Two-Scale Numerical Methods and TSAPS for Tokamak Plasma Physics

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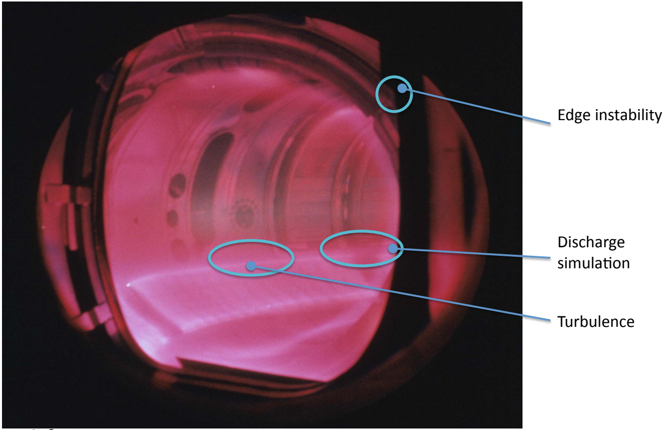
Introduction

Long term target : 10 ms of a Tokamak working

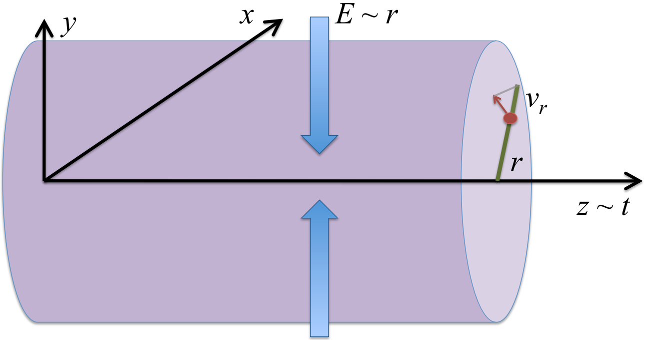


$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \left(\mathbf{E}^\varepsilon + \mathbf{v}\times\left(\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon}\right)\right)\cdot\nabla_v f^\varepsilon = 0$$

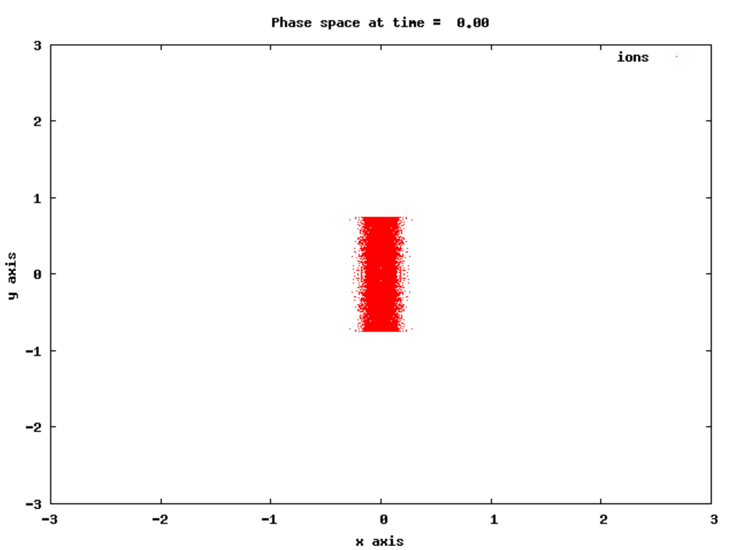
$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon + \frac{\mathbf{v}_\perp}{\varepsilon}\cdot\nabla_x f^\varepsilon + \left(\mathbf{E}^\varepsilon + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon}\right)\cdot\nabla_v f^\varepsilon = 0$$

