Two-dimensional Finite Larmor Radius approximation in canonical gyrokinetic coordinates

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Abstract

In this paper, we present some new results about the approximation of the Vlasov-Poisson system with a strong external magnetic field by the 2D finite Larmor radius model. The proofs within the present work are built by using two-scale convergence tools, and can be viewed as an improvement of previous works of Frénod & Sonnendrücker and Bostan on the 2D finite Larmor Radius model. In a first part, we recall the physical and mathematical contexts. We also recall two main results from previous papers of Frénod & Sonnendrücker and Bostan. Then, we introduce a set of variables which are so-called *canonical gyrokinetic coordinates*, and we write the Vlasov equation in these new variables. Then, we establish some two-scale convergence and weak-* convergence results.

1 Introduction

Nowadays, domestic energy production by using magnetic confinement fusion (MCF) techniques is a huge technological and human challenge, as it is illustrated by the international scientific collaboration around ITER which is under construction in Cadarache (France). Since magnetic confinement, needed to reach nuclear fusion reaction, is a very complex physical phenomenon, the mathematical models which are linked with this plasma physics subject need to be rigorously studied from theoretical and numerical points of view. Such a work programme based on rigorous mathematical studies and high precision numerical simulations can bring some additional informations about the behavior of the studied plasma before the launch of real experiments.

The present paper can be viewed as a part of the recent work programme about the mathematical justification of the mathematical models which are used for numerical simulations of MCF experiments. Indeed, the first tokamak plasma models have been proposed by Littlejohn, Lee *et al.*, Dubin *et al.* or Brizard *et al.* (see [20], [18, 19], [7], [5, 6]) nevertheless most of these models were established by using formal assumptions. For ten years, many mathematicians have been working on mathematical justification of these models, especially the gyrokinetic approaches like guiding-center approximations and finite Larmor radius approximations: many results in this research field are due to Frénod & Sonnendrücker *et al.* [9, 11, 12, 13], Golse & Saint-Raymond [14, 15], Bostan [4] or, more recently, Han-Kwan [17]. These mathematical results mostly rely on two-scale convergence theory (see Allaire [3], Nguetseng [24]) or compactness arguments.

In this paper, we are focused on the 2D finite Larmor radius model and its mathematical justification: more precisely, the goal is to make a synthesis of previous mathematical proofs of the convergence of $(f_\epsilon, \tilde{\mathbf{E}}_\epsilon)$, where

$$f_\epsilon(\mathbf{x},k,\alpha,t) = \tilde{f}_\epsilon\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\sin\alpha, t\big)\,, \qquad (1.1)$$

and where $(\tilde{f}_\epsilon, \tilde{\mathbf{E}}_\epsilon)$ is the solution of the following 2D Vlasov-Poisson system

$$\begin{cases} \partial_t \tilde{f}_\epsilon + \dfrac{1}{\epsilon} \tilde{\mathbf{v}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f}_\epsilon + \left(\tilde{\mathbf{E}}_\epsilon + \dfrac{1}{\epsilon} \begin{pmatrix} \tilde{v}_2 \\ -\tilde{v}_1 \end{pmatrix} \right) \cdot \nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon = 0 \,, \\ \tilde{f}_\epsilon(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, 0) = \tilde{f}^0(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) \,, \\ -\nabla_{\tilde{\mathbf{x}}} \tilde{\phi}_\epsilon = \tilde{\mathbf{E}}_\epsilon \,, \qquad -\Delta_{\tilde{\mathbf{x}}} \tilde{\phi}_\epsilon = \displaystyle\int_{\mathbb{R}^2} \tilde{f}_\epsilon \, d\tilde{\mathbf{v}} - \tilde{n}_e \,, \end{cases} \tag{1.2}$$

towards the couple $(f, \tilde{\mathbf{E}})$ which is the solution of the 2D finite Larmor radius model given in [4] (see also Theorem 2 below). The main results on this model are due to Sonnendrücker, Frénod and Bostan, and indicates that $(f_\epsilon, \tilde{\mathbf{E}}_\epsilon)$ somehow weak-* converges to the solution of the finite Larmor radius model. However the proofs within these articles are based on various assumptions and use various tools. Then, it seems useful to gather these convergence results and to simplify them as more as possible.

The first part of the present paper is devoted to a state-of-the-art about the two-dimensional finite Larmor radius approximation. Firstly, we recall the procedure which allows us to obtain the dimensionless model (1.2) from the complete Vlasov-Poisson model by considering specific assumptions. Then we recall the two-scale convergence theorem of Frénod & Sonnendrücker [13] on the one hand, and the weak-* convergence theorem of Bostan [4] on the other hand.

In a second part, we introduce a set of variables which are so-called *canonical gyrokinetic coordinates* and we reformulate the Vlasov-Poisson system (1.2) in these new variables. Then, we establish a two-scale convergence theorem which only relies on Frénod & Sonnendrücker's assumptions. Finally, we deduce from this result a new justification of the 2D finite Larmor radius model through an almost trivial proof.

2 State-of-the-art

2.1 Scaling of the Vlasov-Poisson model

This paragraph is devoted to the scaling of the following Vlasov-Poisson model:

$$\begin{cases} \partial_t \tilde{f} + \tilde{\mathbf{v}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{f} + \dfrac{e}{m_i} \big(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \tilde{\mathbf{B}} \big) \cdot \nabla_{\tilde{\mathbf{v}}} \tilde{f} = 0 \,, \\ \tilde{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, 0) = \tilde{f}^0(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) \,, \\ -\nabla_{\tilde{\mathbf{x}}} \tilde{\phi} = \tilde{\mathbf{E}} \,, \qquad -\Delta_{\tilde{\mathbf{x}}} \tilde{\phi} = \dfrac{e}{\varepsilon_0} \displaystyle\int_{\mathbb{R}^3_{\tilde{\mathbf{v}}}} \tilde{f} \, d\tilde{\mathbf{v}} - \dfrac{e}{\varepsilon_0} \, \tilde{n}_e \,, \end{cases} \tag{2.3}$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{R}^3_{\tilde{\mathbf{x}}}$ is the position variable, $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \in \mathbb{R}^3_{\tilde{\mathbf{v}}}$ is the velocity variable, $t \in \mathbb{R}_+$ is the time variable, $\tilde{f} = \tilde{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, t)$ is the ion distribution function, $\tilde{n}_e$ is the electron density, $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\tilde{\mathbf{x}}, t)$ is the self-consistent electric field generated by the ions and the electrons, $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}(\tilde{\mathbf{x}}, t)$ is the magnetic field which is applied on the considered plasma, $\tilde{\phi} = \tilde{\phi}(\tilde{\mathbf{x}}, t)$ is the electric potential linked with $\tilde{\mathbf{E}}$, e is the elementary charge and m_i is the elementary mass of an ion.

In this model, the external magnetic field $\tilde{\mathbf{B}}$ is assumed to be uniform and carried by the unit vector $\mathbf{e}_3$. We also assume that the electron density $\tilde{n}_e$ is given for any $(\tilde{\mathbf{x}}, t) \in \mathbb{R}^3_{\tilde{\mathbf{x}}} \times \mathbb{R}_+$. Following the same approach as in Bostan [4], Frénod *et al.* [9, 13], Golse *et al.* [14, 15] and Han-Kwan [17], we add the following assumptions:

(i) The magnetic field is supposed to be strong,

(ii) The finite Larmor radius effects are taken into account,

(iii) The ion gyroperiod is supposed to be small.

We define the dimensionless variables and unknowns $\tilde{\mathbf{x}}' = (\tilde{x}_1', \tilde{x}_2', \tilde{x}_3')$, $\tilde{\mathbf{v}}' = (\tilde{v}_1', \tilde{v}_2', \tilde{v}_3')$, t', $\tilde{f}'$, $\tilde{\mathbf{E}}'$ and $\tilde{\phi}'$ by

$$
\begin{gathered}
\tilde{x}_1 = \overline{L_\perp}\, \tilde{x}_1' \,, \qquad \tilde{x}_2 = \overline{L_\perp}\, \tilde{x}_2' \,, \qquad \tilde{x}_3 = \overline{L_{||}}\, \tilde{x}_3' \,, \qquad t = \overline{t}\, t' \,, \qquad \tilde{\mathbf{v}} = \overline{v}\, \tilde{\mathbf{v}}' \,, \\
\tilde{f}(\overline{L_\perp}\, \tilde{x}_1', \overline{L_\perp}\, \tilde{x}_2', \overline{L_{||}}\, \tilde{x}_3', \overline{v}\, \tilde{v}_1', \overline{v}\, \tilde{v}_2', \overline{v}\, \tilde{v}_3', \overline{t}\, t') = \overline{f}\, \tilde{f}'(\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', \tilde{v}_1', \tilde{v}_2', \tilde{v}_3', t') \,, \\
\tilde{\mathbf{E}}(\overline{L_\perp}\, \tilde{x}_1', \overline{L_\perp}\, \tilde{x}_2', \overline{L_{||}}\, \tilde{x}_3', \overline{t}\, t') = \overline{E}\, \tilde{\mathbf{E}}'(\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', t') \,, \\
\tilde{\phi}(\overline{L_\perp}\, \tilde{x}_1', \overline{L_\perp}\, \tilde{x}_2', \overline{L_{||}}\, \tilde{x}_3', \overline{t}\, t') = \overline{\phi}\, \tilde{\phi}'(\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', t') \,.
\end{gathered}
\tag{2.4}
$$

In these definitions, $\overline{L_\perp}$ is the characteristic length in the direction perpendicular to the magnetic field, $\overline{L_{||}}$ is the characteristic length in the direction of the magnetic field, $\overline{v}$ is the characteristic velocity and $\overline{t}$ is the characteristic time. We also rescale the electron density as follows:

$$
\tilde{n}_e(\overline{L_\perp}\, \tilde{x}_1', \overline{L_\perp}\, \tilde{x}_2', \overline{L_{||}}\, \tilde{x}_3', \overline{t}\, t') = \overline{n}\, \tilde{n}_e'(\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', t') \,. \tag{2.5}
$$

