Solution périodique d'une edp parabolique dégénérée

Existence et unicité de la solution périodique en temps et en espace d'une edp parabolique dégénérée

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Motivations

- Résultat utile pour pour établir l'existence d'une solution d'une edp parabolique dégénérée et singulièrement perturbée
- Résultat utile pour le programme de recherche "Simuler la dynamique des bancs de sable à proximité des côtes dans les zones soumises à la marée"

$$\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla \cdot (\mathcal{A}^\epsilon \nabla z^\epsilon) = \frac{1}{\epsilon}\nabla \cdot \mathcal{C}^\epsilon.$$

$$z^\epsilon_{|t=0} = z_0.$$

Objectifs

- Étudier l'existence et l'unicité de la solution périodique de :

$$\frac{\partial \mathcal{S}}{\partial \theta} - \nabla \cdot (\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)\nabla \mathcal{S}) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

t est un paramètre.

- Établir des estimations fortes sur la dépendance de $\mathcal{S}$ par rapport à t.

Hypothèses

$$\widetilde{\mathcal{A}}_\epsilon \geq 0,\ \theta \longmapsto (\widetilde{\mathcal{A}}_\epsilon, \widetilde{\mathcal{C}}_\epsilon) \text{ périodique de periode } 1.$$

$$x \longmapsto (\widetilde{\mathcal{A}}_\epsilon, \widetilde{\mathcal{C}}_\epsilon) \text{ définie sur } \mathbb{T}^2$$

$$|\widetilde{\mathcal{A}}_\epsilon| \leq \gamma,\ |\widetilde{\mathcal{C}}_\epsilon| \leq \gamma,\ \left|\frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial t}\right| \leq \gamma,\ \left|\frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial t}\right| \leq \gamma, \left|\frac{\partial \nabla \widetilde{\mathcal{A}}_\epsilon}{\partial t}\right| \leq \gamma.$$

$$\left|\frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial \theta}\right| \leq \gamma,\ \left|\frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial \theta}\right| \leq \gamma,\ |\nabla \widetilde{\mathcal{A}}_\epsilon| \leq \gamma,\ |\nabla \cdot \widetilde{\mathcal{C}}_\epsilon| \leq \gamma.$$

$$\exists \widetilde{G}_{thr},\ \theta_\alpha < \theta_\omega \in [0,1] \text{ tels que } \forall\ \theta \in [\theta_\alpha, \theta_\omega] \Longrightarrow \widetilde{\mathcal{A}}_\epsilon(t,\theta,x) \geq \widetilde{G}_{thr}.$$

$$\widetilde{\mathcal{A}}_\epsilon(t,\theta,x) \leq \widetilde{G}_{thr} \Longrightarrow \left\{ \begin{array}{l} \dfrac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial t}(t,\theta,x) = 0,\ \nabla \widetilde{\mathcal{A}}_\epsilon(t,\theta,x) = 0, \\ \dfrac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial t}(t,\theta,x) = 0,\ \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\theta,x) = 0. \end{array} \right.$$

$$|\widetilde{\mathcal{C}}_\epsilon| \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ |\widetilde{\mathcal{C}}_\epsilon|^2 \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ |\nabla \widetilde{\mathcal{A}}_\epsilon| \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ \left|\frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial t}\right| \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|.$$

$$\left|\frac{\partial(\nabla \widetilde{\mathcal{A}}_\epsilon)}{\partial t}\right|^2 \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ \left|\nabla \cdot \widetilde{\mathcal{C}}_\epsilon\right| \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ \left|\frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial t}\right| \leq \gamma|\widetilde{\mathcal{A}}_\epsilon|,\ \left|\frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial t}\right|^2 \leq \gamma^2|\widetilde{\mathcal{A}}_\epsilon|.$$