Example of problem with two scales

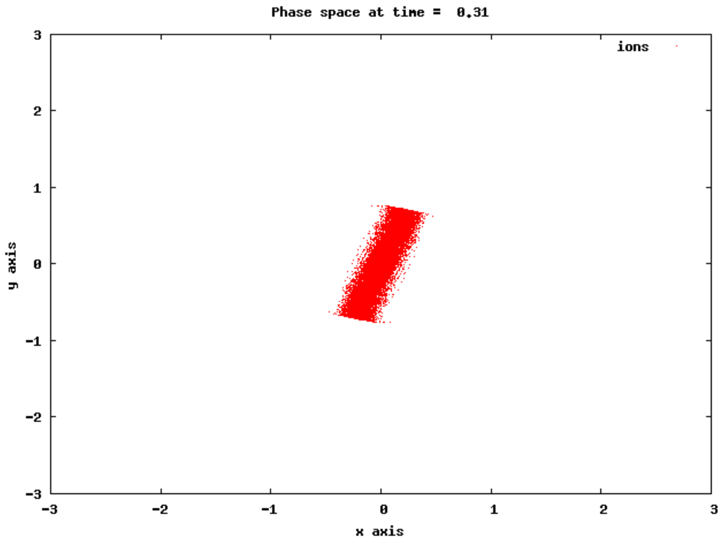
A beam in a focusing channel



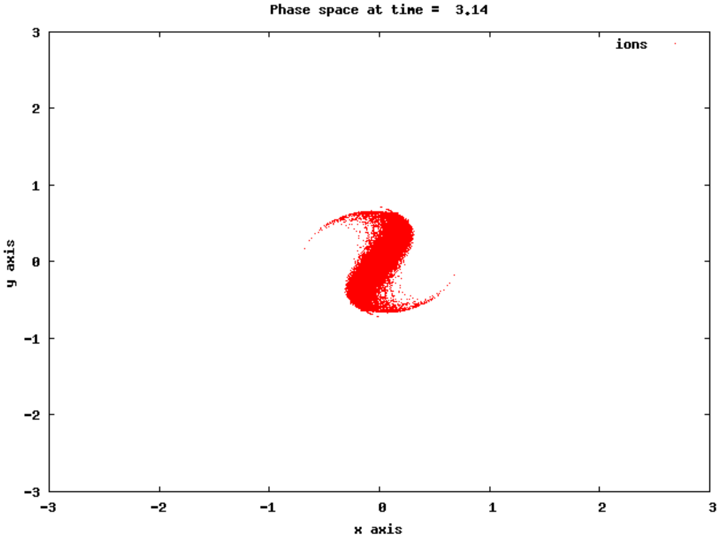
Example of problem with two scales - Simulation



Example of problem with two scales - Simulation



Example of problem with two scales - Simulation



General framework

$$O^\varepsilon \, u^\varepsilon = 0,$$

Essentially : $\varepsilon \sim 0$ but not everywhere everytime
O^ε induces oscillations of period ε and of high amplitude in $u^\varepsilon = u^\varepsilon(z)$.
$\left(O^\varepsilon = \left(NL(z, \frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon} L\right)$

Difficulties

- High frequency oscillations with high amplitude when ε is small
- Manage transition between regions where $\varepsilon \sim 0$ and others

Schemes based on an Approximated Equation

Where ε is small
If $u^\varepsilon(z) \sim u(z)$ strongly for small ε $\quad (\|u^\varepsilon(z) - u(z)\| \xrightarrow{\varepsilon \to 0} 0)$

u^ε solution to
$$O^\varepsilon\, u^\varepsilon = 0$$

$\xrightarrow[\varepsilon \to 0]{}$

u solution to
$$O\, u = 0$$

$\uparrow$ $\Delta z \to 0$

$u_{\Delta z}$ solution to
$$O_{\Delta z}\, u_{\Delta z} = 0$$

$u^\varepsilon(z) \sim u(z)$ strongly —> high-frequency-small-amplitude oscillations.
u good approximation of u^ε.

Schemes based on an Approximated Equation - 2

Where ε is small

If $u^\varepsilon(z) \sim u(z)$ weakly for small ε

$(\int (u^\varepsilon(z) - u(z))\varphi(z)\, dz \xrightarrow{\varepsilon \to 0} 0$ for any test function φ)

u^ε solution to $O^\varepsilon\, u^\varepsilon = 0$ $\quad\xrightarrow[\varepsilon \to 0]{}\quad$ u solution to $O\, u = 0$

$u_{\Delta z}$ solution to $O_{\Delta z}\, u_{\Delta z} = 0$ $\quad\xrightarrow{\Delta z \to 0}\quad$ u solution to $O\, u = 0$

$u^\varepsilon(z) \sim u(z)$ weakly —> high-frequency-NOT-small-amplitude oscillations.

u approximation of u^ε in average only. Do better?

