there exists $\delta > 0$ such that the operator $I - \mathsf{A}_x E_y$ is invertible when $| p | < \delta,$ where $p \in \mathbf{R}\setminus\{0\}.$ Mention that the operator T_g is strongly left and right $\mathcal{A}_r$-linear (see Section 1), while S is the automorphism of the Cayley-Dickson algebra, that is $S[ab] = S[a]S[b]$ and $S[a + b] = S[a] + S[b]$ for each $a, b \in \mathcal{A}_r.$

Since the operator $(I - \mathsf{A}_x E_y)$ is invertible and Conditions 2.2(RS) and either 2(13) or 2(14) are satisfied, then equation 2(12) can be resolved:

(4) $\sum_k {}_k f(x, y)_k g(y) = (I - \mathsf{A}_x E_y)^{-1} u(x, y),$

since if $A : X \to X$ is a bounded R linear operator on a Banach space X with the norm $\| A \| < 1,$ then the inverse of $I - A$ exists:

$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$. Applying Proposition 2.5 and Section 2 we get the statement of this theorem.

4. Remark

If Condition 2.2(RS) is not fulfilled, the corresponding system of PDEs in real components $(\mathsf{A}_x E_y)_{j,s}$, ${}_k f_s$ and ${}_k g_s$ can be considered.

5. Lemma

Let suppositions of Proposition 2.5 *be satisfied and the operator* A_x *be given by Formula* 2(3), *let also* $E = E_y$ *may be depending on the parameter* $y \in U$ *and let* $N(x, z, y) = E_y K(x, z)$ (*see Formulas* 1(1-4)). *Suppose that* $F(x, y) \in Mat_{n\times n}(\mathbf{R})$ *and* $K(x, z) \in Mat_{n\times n}(\mathsf{A}_r)$ *for each* $x, y, z \in U$, *where* $r \geq 2$. *Then*

(1) $A_m(F, E_y K)(x, y) = (I - \mathsf{A}_x E_y)\hat{A}_m(K, E_y K)(x, y)$

$+ P_m(K, E_y K)(x, y),$

(2) $B_m(F, E_y K)(x, y) = (I - \mathsf{A}_x E_y)\hat{B}_m(K, E_y K)(x, y)$

$+ Q_m(K, E_y K)(x, y),$

where

(3) $\hat{A}_{m,x,y}(K(z, y), E_y K(x, z))|_{z=x} = \hat{A}_m(K, E_y K)(x, y) =$

$$- \sum_{j=0}^{m-1} \sigma_x^j K_{1, m-j-1}(x, y)$$

$$+ p \sum_{j=1}^{m-1} \sum_{j_1=0}^{j-1} \sigma_x^{j_1} K_{2, m-j-1, j-j_1-1}(x, y)$$

$$+ p^2 \sum_{j=1}^{m-1} \sum_{j_1=1}^{j-1} \sum_{j_2=0}^{j_1-1} \sigma_x^{j_2} K_{3, m-j-1, j-j_1-1, j_1-j_2-1}(x, y) + \cdots$$

$+\ p^{m-2}K_{m-1,0,\dots,0}(x,\ y),$

(4)

$$\hat{B}_m(K(z,\ y),\ E_yK(x,\ z)) = \sum_{j=1}^{m}(-1)^j\{{}^1\sigma_z^j[K(z,\ y)\sigma_v^{j-1}(E_yK(w,\ v))]$$

$$+\ p\hat{A}_{m-j,z,y}(K(z,\ y),\ E_yK(x,\ z)(\sigma_v^{j-1}E_yK(w,\ v)))\}|_{v=z,\,w=x},$$

(5) $\hat{B}_m(K,\ E_yK)(x,\ y) := \hat{B}_m(K(z,\ y),\ E_yK(x,\ z))|_{z=x},$

(6) $\hat{A}_1(K,\ E_yK)(x,\ y) = -K(z,\ y)(E_yK(x,\ z))|_{z=x},$

(7) $\hat{B}_1(K(z,\ y),\ E_yK(x,\ z)) = -K(z,\ y)(E_yK(x,\ z))$

for each $m \ge 2$ *in* (3, 4), *where* $\sum_{j=l}^{m} a_j := 0$ *for all* $l > m$,

(8) $K_{1,j}(x,\ y|z) := K(x,\ y)\sigma_x^j(E_yK(x,\ z)),$

(9) $K_{m,l_1,\dots,l_m}(x,\ y|z) := K(x,\ y)[\sigma_x^{l_m}E_yK_{m-1,l_1,\dots,l_{m-1}}(x,\ z)],$

(10) $K_{m,l_1,\dots,l_m}(x,\ y) := K_{m,l_1,\dots,l_m}(x,\ y|z)|_{z=x},$

(11) $P_m(K(z,\ y),\ E_yK(x,\ z))|_{z=x} = P_m(K,\ E_yK)(x,\ y)$

$$:= \mathsf{A}_x\Big\{\sum_{j=1}^{m-1}[\sigma_x^j,\ E_y]K_{1,m-j-1}(x,\ y) + p\sum_{j=1}^{m-1}\sum_{j_1=1}^{j-1}[\sigma_x^{j_1},\ E_y]K_{2,m-j-1,j-j_1-1}(x,\ y)$$

$$+ \cdots + p^{m-3}\sum_{j=1}^{m-1}\sum_{j_1=1}^{j-1}\dots\sum_{j_{m-3}=1}^{j_{m-4}-1}[\sigma_x^{j_{m-3}},\ E_y]$$

$$K_{m-2,m-j-1,j-j_1-1,\dots,j_{m-4}-j_{m-3}-1}(x,\ y)\Big\},$$

(12) $Q_m(K, EK)(x, y|z)$

$$:= \sum_{j=1}^{m-1} (-1)^j P_{m-j}(K(\eta, y), E_y K(z, \eta))(\sigma_v^{j-1} E_y K(w, v))|_{\eta=z, v=z, w=x},$$

(13) $Q_m(K,\ EK)(x,\ y) := Q_m(K,\ EK)(x,\ y|z)|_{\ z=x}$.

Proof. Formulas (6) and (7) follow immediately from that of 2.2(2, 3) and 2.5(6, 7). Write A_m for each $m \geq 2$ in the form:

(14)

$$A_m(K,\ E_y K)(x,\ y) = -\sum_{j=1}^{m} \sigma_x^{j-1}\{[\,^2\sigma_x^{m-j} F(z,\ y)(E_y K(x,\ z))]|_{\ z=x}\},$$

where $\sigma^0 = I$. Using that $F(z, y) = (I - \mathsf{A}_z E_y) K(z, y)$ and $F(z, y) \in Mat_{n\times n}(\mathbf{R})$ we get from (14):

(15)

$$A_m(F, E_y K)(x, y) = -\sum_{j=1}^{m} \sigma_x^{j-1}\{[(I - \mathsf{A}_x E_y) K(x, y)(\sigma_x^{m-j} E_y K(x, z))]|_{\ z=x}\}.$$

