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# TWO-SCALE NUMERICAL SIMULATION OF SAND TRANSPORT PROBLEMS

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[1]

Abstract. In this paper we consider the model built in [3] for short term dynamics of dunes in tidal area. We construct a Two-Scale Numerical Method based on the fact that the solution of the equation which has oscillations Two-Scale converges to the solution of a well-posed problem. This numerical method uses on Fourier series.

## 1. Introduction.

This paper deals with numerical simulations of sand transport problems. Its goal is to build a Two-Scale Numerical Method to simulate dynamics of dunes in tidal area.
This paper enters a work program concerning the development of Two-Scale Numerical Methods to solve PDEs with oscillatory singular perturbations linked with physical phenomena. In Ailliot, Frénod and Monbet [2], such a method is used to manage the tide oscillation for long term drift forecast of objects in coastal ocean waters. Frénod, Mouton and Sonnendrücker [5] made simulations of the 1D Euler equation using a Two-Scale Numerical Method. In Frénod, Salvarani and Sonnendrücker [6], such a method is used to simulate a charged particle beam in a periodic focusing channel. Mouton [9, 10] developped a Two-Scale Semi Lagrangian Method for beam and plasma applications.

We consider the following model, valid for short-term dynamics of dunes, built and studied in [3]:

$$\begin{cases} \dfrac{\partial z^\epsilon}{\partial t} - \dfrac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon \nabla z^\epsilon) = \dfrac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon, \\ z^\epsilon_{|t=0} = z_0, \end{cases} \tag{1.1}$$

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where $z^\epsilon = z^\epsilon(t,x)$ is the dimensionless seabed altitude. For a given $T$, $t \in (0,T)$ stands for the dimensionless time and $x \in \mathbb{T}^2$, $\mathbb{T}^2$ being the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, stands for the dimensionless position and $\mathcal{A}^\epsilon$, $\mathcal{C}^\epsilon$ are given by

$$\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}^\epsilon(t,x) + \epsilon \widetilde{\mathcal{A}}_1^\epsilon(t,x), \tag{1.2}$$

and

$$\mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}^\epsilon(t,x) + \epsilon \widetilde{\mathcal{C}}_1^\epsilon(t,x), \tag{1.3}$$

where, for three positive constants $a$, $b$ and $c$,

$$\widetilde{\mathcal{A}}^\epsilon(t,x) = \widetilde{\mathcal{A}}(t,\frac{t}{\epsilon},x) = a\, g_a(|\mathcal{U}(t,\frac{t}{\epsilon},x)|), \tag{1.4}$$

$$\widetilde{\mathcal{C}}^\epsilon(t,x) = \widetilde{\mathcal{C}}(t,\frac{t}{\epsilon},x) = c\, g_c(|\mathcal{U}(t,\frac{t}{\epsilon},x)|)\,\frac{\mathcal{U}(t,\frac{t}{\epsilon},x)}{|\mathcal{U}(t,\frac{t}{\epsilon},x)|}, \tag{1.5}$$

and

$$\widetilde{\mathcal{A}}_1^\epsilon(t,x) = \widetilde{\mathcal{A}}_1(t,\frac{t}{\epsilon},x),\ \widetilde{\mathcal{C}}_1^\epsilon(t,x) = \widetilde{\mathcal{C}}_1(t,\frac{t}{\epsilon},x), \tag{1.6}$$

with

$$\widetilde{\mathcal{A}}_1(t,\theta,x) = -ab\mathcal{M}(t,\theta,x)\, g_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \widetilde{\mathcal{C}}_1(t,\theta,x) = -cb\mathcal{M}(t,\theta,x)\, g_c(|\mathcal{U}(t,\theta,x)|)\,\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}. \tag{1.7}$$

$\mathcal{U}$ and $\mathcal{M}$ are the dimensionless water velocity and height.
The small parameter $\epsilon$ involved in the model is the ratio between the main tide period $\frac{1}{\omega} = 13$ hours and an observation time which is about three months i.e. $\epsilon = \frac{1}{t\omega} = \frac{1}{200}$.
The following hypotheses on $g_a$, $g_c$, $\mathcal{U}$ and $\mathcal{M}$ given in (1.8) and (1.9) are technical assumptions and are needed to prove Theorem 1.1. Functions $g_a$ and $g_c$ are regular functions on $\mathbb{R}^+$ and satisfy

$$\begin{cases} g_a \geq g_c \geq 0,\ g_c(0) = g_c'(0) = 0, \\ \exists\, d \geq 0, \sup_{u\in\mathbb{R}^+} |g_a(u)| + \sup_{u\in\mathbb{R}^+} |g_a'(u)| \leq d, \\ \sup_{u\in\mathbb{R}^+} |g_c(u)| + \sup_{u\in\mathbb{R}^+} |g_c'(u)| \leq d, \\ \exists\, U_{thr} \geq 0,\ \exists\, G_{thr} > 0, \text{ such that } u \geq U_{thr} \Longrightarrow g_a(u) \geq G_{thr}. \end{cases} \tag{1.8}$$

Functions $\mathcal{U}$ and $\mathcal{M}$ are regular and satisfy:

$$\begin{cases} \theta \longmapsto (\mathcal{U},\mathcal{M}) \text{ is periodic of period } 1, \\ |\mathcal{U}|,\ |\frac{\partial \mathcal{U}}{\partial t}|,\ |\frac{\partial \mathcal{U}}{\partial \theta}|,\ |\nabla \mathcal{U}|, \\ |\mathcal{M}|,\ |\frac{\partial \mathcal{M}}{\partial t}|,\ |\frac{\partial \mathcal{M}}{\partial \theta}|,\ |\nabla \mathcal{M}| \text{ are bounded by } d, \\ \forall\,(t,\theta,x) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^2,\ |\mathcal{U}(t,\theta,x)| \leq U_{thr} \Longrightarrow \\ \Big(\frac{\partial \mathcal{U}}{\partial t}(t,\theta,x) = 0,\ \nabla\mathcal{U}(t,\theta,x) = 0, \\ \frac{\partial \mathcal{M}}{\partial t}(t,\theta,x) = 0, \text{ and } \nabla\mathcal{M}(t,\theta,x) = 0\Big), \\ \exists\, \theta_\alpha < \theta_\omega \in [0,1] \text{ such that } \forall\, \theta \in [\theta_\alpha,\theta_\omega] \Longrightarrow |\mathcal{U}(t,\theta,x)| \geq U_{thr}. \end{cases} \tag{1.9}$$

To develop the Two-Scale Numerical Method, we use that in [3] we proved that under assumptions (1.8) and (1.9) the solution $z^\epsilon$ of (1.1) exists, is unique and moreover asymptotically behaves, as $\epsilon \to 0$, the way given by the following theorem.

**Theorem 1.1.** *Under assumptions (1.8) and (1.9), for any $T$, not depending on $\epsilon$, the sequence $(z^\epsilon)$ of solutions to (1.1), with coefficients given by (1.2) coupled with (1.4) and (1.3), (1.5) and (1.6), Two-Scale converges to the profile $Z \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ solution to*

$$\frac{\partial Z}{\partial \theta} - \nabla\cdot(\widetilde{\mathcal{A}}\nabla Z) = \nabla\cdot\widetilde{\mathcal{C}}, \tag{1.10}$$

*where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ are given by*

$$\widetilde{\mathcal{A}}(t,\theta,x) = a\, g_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \widetilde{\mathcal{C}}(t,\theta,x) = c\, g_c(|\mathcal{U}(t,\theta,x)|)\,\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}. \tag{1.11}$$

*Futhermore, if the supplementary assumption*

$$U_{thr} = 0, \tag{1.12}$$

*is done, we have*

$$\widetilde{\mathcal{A}}(t,\theta,x) \geq \widetilde{G}_{thr} \textit{ for any } t,\theta,x \in [0,T]\times\mathbb{R}\times\mathbb{T}^2, \tag{1.13}$$

*and, defining* $Z^\epsilon = Z^\epsilon(t,x) = Z(t,\frac{t}{\epsilon},x)$*, the following estimate holds for* $z^\epsilon - Z^\epsilon$

$$\Big\|\frac{z^\epsilon - Z^\epsilon}{\epsilon}\Big\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \alpha, \tag{1.14}$$

*where* $\alpha$ *is a constant not depending on* $\epsilon$*.*

Because of assumptions (1.8) and (1.9),

$$\widetilde{\mathcal{A}},\ \widetilde{\mathcal{C}},\ \widetilde{\mathcal{A}}_1,\ \widetilde{\mathcal{C}}_1,\ \widetilde{\mathcal{A}}^\epsilon,\ \widetilde{\mathcal{A}}_1^\epsilon,\ \widetilde{\mathcal{C}}^\epsilon, \text{ and } \widetilde{\mathcal{C}}_1^\epsilon \text{ are regular and bounded.} \tag{1.15}$$