Two-Scale Convergence

Approx. of high-frequency-NOT-small-amplitude oscillating functions

$u^{\varepsilon}(z) \longrightarrow$ high-frequency-NOT-small-amplitude oscillations.

$$u^{\varepsilon}(z) \sim U(z, \tfrac{z}{\varepsilon}) \text{ with } U(z,\zeta) \text{ periodic in } \zeta.$$

Illustration $\cdots / \ldots$

function with large scale variation and high-frequency-NOT-small-amplitude oscillations

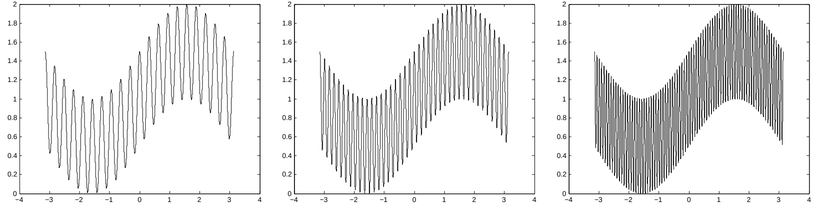


Figure: Graph of $\frac{1}{2}\sin(x)+1+\frac{1}{2}\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

function with high-frequency-modulated-amplitude oscillations

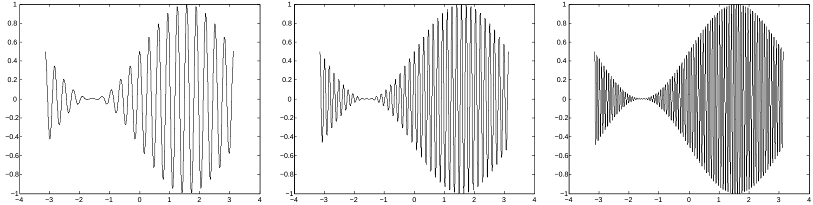


Figure: Graph of $\frac{1}{2}(\sin(x)+1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

function with large scale variation and high-frequency-modulated-amplitude oscillations

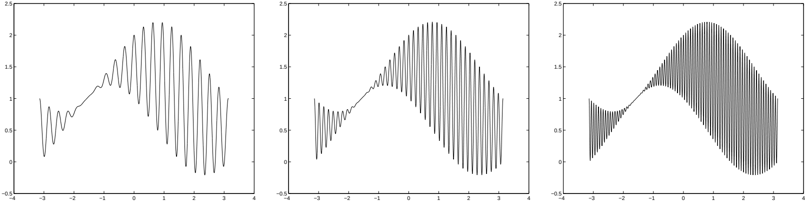


Figure: Graph of $\frac{1}{2}\cos(x) + 1 + \frac{1}{2}(\sin(x) + 1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

Two-Scale Convergence Definition

$(z \in \mathbb{R}^n)$

$(u^\varepsilon(z))$ Two-Scale Converges to $U(z,\zeta)$ periodic of period $[0,1]^n$ in ζ

if

$\forall \psi(z,\zeta)$ regular, compactly supported in z and periodic of period $[0,1]^n$ in ζ

$$\int u^\varepsilon(z)\psi(z,\frac{z}{\varepsilon})\,dz \xrightarrow{\varepsilon\to 0} \int\int_{\zeta\in[0,1]^n} U(z,\zeta)\psi(z,\zeta)\,d\zeta dz$$

MEANS: $u^\varepsilon(z) \sim U(z,\frac{z}{\varepsilon})$

Two-Scale Convergence Results

Two-Scale Convergence

If $\|u^\varepsilon\|$ in bounded, then

$$(u^\varepsilon(z)) \text{ Two-Scale Converges to } U(z,\zeta)$$

$$u^\varepsilon(z) \rightharpoonup u(z) = \int_{\zeta\in[0,1]^n} U(z,\zeta)\, d\zeta \quad \text{weak-}*$$