Following the assumptions on the magnetic field $\tilde{\mathbf{B}}$, we set $\overline{B}$ as being such that

$$
\tilde{\mathbf{B}} = \overline{B}\, \mathbf{e}_3 \,. \tag{2.6}
$$

Then, we set $\overline{L_{||}}$ as the size of the physical device in game. We also link $\overline{f}$, $\overline{E}$ and $\overline{\phi}$ with the characteristic Debye length $\overline{\lambda_D}$ by

$$
\overline{f} = \frac{\overline{n}}{\overline{v}^3}, \qquad \overline{E} = \frac{\overline{\lambda_D}\, e\, \overline{n}}{\varepsilon_0}, \qquad \overline{\phi} = \frac{\overline{\lambda_D}^2\, e\, \overline{n}}{\varepsilon_0}, \tag{2.7}
$$

and we take $\overline{\lambda_D}$ as the characteristic length in the direction perpendicular to the magnetic field, *i.e.*

$$
\overline{L_\perp} = \overline{\lambda_D} \,. \tag{2.8}
$$

Since we want to take into account the smallness of the gyroperiod and the finite Larmor radius effects, we define the characteristic gyrofrequency $\overline{\omega_i}$ and the characteristic Larmor radius $\overline{r_L}$ as

$$
\overline{\omega_i} = \frac{e\, \overline{B}}{m_i}, \qquad \overline{r_L} = \frac{\overline{v}}{\overline{\omega_i}}. \tag{2.9}
$$

With these notations, the Vlasov-Poisson system is rescaled as follows:

$$
\left\{
\begin{aligned}
& \partial_{t'} \tilde{f}' + \overline{t}\, \overline{v} \begin{pmatrix} \frac{\overline{r_L}}{\overline{\lambda_D}}\, \tilde{v}_1' \\ \frac{\overline{r_L}}{\overline{\lambda_D}}\, \tilde{v}_2' \\ \frac{\overline{r_L}}{\overline{L_{||}}}\, \tilde{v}_3' \end{pmatrix} \cdot \nabla_{\tilde{\mathbf{x}}'} \tilde{f}' + \Big(\frac{\overline{t}\, \overline{\lambda_D}\, e^2\, \overline{n}}{\varepsilon_0\, m_i\, \overline{v}} \tilde{\mathbf{E}}' + \overline{t}\, \overline{\omega_i}\, \tilde{\mathbf{v}}' \times \mathbf{e}_3 \Big) \cdot \nabla_{\tilde{\mathbf{v}}'} \tilde{f}' = 0 \,, \\
& \tilde{f}'(\tilde{\mathbf{x}}', \tilde{\mathbf{v}}', 0) = \tilde{f}^{0'}(\tilde{\mathbf{x}}', \tilde{\mathbf{v}}') \,, \\
& \tilde{\mathbf{E}}' = \begin{pmatrix} -\partial_{\tilde{x}_1'} \tilde{\phi}' \\ -\partial_{\tilde{x}_2'} \tilde{\phi}' \\ -\frac{\overline{\lambda_D}}{\overline{L_{||}}}\, \partial_{\tilde{x}_3'} \tilde{\phi}' \end{pmatrix}, \qquad -\Delta_{(\tilde{x}_1', \tilde{x}_2')} \tilde{\phi}' - \frac{\overline{\lambda_D}^2}{\overline{L_{||}}^2} \partial^2_{\tilde{x}_3'} \tilde{\phi}' = \int_{\mathbb{R}^3} \tilde{f}'\, d\tilde{\mathbf{v}}' - \tilde{n}_e' \,.
\end{aligned}
\right.
\tag{2.10}
$$

Taking into account the finite Larmor radius effects consists in considering a regime in which the Larmor radius is of the order of the Debye length. This implies

$$\frac{\overline{r_L}}{\overline{\lambda_D}} = 1\,. \tag{2.11}$$

Since the magnetic field is assumed to be strong, the Larmor radius is small when compared with the size of the physical domain. Then it is natural to take

$$\frac{\overline{r_L}}{\overline{L_{||}}} = \epsilon\,, \tag{2.12}$$

where $\epsilon > 0$ is small.
Assumption (iii) can be translated in terms of characteristic scales by

$$\bar{t}\,\overline{\omega_i} = \frac{1}{\epsilon}. \tag{2.13}$$

Assumption (i) means that the magnetic force is much stronger than the electric force, so we consider

$$\frac{\overline{E}\,e}{\overline{v}\,m_i\,\overline{\omega_i}} = \epsilon\,. \tag{2.14}$$

Then, removing the primes and adding ϵ in subscript, the rescaled Vlasov-Poisson model writes

$$\begin{cases} \partial_t \tilde{f}_\epsilon + \dfrac{1}{\epsilon}\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \cdot \nabla_{(\tilde{x}_1,\tilde{x}_2)} \tilde{f}_\epsilon + \tilde{v}_3\,\partial_{\tilde{x}_3}\tilde{f}_\epsilon + \Big(\tilde{\mathbf{E}}_\epsilon + \dfrac{1}{\epsilon}\tilde{\mathbf{v}}\times\mathbf{e}_3\Big)\cdot\nabla_{\tilde{\mathbf{v}}}\tilde{f}_\epsilon = 0\,, \\ \tilde{f}_\epsilon(\tilde{\mathbf{x}},\tilde{\mathbf{v}},0) = \tilde{f}^0_\epsilon(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\,, \\ \begin{pmatrix} -\nabla_{(\tilde{x}_1,\tilde{x}_2)}\tilde{\phi}_\epsilon \\ -\epsilon\,\partial_{\tilde{x}_3}\tilde{\phi}_\epsilon \end{pmatrix} = \tilde{\mathbf{E}}_\epsilon\,, \qquad -\Delta_{(\tilde{x}_1,\tilde{x}_2)}\tilde{\phi}_\epsilon - \epsilon^2\,\partial^2_{\tilde{x}_3}\tilde{\phi}_\epsilon = \displaystyle\int_{\mathbb{R}^3}\tilde{f}_\epsilon\,d\tilde{\mathbf{v}} - \tilde{n}_e\,, \end{cases} \tag{2.15}$$

which is the model studied in previous works of Frénod & Sonnendrücker [13], Golse & Saint-Raymond [14, 15], and Bostan [4].

2.2 Previous results

In this paragraph, we recall two main results about the asymptotic behavior of the sequences $(\tilde{f}_\epsilon)_{\epsilon>0}$ and $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ when ϵ goes to 0. The first one is based on the use of two-scale convergence and homogenization techniques developed by Allaire [3] and Nguetseng [24], and was established by Frénod and Sonnendrücker in [13]. The second one relies on compactness arguments and was proved by Bostan in [4]. After recalling these two results, we discuss the main differences between them. These differences are the source of the motivation of the present paper.

In order to simplify, we consider that the whole model (2.15) does not depend on $\tilde{x}_3$ nor $\tilde{v}_3$, and we assume that $\tilde{f}^0_\epsilon = \tilde{f}^0$ for all ϵ. Then, it is reduced to a singularly perturbed 2D Vlasov-Poisson model of the form

$$\begin{cases} \partial_t \tilde{f}_\epsilon + \dfrac{1}{\epsilon}\tilde{\mathbf{v}}\cdot\nabla_{\tilde{\mathbf{x}}}\tilde{f}_\epsilon + \Big(\tilde{\mathbf{E}}_\epsilon + \dfrac{1}{\epsilon}\begin{pmatrix} \tilde{v}_2 \\ -\tilde{v}_1 \end{pmatrix}\Big)\cdot\nabla_{\tilde{\mathbf{v}}}\tilde{f}_\epsilon = 0\,, \\ \tilde{f}_\epsilon(\tilde{\mathbf{x}},\tilde{\mathbf{v}},0) = \tilde{f}^0(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\,, \\ -\nabla_{\tilde{\mathbf{x}}}\tilde{\phi}_\epsilon = \tilde{\mathbf{E}}_\epsilon\,, \qquad -\Delta_{\tilde{\mathbf{x}}}\tilde{\phi}_\epsilon = \displaystyle\int_{\mathbb{R}^2}\tilde{f}_\epsilon\,d\tilde{\mathbf{v}} - \tilde{n}_e\,, \end{cases} \tag{2.16}$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2) \in \mathbb{R}^2$.

The following theorem can be attributed to Frénod and Sonnendrücker [13] (see Theorem 1.5) even if the setting of this paper is a charged particle beam Vlasov-Poisson model not involving any electron density. Nonetheless, the proof of [13] works again in the setting of model (2.16) which involves an electron density.

Theorem 1 (Frénod & Sonnendrücker [13])**.** *We assume that, for a fixed $p \geq 2$, $\tilde{f}^0$ is in $L^1(\mathbb{R}^4) \cap L^p(\mathbb{R}^4)$, is positive everywhere and such that*

$$\int_{\mathbb{R}^4} |\tilde{\mathbf{v}}|^2 \, \tilde{f}^0(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) \, d\tilde{\mathbf{x}} \, d\tilde{\mathbf{v}} < +\infty \,. \tag{2.17}$$

We also assume that $\tilde{n}_e$ does not depend on t, is in $L^1(\mathbb{R}^2) \cap L^{3/2}(\mathbb{R}^2)$ and satisfies

$$\int_{\mathbb{R}^4} \tilde{f}^0(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) \, d\tilde{\mathbf{x}} \, d\tilde{\mathbf{v}} = \int_{\mathbb{R}^2} \tilde{n}_e(\tilde{\mathbf{x}}) \, d\tilde{\mathbf{x}} \,. \tag{2.18}$$

Then, the sequence $(\tilde{f}_\epsilon, \tilde{\mathbf{E}}_\epsilon)_{\epsilon > 0}$ is bounded $L^\infty\big(0,T; L^p(\mathbb{R}^4)\big) \times \Big(L^\infty\big(0,T; W^{1,3/2}(\mathbb{R}^2)\big)\Big)^2$ independently of ϵ. Furthermore, by extracting some subsequences,

$$\begin{array}{lcll} \tilde{f}_\epsilon & \longrightarrow & \tilde{F} = \tilde{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, \tau, t) & \textit{two-scale in } L^\infty\Big(0,T; L^\infty_\#\big(0, 2\pi; L^p(\mathbb{R}^4)\big)\Big), \\ \tilde{\mathbf{E}}_\epsilon & \longrightarrow & \tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\tilde{\mathbf{x}}, \tau, t) & \textit{two-scale in } \bigg(L^\infty\Big(0,T; L^\infty_\#\big(0, 2\pi; W^{1,3/2}(\mathbb{R}^2)\big)\Big)\bigg)^2. \end{array} \tag{2.19}$$