In virtue of Proposition 2.5 we deduce from (12) that

(16)

$$A_m(F,\ E_y K)(x,\ y) = -(I - \mathsf{A}_x E_y)\left\{\sum_{j=1}^{m} \sigma_x^{j-1} K_{1,\, m-j}(x,\ y)\right\}$$

$$+ p\sum_{j=2}^{m} A_{j-1}(K(z, y), E_y K_{1, m-j}(x, z))|_{\ z=x} + \mathsf{A}_x \sum_{j=1}^{m-1} [\sigma_x^j, E_y] K_{1, m-j-1}(x, y)$$

$$= -(I - \mathsf{A}_x E_y)\left\{\sum_{j=0}^{m-1} \sigma_x^j K_{1,\, m-j-1}(x,\ y) + p\sum_{j=1}^{m-1}\sum_{j_1=0}^{j-1} \sigma_x^{j_1} K_{2,\, m-j-1,\, j-j_1-1}(x,\ y)\right\}$$

$$+ \mathsf{A}_x \sum_{j=1}^{m-1} [\sigma_x^j, E_y] K_{1, m-j-j}(x, y) + p\mathsf{A}_x \sum_{j=1}^{m-1} \sum_{j_1=1}^{j-1} [\sigma_x^{j_1}, E_y] K_{2, m-j-1, j-j_1-1}(x, y)$$

$$+ p^2 \sum_{j=1}^{m-1} \sum_{j_1=1}^{j-1} A_{j_1}(F(z, y), E_y K_{2, m-j-1, j-j_1-1}(x, z))|_{z=x} = \cdots,$$

since $[\sigma_x^j, E_y] + E_y \sigma_x^j = \sigma_x^j E_y$ for $j \geq 1$,

$A_x E_y K(x, y) = p\ {}_{\sigma}\int_x^{\infty} F(z, y) E_y K(x, z) dz$. Iterating relations (13) we infer by induction Formulas (1, 3, 11). Then we have

(17)

$$B_m(F(z, y), E_y K(x, z)) = \sum_{j=1}^{m} (-1)^{j\ 1} \sigma_z^{m-j\ 2} \sigma_z^{j-1} F(z, y)(E_y K(x, z))$$

and for $F(z, y) \in Mat_{n\times n}(\mathbf{R})$ for each $z, y \in U$ this reduces to:

(18)

$$B_m(F(z, y), E_y K(x, z)) = \sum_{j=1}^{m} (-1)^{j\ 1} \sigma_z^{m-j} F(z, y)(\sigma_z^{j-1}(E_y K(x, z)))$$

$$= \sum_{j=1}^{m} (-1)^{j_1} \sigma_z^{m-j} (I - \mathsf{A}_z E_y) K(z, y)(\sigma_v^{j-1}(E_y K(w, v))|_{v=z, w=x}).$$

Therefore, in view of Proposition 2.5 and Formula (1) the identity

(19)

$$B_m(F(z, y), E_y K(x, z)) = (I - A_z E_y) \left\{ \sum_{j=1}^{m} (-1)^j \{^1 \sigma_z^{m-j} F(z, y)(\sigma_z^{j-1}(E_y K(w, z))) \right.$$

$$+p\hat{A}_{m-j,z,y}(K(\eta, y), E_yK(z, \eta)(\sigma_v^{j-1}E_yK(w, v)))\Big\}\Big|_{\eta=z, v=z, w=x}$$

$$+\sum_{j=1}^{m-1}(-1)^j P_{m-j}(K(\eta, y), E_yK(z, \eta)(\sigma_v^{j-1}E_yK(w, v))|_{\eta=z, v=z, w=x}$$

is valid, since

$$A_zE_yK(z, y)\{\sigma_v^{j-1}(E_yK(w, v))\}|_{v=z, w=x}$$

$$= p\ {}_\sigma\!\int_z^\infty F(\eta, y)\{E_yK(z, \eta)\{\sigma_v^{j-1}(E_yK(w, v))\}\}d\eta|_{v=z, w=x}.$$

6. Proposition

Suppose that

(1) *a PDO* L_j *is a polynomial* $\Omega_j(\sigma_x, \sigma_y)$ *of* σ_x *and* σ_y *for each* $j = 1, \dots, k_0$, *coefficients of* Ω_j *are real and Condition* 1(3) *is fulfilled for all* j;

(2) $L_{s,x,y}K(x, y) - (\mathsf{A}_xE_yL_{s,x,y})K(x, y) =: R_s(K)(x, y)$ *for each* $s = 1, \dots, k_1$; $F(x, y)$ *is in* $Mat_{n\times n}(\mathbf{R})$ *and* $K(x, y) \in Mat_{n\times n}(\mathcal{A}_r)$ *for each* $x, y \in U$ (*see* 2(3)), σ *and* A *are over the Cayley-Dickson algebra* $\mathcal{A}_r$, $2 \le r$, $n \in \mathbf{N}$, $1 \le k_1 \le k_0$;

(3) $L_{j,x,y}F(x, y) = 0$ *for every* $x, y \in U$ *and* $j = 1, \dots, k_0$.

Then there exists a polynomial M_s *of* K, E, σ *and* A *such that*

(4) $R_s(K)(x, y) = (I - \mathsf{A}_xE_y)M_s(K)(x, y)$ *for each* $s = 1, \dots, k_1$.

Proof. Proposition 2.5 and Corollary 2.6 imply that $R_s(K)(x, y)$ can be expressed as a polynomial of $A_m(F, EK)$, $B_m(F, EK)$, σ and $\mathsf{A}EK$, where $m \in \mathbf{N}$.

Take an algebra $\mathcal{B}$ over the real field generated by the operators σ, A, E and I:

$$\mathcal{B} = alg_{\mathbf{R}}(\sigma_x, \sigma_y, \sigma_z, \mathsf{A}_x, \mathsf{A}_y, \mathsf{A}_z, E_y(x, z), \quad \forall x, y, z \in U; I),$$

where I denotes the unit operator. In view of Proposition 3.1 [23], Theorems 2.4.1 and 2.5.2 [22] the algebra $\mathcal{B}$ is associative, since $F(z, y)$ is in $Mat_{n\times n}(\mathbf{R})$ for each $z, y \in U$, the algebra $Mat_{n\times n}(\mathbf{R})$ is associative, also E is given by 1(2, 4). Therefore, there exists the Lie algebra $\mathcal{L}(\mathcal{B})$ generated from $\mathcal{B}$ with the help of commutators $[H, G] := HG - GH$ of elements $H, G \in \mathcal{B}$ (see also about abstract algebras of operators and their Lie algebras in [36]). Then $Y_s := [L_s, E]\mathcal{L}(\mathcal{B})$ is the (two-sided) ideal in $\mathcal{L}(\mathcal{B})$ and hence there exist the quotient algebra $\mathcal{L}_s : \mathcal{L}(\mathcal{B})/Y_s$ and the quotient morphism $\pi_s : \mathcal{L}(\mathcal{B}) \to \mathcal{L}_s$.

Next consider the universal enveloping algebra $\mathcal{U}$ of the Lie algebra $\mathcal{L}(\mathcal{B})$. In virtue of Proposition 2.1.1 [2] there exists a unique homomorphism τ from $\mathcal{U}$ into $\mathcal{B}$. The algebra $C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$ over the real field also has the structure of the left module of the operator ring $\mathcal{B}$ and hence of $\mathcal{L}(\mathcal{B})$ and $\mathcal{U}$ as well, where $C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$ denotes the algebra of all infinitely differentiable functions from $U_{\mathbf{R}}$ into $Mat_{n\times n}(\mathcal{A}_r)$ (see Section 1). Since $C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$ is dense in $C^l(U, Mat_{n\times n}(\mathcal{A}_r))$, then it is sufficient to consider $C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$, where $l = \max_{s=1,\dots,k_0} ord(L_s)$. On the other hand, $[L_s, E]\mathcal{U} =: \mathcal{U}_s$ is the (two-sided) ideal in $\mathcal{U}$.