Théorème et équations régularisées

Théorème

Théorème

Sous les hypothèses, il y a existence et unicité de $\mathcal{S} = \mathcal{S}(t, \theta, x)$ $\in L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$*, périodique de periode 1 en* θ*, solution de*

$$\frac{\partial \mathcal{S}}{\partial \theta} - \nabla \cdot (\widetilde{\mathcal{A}}_\epsilon(t, \cdot, \cdot) \nabla \mathcal{S}) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t, \cdot, \cdot).$$

$$\int_{\mathbb{T}^2} \mathcal{S}(t, \theta, x) dx = 0.$$

Elle satisfait :

$$\|\mathcal{S}\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma.$$

$$\|\frac{\partial \mathcal{S}}{\partial t}\|^2_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma + \gamma^3}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma. \quad (\textit{paramètre } t.)$$

$$(\|f\|^2_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} = \sup_{\theta \in [0,1]} \int_{\mathbb{T}^2} f^2 \, dx, \quad \|f\|^2_{L^2_\#(\mathbb{R} \times \mathbb{T}^2)} = \int_0^1 \int_{\mathbb{T}^2} f^2 \, dx d\theta.)$$

Équations régularisées

t est un paramètre

$$\frac{\partial \mathcal{S}^\nu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu)\nabla \mathcal{S}^\nu) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

$$\mu \mathcal{S}^\nu_\mu + \frac{\partial \mathcal{S}^\nu_\mu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu)\nabla \mathcal{S}^\nu_\mu)) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

$$\left\{ \begin{array}{c} \mu \xi^\nu_\mu + \dfrac{\partial \xi^\nu_\mu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu)\nabla \xi^\nu_\mu) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot), \\ {\xi^\nu_\mu}_{|\theta=0} = \xi. \end{array} \right.$$

Existence et unicité pour les équations régularisées

construction de $\mathcal{S}_\mu^\nu$

$$\mu\xi_\mu^\nu + \frac{\partial \xi_\mu^\nu}{\partial\theta} - \nabla\cdot((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)+\nu)\nabla\xi_\mu^\nu) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot), \quad {\xi_\mu^\nu}_{|\theta=0} = \xi.$$

$$\square : L^2(\mathbb{T}^2) \longrightarrow L^2(\mathbb{T}^2), \quad \xi \longmapsto \xi_\mu^\nu(1,\cdot),$$

ξ_μ^ν, $\widetilde{\xi}_\mu^\nu$ solutions associées aux C.I. ξ et $\widetilde{\xi}$ $(\in L^2(\mathbb{T}^2))$, $\longrightarrow$

$$\mu(\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu) + \frac{\partial(\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu)}{\partial\theta} - \nabla\cdot\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla(\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu)\Big) = 0.$$

$\times(\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu)$ et $\int_{\mathbb{T}^2} dx \longrightarrow$

$$\mu\|\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu\|_2^2 + \frac{1}{2}\frac{\partial\left(\|\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu\|_2^2\right)}{\partial\theta} + \int_{\mathbb{T}^2}(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla(\xi_\mu^\nu - \widetilde{\xi}_\mu^\nu)|^2 dx = 0.$$

Comme $\widetilde{\mathcal{A}}_\epsilon + \nu > 0$, $\|\xi_\mu^\nu(1,\cdot) - \widetilde{\xi}_\mu^\nu(1,\cdot)\|_2^2 \leq e^{-2\mu}\|\xi - \widetilde{\xi}\|_2^2$.

Donc $\square$ contraction stricte. Donc $\exists!$ point fixe à $\square$. Donne $\mathcal{S}_\mu^\nu$.

Propriétés de $\mathcal{S}_\mu^\nu$

$$\mu \mathcal{S}_\mu^\nu + \frac{\partial \mathcal{S}_\mu^\nu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu)\nabla \mathcal{S}_\mu^\nu)) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

Théorème

$$\sup_{\theta \in \mathbb{R}} \left| \int_{\mathbb{T}^2} \mathcal{S}_\mu^\nu(\theta, x) dx \right| = 0.$$

$$\left\| \frac{\partial \mathcal{S}_\mu^\nu}{\partial \theta} \right\|_{L^2_\#(\mathbb{R}\times\mathbb{T}^2)} \leq \gamma_3.$$

$$\|\nabla \mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \gamma_3.$$

$$\|\mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \gamma_3.$$

$$\left\| \frac{\partial \mathcal{S}_\mu^\nu}{\partial t} \right\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \gamma_3. \quad (\textit{paramètre } t.)$$