Moreover, $\tilde{F}$ is linked with $\tilde{G} = \tilde{G}(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, t) \in L^\infty\big(0,T; L^p(\mathbb{R}^4)\big)$ by the relation

$$\tilde{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, \tau, t) = \tilde{G}\big(\tilde{\mathbf{x}} + \mathcal{R}(-\tau)\, \tilde{\mathbf{v}}, R(-\tau)\, \tilde{\mathbf{v}}, t\big) \,, \tag{2.20}$$

and $(\tilde{G}, \tilde{\mathcal{E}})$ is the solution of

$$\begin{cases} \partial_t \tilde{G}(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, t) + \left[\displaystyle\int_0^{2\pi} \mathcal{R}(-\sigma)\, \tilde{\mathcal{E}}\big(\tilde{\mathbf{y}} + \mathcal{R}(\sigma)\, \tilde{\mathbf{u}}, \sigma, t\big) \, d\sigma \right] \cdot \nabla_{\tilde{\mathbf{y}}} \tilde{G}(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, t) \\ \qquad\qquad + \left[\displaystyle\int_0^{2\pi} R(-\sigma)\, \tilde{\mathcal{E}}\big(\tilde{\mathbf{y}} + \mathcal{R}(\sigma)\, \tilde{\mathbf{u}}, \sigma, t\big) \, d\sigma \right] \cdot \nabla_{\tilde{\mathbf{u}}} \tilde{G}(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, t) = 0 \,, \\ \tilde{G}(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}, 0) = \dfrac{1}{2\pi} \, \tilde{f}^0(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}) \,, \\ \tilde{\mathcal{E}}(\tilde{\mathbf{x}}, \tau, t) = -\nabla_{\tilde{\mathbf{x}}} \tilde{\Phi}(\tilde{\mathbf{x}}, \tau, t) \,, \\ -\Delta_{\tilde{\mathbf{x}}} \tilde{\Phi}(\tilde{\mathbf{x}}, \tau, t) = \displaystyle\int_{\mathbb{R}^2} \tilde{G}\big(\tilde{\mathbf{x}} + \mathcal{R}(-\tau)\, \tilde{\mathbf{v}}, R(-\tau)\, \tilde{\mathbf{v}}, t\big) \, d\tilde{\mathbf{v}} - \dfrac{1}{2\pi} \, \tilde{n}_e(\tilde{\mathbf{x}}) \,, \end{cases} \tag{2.21}$$

where

$$\mathcal{R}(\tau) = \begin{pmatrix} \sin\tau & 1 - \cos\tau \\ \cos\tau - 1 & \sin\tau \end{pmatrix} , \quad R(\tau) = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} . \tag{2.22}$$

In this theorem, $L^\infty_\#(0, 2\pi; L^p(\mathbb{R}^4))$ stands for the space of functions $\tilde{h} = \tilde{h}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, \tau)$ being in $L^\infty(0, 2\pi; L^p(\mathbb{R}^4))$ and 2π-periodic with respect to τ.

As a consequence of this theorem, extracting some subsequences, we have

$$\begin{array}{lcll} \tilde{f}_\epsilon & \overset{*}{\rightharpoonup} & \tilde{f} & \text{in } L^\infty\big(0,T; L^p(\mathbb{R}^4)\big), \\ \tilde{\mathbf{E}}_\epsilon & \overset{*}{\rightharpoonup} & \tilde{\mathbf{E}} & \text{in } \Big(L^\infty\big(0,T; W^{1,3/2}(\mathbb{R}^2)\big)\Big)^2, \end{array} \tag{2.23}$$

where

$$\tilde{f}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)=\int_0^{2\pi}\tilde{F}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},\tau,t)\,d\tau\,,\qquad\text{and}\qquad \tilde{\mathbf{E}}(\tilde{\mathbf{x}},t)=\int_0^{2\pi}\tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)\,d\tau\,. \tag{2.24}$$

By using the relation between $\tilde{F}$ and $\tilde{G}$, we can easily remark that $(\tilde{f},\tilde{\mathbf{E}})$ is solution of

$$\left\{\begin{array}{l}
\partial_t\tilde{f}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)\\
\qquad+\displaystyle\int_0^{2\pi}\left[\int_0^{2\pi}\mathcal{R}(\tau-\sigma)\,\tilde{\mathcal{E}}\big(\tilde{\mathbf{x}}+\mathcal{R}(\sigma-\tau)\,\tilde{\mathbf{v}},\sigma,t\big)\,d\sigma\right]\cdot\nabla_{\tilde{\mathbf{x}}}\tilde{F}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},\tau,t)\,d\tau\\
\qquad+\displaystyle\int_0^{2\pi}\left[\int_0^{2\pi}R(\tau-\sigma)\,\tilde{\mathcal{E}}\big(\tilde{\mathbf{x}}+\mathcal{R}(\sigma-\tau)\,\tilde{\mathbf{v}},\sigma,t\big)\,d\sigma\right]\cdot\nabla_{\tilde{\mathbf{v}}}\tilde{F}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},\tau,t)\,d\tau=0\,,\\
\tilde{f}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},0)=\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}^0\big(\tilde{\mathbf{x}}+\mathcal{R}(-\tau)\,\tilde{\mathbf{v}},R(-\tau)\,\tilde{\mathbf{v}}\big)\,d\tau\,,\\
-\nabla_{\tilde{\mathbf{x}}}\tilde{\phi}(\tilde{\mathbf{x}},t)=\tilde{\mathbf{E}}(\tilde{\mathbf{x}},t)\,,\qquad -\Delta_{\tilde{\mathbf{x}}}\tilde{\phi}(\tilde{\mathbf{x}},t)=\displaystyle\int_{\mathbb{R}^2}\tilde{f}(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)\,d\tilde{\mathbf{v}}-\tilde{n}_e(\tilde{\mathbf{x}})\,.
\end{array}\right. \tag{2.25}$$

We notice that these equations still involve $\tilde{F}$ and $\tilde{\mathcal{E}}$. In order to make these dependencies disappear, Bostan has proposed in [4] a reformulation of the Vlasov equation in guiding-center coordinates. Before presenting it, we introduce the sequence $(\breve{f}_\epsilon)_{\epsilon>0}$ defined by

$$\tilde{f}_\epsilon(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)=\breve{f}_\epsilon\Big(\tilde{\mathbf{x}}+\begin{pmatrix}\tilde{v}_2\\-\tilde{v}_1\end{pmatrix},\tilde{\mathbf{v}},t\Big)\,, \tag{2.26}$$

and, in the same spirit, we define the initial guiding-center distribution $\breve{f}^0$ by

$$\tilde{f}^0(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)=\breve{f}^0\Big(\tilde{\mathbf{x}}+\begin{pmatrix}\tilde{v}_2\\-\tilde{v}_1\end{pmatrix},\tilde{\mathbf{v}},t\Big)\,. \tag{2.27}$$

Theorem 2 (Bostan [4]). *We assume that $\tilde{n}_e=1$, and that $\tilde{f}^0$ is 2π-periodic in $\tilde{x}_1$ and $\tilde{x}_2$, is positive everywhere and satisfies*

$$\int_{\mathbb{R}^2}\int_0^{2\pi}\int_0^{2\pi}\tilde{f}^0(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\,d\tilde{\mathbf{x}}\,d\tilde{\mathbf{v}}=1\,,\quad \int_{\mathbb{R}^2}\int_0^{2\pi}\int_0^{2\pi}|\tilde{\mathbf{v}}|^2\,\tilde{f}^0(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\,d\tilde{\mathbf{x}}\,d\tilde{\mathbf{v}}<+\infty\,. \tag{2.28}$$

We also assume that there exists $\tilde{F}^0\in L^\infty(\mathbb{R}_+)\cap L^1(\mathbb{R}_+;r\,dr)$ such that

$$\forall\,(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\in[0,2\pi]^2\times\mathbb{R}^2\,,\qquad \tilde{f}^0(\tilde{\mathbf{x}},\tilde{\mathbf{v}})\le\tilde{F}^0\big(|\tilde{\mathbf{v}}|\big)\,. \tag{2.29}$$

We also assume that $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ admits a strong limit denoted with $\tilde{\mathbf{E}}$ in $\big(L^2\big(0,T;L^2_\#\big([0,2\pi]^2\big)\big)\big)^2$. Then, up to a subsequence, $\breve{f}_\epsilon$ weakly- converges to a function $\breve{f}=\breve{f}(\mathbf{x},\mathbf{v},t)$ in $L^\infty\big([0,T)\times\mathbb{R}^2;L^\infty_\#\big([0,2\pi]^2\big)\big)$ verifying*

$$\breve{f}(\mathbf{x},\mathbf{v},t)=\frac{1}{2\pi}\,g\big(\mathbf{x},\frac{|\mathbf{v}|^2}{2},t\big)\,, \tag{2.30}$$

where $g=g(\mathbf{x},k,t)$ is the solution of

$$\left\{\begin{array}{l}
\partial_t g+\langle\mathcal{E}_2\rangle\,\partial_{x_1}g-\langle\mathcal{E}_1\rangle\,\partial_{x_2}g=0\,,\\
g(\mathbf{x},k,0)=\displaystyle\int_0^{2\pi}\breve{f}^0(\mathbf{x},\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)\,d\alpha\,,\\
\langle\mathcal{E}\rangle(\mathbf{x},k,t)=\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\tilde{\mathbf{E}}\big(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,t\big)\,d\alpha\,,\\
-\nabla_{\tilde{\mathbf{x}}}\tilde{\phi}(\tilde{\mathbf{x}},t)=\tilde{\mathbf{E}}(\tilde{\mathbf{x}},t)\,,\\
-\Delta_{\tilde{\mathbf{x}}}\tilde{\phi}(\tilde{\mathbf{x}},t)=\displaystyle\frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi}g\big(\tilde{x}_1+\sqrt{2k}\,\sin\alpha,\tilde{x}_2-\sqrt{2k}\,\cos\alpha,k,t\big)\,d\alpha\,dk-1\,.
\end{array}\right. \tag{2.31}$$

In this theorem, $L^2_{\#}\big([0,2\pi]^2\big)$ stands for the space of functions $\tilde h = \tilde h(\tilde{\mathbf{x}})$ being in $L^2\big([0,2\pi]^2\big)$ and 2π-periodic with respect to $\tilde x_1$ and $\tilde x_2$.