Strong Two-Scale Convergence

If $\|u^\varepsilon\| \xrightarrow{\varepsilon\to 0} \|U\|$, then $\|u^\varepsilon(z) - U(z,\frac{z}{\varepsilon})\| \xrightarrow{\varepsilon\to 0} 0$

Set of Methods

$$O^\varepsilon\, u^\varepsilon = 0 \longrightarrow \mathcal{O}\, U = 0$$

Deduction of Order 0 Two-Scale Numerical Methods

Order 0 Two-Scale Numerical Methods - 1

Where ε is small

If $u^\varepsilon(z) \sim u(z)$ weakly for small ε

In

u^ε solution to

$$O^\varepsilon \, u^\varepsilon = 0$$

$\xrightarrow{\varepsilon \to 0}$

u solution to

$$O \, u = 0$$

$\uparrow$ $\Delta z \to 0$

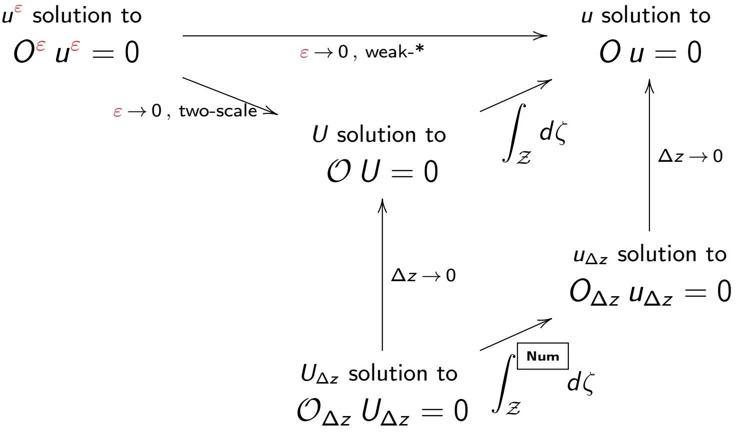
$u_{\Delta z}$ solution to

$$O_{\Delta z} \, u_{\Delta z} = 0$$

it can be better to add a layer

Order 0 Two-Scale Numerical Methods - 2

Where ε is small
If $u^\varepsilon(z) \sim u(z)$ weakly and $u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon})$ more strongly for small ε



Details for passing from $\mathcal{O}^\varepsilon$ to $\mathcal{O}$

Weak Formulation With Oscillating Test Functions (WFWOTF)

- $\left(O^\varepsilon = \left(NL(z, \frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon} L \right)$
  L linear PDE Operator inducing oscillations
  NL non-linear PDE Operator with oscillating coefficients
- Assumptions to have: $(\|u^\varepsilon\|)$ bounded.
- Oscillating Test Functions: $\psi(z,\zeta)$ regular, compactly supported in z and periodic of period $[0,1]^n$ in ζ
  $\longrightarrow \psi(z, \mathcal{P}\zeta)$
  $\longrightarrow [\psi]^\varepsilon(z) = \psi(z, \mathcal{P}\frac{z}{\varepsilon})$
- Weak Formulation With Oscillating Test Functions :
  $O^\varepsilon u^\varepsilon(z) = 0$
  $$\longrightarrow \int \left(\left(\left(NL(z, \frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon} L \right) u^\varepsilon(z) \right) [\psi]^\varepsilon(z)\, dz = 0$$
  $$\longrightarrow \int u^\varepsilon(z) \left(\left(\left(NL(z, \frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon} L \right)^* [\psi]^\varepsilon(z) \right) dz = 0$$