Let $\mathcal{P}(x, y)$ denote the R-linear algebra generated by sums and products of all terms QP so that P are polynomials of functions $K \in C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))$ and Q are acting on them polynomials of operators σ, A, E (or B, S, T_g instead of E, since $E = BST_g$), where coefficients of P and Q are chosen to be real, since coefficients of each polynomial Ω_j are real. Certainly the equality $\alpha T = T\alpha$ is valid for each $T \in Mat_{n\times n}(\mathbf{R})$ and $\alpha \in \mathcal{A}_r$, since $\boldsymbol{R}$ is the center of the Cayley-Dickson algebra $\mathcal{A}_r$, $2 \le r$ and $(\alpha T)_{i,j} = \alpha T_{i,j} = T_{i,j}\alpha = (T\alpha)_{i,j}$ for each (i, j) matrix element $(T\alpha)_{i,j}$ of αT.

The polynomial $R_s(K)(x, y)$ is calculated with the help of Conditions (3), where $s = 1, \ldots, k_1$ (see also Section 2). From formula (2) it follows that the polynomial $R_s(K)(x, y)$ belongs to $(I - \mathsf{A}_x E_y)\mathcal{P}(x, y)$ $+\sum_{j=1}^{k_0}\{\mathcal{U}_j C^\infty(U, Mat_{n\times n}(\mathcal{A}_r))\}$ for each $s = 1, \ldots, k_0$. Applying the quotient mapping π_j for all $j = 1, \ldots, k_0$ and using Proposition 2.3.3 [2] we get Formula (4), since $\pi_j(\mathcal{U}_j) = 0$ and Condition 1(3) is imposed for each $j = 1, \ldots, k_0$.

7. Example

Take two partial differential operators

(1) $L_1 = L_{1,x,y} := \sum_l a_l((-\sigma_x)^l - \sigma_y^l)$ and

(2) $L_2 = L_{2,x,y} := \sum_l a_l(\sigma_x^l - (-\sigma_y)^l)$,

where $a_l \in \mathbf{R}$ for each l when σ is over $\mathcal{A}_r$ with $r \ge 2$, the sum is finite or infinite, $l \in \mathbf{N}$. The functions $F(x, y)$ and $K(x, y)$ of $\mathcal{A}_r$ variables $x, y \in U$ have values in $Mat_{n\times n}(\mathbf{R})$ and $Mat_{n\times n}(\mathcal{A}_r)$ respectively, where $n \ge 1$, $r \ge 2$. A domain U in $\mathcal{A}_r$ satisfies conditions 2.1 $(D1, D2)$ with $\infty \in U$. On a function $F(x, y)$ are imposed two conditions:

(3) $L_{1,x,y}F(x, y) = 0$ and

(4) $L_{2,x,y}F(x, y) = 0$.

Suppose that conditions of Proposition 2.5 are fulfilled and

(5) $K(x,\ y) = F(x,\ y) + p_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz,$

where p is a non-zero real parameter, $N(x,\ z) = EK(x,\ z)$ for each $x,\ z \in U$, while E is a bounded right $\mathcal{A}_r$-linear operator satisfying Conditions 1(2-4) and either 2(13) or 2(14).

Condition (3) is equivalent to

(6) $\sum_l a_l (-\sigma_x)^l F(x,\ y) = \sum_l a_l \sigma_y^l F(x,\ y)$ and (4) to

(7) $\sum_l a_l \sigma_x^l F(x, y) = \sum_l a_l (-1)^l \sigma_y^l F(x, y) = 0$

correspondingly. Acting on both sides of the equality (5) by the operator L_1 and using (6) and Proposition 2.5 we get

(8)

$$L_{1, x, y} K(x,\ y) = p \sum_l a_l ((-\sigma_x)^l - (-{}^1\sigma_z)^l)_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz$$

$$= p \sum_l a_l ((-{}^2\sigma_x)^l - {}^2\sigma_z^l)_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz$$

$$+ p \sum_l a_l (-1)^l (A_l(F;\ N)(x,\ y) - B_l(F;\ N)(x,\ y))$$ and hence

(9) $L_{1, x, y} K(x,\ y) = p\ {}^2L_{1, x, z\sigma} \int_x^\infty F(z,\ y) N(x,\ z) dz$

$+ p \sum_l (-1)^l a_l (A_l(F;\ N)(x,\ y) - B_l(F;\ N)(x,\ y)).$

Then from (5, 7) we infer that

(10) $L_{2, x, y} K(x,\ y) = p \sum_l a_l (\sigma_x^l - {}^1\sigma_z^l)_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz$

$$= p\sum_l a_l({}^2\sigma_x^l - (-{}^2\sigma_x)^l)_\sigma \int_x^\infty F(z, y)N(x, z)dz$$

$+ p\sum_l a_l(A_l(F; N)(x, y) - B_l(F; N)(x, y))$ and consequently,

$$(11)\quad L_{2,x,y}K(x, y) = p\, {}^2L_{2,x,z\sigma}\int_x^\infty F(z, y)N(x, z)dz$$

$$+ p\sum_l a_l(A_l(F; N)(x, y) - B_l(F; N)(x, y)).$$

Then Equalities (9; 11) imply that

$$(12)\quad (L_{1,x,y} \pm L_{2,x,y})K(x, y) = p({}^2L_{1,x,z} \pm {}^2L_{2,x,z})_\sigma \int_x^\infty F(z, y)N(x, z)dz$$

$$+ p\sum_l a_l(\pm 1 + (-1)^l)(A_l(F; N)(x, y) - B_l(F; N)(x, y)).$$

We take into account sufficiently small values of the parameter p, when the operator $I - \mathsf{A}_x E$ is invertible, for example, $\| \mathsf{A}_x E \| < 1$, where

$\mathsf{A}_x K(x, y) = p_\sigma \int_x^\infty F(z, y)K(x, z)dz$. In the case $F \in Mat_{n\times n}(\mathbf{R})$ and $K \in Mat_{n\times n}(\mathcal{A}_r)$ with σ over $\mathcal{A}_r$, from (5, 12), Lemma 5 and Proposition 6 it follows that K satisfies the nonlinear PDE

(13)

$$L^\pm_{x,y}K(x, y) - p\sum_l (\pm 1 + (-1)^l)a_l[\hat{A}_l(K; EK)(x, y) - \hat{B}_l(K; EK)(x, y)] = 0,$$

where

$$(14)\quad L^\pm_{x,y} = \sum_l (\pm 1 + (-1)^l)a_l(\sigma_x^l - \sigma_y^l),$$

$\hat{A}_l$ and $\hat{B}_l$ are given by formulas 5(3-7), since $\langle a, b, c\rangle = 0$ when particularly $a \in Mat_{n\times n}(\mathbf{R})$, where $\langle e, b, c\rangle = (eb)c - e(bc)$ denotes the associator of the Cayley-Dickson matrices $e, b, c \in Mat_{n\times n}(\mathcal{A}_r)$, also since

$\alpha a = a\alpha$ for each $\alpha \in \mathcal{A}_r$. A solution of (13) reduces to linear PDEs and is prescribed by (3-5). Equivalently the function K satisfies also the PDEs

(15)

$$L_{s,x,y}K(x, y) - p\sum_l (-1)^{sl} a_l[\hat{A}_l(K; EK)(x, y) - \hat{B}_l(K; EK)(x, y)] = 0$$

for $s = 1$ and $s = 2$. Instead of this system it is possible also to consider separately PDOs L_1 and L_2 and the corresponding PDEs for F and K as well. Thus with the help of Theorem 3 we get the following.