This last result introduces a mathematical justification of the approximation of the Vlasov-Poisson model (2.16) by the finite Larmor radius model which is exactly (2.30)-(2.31). However, in order to prove this convergence result, Bostan considered stronger assumptions on $\tilde f^0$ and $\tilde n_e$ than needed to get existence of the weak-* limit $(\tilde f, \tilde{\mathbf{E}})$ from Theorem 1: the initial distribution $\tilde f^0$ is supposed to be 2π-periodic in $\tilde x_1$ and $\tilde x_2$ and $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ is supposed to admit a strong limit in some Banach space.

Even if Theorems 1 and 2 induce common results, *i.e.* the convergence in a weak sense of the solution of the 2D Vlasov-Poisson model with a strong magnetic field towards the solution of a 2D finite Larmor radius model, they are built on quite different assumptions. Then, it seems pertinent to gather the two-scale and weak-* convergence results within a unique theorem based on common assumptions which are as weak as possible. This is what we do in the next sections.

3 Synthetic convergence result

This section is devoted to the gathering of two-scale convergence and weak-* convergence results under a common assumption set. For this purpose, we firstly reformulate the Vlasov-Poisson system (2.16) in a new set of variables which are so-called *canonical gyrokinetic coordinates*. Then, we prove a two-scale convergence result from which we are able to deduce a weak-* convergence corollary straightforwardly.

3.1 Reformulation of Vlasov equation

Following the ideas of Littlejohn [20], Lee [18, 19], and Brizard *et al.* [5, 6], we define the variables $(x_1, x_2, k, \alpha) \in \mathbb{R}^2 \times \mathbb{R}_+ \times [0, 2\pi]$ by linking them with $(\tilde x_1, \tilde x_2, \tilde v_1, \tilde v_2) \in \mathbb{R}^4$ by

$$\left\{ \begin{array}{rclrcl} \tilde x_1 & = & x_1 - \tilde v_2\,, & \tilde v_1 & = & \sqrt{2k}\,\cos\alpha\,, \\ \tilde x_2 & = & x_2 + \tilde v_1\,, & \tilde v_2 & = & \sqrt{2k}\,\sin\alpha\,. \end{array} \right. \tag{3.1}$$

This set of variables is so-called *canonical gyrokinetic coordinates*: indeed, if we define the characteristics X_1, X_2, K, A linked with x_1, x_2, k, α by

$$\left\{ \begin{array}{rclrcl} \tilde X_1 & = & X_1 - \tilde V_2\,, & \tilde V_1 & = & \sqrt{2K}\,\cos A\,, \\ \tilde X_2 & = & X_2 + \tilde V_1\,, & \tilde V_2 & = & \sqrt{2K}\,\sin A\,, \end{array} \right. \tag{3.2}$$

where $\tilde X_1, \tilde X_2, \tilde V_1, \tilde V_2$ are the characteristics associated with the Vlasov equation (2.16.a), *i.e.* satisfying

$$\left\{ \begin{array}{rclrcl} \partial_t \tilde X_1(t) & = & \dfrac{1}{\epsilon}\,\tilde V_1(t)\,, & \partial_t \tilde V_1(t) & = & \tilde E_{\epsilon,1}\big(\tilde X_1(t), \tilde X_2(t), t\big) + \dfrac{1}{\epsilon}\,\tilde V_2(t)\,, \\ \\ \partial_t \tilde X_2(t) & = & \dfrac{1}{\epsilon}\,\tilde V_2(t)\,, & \partial_t \tilde V_2(t) & = & \tilde E_{\epsilon,2}\big(\tilde X_1(t), \tilde X_2(t), t\big) - \dfrac{1}{\epsilon}\,\tilde V_1(t)\,, \end{array} \right. \tag{3.3}$$

we have

$$\left\{ \begin{array}{rcl} \partial_t X_1(t) & = & -\partial_{x_2} H_\epsilon\big(X_1(t), X_2(t), K(t), A(t), t\big)\,, \\ \partial_t X_2(t) & = & \partial_{x_1} H_\epsilon\big(X_1(t), X_2(t), K(t), A(t), t\big)\,, \\ \partial_t K(t) & = & \partial_\alpha H_\epsilon\big(X_1(t), X_2(t), K(t), A(t), t\big)\,, \\ \partial_t A(t) & = & -\partial_k H_\epsilon\big(X_1(t), X_2(t), K(t), A(t), t\big)\,, \end{array} \right. \tag{3.4}$$

where the hamiltonian function H_ϵ is defined by

$$H_\epsilon(x_1,x_2,k,\alpha,t) = \frac{k}{\epsilon} + \phi_\epsilon(x_1,x_2,k,\alpha,t)\,, \tag{3.5}$$

and ϕ_ϵ is linked with $\tilde{\phi}_\epsilon$ by the relation

$$\phi_\epsilon(x_1,x_2,k,\alpha,t) = \tilde{\phi}_\epsilon\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, t\big)\,. \tag{3.6}$$

Then it is straightforward to see that, in the gyrokinetic canonical coordinates, the Vlasov-Poisson system (2.16) has the following shape:

$$\begin{cases} \partial_t f_\epsilon + E_{\epsilon,2}\,\partial_{x_1} f_\epsilon - E_{\epsilon,1}\,\partial_{x_2} f_\epsilon + \sqrt{2k}\,(E_{\epsilon,1}\,\cos\alpha + E_{\epsilon,2}\,\sin\alpha)\,\partial_k f_\epsilon \\ \qquad\qquad + \dfrac{E_{\epsilon,2}\,\cos\alpha - E_{\epsilon,1}\,\sin\alpha}{\sqrt{2k}}\,\partial_\alpha f_\epsilon - \dfrac{1}{\epsilon}\,\partial_\alpha f_\epsilon = 0\,, \\ f_\epsilon(\mathbf{x},k,\alpha,0) = \tilde{f}^0\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\sin\alpha\big)\,, \\ \mathbf{E}_\epsilon(\mathbf{x},k,\alpha,t) = \tilde{\mathbf{E}}_\epsilon\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, t\big)\,, \\ \tilde{\mathbf{E}}_\epsilon(\tilde{\mathbf{x}},t) = -\nabla_{\tilde{\mathbf{x}}}\tilde{\phi}_\epsilon(\tilde{\mathbf{x}},t)\,, \\ -\Delta_{\tilde{\mathbf{x}}}\tilde{\phi}_\epsilon(\tilde{\mathbf{x}},t) = \displaystyle\int_0^{2\pi}\!\!\int_0^{+\infty} f_\epsilon\big(\tilde{x}_1 + \sqrt{2k}\,\sin\alpha, \tilde{x}_2 - \sqrt{2k}\,\cos\alpha, k, \alpha, t\big)\,dk\,d\alpha - \tilde{n}_e(\tilde{\mathbf{x}},t)\,, \end{cases} \tag{3.7}$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$ and $\mathbf{x} = (x_1, x_2)$, and where $f_\epsilon = f_\epsilon(\mathbf{x},k,\alpha,t)$ and $\mathbf{E}_\epsilon = \mathbf{E}_\epsilon(\mathbf{x},k,\alpha,t)$ are linked with $\tilde{f}^\epsilon$ and $\tilde{\mathbf{E}}_\epsilon$ by

$$\begin{aligned} f_\epsilon(\mathbf{x},k,\alpha,t) &= \tilde{f}_\epsilon\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\sin\alpha, t\big)\,, \\ \mathbf{E}_\epsilon(\mathbf{x},k,\alpha,t) &= \tilde{\mathbf{E}}_\epsilon\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, t\big)\,. \end{aligned} \tag{3.8}$$

3.2 Two-scale convergence

We set the following notations

$$\Omega = \mathbb{R}^2 \times \mathbb{R}_+ \times [0,2\pi]\,, \qquad \Gamma = \mathbb{R}^2 \times \mathbb{R}_+\,, \qquad \mathcal{S} = \mathbb{R}_+ \times [0,2\pi]\,, \tag{3.9}$$

and we consider the following Banach spaces, involving periodicity with respect to α:

$$\begin{aligned} L^p_\#\big(0,2\pi;L^p(\Gamma)\big) &= \Big\{ f \in L^p(\Omega)\,:\, f \text{ is periodic in } \alpha \Big\}\,, \\ W^{1,p}_\#\big(0,2\pi;W^{1,p}(\Gamma)\big) &= \Big\{ f \in W^{1,p}(\Omega), f(.,.,0) = f(.,.,2\pi)\,:\, f \text{ is periodic in } \alpha \Big\}\,, \\ W^{2,p}_\#\big(0,2\pi;W^{2,p}(\Gamma)\big) &= \Big\{ f \in W^{2,p}(\Omega), f(.,.,0) = f(.,.,2\pi), \partial_\alpha f(.,.,0) = \partial_\alpha f(.,.,2\pi)\,: \\ &\qquad\qquad\qquad\qquad f \text{ is periodic in } \alpha \Big\}\,, \end{aligned}$$

and we can state the following theorem.

Theorem 3. *We assume that, for a fixed $p \geq 2$, $\tilde{f}^0$ and $\tilde{n}_e$ satisfy the assumptions of Theorem 1. Then sequences $(f_\epsilon)_{\epsilon>0}$ and $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ of system (3.7) are bounded independently of ϵ*

in $L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(\Gamma))\big)$ *and* $\big(L^\infty\big(0,T;W^{1,3/2}(\mathbb{R}^2)\big)\big)^2$ *respectively. As a consequence, there exist* $F = F(\mathbf{x},k,\alpha,\tau,t)$ *and* $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)$ *such that, extracting some subsequences,*

$$\begin{array}{lcll} f_\epsilon & \longrightarrow & F & \textit{two-scale in } L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;L^p_\#(0,2\pi;L^p(\Gamma))\big)\big), \\ \tilde{\mathbf{E}}_\epsilon & \longrightarrow & \tilde{\mathcal{E}} & \textit{two-scale in } \big(L^\infty\big(0,T;L^\infty_\#(0,2\pi;W^{1,3/2}(\mathbb{R}^2))\big)\big)^2. \end{array} \tag{3.10}$$