$\cdots / \cdots$

Weak Formulation With Oscillating Test Functions (WFWOTF) - 2

- Weak Formulation With Oscillating Test Functions :

$$O^\varepsilon u^\varepsilon(z) = 0$$

$$\longrightarrow \int \left(\left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)u^\varepsilon(z)\right)[\psi]^\varepsilon(z)\,dz = 0$$

$$\longrightarrow \int u^\varepsilon(z)\left(\left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)^*[\psi]^\varepsilon(z)\right)dz = 0$$

$$\longrightarrow \int u^\varepsilon(z)\left(\left[\left(NL(z,\frac{z}{\varepsilon})\right)_1^*\psi\right]^\varepsilon + \frac{1}{\varepsilon}\left[\left(NL(z,\frac{z}{\varepsilon})\right)_{\mathcal{P}}^*\psi\right]^\varepsilon + \frac{1}{\varepsilon}[L_1^*\psi]^\varepsilon + \underbrace{\frac{1}{\varepsilon^2}[L_{\mathcal{P}}^*\psi]^\varepsilon}_{0}\right)dz = 0$$

Constraint Equation and Consequence

WFWOTF:

$$\int u^\varepsilon(z)\left(\left[\left(NL(z,\frac{z}{\varepsilon})\right)_1^*\psi\right]^\varepsilon + \frac{1}{\varepsilon}\left[\left(NL(z,\frac{z}{\varepsilon})\right)_{\mathcal{P}}^*\psi\right]^\varepsilon + \frac{1}{\varepsilon}L_1^*\left[\psi\right]^\varepsilon\right)dz = 0$$

WFWOTF $\times\varepsilon$ $\longrightarrow$ $\varepsilon\to 0$ $\longrightarrow$

$$\int\int_{[0,1]^n} U(z,\zeta)\left(\left(NL(z,\zeta)\right)_{\mathcal{P}}^*\psi(z,\zeta) + L_1^*\psi(z,\zeta)\right)d\zeta dz = 0$$

$\longrightarrow$

$$\left(\left(NL(z,\zeta)\right)_{\mathcal{P}} + L_1\right)U(z,\zeta) = 0$$

$\longrightarrow$

$$U(z,\zeta) = \left(\mathcal{R}(\zeta)\right)V(z)$$

$$u^\varepsilon(z)\sim U(z,\frac{z}{\varepsilon}) = \left(\mathcal{R}(\frac{z}{\varepsilon})\right)V(z)$$

Equation for V

For $\gamma(y)$, $\psi(z,\zeta) = (\mathcal{R}(\zeta))^* \gamma(z)$ s.t. $\Big(\big(NL(z,\zeta)\big)^*_{\mathcal{P}} + L_1^*\Big)\psi(z,\zeta) = 0$

In WFWOTF:

$$\int u^\varepsilon(z)\left(\left[\Big(NL(z,\frac{z}{\varepsilon})\Big)^*_1 \psi\right]^\varepsilon + \frac{1}{\varepsilon}\left[\Big(NL(z,\frac{z}{\varepsilon})\Big)^*_{\mathcal{P}} \psi\right]^\varepsilon + \frac{1}{\varepsilon} L_1^* \,[\psi]^\varepsilon\right) dz = 0$$

$$\longrightarrow \qquad \int u^\varepsilon(z)\left(\left[\Big(NL(z,\frac{z}{\varepsilon})\Big)^*_1 \psi\right]^\varepsilon\right) dz = 0$$

$$\varepsilon \to 0 \qquad \longrightarrow \int\int_{[0,1]^n} U(z,\zeta)\Big(\big(NL(z,\zeta)\big)^*_1 \psi(z,\zeta)\Big)\, d\zeta dz = 0$$

$U(z,\zeta) = (\mathcal{R}(\zeta))\, V(z)$ and $\psi(z,\zeta) = (\mathcal{R}(\zeta))^* \gamma(z)$

$$\longrightarrow \int V(z)\left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \,\big(NL(z,\zeta)\big)^*_1 \,(\mathcal{R}(\zeta))^* \, d\zeta\right)\gamma(z)\, dz = 0$$

$$\longrightarrow \qquad \left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \,\big(NL(z,\zeta)\big)^*_1 \,(\mathcal{R}(\zeta))^* \, d\zeta\right)^* V = 0$$