2. **Two-Scale Numerical Method Building.** In this section, we develop the Two-Scale Numerical Method in order to approach the solution $z^\epsilon$ of (1.1). The idea is to get a good approximation of $z^\epsilon(t,x)$ seeing Theorem 1.1 content as $z^\epsilon(t,x) \sim Z(t,\frac{t}{\epsilon},x)$.
The strategy is to consider a Fourier expansion of $Z$ solution to (1.10). In this equation, $t$ is only a parameter.
The Fourier expansion of $Z$ is given as follows:

$$Z(t,\theta,x) = \sum_{l,m,n} Z_{l,m,n}(t)\ e^{2i\pi(l\theta+mx_1+nx_2)}, \tag{2.1}$$

where $Z_{l,m,n}(t),\ l=0,1,2,\dots,\ m=0,1,2,\dots,\ n=0,1,2,\dots,$ are the unknown complex coefficients of the Fourier expansion of $Z$. Using (2.1), the Fourier expansion of $\frac{\partial Z}{\partial\theta}$ is given by

$$\frac{\partial Z}{\partial\theta}(t,\theta,x) = \sum_{l,m,n} 2i\pi\, l\, Z_{l,m,n}(t)\ e^{2i\pi(l\theta+mx_1+nx_2)}. \tag{2.2}$$

To obtain the system satisfied by the Fourier expansion (2.1) of $Z$, it is necessary to compute the Fourier expansions of $\nabla\cdot(\widetilde{\mathcal{A}}\nabla Z)$ and $\nabla\cdot\widetilde{\mathcal{C}}$. As $\nabla\cdot(\widetilde{\mathcal{A}}\nabla Z) = \nabla\widetilde{\mathcal{A}}\cdot\nabla Z + \widetilde{\mathcal{A}}\cdot\Delta Z$, let

$$\sum_{l,m,n} \widetilde{\mathcal{A}}_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}, \tag{2.3}$$

and

$$\sum_{l,m,n} \widetilde{\mathcal{A}}^{grad}_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}, \tag{2.4}$$

be respectively the Fourier expansions of $\widetilde{\mathcal{A}}$ and $\nabla\widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}^{grad}_{l,m,n}(t) = 2i\pi\widetilde{\mathcal{A}}_{l,m,n}\begin{pmatrix} m \\ n\end{pmatrix}$ and then the Fourier expansions of $\nabla Z$ and $\Delta Z$ are respectively given by

$$\sum_{l,m,n} 2i\pi\begin{pmatrix} m \\ n\end{pmatrix} Z_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}, \tag{2.5}$$

and

$$-\sum_{l,m,n} 4\pi^2(m^2+n^2) Z_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}. \tag{2.6}$$

In the same way the Fourier expansion of $\nabla\cdot\widetilde{\mathcal{C}}$ is given by

$$\sum_{l,m,n} \widetilde{\mathcal{C}}_{l,m,n} e^{2i\pi(l\theta+mx_1+nx_2)}. \tag{2.7}$$

Using (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7), equation (1.10) becomes

$$\begin{aligned}
&\sum_{l,m,n} 2i\pi\, l\, Z_{l,m,n}(t)\ e^{2i\pi(l\theta+mx_1+nx_2)} \\
&-\Big(\sum_{l,m,n} \widetilde{\mathcal{A}}^{grad}_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}\Big)\cdot\Big(\sum_{l,m,n} 2i\pi\begin{pmatrix} m \\ n\end{pmatrix} Z_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}\Big) \\
&+\Big(\sum_{l,m,n} \widetilde{\mathcal{A}}_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}\Big)\Big(\sum_{l,m,n} 4\pi^2(m^2+n^2) Z_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)}\Big) = \\
&\sum_{l,m,n} \widetilde{\mathcal{C}}_{l,m,n}(t)\, e^{2i\pi(l\theta+mx_1+nx_2)},
\end{aligned} \tag{2.8}$$

which gives after identification, the following algebraic system for $(Z_{l,m,n})$:

$$2i\pi\, l\, Z_{l,m,n}(t) - \sum_{i,j,k} 2i\pi \widetilde{\mathcal{A}}^{grad}_{i,j,k}(t) \cdot \begin{pmatrix} m-j \\ n-k \end{pmatrix} Z_{l-i,m-j,n-k}(t)$$
$$+4\pi^2 \sum_{i,j,k} \widetilde{\mathcal{A}}_{i,j,k}(t)((m-j)^2 + (n-k)^2) Z_{l-i,m-j,n-k}(t) = \widetilde{\mathcal{C}}_{l,m,n}(t). \tag{2.9}$$

In formula (2.1), the integers $m, n$ and $l$ vary from $-\infty$ to $+\infty$. But in practice, we will consider the truncated Fourier series of order $P \in \mathbb{N}$ defined by

$$Z_P(t,\theta,x) = \sum_{0\le l\le P, 0\le m\le P, 0\le n\le P} Z_{l,m,n}(t)\; e^{2i\pi(l\theta + mx_1 + nx_2)}. \tag{2.10}$$

Using (2.10), formula (2.9) becomes:

$$2i\pi\, l\, Z_{l,m,n}(t) - \sum_{0\le i\le P,\, 1\le j\le P,\, 0\le k\le P} 2i\pi \widetilde{\mathcal{A}}^{grad}_{i,j,k}(t) \cdot \begin{pmatrix} m-j \\ n-k \end{pmatrix} Z_{l-i,m-j,n-k}(t)$$
$$+4\pi^2 \sum_{0\le i\le P,\, 0\le j\le P,\, 0\le k\le P} \widetilde{\mathcal{A}}_{i,j,k}(t)((m-j)^2 + (n-k)^2) Z_{l-i,m-j,n-k}(t) = \widetilde{\mathcal{C}}_{l,m,n}(t). \tag{2.11}$$

## 3. **Convergence result.**

*Proof.* of Theorem 1.1. For self-containedness, we recall the proof of Theorem 1.1. Firstly, we obtain an estimate leading to that $z^\epsilon$ is bounded in $L^\infty((0,T); L^2(\mathbb{T}^2))$. Secondly, defining test function $\psi^\epsilon(t,x) = \psi(t, \frac{t}{\epsilon}, x)$ for any $\psi(t,\theta,x)$, regular with a compact support over $[0,T)\times\mathbb{T}^2$ and 1-periodic in $\theta$, multiplying (1.1) by $\psi^\epsilon$ and integrating over $[0,T)\times\mathbb{T}^2$ gives

$$\int_{\mathbb{T}^2}\int_0^T \frac{\partial z^\epsilon}{\partial t}\psi^\epsilon dtdx - \frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot(\mathcal{A}^\epsilon \nabla z^\epsilon)\psi^\epsilon dtdx = \frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx. \tag{3.1}$$

Then integrating by parts in the first integral over $[0,T)$ and using the Green formula in $\mathbb{T}^2$ in the second integral we have

$$-\int_{\mathbb{T}^2} z_0(x)\psi(0,0,x)dx - \int_{\mathbb{T}^2}\int_0^T \frac{\partial\psi^\epsilon}{\partial t} z^\epsilon dtdx$$
$$+\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \mathcal{A}^\epsilon\nabla z^\epsilon \nabla\psi^\epsilon dtdx = \frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx. \tag{3.2}$$

Again using the Green formula in the third integral we obtain

$$-\int_{\mathbb{T}^2} z_0(x)\psi(0,0,x)\,dx - \int_{\mathbb{T}^2}\int_0^T \frac{\partial\psi^\epsilon}{\partial t} z^\epsilon dtdx$$
$$-\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T z^\epsilon\,\nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)\,dtdx = \frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx. \tag{3.3}$$

But

$$\frac{\partial\psi^\epsilon}{\partial t} = \left(\frac{\partial\psi}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial\psi}{\partial\theta}\right)^\epsilon, \tag{3.4}$$

where

$$\left(\frac{\partial\psi}{\partial t}\right)^\epsilon(t,x) = \frac{\partial\psi}{\partial t}(t,\frac{t}{\epsilon},x) \;\text{ and }\; \left(\frac{\partial\psi}{\partial\theta}\right)^\epsilon(t,x) = \frac{\partial\psi}{\partial\theta}(t,\frac{t}{\epsilon},x), \tag{3.5}$$

then we have

$$\int_{\mathbb{T}^2}\int_0^T z^\epsilon\left(\left(\frac{\partial\psi}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial\psi}{\partial\theta}\right)^\epsilon + \frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)\right)dxdt$$
$$+\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx = -\int_{\mathbb{T}^2} z_0(x)\psi(0,0,x)\,dx. \tag{3.6}$$