Furthermore, there exist $G = G(\mathbf{x},k,\alpha,t) \in L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(\Gamma))\big)$ *and* $\mathcal{E} = \mathcal{E}(\mathbf{x},k,\alpha,\tau,t) \in \big(L^\infty\big(0,T;L^\infty_\#(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(\Gamma)))\big)\big)^2$ *such that*

$$F(\mathbf{x},k,\alpha,\tau,t) = G(\mathbf{x},k,\alpha+\tau,t)\,, \tag{3.11}$$

$$\mathcal{E}(\mathbf{x},k,\alpha,\tau,t) = \tilde{\mathcal{E}}\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha,\tau,t\big)\,, \tag{3.12}$$

and verifying

$$\left\{\begin{array}{l} \partial_t G + \langle\mathcal{E}_2\rangle\,\partial_{x_1} G - \langle\mathcal{E}_1\rangle\,\partial_{x_2} G + \langle\mathcal{F}_\alpha\rangle\,\partial_\alpha G = 0\,, \\[2mm] G(\mathbf{x},k,\alpha,0) = \dfrac{1}{2\pi}\tilde{f}^0\big(x_1 - \sqrt{2k}\,\sin\alpha, x_2 + \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\cos\alpha, \sqrt{2k}\,\sin\alpha\big)\,, \\[2mm] -\nabla_{\tilde{\mathbf{x}}}\tilde{\Phi}(\tilde{\mathbf{x}},\tau,t) = \tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)\,, \\[2mm] -\Delta_{\tilde{\mathbf{x}}}\tilde{\Phi}(\tilde{\mathbf{x}},\tau,t) = \displaystyle\int_{\mathcal{S}} G\big(\tilde{x}_1 + \sqrt{2k}\,\sin\alpha, \tilde{x}_2 - \sqrt{2k}\,\cos\alpha, k, \alpha+\tau, t\big)\,dk\,d\alpha \\ \hfill -\dfrac{1}{2\pi}\tilde{n}_e(\tilde{\mathbf{x}})\,, \\[2mm] \mathcal{F}_\alpha(\mathbf{x},k,\alpha,\tau,t) = \dfrac{\mathcal{E}_2(\mathbf{x},k,\alpha,\tau,t)\,\cos\alpha - \mathcal{E}_1(\mathbf{x},k,\alpha,\tau,t)\,\sin\alpha}{\sqrt{2k}}\,, \end{array}\right. \tag{3.13}$$

where the notation $\langle\cdot\rangle$ *stands for*

$$\langle u\rangle(\mathbf{x},k,t) = \int_0^{2\pi} u(\mathbf{x},k,-\tau,\tau,t)\,d\tau\,. \tag{3.14}$$

Proof of Theorem 3. Several parts of this proof are only sketched since they can be redundant with [13]. However, more details can be found in Mouton [22].

Following the same way as in [13], we prove that, under the assumptions of Theorem 1, we have

$$\big\|f_\epsilon(\cdot,t)\big\|_{L^p_\#(0,2\pi;L^p(\Gamma))} = \|\tilde{f}^0\|_{L^p(\mathbb{R}^4)}\,, \forall\, t \geq 0\,, \tag{3.15}$$

and, defining $\tilde{\rho}_\epsilon$ as

$$\tilde{\rho}_\epsilon(\tilde{\mathbf{x}},t) = \int_{\mathcal{S}} f_\epsilon\big(\tilde{x}_1 + \sqrt{2k}\,\sin\alpha, \tilde{x}_2 - \sqrt{2k}\,\cos\alpha, k, \alpha, t\big)\,dk\,d\alpha\,, \tag{3.16}$$

that the sequence $(\tilde{\rho}_\epsilon)_{\epsilon>0}$ is bounded in $L^\infty\big(0,T;L^{3/2}(\mathbb{R}^2)\big)$ independently of ϵ. Then, we deduce that $(\tilde{\phi}_\epsilon)_{\epsilon>0}$ and $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ are bounded independently of ϵ in $L^\infty\big(0,T;W^{2,3/2}(\mathbb{R}^2)\big)$ and $\big(L^\infty\big(0,T;W^{1,3/2}(\mathbb{R}^2)\big)\big)^2$ respectively. As a consequence, there exist $F = F(\mathbf{x},k,\alpha,\tau,t)$, $\tilde{\Phi} = \tilde{\Phi}(\tilde{\mathbf{x}},\tau,t)$ and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)$ such that

$$\begin{array}{lcll} f_\epsilon & \longrightarrow & F & \textit{two-scale in } L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;L^p_\#(0,2\pi;L^p(\Gamma))\big)\big)\,, \\ \tilde{\phi}_\epsilon & \longrightarrow & \tilde{\Phi} & \textit{two-scale in } L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;W^{2,3/2}(\mathbb{R}^2)\big)\big)\,, \\ \tilde{\mathbf{E}}_\epsilon & \longrightarrow & \tilde{\mathcal{E}} & \textit{two-scale in } \big(L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;W^{1,3/2}(\mathbb{R}^2)\big)\big)\big)^2\,. \end{array} \tag{3.17}$$

Considering a compact set $K \subset \Gamma$, we easily remark that the sequence $(\phi_\epsilon)_{\epsilon>0}$ defined by (3.6) is bounded in $L^\infty\big(0,T;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)$ and that all its second order derivatives except $\partial_k^2\phi_\epsilon$ are bounded independently of ϵ in $L^\infty\big(0,T;L^{3/2}_\#\big(0,2\pi;L^{3/2}(K)\big)\big)$. The sequence $(\mathbf{E}_\epsilon)_{\epsilon>0}$ defined by (3.8.b) is bounded in $\big(L^\infty\big(0,T;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)\big)^2$ independently of ϵ. As a consequence, we claim that there exist $\Phi = \Phi(\mathbf{x},k,\alpha,\tau,t)$ and $\mathcal{E} = \mathcal{E}(\mathbf{x},k,\alpha,\tau,t)$ such that

$$\begin{array}{llll} \phi_\epsilon & \longrightarrow & \Phi & \textit{two-scale in } L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)\big)\,, \\ \mathbf{E}_\epsilon & \longrightarrow & \mathcal{E} & \textit{two-scale in } \big(L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)\big)\big)^2\,. \end{array} \tag{3.18}$$

Furthermore, we remark that Φ and $\mathcal{E}$ are linked with $\tilde{\Phi}$ and $\tilde{\mathcal{E}}$ by the formula

$$\begin{aligned} \mathcal{E}(\mathbf{x},k,\alpha,\tau,t) &= \tilde{\mathcal{E}}\big(x_1-\sqrt{2k}\,\sin\alpha, x_2+\sqrt{2k}\,\cos\alpha,\tau,t\big)\,, \\ \Phi(\mathbf{x},k,\alpha,\tau,t) &= \tilde{\Phi}\big(x_1-\sqrt{2k}\,\sin\alpha, x_2+\sqrt{2k}\,\cos\alpha,\tau,t\big)\,. \end{aligned} \tag{3.19}$$

Then the vector function $\mathbf{A}_\epsilon$ defined by

$$\mathbf{A}_\epsilon = \begin{pmatrix} -\partial_{x_2}\phi_\epsilon \\ \partial_{x_1}\phi_\epsilon \\ \partial_\alpha\phi_\epsilon \\ -\partial_k\phi_\epsilon \end{pmatrix} = \begin{pmatrix} E_{\epsilon,2} \\ -E_{\epsilon,1} \\ \sqrt{2k}\,\big(E_{\epsilon,1}\,\cos\alpha + E_{\epsilon,2}\,\sin\alpha\big) \\ \dfrac{E_{\epsilon,2}\,\cos\alpha - E_{\epsilon,1}\,\sin\alpha}{\sqrt{2k}} \end{pmatrix}\,, \tag{3.20}$$

has its three first components which are bounded in $L^\infty\big(0,T;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)$, independently of ϵ and its fourth one in $L^\infty\big(0,T;L^{3/2}_\#(0,2\pi;L^{3/2}(K))\big)$ and admits a two-scale limit denoted $\mathcal{A} = \mathcal{A}(\mathbf{x},k,\alpha,\tau,t)$ in $\big(L^\infty\big(0,T;L^\infty_\#(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K)))\big)\big)^3 \times L^\infty\big(0,T;L^\infty_\#(0,2\pi;L^{3/2}_\#(0,2\pi;L^{3/2}(K))\big)$. The convergence of the three first components is the consequence of classical embedding of Sobolev spaces in L^p spaces. Concerning the convergence of the fourth one we need to use that $E_{\epsilon,1}$ and $E_{\epsilon,2}$ are bounded in $L^\infty\big(0,T;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K)\big)$ and consequently in $L^\infty\big(0,T;W^{1,3/2}_\#(0,2\pi;L^6(K)\big)$ and that $(2k)^{-1/2}$ is, for any $k_{\max}$ and any $q<2$, in $L^q(0,k_{\max})$.

This vector function is linked with Φ and $\mathcal{E}$ as follows:

$$\mathcal{A} = \begin{pmatrix} -\partial_{x_2}\Phi \\ \partial_{x_1}\Phi \\ \partial_\alpha\Phi \\ -\partial_k\Phi \end{pmatrix} = \begin{pmatrix} \mathcal{E}_2 \\ -\mathcal{E}_1 \\ \sqrt{2k}\,\big(\mathcal{E}_1\,\cos\alpha + \mathcal{E}_2\,\sin\alpha\big) \\ \dfrac{\mathcal{E}_2\,\cos\alpha - \mathcal{E}_1\,\sin\alpha}{\sqrt{2k}} \end{pmatrix}\,. \tag{3.21}$$

In order to establish the two-scale limit model, we cannot simply apply Theorem 1.3 of [13]: indeed, the formulation (3.7.a) of Vlasov equation does not fit with the assumptions which are needed for applying this theorem since the differential operator $f \mapsto -\dfrac{1}{\epsilon}\partial_\alpha f$ cannot be written under the form

$$f \mapsto \frac{1}{\epsilon}\Big(\mathbb{M}\begin{pmatrix} x_1 \\ x_2 \\ k \\ \alpha \end{pmatrix} + \mathbf{N}\Big)\cdot\nabla f\,, \tag{3.22}$$

where $\mathbb{M}$ is a constant square matrix satisfying $Tr(\mathbb{M}) = 0$, and $\mathbf{N} \in Im(\mathbb{M})$. However, the approach which is considered in [13] can be adapted to the present case.