Implication on Order 0 Two-Scale Numerical Methods

Order 0 Two-Scale Numerical Methods - 3

In

u^ε solution to $\quad O^\varepsilon\, u^\varepsilon = 0$

$\xrightarrow{\ \varepsilon \to 0\ ,\ \text{weak-*}\ }$

u solution to $\quad O\, u = 0$

$\varepsilon \to 0$, two-scale $\searrow$

U solution to $\quad \mathcal{O}\, U = 0$

$\nearrow \displaystyle\int_{\mathcal{Z}} d\zeta$

$\uparrow\ \Delta z \to 0$

$u_{\Delta z}$ solution to $\quad O_{\Delta z}\, u_{\Delta z} = 0$

$\uparrow\ \Delta z \to 0$

$U_{\Delta z}$ solution to $\quad \mathcal{O}_{\Delta z}\, U_{\Delta z} = 0$

$\nearrow \displaystyle\int_{\mathcal{Z}}^{\boxed{\textbf{Num}}} d\zeta$

solution $\ \mathcal{O}\, U = 0$ means: $\qquad \cdots / \ldots$

Order 0 Two-Scale Numerical Methods - 4

solution:

$$\left(\int_{[0,1]^{\boldsymbol{n}}} (\mathcal{R}(\zeta))^* \left(NL(z,\zeta)\right)_1^* (\mathcal{R}(\zeta))^* \, d\zeta\right)^* V = 0$$

and compute:

$$U(z,\zeta) = (\mathcal{R}(\zeta))\, V(z)$$

Order 0 Two-Scale Numerical Method - 5 : Algorithm

To compute u^ε solution to

$$O^\varepsilon \, u^\varepsilon = 0$$

for ε small:

Compute V solution to

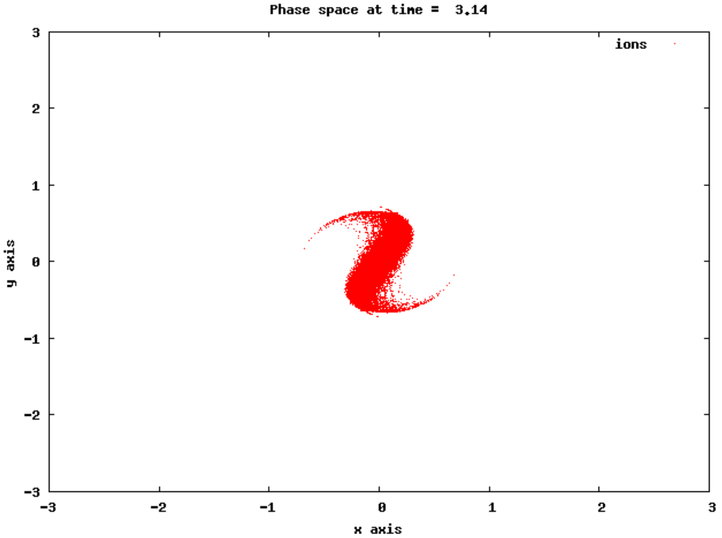
$$\left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \left(NL(z,\zeta) \right)_1^* (\mathcal{R}(\zeta))^* \, d\zeta \right)^* V = 0$$

And use

$$u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon}) \qquad U(z, \frac{z}{\varepsilon}) = \left(\mathcal{R}(\frac{z}{\varepsilon}) \right) V(z)$$

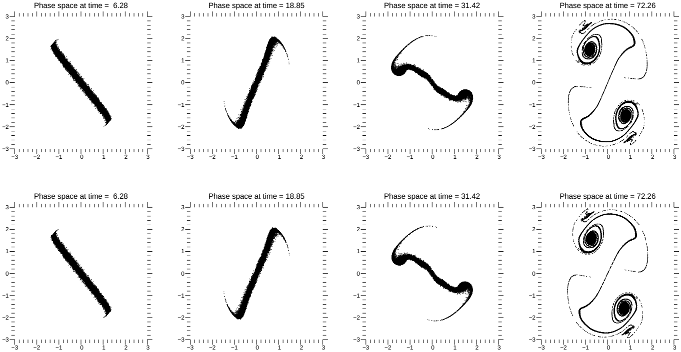
Tests

Order 0 Two-Scale Numerical Method : Simulations



Order 0 Two-Scale Numerical Method : Simulations

In : Frénod - Salvarani - Sonnendrücker, *Math. Mod. Meth. Appl. Sc.*, 2009.