7.1. Theorem

Suppose that conditions of Theorem 3 *and Example* 7 *are fulfilled, then a solution of PDE* (15) *is given by* (3, 5), *where* L_1 *is prescribed by Formula* (1), $s = 1$.

8. Example

Let PDOs be

(1) $L_1 = L_{1;x,y} = \sigma_x - \sigma_y$,

(2) $L_{2,j} = L_{2,j;x,y} = \sum_l (a_l\sigma_x^l + (-1)^{jl} b_l\sigma_y^l)$,

where $a_l, b_l \in \mathbf{R}$ for each l when σ is over $\mathcal{A}_m$ with $m \geq 2$, the sum is finite or infinite, $j = 1$ or $j = 2$. It is also supposed that the functions $F(x, y)$ and $K(x, y)$ of $\mathcal{A}_m$ variables $x, y \in U$ have values in $Mat_{n\times n}(\mathbf{R})$ and $Mat_{n\times n}(\mathcal{A}_m)$ correspondingly, where $n \geq 1$, $m \geq 2$. A domain U in $\mathcal{A}_m$ satisfies Conditions $2.1(D1, D2)$ with $\infty \in U$. Suppose that

(3) $L_{1;x,y}F(x, y) = 0$ and consider the integral relation:

(4) $K(x, y) = F(x, y) + p_\sigma\int_x^\infty F(z, y)N(x, z)dz$,

where N and K are related by Formulas 1(1, 2).

Using condition (3) we can write $F(x,\ y) = F\left(\frac{x+y}{2}\right)$. Therefore we deduce that

(5)

$$L_{2,\ j;\ x,\ y}K(x,\ y) = L_{2,\ j;\ x,\ y}F\left(\frac{x+y}{2}\right) + pL_{2,\ j;\ x,\ y\ \sigma}\int_x^\infty F\left(\frac{z+y}{2}\right)N(x,\ z)dz$$

$$= L_{2,\ j;\ x,\ y}F\left(\frac{x+y}{2}\right) + p\sum_l (a_l\sigma_x^l + (-1)^{jl}b_l\ {}^1\sigma_z^l)_\sigma\int_x^\infty F\left(\frac{z+y}{2}\right)N(x,\ z)dz$$

$$= L_{2,\ j;\ x,\ y}F\left(\frac{x+y}{2}\right) + p\sum_l \left\{\left[(a_l\ {}^2\sigma_x^l + (-1)^{(j+1)l}b_l\ {}^2\sigma_z^l)_\sigma\int_x^\infty F\left(\frac{z+y}{2}\right)N(x,\ z)dz\right]\right.$$

$$\left. + a_lA_l(F;\ N)(x,\ y) + (-1)^{jl}b_lB_l(F;\ N)(x,\ y)\right\}.$$

Imposing the condition

(6) $L_{2;\ x,\ y}F(x,\ y) = 0,$ where

(7) $L_{2;\ x,\ y} = L_{2,1;\ x,\ y} + L_{2,2;\ x,\ y} = \sum_l (2a_l\sigma_x^l + (1+(-1)^l)b_l\sigma_y^l),$ we get the nonlinear PDE with the help of Lemma 5 and Proposition 6

(8)

$$L_{2;\ x,\ y}K(x,\ y) - p\sum_l \{2a_l\hat{A}_l(K;\ N)(x,\ y) + (1+(-1)^l)b_l\hat{B}_l(K;\ N)(x,\ y)\} = 0,$$

since the center of the Cayley-Dickson algebra $\mathcal{A}_m$ is the real field R and so the commutator of bI and ${}_\sigma\int$ is zero, $[bI,\ {}_\sigma\int] = 0,$ also $(bI)(FK) = F(bIK) = bFK,$ when b is a real constant. Making the variable change $y \mapsto -y$ one gets the PDO $\sigma_x + \sigma_y$ instead of $\sigma_x - \sigma_y$ and the corresponding changes in the PDO L_2. Then Theorem 3 implies the following.

8.1. Theorem

Let conditions of Theorem 3 *and Example* 8 *be fulfilled, then a solution of PDE* (8) *is described by formulas* (3), (4), (6), *where PDOs* L_1 *and* L_2 *are provided by expressions* (1, 7).

9. Example

Consider now the generalization of PDOs from Section 5 with $k \geq 2$:

(1) $L_1 = L_{1;\,x,\,y} = \sigma_x^k - \sigma_y^k,$

(2) $L_{2,\,y} = L_{2,\,j;\,x,\,y} = \sum_l (a_l \sigma_x^{kl} + (-1)^{jkl} b_l \sigma_y^{kl}),$ where k is a natural number, $a_l,\, b_l \in \mathbf{R}$ for each l when the Dirac type operator σ is over $\mathcal{A}_m$ with $m \geq 2,$ the sum is finite or infinite, $j = 1$ or $j = 2.$ Other suppositions are as in Section 8. Let

(3) $L_{1;\,x,\,y} F(x,\, y) = 0$ and

(4) $K(x,\, y) = F(x,\, y) + p_\sigma \int_x^\infty F(z,\, y) N(x,\, z) dz,$

where N is expressed through K by 1(1, 2).

Then we deduce the identities:

(5) $L_{2,\,j;\,x,\,y} F(x,\, y) = L_{2,\,j;\,x,\,y} F(x,\, y) + p L_{2,\,j;\,x,\,y\sigma} \int_x^\infty F(z,\, y) N(x,\, z) dz$

$$= L_{2,\,j;\,x,\,y} F(x,\, y) + p \sum_l (a_l \sigma_x^{kl} + (-1)^{jkl} b_l\ {}^1\sigma_z^{kl})_\sigma \int_x^\infty F(z,\, y) N(x,\, z) dz$$

$$= L_{2,\,j;\,x,\,y} F(x,\, y) + p \sum_l \left\{ \left[(a_l\ {}^2\sigma_x^{kl} + (-1)^{(j+1)kl} b_l\ {}^2\sigma_z^l)_\sigma \int_x^\infty F(z,\, y) N(x,\, z) dz \right] \right.$$

$$\left. + a_l \hat{A}_{kl}(F;\, N)(x,\, y) + (-1)^{jkl} b_l \hat{B}_{kl}(F;\, N)(x,\, y) \right\}.$$

From the condition

(6) $L_{2;\, x,\, y}F(x,\ y) = 0$ with the PDO

(7) $L_{2;\, x,\, y} = L_{2,1;\, x,\, y} + L_{2,\, 2;\, x,\, y} = \sum_l (2a_l\sigma_x^{kl} + (1 + (-1)^{kl})b_l\sigma_y^{kl})$

we infer that a function K is a solution of the nonlinear PDE of the form:

(8)

$$L_{2;\, x,\, y}K(x,\ y) - p\sum_l \{2a_l\hat{A}_{kl}(K;\ N)(x,\ y) + (1 + (-1)^{kl})b_l\hat{B}_{kl}(K;\ N)(x,\ y)\} = 0.$$

9.1. Theorem

Let conditions of Theorem 3 *and Example* 9 *be satisfied, then a solution of PDE* (8) *is provided by Formulas* (3), (4), (6), *where PDOs* L_1 *and* L_2 *are given by* (1, 7).