Firstly, we prove that there exists a function G such that $F(\mathbf{x},k,\alpha,\tau,t) = G(\mathbf{x},k,\alpha+\tau,t)$. To reach such a result, we consider a test function $\psi = \psi(\mathbf{x},k,\alpha,\tau,t)$ on $\Omega\times[0,2\pi]\times[0,T]$ which is 2π-periodic in α and τ. If we multiply (3.7.a) by $\psi(\mathbf{x},k,\alpha,\frac{t}{\epsilon},t)$ and integrate over $\Omega\times[0,T]$, we obtain

$$\begin{aligned}\int_0^T\int_\Omega f_\epsilon(\mathbf{x},k,\alpha,t)\Big[\partial_t\psi\big(\mathbf{x},k,\alpha,\frac{t}{\epsilon},t\big) + \frac{1}{\epsilon}\,\partial_\tau\psi\big(\mathbf{x},k,\alpha,\frac{t}{\epsilon},t\big)\\ + \mathbf{A}_\epsilon(\mathbf{x},k,\alpha,t)\cdot\nabla f_\epsilon(\mathbf{x},k,\alpha,t) - \frac{1}{\epsilon}\,\partial_\alpha\psi\big(\mathbf{x},k,\alpha,\frac{t}{\epsilon},t\big)\Big]\,d\mathbf{x}\,dk\,d\alpha\,dt\\ = -\int_\Omega \tilde{f}^0(x_1-\sqrt{2k}\,\sin\alpha, x_2+\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)\\ \times\,\psi(x_1,x_2,k,\alpha,0,0)\,dx_1\,dx_2\,dk\,d\alpha\,.\end{aligned} \tag{3.23}$$

Multiplying (3.23) by ϵ and letting $\epsilon\to 0$, we obtain the weak formulation of $\partial_\tau F - \partial_\alpha F = 0$, which indicates that there exists a function $G\in L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(\Gamma))\big)$ such that

$$F(\mathbf{x},k,\alpha,\tau,t) = G(\mathbf{x},k,\alpha+\tau,t)\,. \tag{3.24}$$

Secondly, we introduce the sequence $(g_\epsilon)_{\epsilon>0}$ defined by

$$g_\epsilon(\mathbf{x},k,\alpha,t) = f_\epsilon\Big(\mathbf{x},k,\alpha-\frac{t}{\epsilon},t\Big)\,. \tag{3.25}$$

In the spirit of [13], we prove that g_ϵ strongly converges to $2\pi\,G$ in a given Banach space. For that, we notice that, up to a subsequence, g_ϵ two-scale converges to G in $L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;L^p_\#(0,2\pi;L^p(\Gamma))\big)\big)$ since we have

$$\begin{aligned}&\int_0^T\int_\Omega g_\epsilon(\mathbf{x},k,\alpha,t)\,\psi\big(\mathbf{x},k,\alpha,\frac{t}{\epsilon},t\big)\,d\mathbf{x}\;dk\,d\alpha\,dt\\ &= \int_0^T\int_\Omega f_\epsilon(\mathbf{x},k,\alpha,t)\,\psi\big(\mathbf{x},k,\alpha+\frac{t}{\epsilon},\frac{t}{\epsilon},t\big)\,d\mathbf{x}\;dk\,d\alpha\,dt\\ &\to \int_0^{2\pi}\int_0^T\int_\Omega F(\mathbf{x},k,\alpha,\tau,t)\,\psi(\mathbf{x},k,\alpha+\tau,\tau,t)\,d\mathbf{x}\;dk\,d\alpha\,dt\,d\tau\\ &= \int_0^{2\pi}\int_0^T\int_\Omega G(\mathbf{x},k,\alpha,t)\,\psi(\mathbf{x},k,\alpha,\tau,t)\,d\mathbf{x}\;dk\,d\alpha\,dt\,d\tau\,,\end{aligned} \tag{3.26}$$

for any test function ψ on $\Omega\times[0,2\pi]\times[0,T]$ which is 2π-periodic in α and τ. As a consequence, g_ϵ weakly-* converges to $2\pi\,G$ in $L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(\Gamma))\big)$.

Let us prove that this weak-* convergence is a strong convergence in a given Banach space. This is the aim of the following lemma:

Lemma 4. *For any compact subset K of Γ, and up to a subsequence, g_ϵ strongly converges to $2\pi G$ in $L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*\big)$.*

In this Lemma $(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*$ stands for the dual of $W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K))$.

Proof of Lemma 4. For any compact subset K of Γ, $(g_\epsilon)_{\epsilon>0}$ and $(\mathbf{A}^\epsilon)_{\epsilon>0}$ are respectively bounded in $L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(K))\big)$ and $\Big(L^\infty\big(0,T;L^\infty_\#(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K)))\big)\Big)^3$ $\times L^\infty\big(0,T;L^\infty_\#(0,2\pi;L^{3/2}_\#(0,2\pi;L^{3/2}(K))\big)$, independently of ϵ. Then, remarking that g_ϵ is solution of

$$\left\{\begin{aligned}&\partial_t g_\epsilon(\mathbf{x},k,\alpha,t) + \mathbf{A}_\epsilon\big(\mathbf{x},k,\alpha-\frac{t}{\epsilon},t\big)\cdot\nabla g_\epsilon(\mathbf{x},k,\alpha,t) = 0\,,\\ &g_\epsilon(\mathbf{x},k,\alpha,0) = \tilde{f}^0(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)\,,\end{aligned}\right. \tag{3.27}$$

we use similar arguments as the ones given after equation (3.20) to deduce

1. $(\mathbf{A}_\epsilon)_{\epsilon>0}$ is bounded independently of ϵ in $\big(L^\infty\big(0,T;L^q_\#(0,2\pi;L^q(K))\big)\big)^4$ for any $q\in[1,\frac{3}{2}[$,
2. $\big(g_\epsilon(\mathbf{x},k,\alpha,t)\,\mathbf{A}_\epsilon(\mathbf{x},k,\alpha-\frac{t}{\epsilon},t)\big)_{\epsilon>0}$ is bounded in $\big(L^\infty\big(0,T;L^r_\#(0,2\pi;L^r(K))\big)\big)^4$ independently of ϵ with r defined by $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ $(r\in]1,\frac{3}{2}[)$,
3. $(\partial_t g_\epsilon)_{\epsilon>0}$ is bounded in $L^\infty\big(0,T;\big(W^{1,r^*}_\#(0,2\pi;W^{1,r^*}_0(K))\big)^*\big)$ independently of ϵ with $\frac{1}{r^*}+\frac{1}{r}=1$.

If $r^*\geq 3/2$, the embedding $W^{1,r^*}_\#(0,2\pi;W^{1,r^*}_0(K))\subset W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K))$ is compact with density. Furthermore, Rellich-Kondrakov's theorem (see [1]) gives the compact embedding $L^p_\#(0,2\pi;L^p(K))\subset(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*$ since $p\geq 2$. Then, we apply Aubin-Lions' lemma (see [21]) and we prove that the functional space $\mathcal{U}$ defined by

$$\begin{aligned}\mathcal{U}=\Big\{u\in L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(K))\big)\ :&\\ \partial_t u\in L^\infty\big(0,T;\big(W^{1,r^*}_\#(0,2\pi;W^{1,r^*}_0(K))\big)^*\big)\Big\}&,\end{aligned}\tag{3.28}$$

is compactly embedded in $L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*\big)$. Since $g_\epsilon\in\mathcal{U}$ for all ϵ, we deduce that the weak-* convergence of g_ϵ to $2\pi\,G$ in $L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(K))\big)$ is a strong convergence in $L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{3/2}_0(K)))^*\big)$.
If $r^*<3/2$, the compact embedding $\big(W^{1,r^*}_\#(0,2\pi;W^{1,r^*}_0(K))\big)^*\subset\big(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K))\big)^*$ is gotten directly. If we introduce the functional space $\mathcal{U}'$ defined by

$$\begin{aligned}\mathcal{U}'=\Big\{u\in L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(K))\big)\ :&\\ \partial_t u\in L^\infty\big(0,T;\big(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K))\big)^*\big)\Big\}&,\end{aligned}\tag{3.29}$$

we remark that the sequence $(g_\epsilon)_{\epsilon>0}$ is bounded in $\mathcal{U}'$ independently of ϵ. By using Aubin-Lions' lemma, we prove that $\mathcal{U}'\subset L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*\big)$ is a compact embedding. Then, g_ϵ strongly converges to $2\pi\,G$ in $L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*\big)$. □

Let us finish the proof of Theorem 3 by establishing a transport equation satisfied by G. Let us consider a test function $\psi=\psi(\mathbf{x},k,\alpha,t)$ on Ω. We denote its compact support in Γ by K and we assume that ψ and its first order derivatives are 2π-periodic in the α direction. Then we have

$$\begin{aligned}\int_0^T\int_0^{2\pi}\int_K g_\epsilon(\mathbf{x},k,\alpha,t)\,\partial_t\psi(\mathbf{x},k,\alpha,t)d\mathbf{x}\,dk\,d\alpha\,dt&\\ +\int_0^T\int_0^{2\pi}\int_K g_\epsilon(\mathbf{x},k,\alpha,t)\,\mathbf{A}_\epsilon\big(\mathbf{x},k,\alpha-\frac{t}{\epsilon},t\big)\cdot\nabla\psi(\mathbf{x},k,\alpha,t)\big]\,d\mathbf{x}\,dk\,d\alpha\,dt&\\ +\int_0^{2\pi}\int_K\tilde{f}^0(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)&\\ \times\,\psi(\mathbf{x},k,\alpha,0)\,d\mathbf{x}\,dk\,d\alpha=0\,.&\end{aligned}\tag{3.30}$$

Since g_ϵ converges to $2\pi\,G$ strongly in $L^\infty\big(0,T;(W^{1,3/2}_\#(0,2\pi;W^{1,3/2}_0(K)))^*\big)$ and weakly-* in $L^\infty\big(0,T;L^p_\#(0,2\pi;L^p(K))\big)$, and $\mathbf{A}_\epsilon(\mathbf{x},k,\alpha-\frac{t}{\epsilon},t)$ two-scale converges to $\mathcal{A}(\mathbf{x},k,\alpha-\tau,\tau,t)$ in

$\big(L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;W^{1,3/2}_\#(0,2\pi;W^{1,3/2}(K))\big)\big)\big)^3\times L^\infty\big(0,T;L^\infty_\#\big(0,2\pi;L^{3/2}_\#(0,2\pi;L^{3/2}(K))\big)\big)$, we obtain the weak formulation of

$$\left\{\begin{array}{l} \partial_t G(\mathbf{x},k,\alpha,t)+\left[\displaystyle\int_0^{2\pi}\mathcal{A}(\mathbf{x},k,\alpha-\tau,\tau,t)\,d\tau\right]\cdot\nabla G(\mathbf{x},k,\alpha,t)=0\,, \\ G(\mathbf{x},k,\alpha,0)=\dfrac{1}{2\pi}\tilde f^0(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)\,, \end{array}\right. \tag{3.31}$$

when $\epsilon\to 0$.