Beam simulation (at times 6.26, 18.85, 31.42 et 72.26) by a usual PIC Code (top) and a TSPICC (bottom).

Theoretical basis for Order 1 Two-Scale Numerical Methods

Order 1 Two-Scale Numerical Methods - 1

$u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon}) + \varepsilon U^1(z, \frac{z}{\varepsilon})$ with $U(z,\zeta)$ and $U^1(z,\zeta)$ periodic in ζ.

$(u^\varepsilon(z))$ Two-Scale Converges to $U(z,\zeta)$

$\left(\frac{1}{\varepsilon}\left(u^\varepsilon(z) - U(z, \frac{z}{\varepsilon})\right)\right)$ Two-Scale Converges to $U^1(z,\zeta)$

Set of Methods

$$O^\varepsilon\, u^\varepsilon = 0 \quad \longrightarrow \quad \mathcal{O}\, U = 0 \text{ and } \mathcal{O}^1\, U^1 = 0$$

Where ε is small

If $u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon}) + \varepsilon U^1(z, \frac{z}{\varepsilon})$ for small ε ($u^\varepsilon(z) \sim u(z) + \varepsilon u^1(z)$ weakly)

u^ε solution to

$$O^\varepsilon\, u^\varepsilon = 0$$

$\varepsilon \to 0$, weak-*

u, u_1 solutions to

$$O\, u = 0$$
$$O^1 u^1 = 0$$

$\varepsilon \to 0$, two-scale

U, U^1 solutions to

$$\mathcal{O}\, U = 0$$
$$\mathcal{O}^1 U^1 = 0$$

$\int_{\mathcal{Z}} d\zeta$

$\Delta z \to 0$

$\Delta z \to 0$

$u_{\Delta z}$, $u^1_{\Delta z}$ solutions to

$$O_{\Delta z}\, u_{\Delta z} = 0$$
$$O^1_{\Delta z} u^1_{\Delta z} = 0$$

$U_{\Delta z}$, $U^1_{\Delta z}$ solutions to

$$\mathcal{O}_{\Delta z}\, U_{\Delta z} = 0$$
$$\mathcal{O}^1_{\Delta z} U^1_{\Delta z} = 0$$

$\int_{\mathcal{Z}}^{\boxed{\text{Num}}} d\zeta$

Details for deducing $\mathcal{O}_1$ from $\mathcal{O}^\varepsilon$

Characterization of U^1 - 1

- $$\left(O^\varepsilon = \left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)$$
- $$O^\varepsilon u^\varepsilon(z) = 0 : \left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)u^\varepsilon(z) = 0$$
- $$\left(\left(NL(z,\zeta)\right)_{\mathcal{P}} + L_1\right)U(z,\zeta) = 0 \text{ and}$$
$$\left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \left(NL(z,\zeta)\right)_1^* (\mathcal{R}(\zeta))^* \, d\zeta\right)^* V = 0$$

$$\left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)(u^\varepsilon - [U]^\varepsilon) = \left(B(z,\frac{z}{\varepsilon})\right)[U]^\varepsilon$$

$$[U]^\varepsilon(z) = U(z,\tfrac{z}{\varepsilon})$$

$$\times\frac{1}{\varepsilon} \longrightarrow \quad \left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)\frac{(u^\varepsilon - [U]^\varepsilon)}{\varepsilon} = \frac{1}{\varepsilon}\left(B(z,\frac{z}{\varepsilon})\right)[U]^\varepsilon$$

Characterization of U^1 - 2

$$\left(\left(NL(z,\frac{z}{\varepsilon})\right)+\frac{1}{\varepsilon}L\right)\frac{(u^\varepsilon-[U]^\varepsilon)}{\varepsilon}=\frac{1}{\varepsilon}\left(B(z,\frac{z}{\varepsilon})\right)[U]^\varepsilon$$