10. Example

Let now the pair of PDOs be

(1) $L_1 = L_{1;\, x,\, y} = \sigma_x^k + \sigma_y^k,$

(2) $L_{2,\, j} = L_{2,\, j;\, x,\, y} = \sum_l (a_l\sigma_x^{kl} + (-1)^{lj(k+1)}b_l\sigma_y^{kl}),$

where k is a natural number, $k \ge 2,$ $K,$ $F,$ U and σ have the same meaning as in Sections 1 and 2, $a_l,\, b_l \in \mathbf{R}$ for each l when σ are over $\mathcal{A}_m$ with $m \ge 2,$ $F \in Mat_{n\times n}(\mathbf{R})$ and $K \in Mat_{n\times n}(\mathcal{A}_r),$ the sum is finite or infinite, $j = 1$ or $j = 2.$ Imposing the conditions

(3) $L_{1;\, x,\, y}F(x,\ y) = 0$ and

(4) $L_{2;\, x,\, y}F(x,\ y) = 0$ with

(5) $L_{2;\, x,\, y} = L_{2,1;\, x,\, y} + L_{2,\, 2;\, x,\, y} = \sum_l (2a_l\sigma_x^{kl} + (1 + (-1)^{l(k+1)})b_l\sigma_y^{kl})$

and considering the integral transform

$$(6)\ \ K(x,\ y) = F(x,\ y) + p_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz,$$

where N is related with K by expressions 1(1, 2), we infer that

$$(7)\ \ L_{2,\ j;\, x,\, y} K(x,\, y) = L_{2,\ j;\, x,\, y} F(x,\, y) + p L_{2,\ j;\, x,\, y\sigma} \int_x^\infty F(z,\, y) N(x,\, z) dz$$

$$= L_{2,\ j;\, x,\, y} F(x,\ y) + p \sum_l (a_l \sigma_x^{kl} + (-1)^{l(j+1)+jkl} b_l\ {}^1\sigma_z^{kl})_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz$$

$$= L_{2,\ j;\, x,\, y} F(x,\ y) + p \sum_l \left\{ \left[(a_l\ {}^2\sigma_x^{kl} + (-1)^{l(j+1)(k+1)} b_l\ {}^2\sigma_z^{l})_\sigma \int_x^\infty F(z,\ y) N(x,\ z) dz \right] \right.$$

$$\left. + a_l A_{kl}(F;\ N)(x,\ y) + (-1)^{l(j+1)+jkl} b_l B_{kl}(F;\ N)(x,\ y) \right\}.$$

Thus in virtue of Lemma 5 and Proposition 6:

(8)

$$L_{2;\, x,\, y} K(x,\, y) - p \sum_l \{2a_l \hat{A}_{kl}(K;\, N)(x,\, y) + ((-1)^l + (-1)^{kl}) b_l \hat{B}_{kl}(K;\, N)(x,\, y)\} = 0.$$

10.1. Theorem

If conditions of Theorem 3 *and Example* 10 *are satisfied, then a solution of PDE* (8) *is given by Formulas* (3, 4, 6), *where PDOs* L_1 *and* L_2 *are as in* (1, 5).

10.2. Remark

Transformation groups related with the quaternion skew field are described in [31]. Automorphisms and derivations of the quaternion skew field and the octonion algebra are contained in [36], that of Lie algebras and groups in [6].

11. Example

Consider now the term N in the integral operator

(1)

$$f(y)(g(x)K(x, y)) = K(x, y) + p_\sigma \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz$$

with multiplier functions $f(z)$ and $g(x)$ satisfying definite conditions (see below), where F, K and $N(x, z) = f(z)(g(x)EK(x, z))$, p have the meaning of the preceding paragraphs, E is an operator fulfilling Conditions 1(2, 3) and either 2(13) or 2(14) also. Suppose that

(2) $\sigma_z f(z) = \sum_j i_j \psi_j \partial f(z)/\partial z_{\xi(j)} = \lambda f(z)$ and

(3) $\sigma_x g(x) = \sum_j i_j \psi_j \partial g(x)/\partial x_{\xi(j)} = \mu g(x),$ where

$\lambda = \sum_j i_j \psi_j \lambda_j$ and

$\mu = \sum_j i_j \psi_j \mu_j$ with $\lambda_j, \mu_j \in \mathbf{R}$ for each j. We choose the functions $f(z) = C_1 \exp\left(\sum_j z_j \lambda_j\right)$ and $g(x) = C_2 \exp\left(\sum_j x_j \mu_j\right)$ satisfying PDEs (2) and (3) correspondingly, where C_1 and C_2 are real non-zero constants, $x_j, z_j \in R$, $x = \sum_j i_j x_j$, $x, z \in U$. The first PDO we take as

(4) $L_1 = L_{1, x, y} = \sigma_x^k + s\sigma_y^k,$

where $k \geq 1$, either $s = 1$ or $s = -1$. Then the condition

(5) $L_{1, x, y} F(x, y) = 0$ is equivalent to

(6) $\sigma_x^k F(x, y) = -s\sigma_y^k F(x, y).$

Therefore we get from Proposition 2.5 with $N(x, z) = f(z)(g(x)EK(x, z))$ that

$$(7)\ \sigma_y^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz = (-s)^{l\ 1}\sigma_z^{kl}$$

$$\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$= s^l(-1)^{l(k+1)}[{}^2\sigma_z + {}^4\sigma_z]^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$+ (-s)^l B_{kl}(F(z,\ y);\ [f(z)(g(x)EK(x,\ z))])|_{z=x},$$

where F stands on the first place, f on the second, g on the third and (EK) on the fourth place. Then from (2) and (7) it follows that

$$(8)\ \sigma_y^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$= s^l(-1)^{l(k+1)}[{}^4\sigma_z + \lambda]^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$+ (-s)^l B_{kl}(F(z,\ y);\ [f(z)(g(x)EK(x,\ z))])|_{z=x}.$$

Evaluation of the other integral with the help of Proposition 2.5 and Formula (3) leads to:

$$(9)\ \sigma_y^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$= [{}^3\sigma_x + {}^4\sigma_x]^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$+ A_{kl}(F(z,\ y);\ [f(z)(g(x)EK(x,\ z))])|_{z=x},$$

$$= [{}^4\sigma_x + \mu]^{kl}\sigma\int_x^\infty F(z,\ y)[f(z)(g(x)EK(x,\ z))]dz$$

$$+ A_{kl}(F(z,\ y);\ [f(z)(g(x)EK(x,\ z))])|_{z=x}.$$

Thus in this particular case PDEs of Examples 7-10 change. For example, PDE 10(8) takes the form:

$$(10)\ \sum_l (2a_l(\sigma_x + \mu)^{kl} + (1 + (-1)^{l(k+1)})b_l(\sigma_y + \lambda)^{kl})K(x,\ y)$$

$$- \frac{p}{f(y)g(x)} \sum_l \{2a_l \hat{A}_{kl}([f(y)(g(z)K(z,\ y))];\ [f(z)(g(x)EK(x,\ z))])|_{z=x}$$

$$+ (1 + (-1)^{l(k+1)})b_l \hat{B}_{kl}([f(y)(g(z)K(z,\ y))];\ [f(z)(g(x)EK(x,\ z))])|_{z=x}\} = 0,$$

when $F \in Mat_{n\times n}(\mathbf{R})$ and $K \in Mat_{n\times n}(\mathcal{A}_r)$ with $2 \le r$ and E is the right linear operator over $\mathcal{A}_r$, since the operator E satisfies Conditions 1(2, 3) and either 2(13) or 2(14); the functions $f(y)$ and $g(x)$ have values in $\mathbf{R}\setminus\{0\}$ for each $x, y \in U$, whilst R is the center of the Cayley-Dickson algebra. Analogous changes will be in Examples 7-9.