Using the Two-Scale convergence due to Nguetseng [11] and Allaire [1] (see also Frénod Raviart and Sonnendrücker [7]), since $z^\epsilon$ is bounded in $L^\infty([0,T), L^2(\mathbb{T}^2))$, there exists a profile $Z(t,\theta,x)$, periodic of period 1 with respect to $\theta$, such that for all $\psi(t,\theta,x)$, regular with a compact support with respect to $(t,x)$ and 1-periodic with respect to $\theta$, we have

$$\int_{\mathbb{T}^2}\int_0^T z^\epsilon\psi^\epsilon dtdx \longrightarrow \int_{\mathbb{T}^2}\int_0^T\int_0^1 Z\psi\,d\theta dtdx, \;\text{ as } \epsilon \text{ tends to zero,} \tag{3.7}$$

for a subsequence extracted from $(z^\epsilon)$.
Multiplying (3.6) by $\epsilon$, passing to the limit as $\epsilon \to 0$ and using (3.7) we have

$$\int_{\mathbb{T}^2}\int_0^T\int_0^1 Z\frac{\partial \psi}{\partial \theta}\,d\theta dt dx + \lim_{\epsilon\to 0}\int_{\mathbb{T}^2}\int_0^T z^\epsilon \nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)\,dtdx = \lim_{\epsilon\to 0}\int_{\mathbb{T}^2}\int_0^T \mathcal{C}^\epsilon\cdot\nabla\psi^\epsilon dtdx, \tag{3.8}$$

for an extracted subsequence. As $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ are bounded and $\psi^\epsilon$ is a regular function, $\mathcal{A}^\epsilon\nabla\psi^\epsilon$ and $\nabla\psi^\epsilon$ can be considered as test functions. Using (3.7) we have

$$\int_{\mathbb{T}^2}\int_0^T z^\epsilon\,\nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)dtdx \longrightarrow \int_{\mathbb{T}^2}\int_0^T\int_0^1 Z\nabla\cdot(\widetilde{\mathcal{A}}\nabla\psi)\,d\theta dtdx, \tag{3.9}$$

and

$$\int_{\mathbb{T}^2}\int_0^T \mathcal{C}^\epsilon\cdot\nabla\psi^\epsilon dtdx \text{ Two-Scale converges to } \int_{\mathbb{T}^2}\int_0^T\int_0^1 \widetilde{\mathcal{C}}\cdot\nabla\psi\,d\theta dtdx. \tag{3.10}$$

Passing to the limit as $\epsilon\to 0$ we obtain from (3.8) a weak formulation of the equation (1.10) satisfied by $Z$.
Using (1.2) and (1.3) equation (1.1) becomes

$$\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot(\widetilde{\mathcal{A}}^\epsilon\nabla z^\epsilon) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon) + \nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon. \tag{3.11}$$

For $Z^\epsilon$, we have

$$\frac{\partial Z^\epsilon}{\partial t} = \left(\frac{\partial Z}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial Z}{\partial \theta}\right)^\epsilon, \tag{3.12}$$

where

$$\left(\frac{\partial Z}{\partial t}\right)^\epsilon(t,x) = \frac{\partial Z}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \left(\frac{\partial Z}{\partial \theta}\right)^\epsilon(t,x) = \frac{\partial Z}{\partial \theta}(t,\frac{t}{\epsilon},x). \tag{3.13}$$

Using (1.10), $Z^\epsilon$ is solution to

$$\frac{\partial Z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla Z^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \left(\frac{\partial Z}{\partial t}\right)^\epsilon. \tag{3.14}$$

Formulas (3.11) and (3.14) give

$$\frac{\partial(z^\epsilon - Z^\epsilon)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla(z^\epsilon - Z^\epsilon)\right) = \nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon + \left(\frac{\partial Z}{\partial t}\right)^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon). \tag{3.15}$$

Multiplying equation (3.15) by $\frac{1}{\epsilon}$ and using the fact that $z^\epsilon = z^\epsilon - Z^\epsilon + Z^\epsilon$ in the right hand side of equation (3.15), $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is solution to:

$$\frac{\partial\left(\dfrac{z^\epsilon - Z^\epsilon}{\epsilon}\right)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left((\widetilde{\mathcal{A}}^\epsilon + \epsilon\widetilde{\mathcal{A}}_1^\epsilon)\nabla(\frac{z^\epsilon - Z^\epsilon}{\epsilon})\right) = \frac{1}{\epsilon}\left(\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon + (\frac{\partial Z}{\partial t})^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon)\right). \tag{3.16}$$

Our aim here is to prove that $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is bounded by a constant $\alpha$ not depending on $\epsilon$. For this let us use that $\widetilde{\mathcal{A}}^\epsilon$, $\widetilde{\mathcal{A}}_1^\epsilon$, $\widetilde{\mathcal{C}}^\epsilon$ and $\widetilde{\mathcal{C}}_1^\epsilon$ are regular and bounded coefficients (see (1.15)) and that $\widetilde{\mathcal{A}}^\epsilon \geq G_{thr}$ (see (1.13)). Hence, $\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon$ is bounded, $\nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon)$ is also bounded. Since $Z^\epsilon$ is solution to (3.14), $\frac{\partial Z}{\partial t}$ satisfies the following equation

$$\frac{\partial\left(\dfrac{\partial Z}{\partial t}\right)}{\partial \theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla\frac{\partial Z}{\partial t}\right) = \frac{\partial\nabla\cdot\widetilde{\mathcal{C}}}{\partial t} + \nabla\cdot\left(\frac{\partial\widetilde{\mathcal{A}}}{\partial t}\nabla Z\right). \tag{3.17}$$

Equation (3.17) is linear with regular and bounded coefficients. Using a result of Ladyzenskaja, Solonnikov and Ural'Ceva [8], $\frac{\partial Z}{\partial t}$ is regular and bounded and so the coefficients of equations (3.16) are regular and bounded. Then, using the same arguments as in the proof of Theorem 1.1 in [3] we obtain that $\left(\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right)$ is bounded.
To determine the value of the constant $\alpha$, we proceed in the same way as in the proof of Theorem 3.16 of [3]. Since the coefficients $\left(\widetilde{\mathcal{A}}^\epsilon,\ \widetilde{\mathcal{A}}_1^\epsilon,\ \widetilde{\mathcal{C}}^\epsilon \text{ and } \widetilde{\mathcal{C}}_1^\epsilon,\ \nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon,\ \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon), \text{ and } \frac{\partial Z}{\partial t}\right)$ are bounded by constants, let $\beta$ denotes the maximum between all these constants. Then we use the same argument as in the proof of Theorems 1.1 and 3.16 and we get:

$$\left\|\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \|z_0(\cdot) - Z(0,0,\cdot)\|_2\sqrt{\frac{\beta+\beta^3}{\sqrt{\widetilde{G}_{thr}}} + 2\beta}\,T. \tag{3.18}$$

□

**Theorem 3.1.** *Let $\epsilon$ be a positive real, $z^\epsilon$ be the solution to (1.1), $Z_P$ be the truncated Fourier series (defined by (2.10)) of $Z$ solution to (1.10) and $Z_P^\epsilon$ defined by $Z_P^\epsilon(t,x) = Z_P(t, \frac{t}{\epsilon}, x)$. Then, under assumptions (1.8), (1.9) and (1.12), $z^\epsilon - Z_P^\epsilon$ satisfies the following estimate:*

$$\|z^\epsilon - Z_P^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \epsilon \|z_0(\cdot) - Z(0,0,\cdot)\|_2 \sqrt{\frac{\beta+\beta^3}{\sqrt{\widetilde{G}_{thr}}} + 2\beta\, T} + f(P), \tag{3.19}$$

*where $f$ is a non-negative function of $P$ not depending on $\epsilon$ and satisfying $\lim_{P\to+\infty} f(P) = 0$.*

*Proof.* We can write:

$$\begin{aligned} \|z^\epsilon - Z_P^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} &= \|z^\epsilon - Z^\epsilon + Z^\epsilon - Z_p^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \\ &\leq \|z^\epsilon - Z^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} + \|Z^\epsilon - Z_p^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))}. \end{aligned} \tag{3.20}$$

Using (3.18), the first term in the right hand side of (3.20) is bounded by

$$\|z^\epsilon - Z^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \epsilon \|z_0(\cdot) - Z(0,0,\cdot)\|_2 \sqrt{\frac{\beta+\beta^3}{\sqrt{\widetilde{G}_{thr}}} + 2\beta T}. \tag{3.21}$$

For the second term of (3.20), using classical results of Fourier series theory, since $Z - Z_P$ is nothing but the rest of the Fourier series of order $P$ of $Z$ and since $Z$ is regular (because it is the solution of (1.10) which has regular coefficients), the non-negative function $f$ satisfying $\lim_{P\to+\infty} f(P) = 0$ such that

$$\|Z - Z_p\|_{L^\infty([0,T],L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))} \leq f(P), \tag{3.22}$$

exists. From this last inequality,

$$\|Z^\epsilon - Z_p^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq f(P), \tag{3.23}$$

follows and coupling this with (3.21) and (3.20) gives inequality (3.19). □

## 4. **Numerical illustration of Theorem 3.1.**

### 4.1. **Reference solution.**

Having Fourier coefficients of $Z$ on hand, we will do the same for function $z^\epsilon(t,x)$ solution to (1.1) in order to compare it to the profile $Z$ for a given $\epsilon$, in a fixed time. The Fourier expansion of $z^\epsilon$ is given by

$$z^\epsilon(t,x_1,x_2) = \sum_{m,n} z_{m,n}(t)\; e^{2\pi i(mx_1+nx_2)}, \tag{4.1}$$

where $m = 0,1,2,\dots$ and $n = 0,1,2,\dots$, then the Fourier expansion of $\frac{\partial z^\epsilon}{\partial t}$ is

$$\frac{\partial z^\epsilon}{\partial t} = \sum_{m,n} \dot{z}_{m,n}(t)\; e^{2\pi i(mx_1+nx_2)}. \tag{4.2}$$