In order to obtain Poisson type equations (3.13.c) and (3.13.d), we consider a test function $\tilde\psi=\tilde\psi(\tilde{\mathbf{x}},\tau,t)$ on $\mathbb{R}^2\times[0,2\pi]\times[0,T]$ which is 2π-periodic in τ, we multiply $\tilde{\mathbf{E}}_\epsilon(\tilde{\mathbf{x}},t)$, $\nabla_{\tilde{\mathbf{x}}}\tilde\phi_\epsilon(\tilde{\mathbf{x}},t)$ and $\Delta_{\tilde{\mathbf{x}}}\tilde\phi_\epsilon(\tilde{\mathbf{x}},t)$ by $\tilde\psi(\tilde{\mathbf{x}},\frac{t}{\epsilon},t)$ and we integrate in $\tilde{\mathbf{x}}$ and t. We obtain

$$\int_0^T\int_{\mathbb{R}^2}\tilde{\mathbf{E}}_\epsilon(\tilde{\mathbf{x}},t)\,\psi\big(\tilde{\mathbf{x}},\frac{t}{\epsilon},t\big)\,d\tilde{\mathbf{x}}\,dt\to\int_0^{2\pi}\int_0^T\int_{\mathbb{R}^2}\tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)\,\psi(\tilde{\mathbf{x}},\tau,t)\,d\tilde{\mathbf{x}}\,dt\,d\tau\,, \tag{3.32}$$

$$\int_0^T\int_{\mathbb{R}^2}\nabla_{\tilde{\mathbf{x}}}\tilde\phi_\epsilon(\tilde{\mathbf{x}},t)\,\psi\big(\tilde{\mathbf{x}},\frac{t}{\epsilon},t\big)\,d\tilde{\mathbf{x}}\,dt\to\int_0^{2\pi}\int_0^T\int_{\mathbb{R}^2}\nabla_{\tilde{\mathbf{x}}}\tilde\Phi(\tilde{\mathbf{x}},\tau,t)\,\psi(\tilde{\mathbf{x}},\tau,t)\,d\tilde{\mathbf{x}}\,dt\,d\tau\,, \tag{3.33}$$

$$\int_0^T\int_{\mathbb{R}^2}\Delta_{\tilde{\mathbf{x}}}\tilde\phi_\epsilon(\tilde{\mathbf{x}},t)\,\psi\big(\tilde{\mathbf{x}},\frac{t}{\epsilon},t\big)\,d\tilde{\mathbf{x}}\,dt\to\int_0^{2\pi}\int_0^T\int_{\mathbb{R}^2}\Delta_{\tilde{\mathbf{x}}}\tilde\Phi(\tilde{\mathbf{x}},\tau,t)\,\psi(\tilde{\mathbf{x}},\tau,t)\,d\tilde{\mathbf{x}}\,dt\,d\tau\,, \tag{3.34}$$

when ϵ converges to 0. Since F is the two-scale limit of $(f_\epsilon)_{\epsilon>0}$, we also have

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^2}\left(\int_{\mathcal{S}}f_\epsilon\big(\tilde x_1+\sqrt{2k}\,\sin\alpha,\tilde x_2-\sqrt{2k}\,\sin\alpha,k,\alpha,t\big)\,dk\,d\alpha\right)\psi\big(\tilde{\mathbf{x}},\frac{t}{\epsilon},t\big)\,d\tilde{\mathbf{x}}\,dt \\ &\to\int_0^{2\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathcal{S}}F\big(\tilde x_1+\sqrt{2k}\,\sin\alpha,\tilde x_2-\sqrt{2k}\,\sin\alpha,k,\alpha,\tau,t\big) \\ &\qquad\qquad\times\psi(\tilde{\mathbf{x}},\tau,t)\,dk\,d\alpha\,d\tilde{\mathbf{x}}\,dt\,d\tau\,, \end{aligned} \tag{3.35}$$

when $\epsilon\to 0$. Then, gathering convergence results (3.32)-(3.35), we obtain the weak formulation of (3.13.c) and (3.13.d).

To summarize the work which has already been done, we have proved that, up to a subsequence the solution $(f_\epsilon,\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ two-scale converges to a couple $(F,\tilde{\mathcal{E}})$ such that

$$F(\mathbf{x},k,\alpha,\tau,t)=G(\mathbf{x},k,\alpha+\tau,t)\,, \tag{3.36}$$

where G is solution of

$$\left\{\begin{array}{l} \partial_t G(\mathbf{x},k,\alpha,t)+\left[\displaystyle\int_0^{2\pi}\mathcal{A}(\mathbf{x},k,\alpha-\tau,\tau,t)\,d\tau\right]\cdot\nabla G(\mathbf{x},k,\alpha,t)=0\,, \\ G(\mathbf{x},k,\alpha,0)=\dfrac{1}{2\pi}\tilde f^0(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\cos\alpha,\sqrt{2k}\,\sin\alpha)\,, \\ \mathcal{E}(\mathbf{x},k,\alpha,\tau,t)=\tilde{\mathcal{E}}(x_1-\sqrt{2k}\,\sin\alpha,x_2+\sqrt{2k}\,\cos\alpha,\tau,t)\,, \\ -\nabla_{\tilde{\mathbf{x}}}\tilde\Phi(\tilde{\mathbf{x}},\tau,t)=\tilde{\mathcal{E}}(\tilde{\mathbf{x}},\tau,t)\,, \\ -\Delta_{\tilde{\mathbf{x}}}\tilde\Phi(\tilde{\mathbf{x}},\tau,t)=\displaystyle\int_{\mathcal{S}}G(\tilde x_1+\sqrt{2k}\,\sin\alpha,\tilde x_2-\sqrt{2k}\,\cos\alpha,k,\alpha,t)\,dk\,d\alpha-\frac{1}{2\pi}\tilde n_e(\tilde{\mathbf{x}})\,. \end{array}\right. \tag{3.37}$$

In order to complete the proof of the present theorem, we have to prove that the function $(\mathbf{x}, k, \alpha, t) \mapsto \int_0^{2\pi} \mathcal{A}(\mathbf{x}, k, \alpha - \tau, \tau, t)\, d\tau$ does not depends on α, and this can be viewed as a direct consequence of the following lemma:

Lemma 5. *For all* $(\mathbf{x}, k, \alpha, t) \in \Omega \times [0, T]$*, we have*

$$\partial_\alpha \left(\int_0^{2\pi} \Phi(\mathbf{x}, k, \alpha - \tau, \tau, t)\, d\tau \right) = 0\,. \tag{3.38}$$

Indeed, this lemma allows us to claim that

$$\int_0^{2\pi} \mathcal{A}(\mathbf{x}, k, \alpha - \tau, \tau, t)\, d\tau = \int_0^{2\pi} \mathcal{A}(\mathbf{x}, k, -\tau, \tau, t)\, d\tau = \langle \mathcal{A} \rangle(\mathbf{x}, k, t)\,, \tag{3.39}$$

where the $\langle \cdot \rangle$ notation is defined by (3.14), and that

$$\int_0^{2\pi} \sqrt{2k}\, \big(\mathcal{E}_1(\mathbf{x}, k, \alpha - \tau, \tau, t)\, \cos(\alpha - \tau) + \mathcal{E}_2(\mathbf{x}, k, \alpha - \tau, \tau, t)\, \sin(\alpha - \tau) \big)\, d\tau = 0\,, \tag{3.40}$$

for all $(\mathbf{x}, k, \alpha, t) \in \Omega \times [0, T]$. Then model (3.37) reduces itself to (3.13), which concludes the proof of Theorem 3. ☐

Proof of Lemma 5. We consider a compact subset K of Γ. As it has been previously mentioned, ϕ_ϵ two-scale converges to Φ in $L^\infty\big(0, T; L^\infty_\#\big(0, 2\pi; W^{1,3/2}_\#(0, 2\pi; W^{1,3/2}(K))\big)\big)$, so we have

$$\begin{aligned} &\int_0^T \int_\Omega \partial_\alpha \phi_\epsilon\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big)\, \psi(\mathbf{x}, k, \alpha, t)\, d\mathbf{x}\, dk\, d\alpha\, dt \\ &\qquad \to \int_0^T \int_\Omega \int_0^{2\pi} \partial_\alpha \Phi(\mathbf{x}, k, \alpha - \tau, t)\, \psi(\mathbf{x}, k, \alpha, t)\, d\tau\, d\mathbf{x}\, dk\, d\alpha\, dt\,, \end{aligned} \tag{3.41}$$

for any regular test function ψ which support in Γ is included in K, and which is 2π-periodic in α. It means that we have the following weak-* convergence result

$$\partial_\alpha \phi_\epsilon\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big) \overset{*}{\rightharpoonup} \int_0^{2\pi} \partial_\alpha \Phi(\mathbf{x}, k, \alpha - \tau, t)\, d\tau\,, \tag{3.42}$$

in $L^\infty\big(0, T; L^{3/2}_\#(0, 2\pi; L^{3/2}(K))\big)$. Considering such a test fonction ψ, we define $\bar\psi$ by

$$\bar\psi(\mathbf{x}, k, \alpha, \tau, t) = \psi(\mathbf{x}, k, \alpha + \tau, t)\,. \tag{3.43}$$

Then we have

$$\begin{aligned} &\int_0^T \int_\Omega \partial_\alpha \phi_\epsilon\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big)\, \psi(\mathbf{x}, k, \alpha, t)\, d\mathbf{x}\, dk\, d\alpha\, dt \\ &\qquad = -\epsilon \int_0^T \int_\Omega \phi_\epsilon(\mathbf{x}, k, \alpha, t)\, \partial_\tau \bar\psi\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big)\, d\mathbf{x}\, dk\, d\alpha\, dt\,. \end{aligned} \tag{3.44}$$

Since $(\phi_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in $L^\infty\big(0, T; W^{1,3/2}_\#(0, 2\pi; W^{1,3/2}(K))\big)$, there exists a constant $C > 0$ which only depends on the initial data $\tilde f^0$ and $\tilde n_e$ such that

$$\left| \int_0^T \int_\Omega \partial_\alpha \phi_\epsilon\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big)\, \psi(\mathbf{x}, k, \alpha, t)\, d\mathbf{x}\, dk\, d\alpha\, dt \right| \leq C\, \epsilon \times \max_{K \times [0, 2\pi] \times [0, T]} |\partial_\alpha \psi|\,. \tag{3.45}$$

We deduce that

$$\partial_\alpha \phi_\epsilon\Big(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t\Big) \stackrel{*}{\rightharpoonup} 0 \qquad \text{in } L^\infty\big(0,T; L^{3/2}_\#(0,2\pi; L^{3/2}(K))\big). \tag{3.46}$$

When coupled with (3.42), this result allows us to finish the proof of the lemma by using the uniqueness of the weak-* limit of $\big(\partial_\alpha \phi_\epsilon(\mathbf{x}, k, \alpha - \frac{t}{\epsilon}, t)\big)_{\epsilon > 0}$. □

3.3 Weak-* convergence

As it has been announced previously, we have chosen the variables $(\mathbf{x}, k, \alpha)$ in order to present the weak-* convergence of $(f_\epsilon, \tilde{\mathbf{E}}_\epsilon)_{\epsilon > 0}$ to the solution of the 2D finite Larmor radius model as a direct consequence of the two-scale convergence result we have just proved.