12. Example

Let the non-commutative integral operator be

(1) $K(x,\ y) = F(x,\ y) + \mathsf{B}_x K(x,\ y)$ with

(2) $\mathsf{B}_x K(x,\ y) = p_\sigma \int_x^\infty F(z,\ y) N(x,\ z,\ y) dz,$

where $F, K, N(x, z, y)$ are as in Proposition 2.5 and Theorem 3, $F \in Mat_{n\times n}(\mathbf{R})$, K and N are in $Mat_{n\times n}(\mathcal{A}_m)$, while p is a sufficiently small non-zero real parameter, N is an operator function right linear in K as in Section 1. Put

(3) $N(x,\ z,\ y) = E_y K(x,\ z)$ for every x, y and z in U,

where $[E_y,\ L_j] = 0$ for each $j = 1, ..., k_0$ and $y \in U$, $E = E_y$ may depend on the variable $y \in U$ also, E is an operator satisfying Conditions 1(2, 3) and either 2(13) or 2(14), $m \ge 2.$ Choose two PDO

(4) $L_1 = L_{1,x,y} = \sigma_x + \sigma_y,$

(5) $L_2 = L_{2,x,y} = \left(\sum_l a_l \sigma_x^l\right) + s\sigma_y,$

where $s \in \mathbf{R}$, s is a non-zero real constant, $a_l \in \mathbf{R}$ for each l when the Dirac type operator σ is over $\mathcal{A}_m$ with $m \geq 2$. We impose the conditions:

(6) $L_{j,x,y}F(x, y) = 0$ for $j = 1$ and $j = 2$, for all $x, y \in U$. Then it is possible to write $F(z, y) = F\left(\frac{z-y}{2}\right)$. Applying the PDO L_2 to both sides of (1) and using (2), Proposition 2.5 and Conditions (3-6) we deduce that

(7)

$$L_{2,x,y}K(x, y) = p\left\{\sum_l a_l\sigma_x^l - s^1\sigma_z + s^2\sigma_y\right\}_\sigma \int_x^\infty F(z, y)N(x, z, y)dz$$

$$= \left[p\left\{\sum_l a_l^2\sigma_x^l + s^2\sigma_z + s^2\sigma_y\right\}_\sigma \int_x^\infty F(z, y)N(x, z, y)dz\right]$$

$$+ \left[p\sum_l a_l A_l(F; N)(x, y)\right] - psB_1(F; N)(x, y).$$

For sufficiently small non-zero real values of p the operator $I - \mathsf{B}_x$ is invertible and hence Equality (7), Lemma 5 and Proposition 6 imply that K satisfies the nonlinear partial integro-differential equation:

(8) $$L_{2,x,y}K(x, y) - \left[p\sum_l a_l \hat{A}_l(K; N)(x, y)\right] - psK(x; y)(x, x, y)$$

$$- ps_\sigma \int_x^\infty K(z, y)\sigma_y N(x, z, y)dz = 0.$$

12.1. Theorem

A solution of PIDE (8) *is described by* (1, 2, 3, 6), *where* PDOs L_1 *and*

L_2 *are given by* (4, 5), *provided that conditions of Theorem* 3 *and Example* 12 *are satisfied*.

13. Example

Suppose that functions K and F are related by equations 12(1, 2) and take two PDOs

(1) $L_{1,x,y} = \sigma_x - \sigma_y$ and

(2) $L_{2,x,y} = \Delta_x + s\Delta_y,$

where the coefficient ψ_0 is null in σ and hence the Laplace operator is expressed as $\Delta = -\sigma^2,$ while $s \in \mathbf{R}\setminus\{0\}$. Now we take a function N in the form

(3) $N(x, z, y) = EK(x, ay + bz),$

where a and b real parameters to be calculated below such that $a^2 + b^2 > 0,$ b is non-zero. Then from the conditions

(4) $L_{j,x,y}F(x, y) = 0$ for $j = 1$ and $j = 2,$

Proposition 2.5 and Corollary 2.6 it follows that

(5) $L_{2,x,y}K(x, y) = -p(\sigma_x^2 + s\sigma_y^2)_\sigma \int_x^\infty F(z, y)EK(x, ay + bz)dz$

$= p({}^2\Delta_x - s({}^1\sigma_y + {}^2\sigma_y)^2)_\sigma \int_x^\infty F(z, y)EK(x, ay + bz)dz$

$- pA_2(F(z, y), EK(x, ay + bz))|_{z=x}$

and

(6) $({}^1\sigma_y + {}^2\sigma_y)^2{}_\sigma \int_x^\infty F(z, y)EK(x, ay + bz)dz$

$$= [{}^1\sigma_z^2 + ab^{-1}\,{}^1\sigma_z\,{}^2\sigma_z + ab^{-1}\,{}^2\sigma_z\,{}^1\sigma_z + a^2b^{-2}\,{}^2\sigma_z^2]_\sigma\int_x^\infty F(z, y)EK(x, ay+bz)dz$$

$$= [ab^{-1}({}^1\sigma_z + {}^2\sigma_z)^2 + (1-ab^{-1})^1\sigma_z^2 + (a^2b^{-2} - ab^{-1})^2\sigma_z^2]_\sigma\int_x^\infty F(z, y)EK(x, ay+bz)dz$$

$$= (1-ab^{-1})^2\,{}^2\sigma_{z\sigma}^2\int_x^\infty F(z, y)EK(x, ay+bz)dz$$

$$-ab^{-1}[\sigma_z(F(z, y)EK(x, ay+bz))]|_{z=x} + (1-ab^{-1})B_2(F, EK)(x, y)$$

$$= p^{-1}(1-ab^{-1})^2b^2\mathsf{B}_x(\sigma_y^2K(x, y))$$

$$-ab^{-1}[\sigma_z(F(z, y)EK(x, ay+bz))]|_{z=x} + (1-ab^{-1})B_2(F, EK)(x, y),$$

since

$$\sigma_{z\sigma}^2\int_x^\infty F(z, y)N(x, z, y)dz = {}_\sigma\int_x^\infty \sigma_z^2[F(z, y)N(x, z, y)]dz$$

$$= -\sigma_z[(F(z, y)EK(x, ay+bz))]|_{z=x}.$$

Then identities (5; 6) imply that

(7) $L_{2,x,y}K(x, y) = \mathsf{B}_x[L_{2,x,y}K(x, y)] - pA_2(F(z, y), EK(x, ay+bz))|_{z=x}$

$$+ psab^{-1}[\sigma_z(F(z, y)EK(x, ay+bz))]|_{z=x} - ps(1-ab^{-1})B_2(F, EK)(x, y)$$

when $(b-a)^2 = 1$ and b is non-zero, that is either $a = b+1$ or $a = b-1$. In virtue of Lemma 5 and Proposition 6 this gives the nonlinear PDE for K.

(8) $L_{2,x,y}K(x, y) + p\hat{A}_2(K(z, y), EK(x, ay+bz))|_{z=x}$

$- ab^{-1}ps[\sigma_z(K(z, y)EK(x, ay+bz))]|_{z=x} + (1 -$

$ab^{-1})\,ps\hat{B}_2(K(z, y), EK(x, ay+bz))|_{z=x} = 0.$

13.1. Theorem

A solution of PDE (8) *is given by* (3, 4) *and* 12(1, 2), *where PDOs* L_1 *and* L_2 *are prescribed by* (1, 2), *whenever conditions of Theorem* 3 *and Example* 13 *are satisfied*.