Using the same idea as in the Fourier expansion of $Z$, we obtain the following infinite system of Ordinary Differential Equations

$$\begin{aligned} &\frac{\partial z_{m,n}}{\partial t}(t) - \frac{1}{\epsilon}\sum_{i,j} 2i\pi \mathcal{A}^{grad}_{i,j}(t)\cdot \begin{pmatrix} m-i \\ n-j \end{pmatrix} z_{m-i,n-j}(t) \\ &+\frac{1}{\epsilon}4\pi^2 \sum_{i,j} \mathcal{A}_{i,j}(t)((m-i)^2+(n-j)^2) z_{m-i,n-j}(t) = \frac{1}{\epsilon}\mathcal{C}_{m,n}(t), \end{aligned} \tag{4.3}$$

where $\mathcal{A}^{grad}_{i,j}(t)$, $\mathcal{A}_{i,j}(t)$ and $\mathcal{C}_{m,n}(t)$ are respectively the Fourier coefficients of $\nabla\mathcal{A}^\epsilon$, $\mathcal{A}^\epsilon$ and $\nabla\cdot\mathcal{C}^\epsilon$.
In the same way, the truncated Fourier series of order $P \in \mathbb{N}$ of $z^\epsilon$ is given by

$$z_P^\epsilon(t,x_1,x_2) = \sum_{m,n=0}^{P} z_{m,n}(t)\; e^{2\pi i(mx_1+nx_2)}, \tag{4.4}$$

which gives from (4.3) the following system Ordinary Differential Equations

$$\begin{aligned} &\frac{\partial z_{m,n}}{\partial t}(t) - \frac{1}{\epsilon}\sum_{i,j=0}^{P} 2i\pi \mathcal{A}^{grad}_{i,j}(t)\cdot \begin{pmatrix} m-i \\ n-j \end{pmatrix} z_{m-i,n-j}(t) \\ &+\frac{1}{\epsilon}4\pi^2 \sum_{i,j=0}^{P} \mathcal{A}_{i,j}(t)((m-i)^2+(n-j)^2) z_{m-i,n-j}(t) = \frac{1}{\epsilon}\mathcal{C}_{m,n}(t). \end{aligned} \tag{4.5}$$

In (4.5), we will use an initial condition $z_{m,n}(0,x)$. To solve (4.5) we use, for the discretization in time, a Runge-Kutta method (ode45).

4.2. **Comparison Two-Scale Numerical Solution and reference solution.** In this paragraph, we consider the truncated solution $z_P^\epsilon(t,x_1,x_2)$ and $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$. The objective here is to compare for a fixed $\epsilon$ and a given time, the quantity $|z_P^\epsilon(t,x_1,x_2) - Z_P(t,\frac{t}{\epsilon},x_1,x_2)|$ when the water velocity $\mathcal{U}$ is given.

4.2.1. *Comparisons of $z_P^\epsilon(t,x)$ and $Z_P(t,\frac{t}{\epsilon},x)$ with $\mathcal{U}$ given by (4.6).* For the numerical simulations, concerning $z^\epsilon$, we take $z_0(x_1,x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ and $z_0(x_1,x_2) = Z(0,0,x_1,x_2)$. In what concerns the water velocity field, we consider the function

$$\mathcal{U}(t,\theta,x_1,x_2) = \sin\pi x_1 \sin 2\pi\theta\, \mathbf{e}_1, \tag{4.6}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are respectively the first and the second vector of the canonical basis of $\mathbb{R}^2$ and $x_1,\ x_2$ are the first and the second components of $x$.
In Figure 1 , we can see the space distribution of the first component of the velocity $\mathcal{U}$ for a given time $t=1$ and for various values of $\theta$: 0.3, 0.55, and 0.7. In Figure 2, we see, for a fixed point $x=(x_1,x_2)$, how the water velocity $\widetilde{\mathcal{U}}(\theta)$

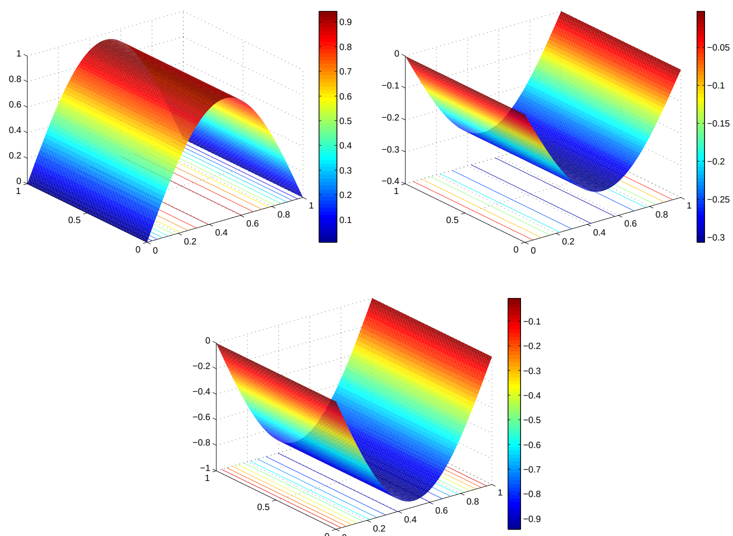

FIGURE 1. Space distribution of the first component of $\mathcal{U}(1,0.3,(x_1,x_2))$, $\mathcal{U}(1,0.55,(x_1,x_2))$ and $\mathcal{U}(1,0.7,(x_1,x_2))$ when $\mathcal{U}$ is given by (4.6).

evolves with respect to $\theta$. In Figure 3, the $\theta$-evolution of $\widetilde{\mathcal{A}}(\theta)$ is also given in various points $(x_1,x_2)\in\mathbb{R}^2$.

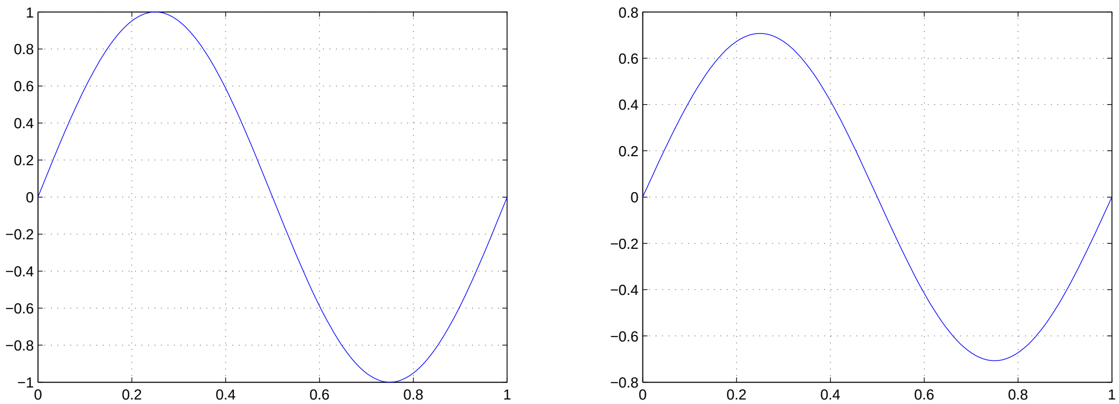


FIGURE 2. $\theta$-evolution of $\widetilde{\mathcal{U}}(\theta,(1/2,0))$ and $\widetilde{\mathcal{U}}(\theta,(1/4,0))$ when $\mathcal{U}$ is given by (4.6)

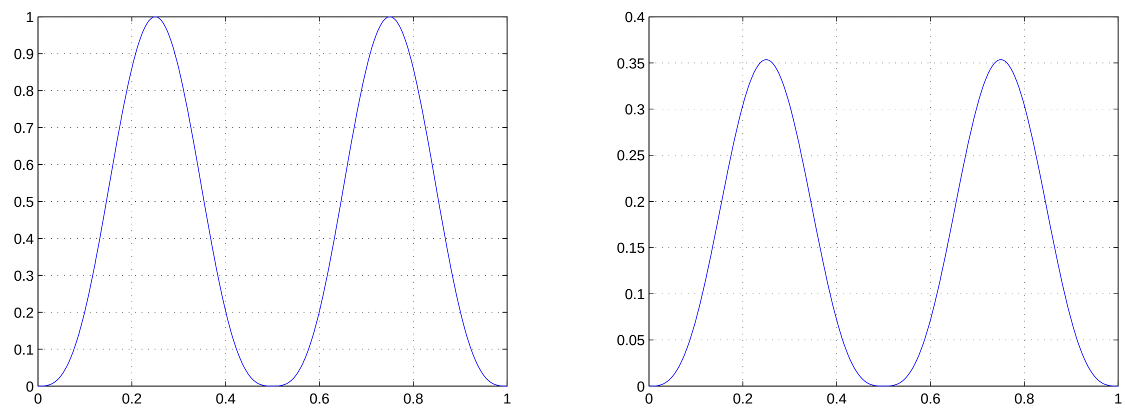


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FIGURE 3. $\theta$-evolution of $\widetilde{\mathcal{A}}(\theta,(1/2,0))$ and $\widetilde{\mathcal{A}}(\theta,(1/4,0))$ when $\mathcal{U}$ is given by (4.6)

In this paragraph, we present numerical simulations in order to validate the Two-Scale convergence presented in Theorem 1.1. For a given $\epsilon$, we compare $Z_P(t,\frac{t}{\epsilon},x)$, where $Z_P$ is the Fourier expansion of order $P$ of the solution to (1.10) and $z_P^\epsilon(t,x)$ the Fourier expansion of order $P$ of the solution to the reference problem. The simulations presented are given for $P=4$. The calculation of $z_P^\epsilon(t,x)$ implies knowledge of $z_0(x)$. For an initial condition $z_0(x)$ well prepared and equal to $Z(0,0,x)$, we obtain the results of Figure 4 and we remark that the results obtained are the same for $z_P^\epsilon(t,x)$ and $Z_P(t,\frac{t}{\epsilon},x)$.