Firstly, we have a direct corollary of Theorem 3:

Corollary 6. *Up to some subsequences,*

- f_ϵ *weakly-* converges to* $f \in L^\infty\big(0,T; L^p_\#(0,2\pi; L^p(\Gamma))\big)$ *with* f *defined by*

$$f(\mathbf{x}, k, \alpha, t) = \int_0^{2\pi} F(\mathbf{x}, k, \alpha, \tau, t)\, d\tau\,, \tag{3.47}$$

- $\tilde{\mathbf{E}}_\epsilon$ *weakly-* converges to* $\tilde{\mathbf{E}} \in \big(L^\infty\big(0,T; W^{1,3/2}(\mathbb{R}^2)\big)\big)^2$ *with* $\tilde{\mathbf{E}}$ *defined by*

$$\tilde{\mathbf{E}}(\mathbf{x}, t) = \int_0^{2\pi} \tilde{\mathcal{E}}(\mathbf{x}, \tau, t)\, d\tau\,, \tag{3.48}$$

- $\tilde{\phi}_\epsilon$ *weakly-* converges to* $\tilde{\phi} \in L^\infty\big(0,T; W^{2,3/2}(\mathbb{R}^2)\big)$ *with* $\tilde{\phi}$ *defined by*

$$\tilde{\phi}(\mathbf{x}, t) = \int_0^{2\pi} \tilde{\Phi}(\mathbf{x}, \tau, t)\, d\tau\,. \tag{3.49}$$

Then, we have the following theorem:

Theorem 7. *There exists a function* g *defined on* $\Gamma \times [0,T]$ *such that*

$$f(\mathbf{x}, k, \alpha, t) = \frac{1}{2\pi}\, g(\mathbf{x}, k, t)\,, \qquad \forall\, (\mathbf{x}, k, \alpha, t) \in \Omega \times [0,T]\,, \tag{3.50}$$

and verifying

$$\begin{cases} \partial_t g + \langle \mathcal{E}_2 \rangle\, \partial_{x_1} g - \langle \mathcal{E}_1 \rangle\, \partial_{x_2} g = 0\,, \\[2mm] g(\mathbf{x}, k, 0) = \displaystyle\int_0^{2\pi} \tilde{f}^0(x_1 - \sqrt{2k}\, \sin\alpha, x_2 + \sqrt{2k}\, \cos\alpha, \sqrt{2k}\, \cos\alpha, \sqrt{2k}\, \sin\alpha)\, d\alpha\,, \\[2mm] \langle \mathcal{E} \rangle(\mathbf{x}, k, t) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \tilde{\mathbf{E}}\big(x_1 - \sqrt{2k}\, \sin\alpha, x_2 + \sqrt{2k}\, \cos\alpha, t\big)\, d\alpha\,, \\[2mm] -\nabla_{\tilde{\mathbf{x}}} \tilde{\phi}(\tilde{\mathbf{x}}, t) = \tilde{\mathbf{E}}(\tilde{\mathbf{x}}, t)\,, \\[2mm] -\Delta_{\tilde{\mathbf{x}}} \tilde{\phi}(\tilde{\mathbf{x}}, t) = \dfrac{1}{2\pi} \displaystyle\int_{\mathcal{S}} g\big(\tilde{x}_1 + \sqrt{2k}\, \sin\alpha, \tilde{x}_2 - \sqrt{2k}\, \cos\alpha, k, t\big)\, dk\, d\alpha - \tilde{n}_e(\tilde{\mathbf{x}})\,. \end{cases} \tag{3.51}$$

Proof of Theorem 7. Since F and G are linked by the relation (3.11), f does not depend on α. Indeed, we have

$$\begin{aligned}\partial_\alpha f(\mathbf{x},k,\alpha,t) &= \partial_\alpha \int_0^{2\pi} F(\mathbf{x},k,\alpha,\tau,t)\,d\tau \\ &= \partial_\alpha \int_0^{2\pi} G(\mathbf{x},k,\alpha+\tau,t)\,d\tau \\ &= \partial_\alpha \int_0^{2\pi} G(\mathbf{x},k,\tau,t)\,d\tau \\ &= 0\,.\end{aligned} \tag{3.52}$$

Then, if we integrate (3.13.a) in α, we obtain

$$\partial_t f + \langle \mathcal{E}_2\rangle\,\partial_{x_1} f - \langle \mathcal{E}_1\rangle\,\partial_{x_2} f = 0\,, \tag{3.53}$$

and the equation (3.51.a).

By integrating the initial condition (3.13.b) in α and dividing it by 2π, we obtain the initial condition (3.51.b).

Since $\langle\mathcal{E}\rangle$ does not depend on α, we can integrate it in α and divide it by 2π: then, we have

$$\begin{aligned}\langle\mathcal{E}\rangle(\mathbf{x},k,t) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{E}(\mathbf{x},k,\alpha-\tau,\tau,t)\,d\tau\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{E}(\mathbf{x},k,\alpha,\tau,t)\,d\tau\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \tilde{\mathbf{E}}(x_1-\sqrt{2k}\,\sin\alpha, x_2+\sqrt{2k}\,\cos\alpha,t)\,d\alpha\,.\end{aligned} \tag{3.54}$$

By using similar techniques, we integrate (3.13.d) in τ and we obtain

$$-\Delta_{\tilde{\mathbf{x}}}\tilde{\phi}(\tilde{\mathbf{x}},t) = \frac{1}{2\pi}\int_{\mathcal{S}} g\big(\tilde{x}_1+\sqrt{2k}\,\sin\alpha, \tilde{x}_2-\sqrt{2k}\,\cos\alpha,k,t)\,dk\,d\alpha - \tilde{n}_e(\tilde{\mathbf{x}})\,. \tag{3.55}$$

Finally, we integrate (3.13.c) in τ and we obtain (3.51.d), which concludes the proof. □

We can remark that Bostan's results on the mathematical justification of the 2D finite Larmor radius approximation have been improved: indeed, we proved that Frénod & Sonnendrücker's assumptions are sufficient not only for proving the existence of the weak-* limit $(f,\tilde{\mathbf{E}})$ but also for establishing a system for $(f,\tilde{\mathbf{E}})$ which is exactly the 2D finite Larmor radius model.

4 Conclusions and perspectives

In a first part, we recalled a two-scale convergence result for a 2D Vlasov-Poisson model for a charged particle beam which is due to Frénod and Sonnendrücker. After adapting it to plasma modeling by adding an electron density $\tilde{n}_e$ and some reasonable compatibility conditions, we recalled that such a result trivially implies, up to some subsequences, the weak-* convergence of $(\tilde{f}_\epsilon,\tilde{\mathbf{E}}_\epsilon)$ to a couple $(\tilde{f},\tilde{\mathbf{E}})$. Then we compared this conclusion to Bostan's results which was established in [4] under stronger assumptions.

In a second part, we introduced a new set of variables involving the guiding-center position coordinates and the transverse part of the kinetic energy which is denoted with k. After rewriting the Vlasov equation in these new coordinates, we established a two-scale convergence result by using Frenod & Sonnendrücker's assumptions and the compatibility conditions for $\tilde{n}_e$ which were added in Theorem 1. Then, we proved that $(f_\epsilon, \tilde{\mathbf{E}}_\epsilon)$ weakly-* converges to the solution of the 2D finite Larmor radius model and we established the limit model itself through a few computations lines and without adding any assumption.

The first main remark we can do about the present work concerns the formulation of the two-scale limit model in canonical gyrokinetic coordinates: indeed, the transport equations (3.13.a) and (3.51.a) indicate that k is an adiabatic invariant not only for the weak-* limit system but also for the two-scale limit system. Since k is the dimensionless transverse kinetic energy, this remark confirms previous results about the adiabatic invariant property of the magnetic momentum (see Littlejohn [20], Lee [18, 19], Brizard *et al.* [5, 6], Grandgirard *et al.* [16]).

The second remark which can be done is about the mathematical results which have established in this paper: by gathering previous two-scale and weak-* convergence under common assumptions, we have improved Bostan's results by weakening the assumptions used in Theorem 2, especially those which were set on $(\tilde{\mathbf{E}}_\epsilon)_{\epsilon>0}$ and which are no longer necessary.

Since the canonical gyrokinetic variables allow us to simplify two-scale or weak-* convergence results for the finite Larmor radius approximation by highlighting the adiabatic invariant property of the magnetic momentum, they may be a useful tool for a mathematical justification of the full 3D finite Larmor radius model.

From a numerical point of view, the model (3.13) can be used to build a two-scale numerical method in order to simulate the high frequency oscillations of the solution $(\tilde{f}_\epsilon, \tilde{\mathbf{E}}_\epsilon)$ of the Vlasov-Poisson model (2.16), such as it has been done before for low Mach number problems (see [8]), charged particle beams problems (see [10] and [23]) or drift problems in the ocean (see [2]). As an example, a numerical method based on the computation of the characteristics associated the limit transport equation will be simpler to be developed on the formulation (3.13.a) than the formulation (2.21.a) since the advection terms do not depend on the azimutal velocity α when the transport equation is written in canonical gyrokinetic coordinates. Furthermore, such an advection is tridimensional when we work on (3.13.a) whereas it is 4-dimensional when we work on (2.21.a).

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