γ_3 *dépend seulement de* γ *et* ν *(et pas de* μ*).*

Demonstration de $\sup_{\theta\in\mathbb{R}} \left|\int_{\mathbb{T}^2} \mathcal{S}_\mu^\nu(\theta,x)dx\right| = 0$

$\displaystyle\int_{\mathbb{T}^2}\left[\mu\mathcal{S}_\mu^\nu + \frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta} - \nabla\cdot((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)+\nu)\nabla\mathcal{S}_\mu^\nu)) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot)\right]dx \longrightarrow$

$$\mu\int_{\mathbb{T}^2}\mathcal{S}_\mu^\nu dx + \frac{d\left(\int_{\mathbb{T}^2}\mathcal{S}_\mu^\nu dx\right)}{d\theta} = 0.$$

$\longrightarrow$

$$\int_{\mathbb{T}^2}\mathcal{S}_\mu^\nu(\theta,x)dx = \int_{\mathbb{T}^2}\mathcal{S}_\mu^\nu(\tilde\theta,x)dx\; e^{-\mu(\theta-\tilde\theta)}, \quad \forall\theta,\tilde\theta.$$

Ceci + la périodicité en θ de $\mathcal{S}_\mu^\nu$ $\longrightarrow$

$$\int_{\mathbb{T}^2}\mathcal{S}_\mu^\nu(\theta,x)dx = 0 \quad \forall\theta.$$

Demonstration de $\|\nabla \mathcal{S}^{\nu}_{\mu}\|_{L^{\infty}_{\#}(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_3$ - 1

$$\int_{\mathbb{T}^2} \left(S^{\nu}_{\mu}\right)\left[\mu \mathcal{S}^{\nu}_{\mu} + \frac{\partial \mathcal{S}^{\nu}_{\mu}}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_{\epsilon}(t,\cdot,\cdot) + \nu)\nabla \mathcal{S}^{\nu}_{\mu})) = \nabla \cdot \widetilde{\mathcal{C}}_{\epsilon}(t,\cdot,\cdot)\right] dx \longrightarrow$$

$$\mu \|\mathcal{S}^{\nu}_{\mu}(\theta,\cdot)\|_2^2 + \frac{1}{2}\frac{d(\|\mathcal{S}^{\nu}_{\mu}(\theta,\cdot)\|_2^2)}{d\theta} + \int_{\mathbb{T}^2} (\widetilde{\mathcal{A}}_{\epsilon} + \nu)|\nabla \mathcal{S}^{\nu}_{\mu}(\theta,\cdot)|^2 dx = -\int_{\mathbb{T}^2} \widetilde{\mathcal{C}}_{\epsilon} \nabla \mathcal{S}^{\nu}_{\mu}(\theta,\cdot)\, dx.$$

$\int_0^1 \; d\theta \longrightarrow$

$$\mu \|\mathcal{S}^{\nu}_{\mu}\|^2_{L^2_{\#}(\mathbb{R}\times\mathbb{T}^2)} + \int_0^1 \int_{\mathbb{T}^2} (\widetilde{\mathcal{A}}_{\epsilon} + \nu)|\nabla \mathcal{S}^{\nu}_{\mu}|^2 \, dx d\theta = -\int_0^1 \int_{\mathbb{T}^2} \widetilde{\mathcal{C}}_{\epsilon} \nabla \mathcal{S}^{\nu}_{\mu}(\theta,\cdot)\, dx \leq \gamma \|\nabla \mathcal{S}^{\nu}_{\mu}\|_{L^2_{\#}(\mathbb{R}\times\mathbb{T}^2)}.$$

$$\widetilde{\mathcal{A}}_{\epsilon} + \nu \geq \nu \to \nu \|\nabla \mathcal{S}^{\nu}_{\mu}\|^2_{L^2_{\#}(\mathbb{R}\times\mathbb{T}^2)} \leq \gamma \|\nabla \mathcal{S}^{\nu}_{\mu}\|_{L^2_{\#}(\mathbb{R}\times\mathbb{T}^2)} \to \|\nabla \mathcal{S}^{\nu}_{\mu}\|_{L^2_{\#}(\mathbb{R}\times\mathbb{T}^2)} \leq \frac{\gamma}{\nu}.$$