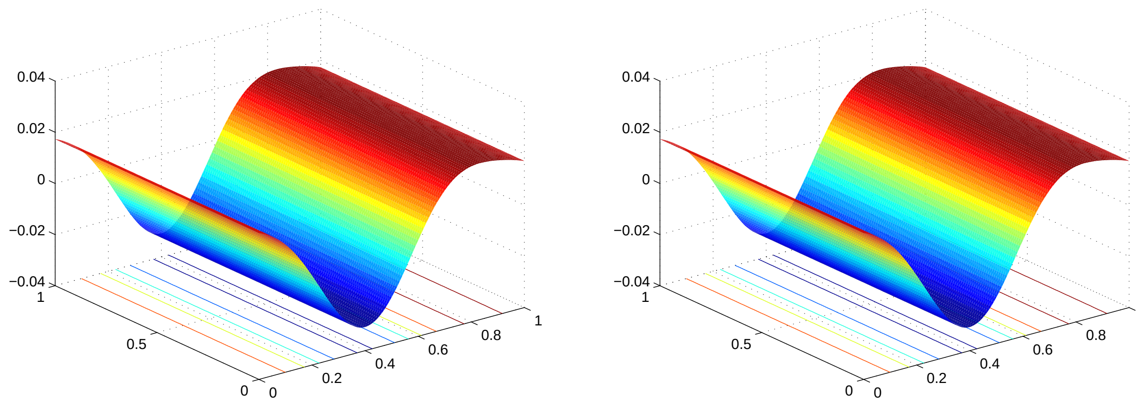


FIGURE 4. Comparison of $z_P^\epsilon(t,\cdot)$ and $Z_P(t,\frac{t}{\epsilon},\cdot)$, $P=4$, at time $t=1$, $\epsilon=0.001$, when $\mathcal{U}$ is given by (4.6) and when $z_0(\cdot)=Z(0,0,\cdot)$. On the left $z_P^\epsilon(t,\cdot)$, on the right $Z_P(t,\frac{t}{\epsilon},\cdot)$.

In practice, the solution $Z_P$, $P\in\mathbb{N}$ evolves according to $P$. For the simulations, we made the value of the integer $P$ vary and we saw that this variation is very low from $P\geq 6$.
To better show that $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$ is close to the reference solution $z_P^\epsilon(t,x_1,x_2)$, we plot and compare $Z_P(t,\frac{t}{\epsilon},x_1,0)$ and $z_P^\epsilon(t,x_1,0)$, at different times $t$. In these comparisons the initial condition $z_0(x_1,x_2)=\cos 2\pi x_1+\cos 4\pi x_1$ is different from $Z(0,0,x_1,x_2)$. The results are shown in Figure 5 and Figure 6. We see in these figures that the solution $z_P^\epsilon(t,x)$ get closer and closer to $Z_P(t,\frac{t}{\epsilon},x)$ with time of order $\epsilon$.

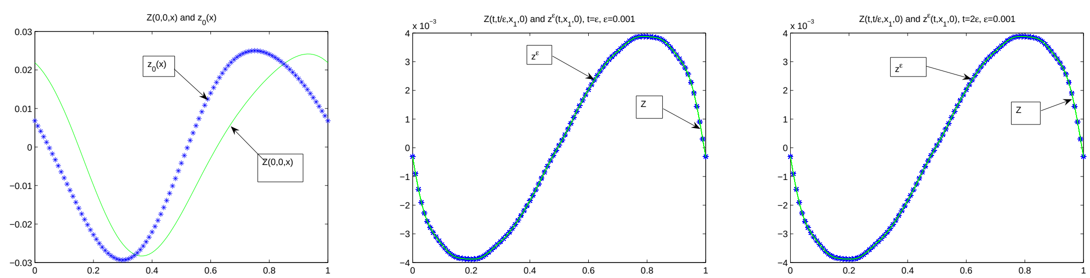


FIGURE 5. Comparison of $z_P^\epsilon(t,x_1,0)$ and $Z_P(t,\frac{t}{\epsilon},x_1,0))$, $P=4$. On the left $t=0$, in the middle $t=\epsilon$ and $t=2\epsilon$ on the right, $\epsilon=0.001$.

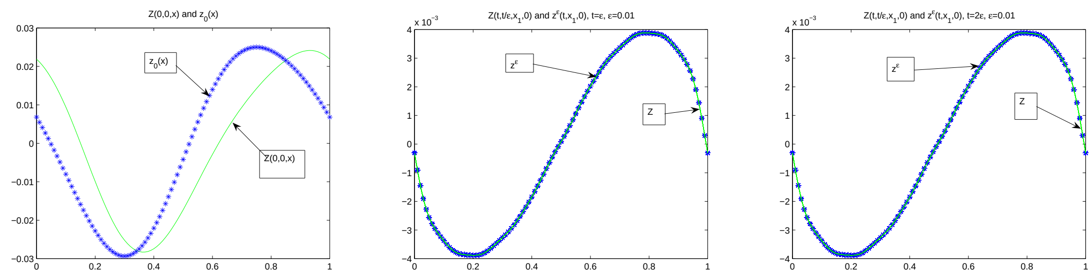


FIGURE 6. Comparison of $z_P^\epsilon(t,x_1,0)$ and $Z_P(t,\frac{t}{\epsilon},x_1,0))$. On the left $t=0$, in the middle $t=\epsilon$ and $t=2\epsilon$ on the right, $\epsilon=0.01$.

So we can see from these figures that the solution $Z$ of the Two-Scale limit problem is such that $Z(t,\frac{t}{\epsilon},\cdot,\cdot)$ is close to the solution $z^\epsilon(t,\cdot,\cdot)$ of the reference problem. In the presently considered case where the initial condition for $z^\epsilon$ is not $Z(0,0,\cdot,\cdot)$, we saw in Figure 5 and Figure 6 that $z_P^\epsilon$ tends to reach a steady state. This steady state is an oscillatory one in the sense that for large $t$, $z_P^\epsilon(t,\cdot,\cdot)$ behaves like $Z_P(t,\frac{t}{\epsilon},\cdot,\cdot)$. This is illustrated by Figure 7 where $z_P^\epsilon(t,x_1,0)$ and $Z_P(t,\frac{t}{\epsilon},x_1,0)$ are given for various value of $t$ in a period of lenght $\epsilon$.
More precisely, in this figure we see that within a period of time of lenght $\epsilon$, $z_P^\epsilon(t,\cdot,\cdot)$ and $Z_P(t,\frac{t}{\epsilon},\cdot,\cdot)$ do not glue together completly. Nevertheless, despite this phenomenon which is linked with the fact that the Two-Scale approximation of $z^\epsilon(t,\cdot,\cdot)$ by $Z(t,\frac{t}{\epsilon},\cdot,\cdot)$ is only of order 1 in $\epsilon$, the two solutions re-glue well together at the end of the period.