14. Nonlinear PDE with parabolic terms

Let

(1) $\partial_t := \sum_{k=1}^{v} \partial/\partial t_k$

be the first order PDO, where $t_1, \ldots, t_v$ are real variables independent of other variables $x, y, z \in U$, $t = (t_1, \ldots, t_v) \in W$, $W := \{t \in \mathbf{R}^v : \forall k = 1, \ldots, v\ 0 \leq t_k < T_k\}$, where T_k is a constant, $0 < T_k \leq \infty$ for each k.

Suppose that

(2) F and K are continuously differentiable functions by t_k for each k so that ${}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz$ converges for some $t \in W$ and

(3) the integrals ${}_\sigma\int_x^\infty (\partial_t F(z, y)) N(x, z, y) dz$ and

$${}_\sigma\int_x^\infty F(z, y)(\partial_t N(x, z, y)) dz$$

converge uniformly on W in the parameter t.

In virtue of the theorem about differentiation of an improper integral by a parameter the equality is valid:

(4) $\partial_t {}_\sigma\int_x^\infty F(z, y) N(x, z, y) dz$

$$ {}_{\sigma}\int_{x}^{\infty} (\partial_t F(z, y)) N(x, z, y) dz + {}_{\sigma}\int_{x}^{\infty} F(z, y)(\partial_t N(x, z, y)) dz. $$

Using (4) the commutator $(I - \mathsf{A}_x E_y)((\partial_t + L_s) f) - (\partial_t + L_s)[(I - \mathsf{A}_x E_y) f]$ can be calculated, when there is possible to evaluate the commutator $(I - \mathsf{A}_x E_y)(L_s f) - L_s[(I - \mathsf{A}_x E_y) f] = R_s(f)$ for suitable functions f and a PDO $L_s = L_{s, x, y}$ (see also Section 2).

14.1. Example

Let a PDO be

(1) $L_1 = \partial_t + \sum_l a_l(\sigma_x^l + (-1)^{l+1}\sigma_y^l)$ and let

(2) $N(x, z, y) = E_y K(x, z)$,

where $a_l \in \mathbf{R}$ for all $l = 0, 1, 2, \ldots,$ so that conditions 1(2, 3) and 2(1, 5) are fulfilled, $F \in Mat_{n\times n}(\mathbf{R})$, $K \in Mat_{n\times n}(\mathcal{A}_r)$, $2 \le r$, the first order PDO σ is over the Cayley-Dickson algebra $\mathcal{A}_r$ (see Sections 1, 2 and 14). Then

$$ \left(\partial_t + \sum_l a_l(-1)^{l+1}\sigma_y^l\right) F(x, y) = -\left(\sum_l a_l \sigma_x^l\right) F(x, y) $$

for all $x, y \in U$. Therefore, we infer from Proposition 2.5 that

(3) $$ L_1 K(x, y) = p\left({}^2\partial_t + \sum_l a_l(\sigma_x^l - {}^1\sigma_z^l)\right) {}_{\sigma}\int_x^{\infty} F(z, y) E_y N(x, z) dz $$

$$ = p\left({}^2\partial_t + \sum_l a_l({}^2\sigma_x^l - (-1)^{l+1}\, {}^2\sigma_z^l)\right) {}_{\sigma}\int_x^{\infty} F(z, y) E_y N(x, z) dz $$

$$ + p\sum_l a_l(A_l(F; N)(x, y) - B_l(F; N)(x, y)). $$

Hence we deduce a nonlinear PDE

(4) $L_1 K(x, y) - p\sum_l a_l(\hat{A}_l(K; EK)(x, y) - \hat{B}_l(K; EK)(x, y)) = 0.$

Its solution reduces to the linear problem 2(2, 5). We mention that PDE (4) corresponds to some kinds of Sobolev type nonlinear PDEs.

14.2. Generalized approach

Let $L_1, \ldots, L_k$ and $S_1, \ldots, S_k$ be PDOs which are polynomials or series of σ_x and σ_y so that

(1) $[L_j, S_j] = 0$

for each $j = 1, \ldots, k,$ where x and y are in a domain U in the Cayley-Dickson algebra $\mathcal{A}_r, 2 \le r$ (see Subsection 2.3). Instead of the conditions $L_j F = 0$ it is possible to consider more generally

(2) $L_j F = G_j,$ where G_j are some functions known or defined by some relations, while functions F, G_j and K may also depend on a parameter $t \in W$ (see Section 14) so that $F \in Mat_{n\times n}(\mathbf{R}), G_j$ for all j and K have values in $Mat_{n\times n}(\mathcal{A}_r).$ It is also supposed that F and K are related by the integral equation 2(1) and $N(x, y, z) = E_y K(x, z)$ and Conditions 1(2, 3) are satisfied. In particular, if

(3) $G_j = L_j(I + S_j)K,$ then a solution of the linear system of PIDEs

(4) $L_j F(x, y) = L_j(I + S_j)K(x, y)$ and

(5) $(I - \mathsf{A}_x E)K(x, y) = F(x, y)$

would also be a solution of nonlinear PIDEs

(6) $S_j L_j K(x, y) + M_j(K) = 0,$

where M_j corresponds to L_j for each $j = 1, \ldots, k_0$ with $1 \le k_0 \le k$ as in Section 2. Thus, this generalizes PIDEs 2(12).

15. Theorem

Let $\{L_s : s = 1, ..., k_0\}$ *be a set of PDOs which are polynomials* $\Omega_s({}_1\sigma_x, {}_2\sigma_y)$ over $\mathcal{A}_r$ *or* $\mathbf{R}$. *Let also G be the family of all operators* $E = BST_g$ *satisfying the condition* $[L_s, E] = 0$ *for each* $s = 1, ..., k_0$, *where* $B \in SL_n(\mathbf{R})$, $S \in Aut(Mat_{n\times n}(\mathcal{A}_r))$, $g \in Diff^{\infty}(U)$, T_g *is prescribed by Formula* 1(4), ${}_1\sigma_x$ *and* ${}_2\sigma_y$ *are over the Cayley-Dickson algebra* $\mathcal{A}_r$, $r \geq 2$. *Then the family G forms the group and there exists an embedding of G into* $SL_n(\mathbf{R}) \times Aut(Mat_{n\times n}(\mathcal{A}_r)) \times Diff^{\infty}(U)$.

Proof. The composition (set theoretic) in the family G of the aforementioned operators is associative. Then the inverse $E^{-1} = T_g^{-1}S^{-1}B^{-1}$ of $E = BST_g$ exists, since B, S and T_g are invertible for every $B \in SL_n(R)$, $S \in Aut(Mat_{n\times n}(\mathcal{A}_r))$ and $g \in Diff^{\infty}(U)$ so that $T_g^{-1} = T_{g^{-1}}$. On the other hand, the identity $E^{-1}[L_s, E]E^{-1} = -[L_s, E^{-1}]$ is valid. Thus, the equality $[L_s, E] = 0$ implies that L_s and E^{-1} commute, $[L_s, E^{-1}] = 0$, as well. Therefore, from $E \in G$ the inclusion $E^{-1} \in G$ follows. The identity $[L_s, E_1E_2] = [L_s, E_1]E_2 + E_1[L_s, E_2]$ implies that $E_1E_2 \in G$ whenever $E_1 \in G$ and $E_2 \in G$.. Thus the family G has the group structure. There exists the bijective correspondence between diffeomorphisms $g \in Diff^{\infty}(U)$ and operators T_g acting on functions defined on U with values in $Mat_{n\times n}(\mathcal{A}_r)$ according to Formula 1(4). Each element E in G is of the form $E = BST_g$, where $B \in SL_n(\mathbf{R})$, $S \in Aut(Mat_{n\times}(\mathcal{A}_r))$, $g \in Diff^{\infty}(U)$, consequently, an embedding $\omega : G \hookrightarrow SL_n(\mathbf{R}) \times Aut(Mat_{n\times n}(\mathcal{A}_r)) \times Diff^{\infty}(U)$ exists.