Demonstration de $\|\nabla \mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_3$ - 2

$$\int_{\mathbb{T}^2} \Big(-\Delta S_\mu^\nu \Big) \Big[\mu \mathcal{S}_\mu^\nu + \frac{\partial \mathcal{S}_\mu^\nu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)+\nu)\nabla \mathcal{S}_\mu^\nu)) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot) \Big] dx \longrightarrow$$

$$\mu \|\nabla \mathcal{S}_\mu^\nu\|_2^2 + \frac{1}{2}\frac{d\left(\|\nabla \mathcal{S}_\mu^\nu\|_2^2\right)}{d\theta} + \int_{\mathbb{T}^2} (\widetilde{\mathcal{A}}_\epsilon + \nu)|\Delta \mathcal{S}_\mu^\nu|^2 dx = - \int_{\mathbb{T}^2} \nabla \widetilde{\mathcal{A}}_\epsilon \cdot \nabla \mathcal{S}_\mu^\nu \Delta \mathcal{S}_\mu^\nu dx - \int_{\mathbb{T}^2} \nabla \cdot \widetilde{\mathcal{C}}_\epsilon \, \Delta \mathcal{S}_\mu^\nu dx.$$

Formule : $\forall U$ et V : $|UV| \leq \frac{\widetilde{\mathcal{A}}_\epsilon+\nu}{4} U^2 + \frac{1}{\widetilde{\mathcal{A}}_\epsilon+\nu} V^2$, avec $U = \Delta \mathcal{S}_\mu^\nu$ et $V = \nabla \widetilde{\mathcal{A}}_\epsilon \cdot \nabla \mathcal{S}_\mu^\nu \longrightarrow$

$$\left| \int_{\mathbb{T}^2} \nabla \widetilde{\mathcal{A}}_\epsilon \cdot \nabla \mathcal{S}_\mu^\nu \, \Delta \mathcal{S}_\mu^\nu dx \right| \leq \int_{\mathbb{T}^2} \frac{\widetilde{\mathcal{A}}_\epsilon + \nu}{4} |\Delta \mathcal{S}_\mu^\nu|^2 dx + \int_{\mathbb{T}^2} \frac{|\nabla \widetilde{\mathcal{A}}_\epsilon|^2}{\widetilde{\mathcal{A}}_\epsilon + \nu} |\nabla \mathcal{S}_\mu^\nu|^2 dx.$$

$$\longrightarrow$$

$$\mu \|\nabla \mathcal{S}_\mu^\nu\|_2^2 + \frac{1}{2}\frac{d\left(\|\nabla \mathcal{S}_\mu^\nu\|_2^2\right)}{d\theta} + \int_{\mathbb{T}^2} \frac{\widetilde{\mathcal{A}}_\epsilon + \nu}{2} |\Delta \mathcal{S}_\mu^\nu|^2 dx \leq \frac{\gamma^2}{\nu}\Big(\int_{\mathbb{T}^2} |\nabla \mathcal{S}_\mu^\nu|^2 dx + 1\Big).$$

/dots

Demonstration de $\|\nabla \mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_3$ - 3

$\|\nabla \mathcal{S}_\mu^\nu\|_{L^2_\#(\mathbb{R}\times\mathbb{T}^2)} \leq \dfrac{\gamma}{\nu} \Rightarrow \exists \theta_0 \in [0,1]$ tel que $\|\nabla \mathcal{S}_\mu^\nu(\theta_0,\cdot)\|_2 \leq \frac{\gamma}{\nu}$