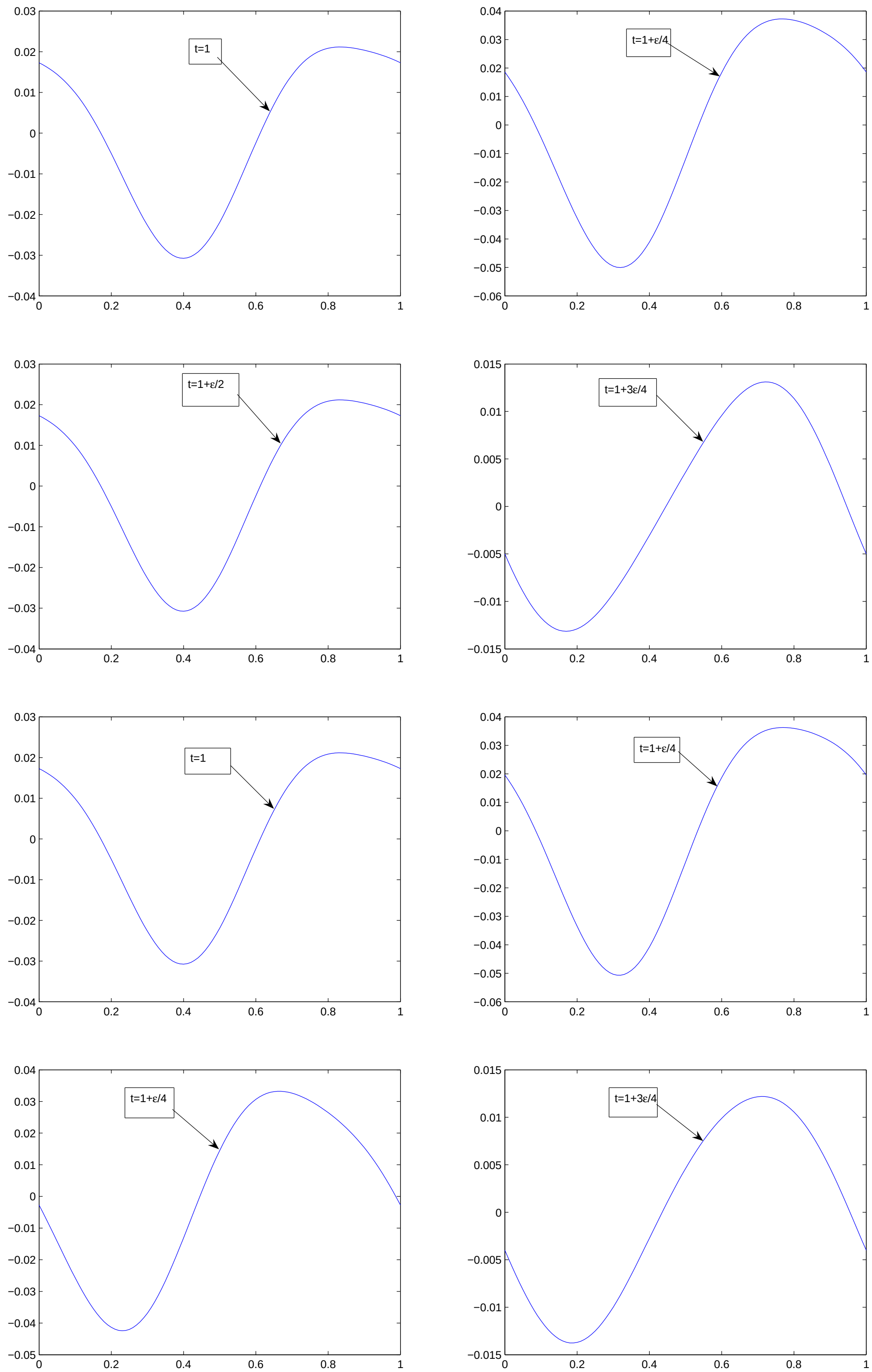


FIGURE 7. Evolution of $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$ in the top and $z_P^{\epsilon}(t, x_1, 0)$ in the bottom, $t = 1 + \frac{n\epsilon}{4},\ n = 0, 1, 2, 3.$

4.2.2. *Comparisons of $z^\epsilon(t,x)$ and $Z(t,\frac{t}{\epsilon},x)$ with $\mathcal{U}$ is given by (4.7).* In this subsection, we do the same as in the precedent one, but when the velocity fields $\mathcal{U}$ given by (4.7). The results are all identical to the precedent one i.e. the Two-Scale limit $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$ is very close to the solution $z_P^\epsilon(t,x_1,x_2)$ to the reference problem when $P=4$. The initial condition $z_0(x_1,x_2)\neq Z(0,0,x_1,x_2)$ and is the same as in the subsection 4.2.1. The results are given for $\epsilon=0.1$ and $\epsilon=0.005$ and for various time $t$. We notice that $z^\epsilon$ comes very close to $Z(t,\frac{t}{\epsilon},x_1,x_2)$ when $\epsilon$ is very small. We begin by giving the space distribution of $\mathcal{U}$ at various time and the $\theta-$evolution of $\mathcal{U}$ and $\widetilde{\mathcal{A}}$. The second velocity fields is given by

$$\mathcal{U}(t,\theta,x_1,x_2)=\mathcal{U}(t,\theta,x)=\begin{cases} 0 \text{ in } [0,\theta_1],\\ \frac{\theta-\theta_1}{\theta_2-\theta_1}U_{thr}\mathbf{e}_2 \text{ in } [\theta_1,\theta_2],\\ U_{thr}\mathbf{e}_2+\phi(\frac{\theta-\theta_2}{\theta_3-\theta_2})\psi(t,x) \text{ in } [\theta_2,\theta_3],\\ \frac{\theta-\theta_3}{\theta_4-\theta_3}U_{thr}\mathbf{e}_2 \text{ in } [\theta_3,\theta_4],\\ 0 \text{ in } [\theta_4,\theta_5],\\ \frac{\theta-\theta_5}{\theta_6-\theta_5}U_{thr}\mathbf{e}_2 \text{ in } [\theta_5,\theta_6],\\ -U_{thr}\mathbf{e}_2-\phi(\frac{\theta-\theta_6}{\theta_7-\theta_6})\psi(t,x) \text{ in } [\theta_6,\theta_7],\\ -\frac{\theta-\theta_7}{\theta_8-\theta_7}U_{thr}\mathbf{e}_2 \text{ in } [\theta_7,\theta_8],\\ 0 \text{ in } [\theta_8,1], \end{cases} \tag{4.7}$$

where $U_{thr}>0$, $\phi$ is a regular positive function satisfying $\phi(s)=s(1-s)$ and $\psi(t,x_1)=(1+\sin\frac{\pi}{30}t)(U_{thr}\mathbf{e}_2+\frac{1}{10}(1+\sin 2\pi x_1)\mathbf{e}_1)$, $\theta_i=\frac{i+1}{10}$, $i=1,\dots,8$.
The $\theta$-evolution of $\mathcal{U}$, given by (4.7), is given in Figure 9 for various position in $[0,1]^2$.
Function $g_a(\mathbf{u})=g_c(\mathbf{u})=|\mathbf{u}|^3, a=c=1$ and $\mathcal{M}(t,\theta,x)=0$ which yields a $\theta$-evolution of $\widetilde{\mathcal{A}}(\theta)$ which is drawn for various positions in Figure 10.

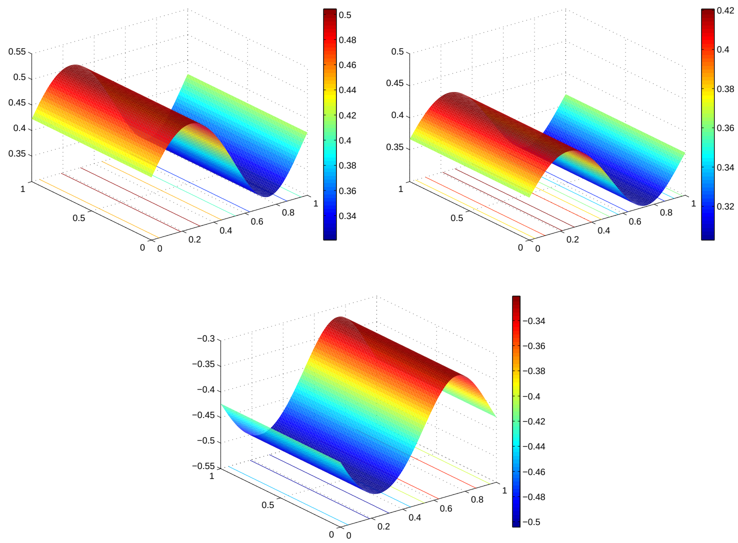

FIGURE 8. Space distribution of the first component of $\mathcal{U}(1, 0.25, (x_1, x_2))$, $\mathcal{U}(1, 0.275, (x_1, x_2))$ and $\mathcal{U}(1, 0.75, (x_1, x_2))$ when $\mathcal{U}$ is given by (4.7).

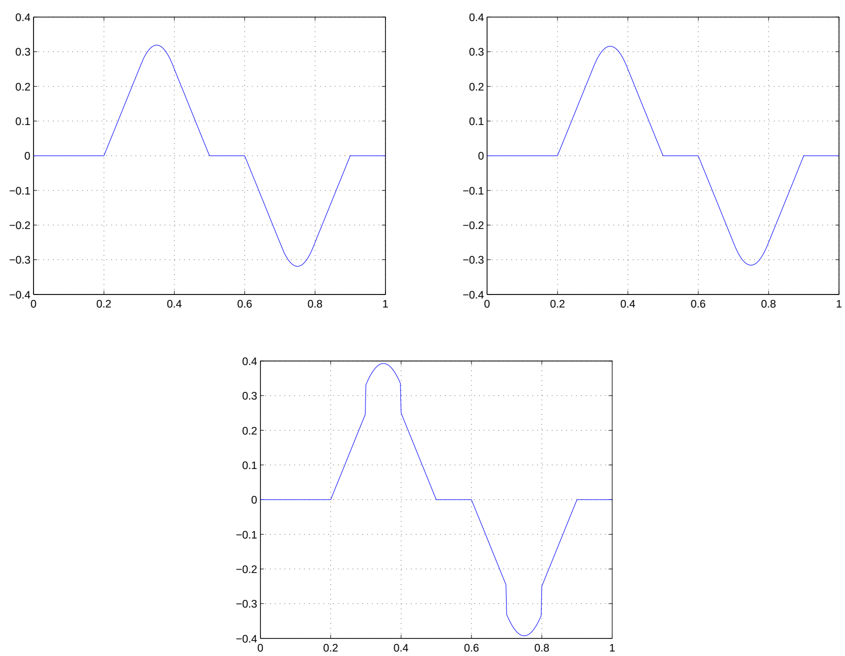

FIGURE 9. $\theta$-evolution of $\mathcal{U}(1, \theta, (1, 0))$, $\mathcal{U}(1, \theta, (4, 0))$ and $\mathcal{U}(1, \theta, (1/3, 1/3))$ when $\mathcal{U}$ is given by (4.7).