4. Nonlinear PDEs Used in Hydrodynamics

1. Remark

In the previous article [23] vector hydrodynamical PDEs were investigated. Using results of Sections 2 and 3 we generalize the approach using transformations of functions by operators E of the form 3.1(2).

2. Example

Generalized Korteweg-de-Vries' type PDE. Let

(1) $N(x, z, y) = EK(x, z)$ as in 3.1(1) and let A_x be given by 3.2 (3), where E satisfies conditions 3.1(2, 4),

(2) $L_1 = {}_1\sigma_x^2 - {}_2\sigma_y^2$ and

(3) $L_2 = {}_3\sigma_t + {}_1\sigma_x^3 + 3\,{}_2\sigma_y\,{}_1\sigma_x^2 + 3\,{}_2\sigma_y^2\,{}_1\sigma_x + {}_2\sigma_y^3$,

where ${}_1\psi_0 = {}_2\psi_0 = 0$,

(4) $L_1 F = 0$ and $L_{2,j} F = 0$ for each $j = 0, \ldots, 2^{r-1}$.

Taking into account symmetry operators E and transforming corresponding equations from example 4.2 [23] we get the equality

(5) $({}_3\sigma_t + {}_1\sigma_x^3 + 3\,{}_2\sigma_y\,{}_1\sigma_x^2 + 3\,{}_2\sigma_y^2\,{}_1\sigma_x + {}_2\sigma_y^3)K(x, y)$

$+ 6({}_1^1\sigma_x + {}_2^1\sigma_y)[K(x, y)({}_1\sigma_x EK(x, x))] - K(x, y)\{[{}_1\sigma_z, {}_1\sigma_x]EK(x, z)|_{z=x}\}$

$- [{}_1^1\sigma_x, {}_2^1\sigma_x][K(x, y)EK(x, x)] = 0$

follows, when the operator $(I - \mathsf{A}_x E)$ is invertible.

2.1. Theorem

If suppositions of Theorem 3.3 *and Example* 2 *are satisfied*. *Then a solution of PDE* (5) *with* ${}_1\psi_0 = {}_2\psi_0 = 0$ *over the Cayley-Dickson algebra* $\mathcal{A}_r$ *with* $2 \le r \le 3$ *is given by Formulas* (2-4) *and* 3.2(1), *when* $p = 1$.

2.2. Example

Korteweg-de-Vries' type PDE. Continuing Example 2 mention that on the diagonal $x = y$ the operators are: $L_{1,x,x} = 0$, $L_{2,x,x} = \partial/\partial t + 8\,{}_1\sigma_x^3$. Therefore, $[L_{1,x,x}, E] = 0$ is valid. Let E be independent of the parameter t, then $[\partial/\partial t, E] = 0$, since $t \in \mathbf{R}$ and $\boldsymbol{R}$ is the center of the Cayley-Dickson algebra $\mathcal{A}_r$. To the term ${}_1\sigma_x^3$ the cubic form $(\operatorname{Im} w)^3 = -|w|^2 w$ corresponds, since ${}_1\psi_0 = 0$, where $\operatorname{Im} w = (w - w^*)/2$, $w = i_1 x_1 \psi_1 + \cdots + i_{2^r-1} x_{2^r-1} \psi_{2^r-1}$, $x_j \in \mathbf{R}$ for each j. That is for $E = ST_g$ the restriction is $[|w|^2 w, E] = 0$, where $n = 1$ and $B = 1$. Geometrically in the real shadow of $\operatorname{Im}(\mathcal{A}_r)$ such $E = E(x)$ permits any rotations along the axis J_w parallel to w such that J_w crosses the origin of the coordinate system. Evidently $[w^3, E] = 0$ is satisfied if $[w, E] = 0$, that is $[{}_1\sigma_x, E(x)] = 0$. In the latter case and when $n = 1$, ${}_1\sigma = {}_2\sigma$, ${}_1\psi_0 = 0$ and ${}_3\sigma_t = \partial/\partial t_0$ the differentiation of 2(12) with the operator ${}_1\sigma_x$ and the restriction on the diagonal $x = y$ provides the PDE

(1) $v_t(t, x) + 6\,{}_1\sigma_x[v(t, x)Ev(t, x)] + {}_1\sigma_x^3 v(t, x) = 0$

of Korteweg-de-Vries' type, where $v(t, x) = 2\,{}_1\sigma_x K(x, x)$. Particularly there are solutions having the symmetry property $Ev(t, x) = v(t, x)$.

3. Example

Non-isothermal flow of a non-compressible Newtonian liquid with a dissipative heating. Take the pair of PDOs

(1) $L_1 = \sigma_x + \sigma_y$ and

(2) $L_2 = {}_1\sigma_t + \sigma_x^2 + q\sigma_y\sigma_x + \sigma_y^2$,

where $q \in \mathbf{R}$ is a real constant, and consider the integral equation 3.2(1) with N of the form 3.1(1), so that

(3) $L_1 F(x, y) = 0$ and

(4) $L_{2,j} F(x, y) = 0$ for each j, (see also (4.81) and (4.82) in [23]). Therefore, in (4.83) [23] the term K changes into EK. Transforming the corresponding equations from [23] with the help of the operator E we deduce that

(5) $({}_1\sigma_t + \sigma_x^2 + 2\,{}^2\sigma_y\,{}^2\sigma_x + \sigma_y^2)K(x, y) = -2pK(x, y)[\sigma_x EK(x, x)]$,

where K depends on the parameter t.

Let $g(x, t) = K(x, x)$, then on the diagonal $x = y$ this implies the PDE:

(6) $({}_1\sigma_t + \sigma_x^2)g(x, t) = -2pg(x, t)[\sigma_x Eg(x, t)]$.

Therefore $K(x, y) = K\left(\frac{\langle\psi, x - y\rangle}{2}\right)$ and $[L_1, E]K = 0$ is fulfilled if $E(x, y) = E\left(\frac{\langle\psi, x - y\rangle}{2}\right)$. For E independent of t the condition $[L_{2,x,x}, E(x, x)] = 0$ means that $[L_{2,x,x}, E(0)] = 0$. Thus in the real shadow of $\mathrm{Im}(\mathcal{A}_r)$ this $E(0)$ induces any element of the orthogonal group $O(2^r - 1)$.

3.1. Theorem

Suppose that conditions of Theorem 3.3 *and Example* 3 *are satisfied, then PDE* (5) *over the Cayley-Dickson algebra* $\mathcal{A}_r$ *with* $2 \le r \le 3$ *has a solution given by Formulas* (3, 4), 3.1(1) *and* 3.2(1), *where PDOs* L_1 *and* L_2 *are given by* (1,2), $F \in Mat_{n\times n}(\mathbf{R})$ *and* $K \in Mat_{n\times n}(\mathcal{A}_r)$, $n \in \mathbf{N}$ *for* $r = 2$, $n = 1$ *for* $r = 3$.

5. Conclusion

The results of this paper can be applied for analysis and solution of nonlinear PDE mentioned in the introduction and for dynamical nonlinear processes [14, 15] and air target range radar measurements [40].

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