Pour θ_0 et $\theta_1 \in [0,1]$,

$\int_{\theta_0}^{\theta_1} \Big[\mu\|\nabla \mathcal{S}_\mu^\nu\|_2^2 + \frac{1}{2}\frac{d\big(\|\nabla \mathcal{S}_\mu^\nu\|_2^2\big)}{d\theta} + \int_{\mathbb{T}^2} \frac{\widetilde{\mathcal{A}}_\epsilon+\nu}{2}|\Delta \mathcal{S}_\mu^\nu|^2 dx \leq \frac{\gamma^2}{\nu}\big(\int_{\mathbb{T}^2}|\nabla \mathcal{S}_\mu^\nu|^2 dx + 1\big)\Big] d\theta$

$\longrightarrow$

$$\begin{aligned}\|\nabla \mathcal{S}_\mu^\nu(\theta_1,\cdot)\|_2^2 - \|\nabla \mathcal{S}_\mu^\nu(\theta_0,\cdot)\|_2^2 &\leq \frac{2\gamma^2}{\nu}\int_{\theta_0}^{\theta_1}\left(\int_{\mathbb{T}^2}|\nabla \mathcal{S}_\mu^\nu|^2 dx + 1\right) d\theta \\ &\leq \frac{2\gamma^2}{\nu}\left(\|\nabla \mathcal{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R}\times\mathbb{T}^2)} + 1\right),\end{aligned}$$

Ceci + périodicité $\longrightarrow \|\nabla \mathcal{S}_\mu^\nu(\theta_1,\cdot)\|_2 \leq \gamma_3 \ \forall \theta_1$, i.e.

$$\|\nabla \mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_3.$$

construction de $\mathcal{S}^\nu$

Estimations indépendantes de μ. Donc en faisant $\mu \to 0$,

Théorème

$\exists! \mathcal{S}^\nu \in L^2(\mathbb{R} \times \mathbb{T}^2)$, *périodique de période 1 en θ solution de*

$$\frac{\partial \mathcal{S}^\nu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu) \nabla \mathcal{S}^\nu) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

$$\sup_{\theta \in \mathbb{R}} \left| \int_{\mathbb{T}^2} \mathcal{S}^\nu(\theta, x)\, dx \right| = 0.$$

Estimations :

$$\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \gamma_4.$$

$$\left\| \frac{\partial \mathcal{S}^\nu}{\partial t} \right\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \gamma_4.$$

γ_4 *dépend seulement de* γ *et* $\widetilde{G}_{thr}$ *(et pas de* ν*).*

Démonstration de $\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_4$ - 1

$$\int_0^1 \left(\mathcal{S}^\nu\right)\left[\frac{\partial \mathcal{S}^\nu}{\partial\theta} - \nabla\cdot((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)+\nu)\nabla\mathcal{S}^\nu) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot)\right]dx \longrightarrow$$

$$\begin{aligned}
\left(\int_0^1\int_{\mathbb{T}^2}\sqrt{\widetilde{\mathcal{A}}_\epsilon}\,|\nabla\mathcal{S}^\nu|dxd\theta\right)^2 &\leq \int_0^1\int_{\mathbb{T}^2}\left(\sqrt{\widetilde{\mathcal{A}}_\epsilon}\,|\nabla\mathcal{S}^\nu|\right)^2 dxd\theta \\
&\leq \int_0^1\int_{\mathbb{T}^2}(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla\mathcal{S}^\nu|^2 dxd\theta \\
&\leq \int_0^1\int_{\mathbb{T}^2}|\widetilde{\mathcal{C}}_\epsilon|\,|\nabla\mathcal{S}^\nu|dxd\theta \leq \gamma\int_0^1\int_{\mathbb{T}^2}\sqrt{\widetilde{\mathcal{A}}_\epsilon}\,|\nabla\mathcal{S}^\nu|dxd\theta.
\end{aligned}$$

$\longrightarrow$

$$\left\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\,|\nabla\mathcal{S}^\nu|\right\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma.$$