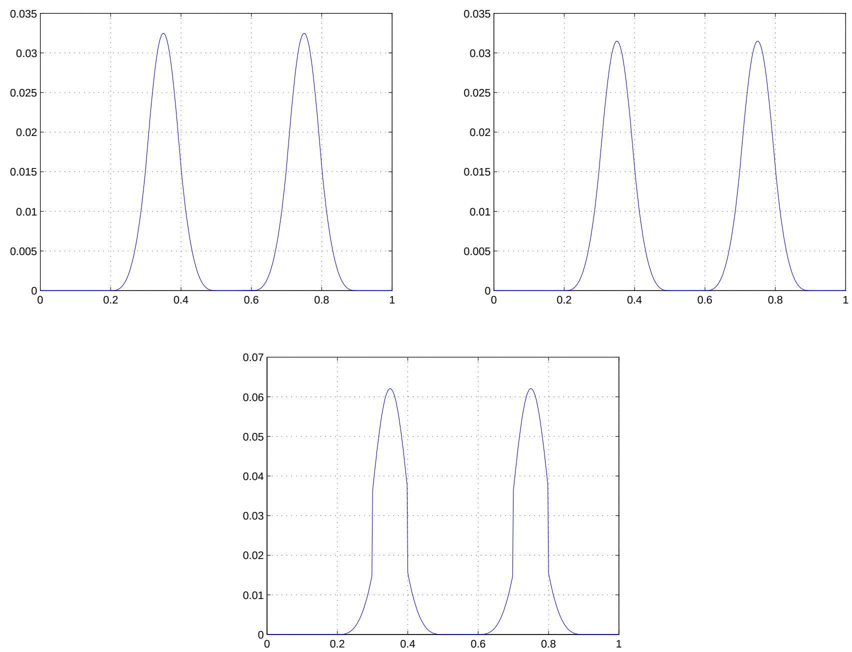


FIGURE 10. $\theta$-evolution of $\widetilde{\mathcal{A}}(1,\theta,(1,0))$, $\widetilde{\mathcal{A}}(1,\theta,(4,0))$ and $\widetilde{\mathcal{A}}(1,\theta,(1/3,1/3))$ when $\mathcal{U}$ is given by (4.7).

Using this, we compute $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$ and $z^\epsilon_P(t,x)$ for $P=4$. To compute $z^\epsilon_P(t,x)$ we take $z_0(x_1,x_2)=\cos 2\pi x_1 + \cos 4\pi x_1$ which is not $Z(0,0,x_1,x_2)$. First we study the errors $Z_P(t,\frac{t}{\epsilon},x_1,x_2)-z^\epsilon_P(t,x)$ at $t=1$. This quantity decreases when $\epsilon$ decreases as illustrated in the following tabular.

| value of $\epsilon$ | norm $L^1$ | norm $L^2$ | norm $L^\infty$ |
|---|---|---|---|
| 0.01 | 0.012212 | 0.00048013 | 0.003376 |
| 0.03 | 0.019082 | 0.0005753 | 0.0017347 |
| 0.05 | 0.030769 | 0.01348 | 0.0069818 |
| 0.07 | 0.045123 | 0.029055 | 0.009 |
| 0.09 | 0.17067 | 0.10562 | 0.038790 |
| 0.1 | 0.3053 | 0.10562 | 0.04878 |

Table: Errors norm $Z_P(t,\frac{t}{\epsilon},x_1,x_2)-z^\epsilon_{\tilde{P}}(t,x_1,x_2),\ \tilde{P}=(4,4),\ P=(4,4,4),\ t=1.$

The results given in this table show that, at time $t=1$, $z^\epsilon(t,x)$ is closer to $Z(t,\frac{t}{\epsilon},x)$ when $\epsilon$ is very small. These results validate the results obtained in Theorem 1.1.
In Figures 11 and 12, we present simulations at times $t=0.75$ and $t=0.775$. We see that $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$ is close to $z^\epsilon_P(t,x_1,x_2)$. The numerical results shown in these figures are made with $\epsilon=0.1$.

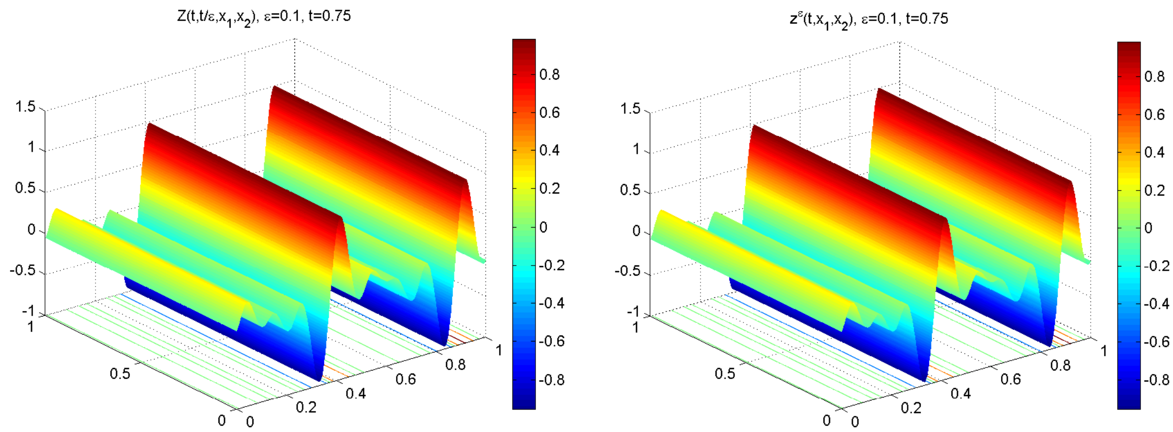


FIGURE 11. Comparison of $z^{\epsilon}_{P}(t,x_1,x_2)$ and $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, $P=4$; $t=0.75$, $\epsilon=0.1$, $z_0(x_1,x_2)=\cos 2\pi x_1+\cos 4\pi x_1$. On the left $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, on the right $z^{\epsilon}_{P}(t,x_1,x_2)$.

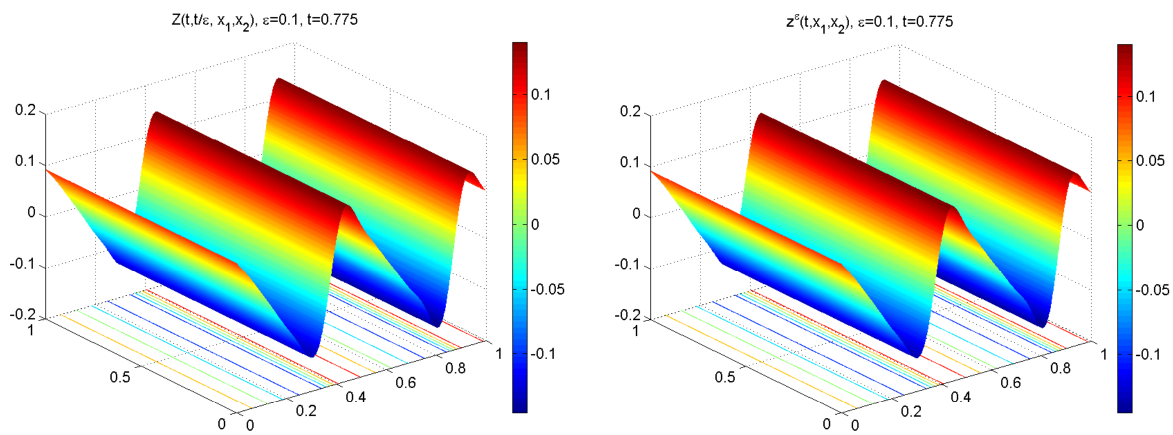


FIGURE 12. Comparison of $z^{\epsilon}_{P}(t,x_1,x_2)$ and $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, $P=4$; $t=0.775$, $\epsilon=0.1$, $z_0(x_1,x_2)=\cos 2\pi x_1+\cos 4\pi x_1$. On the left $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, on the right $z^{\epsilon}_{P}(t,x_1,x_2)$.

In Figure 13 and 14, we do the same but for $\epsilon=0.005$. The numerical results show that $z^{\epsilon}_{P}(t,x)$ is also very close to $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$.

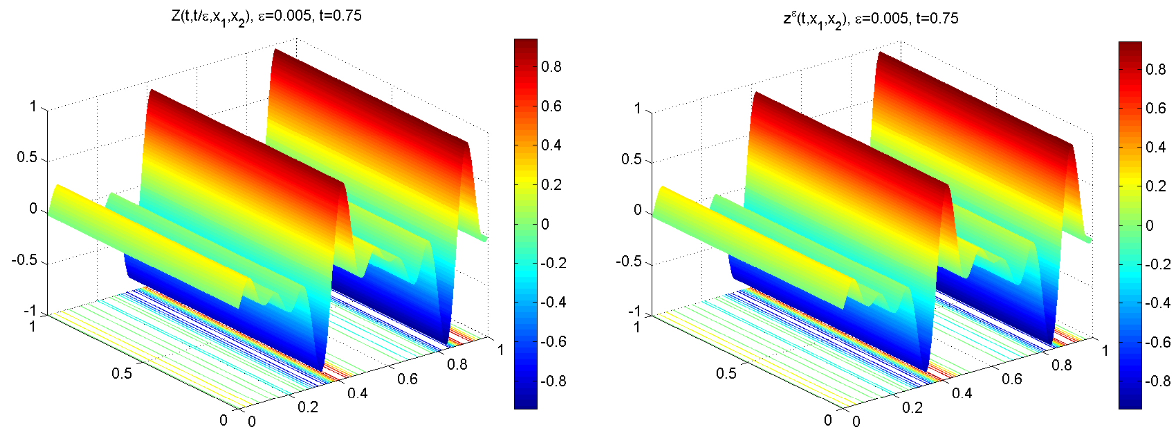


FIGURE 13. Comparison of $z_P^\epsilon(t,x_1,x_2)$ and $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, $P=4$; $t=0.75$, $\epsilon=0.005$, $z_0(x_1,x_2)=\cos 2\pi x_1+\cos 4\pi x_1$. On the left $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, on the right $z_P^\epsilon(t,x_1,x_2)$.

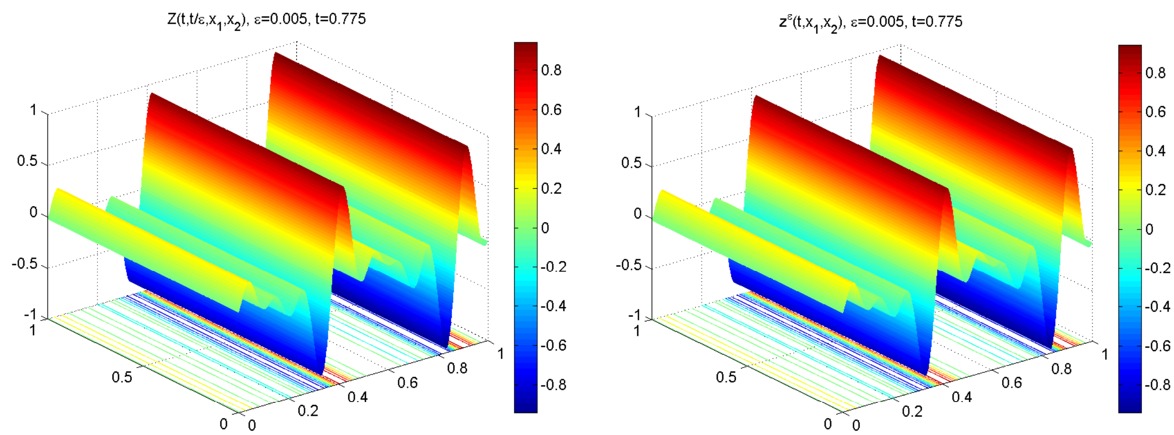


FIGURE 14. Comparison of $z_P^\epsilon(t,x_1,x_2)$ and $Z(t,\frac{t}{\epsilon},x_1,x_2)$, $P=4$; $t=0.775$, $\epsilon=0.005$, $z_0(x_1,x_2)=\cos 2\pi x_1+\cos 4\pi x_1$. On the left $Z_P(t,\frac{t}{\epsilon},x_1,x_2)$, on the right $z_P^\epsilon(t,x_1,x_2)$.

We remark that for $\epsilon=0.1$ and $\epsilon=0.005$, the solution $z_P^\epsilon(t,x)$ is very close to $Z_P(t,\frac{t}{\epsilon},x)$. But the approximation $z_P^\epsilon(t,x)\sim Z_P(t,\frac{t}{\epsilon},x)$ is very good when $\epsilon$ is very small.
To show that $z_P^\epsilon$ is very close to $Z_P$, we construct the same figures as previously but in dimension 2 with $\epsilon=0.005$ i.e. we construct $z_P^\epsilon(t,x_1,0)$ and $Z_P(t,\frac{t}{\epsilon},x_1,0)$ for $\epsilon=0.005$ at time $t=0.775$. This is given in Figure 15.

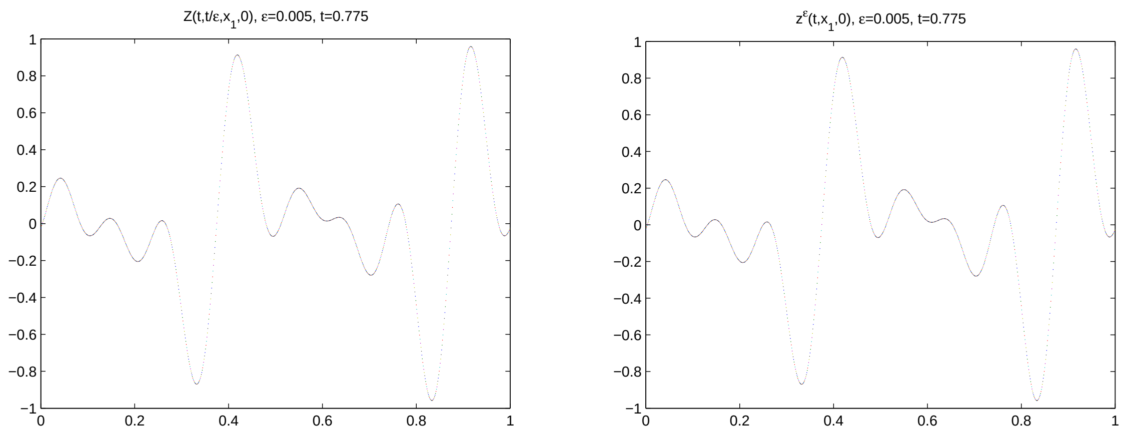


FIGURE 15. Comparison of $z^\epsilon_P(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0),\ t = 0.775,\ \epsilon = 0.005$. On the left $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$, on the right $z^\epsilon_P(t, x_1, 0)$.

The results in Figure 16 show that $Z_P$ and $z^\epsilon_P$ have the same behavior in the same period and $Z_P$ is very close to $z^\epsilon_P$. We also notice that, despite the small shifts that occur during a period, the two solutions glue together.

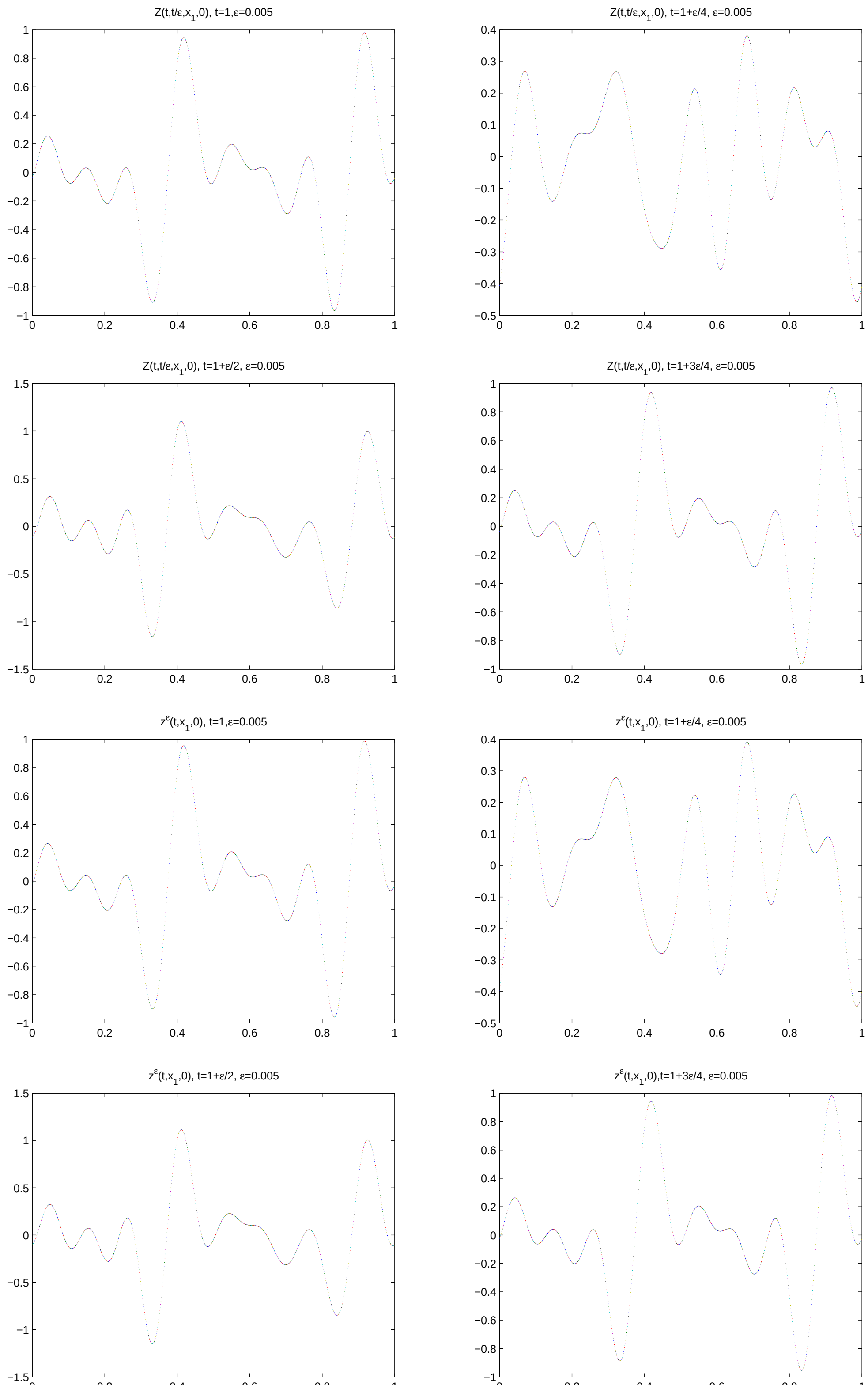


FIGURE 16. Evolution of $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$(top) and $z_P^\epsilon(t, x_1, 0)$(bottom), $t = 1 + n\epsilon/4,\ n = 0, 1, 2, 3.$

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