Démonstration de $\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_4$ - 2

$$\sqrt{\widetilde{G}_{thr}}\left(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla\mathcal{S}^\nu|^2 dxd\theta\right)^{1/2} \leq \left(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}\widetilde{\mathcal{A}}_\epsilon|\nabla\mathcal{S}^\nu|^2 dxd\theta\right)^{1/2} \leq \left\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\,|\nabla\mathcal{S}^\nu|\right\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}.$$

$\longrightarrow$

$$\left(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla\mathcal{S}^\nu|^2 dxd\theta\right)^{1/2} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}.$$

$\longrightarrow$

$$\exists\theta_0 \in [\theta_\alpha,\theta_\omega] \text{ tel que } \|\nabla\mathcal{S}^\nu(\theta_0,\cdot)\|_2 \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}.$$

Démonstration de $\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \gamma_4$ - 3

$$\|\nabla\mathcal{S}^\nu(\theta_0,\cdot)\|_2 \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} \Rightarrow \|\mathcal{S}^\nu(\theta_0,\cdot)\|_2 \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}.$$

$$\int_{\mathbb{T}^2}\left(S^\nu\right)\left[\frac{\partial\mathcal{S}^\nu}{\partial\theta} - \nabla\cdot((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)+\nu)\nabla\mathcal{S}^\nu)) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot)\right]dx \longrightarrow$$

$$\frac{1}{2}\frac{d\|\mathcal{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} + \int_{x\in\mathbb{T}^2,\ \widetilde{\mathcal{C}}_\epsilon(\theta,x)=0}(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla\mathcal{S}^\nu(\theta,\cdot)|^2dx$$

$$\leq \int_{x\in\mathbb{T}^2,\ \widetilde{\mathcal{C}}_\epsilon(\theta,x)\neq 0}\frac{\widetilde{\mathcal{A}}_\epsilon+\nu}{4}|\nabla\mathcal{S}^\nu(\theta,\cdot)|^2dx + \int_{\mathbb{T}^2}\gamma\,dx.$$

Donc :

$$\frac{d\|\mathcal{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} \leq 2\gamma \Rightarrow \|\mathcal{S}^\nu(\theta,\cdot)\|_2^2 \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma = \gamma_4,\ \forall\theta\in[\theta_0,\theta_0+1].$$

i.e.

$$\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma = \gamma_4.$$

Existence et unicité de $\mathcal{S}$

Existence et unicité de $\mathcal{S}$

On vient de prouver :

Théorème

$\exists! \mathcal{S}^\nu \in L^2(\mathbb{R} \times \mathbb{T}^2)$, *périodique de période 1 en θ solution de*

$$\frac{\partial \mathcal{S}^\nu}{\partial \theta} - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot) + \nu)\nabla \mathcal{S}^\nu) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

$$\sup_{\theta \in \mathbb{R}} \left| \int_{\mathbb{T}^2} \mathcal{S}^\nu(\theta, x) dx \right| = 0.$$

Estimations :

$$\|\mathcal{S}^\nu\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma.$$

$$\left\| \frac{\partial \mathcal{S}^\nu}{\partial t} \right\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma + \gamma^3}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma.$$

En faisant $\nu \to 0$ on a : $\cdots / \ldots$

Théorème

Théorème

$\exists!\ \mathcal{S} = \mathcal{S}(t,\theta,x) \in L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$, périodique de periode 1 en θ, solution de

$$\frac{\partial \mathcal{S}}{\partial \theta} - \nabla \cdot (\widetilde{\mathcal{A}}_\epsilon(t,\cdot,\cdot)\nabla \mathcal{S}) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon(t,\cdot,\cdot).$$

$$\int_{\mathbb{T}^2} \mathcal{S}(t,\theta,x)dx = 0.$$

Elle satisfait :

$$\|\mathcal{S}\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma.$$

$$\|\frac{\partial \mathcal{S}}{\partial t}\|^2_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \frac{\gamma + \gamma^3}{\sqrt{\widetilde{G}_{thr}}} + 2\gamma.$$