$W(z,\zeta)$ periodic of period $[0,1]^n$ in ζ such that:

$$\left(\left(NL(z,\frac{z}{\varepsilon})\right)+\frac{1}{\varepsilon}L\right)[(\mathcal{R})W]^\varepsilon=\frac{1}{\varepsilon}\left(B(z,\frac{z}{\varepsilon})\right)[U]^\varepsilon-\left(C(z,\frac{z}{\varepsilon})\right)[W]^\varepsilon$$

$$\boldsymbol{U}(\boldsymbol{z},\zeta)=(\mathcal{R}(\zeta))\boldsymbol{V}(\boldsymbol{z}),\ [(\mathcal{R})\boldsymbol{W}]^\varepsilon(\boldsymbol{z})=\left(\mathcal{R}(\frac{\boldsymbol{z}}{\varepsilon})\right)\boldsymbol{W}(\boldsymbol{z},\frac{\boldsymbol{z}}{\varepsilon})$$

$$\left(\left(NL(z,\frac{z}{\varepsilon})\right)+\frac{1}{\varepsilon}L\right)\left(\frac{(u^\varepsilon-[U]^\varepsilon)}{\varepsilon}-[(\mathcal{R})W]^\varepsilon\right)=\left(C(z,\frac{z}{\varepsilon})\right)[W]^\varepsilon$$

Estimates $\longrightarrow$
$\left(\frac{(u^\varepsilon-[U]^\varepsilon)}{\varepsilon}-[(\mathcal{R})W]^\varepsilon\right)$ Two-Scale Converges to $\left(U^1-(\mathcal{R})W\right)$

Characterization of U^1 - 3

From:

$$\left(\left(NL(z,\frac{z}{\varepsilon})\right)+\frac{1}{\varepsilon}L\right)\left(\frac{(u^\varepsilon-[U]^\varepsilon)}{\varepsilon}-[(\mathcal{R})W]^\varepsilon\right)=\left(C(z,\frac{z}{\varepsilon})\right)[W]^\varepsilon$$

As for $U \longrightarrow$ Constraint: $\left(\left(NL(z,\zeta)\right)_{\mathcal{P}}+L_1\right)\left(U^1-(\mathcal{R})W\right)=0$

$$\left(U^1(z,\zeta)-(\mathcal{R}(\zeta))W(z,\zeta)\right)=(\mathcal{R}(\zeta))\,V^1(z) \text{ i.e:}$$
$$U^1(z,\zeta)=(\mathcal{R}(\zeta))\,V^1(z)+(\mathcal{R}(\zeta))W(z,\zeta)$$

$$u^\varepsilon(z)\sim U(z,\frac{z}{\varepsilon})+\varepsilon U^1(z,\frac{z}{\varepsilon})=\left(\mathcal{R}(\frac{z}{\varepsilon})\right)\left(V(z)+\varepsilon(V^1(z)+W(z,\frac{z}{\varepsilon}))\right)$$

As for $U \longrightarrow$

$$\left(\int_{[0,1]^n}(\mathcal{R}(\zeta))^*\left(NL(z,\zeta)\right)_1^*(\mathcal{R}(\zeta))^*\,d\zeta\right)^* V^1=RHS$$

In

u^{ε} solution to

$$O^{\varepsilon}\, u^{\varepsilon} = 0$$

$\varepsilon \to 0$, weak-*

u, u_1 solutions to

$$O\, u = 0$$
$$O^1 u^1 = 0$$

$\varepsilon \to 0$, two-scale

U, U^1 solutions to

$$\mathcal{O}\, U = 0$$
$$\mathcal{O}^1 U^1 = 0$$

$\int_{\mathcal{Z}} d\zeta$

$\Delta z \to 0$

$\Delta z \to 0$

$u_{\Delta z}$, $u^1_{\Delta z}$ solutions to

$$O_{\Delta z}\, u_{\Delta z} = 0$$
$$O^1_{\Delta z} u^1_{\Delta z} = 0$$

$U_{\Delta z}$, $U^1_{\Delta z}$ solutions to

$$\mathcal{O}_{\Delta z}\, U_{\Delta z} = 0$$
$$\mathcal{O}^1_{\Delta z} U^1_{\Delta z} = 0$$

$\int_{\mathcal{Z}}^{\boxed{\text{Num}}} d\zeta$

solution $\mathcal{O}\, U = 0$ and $\mathrm{O}^1 U^1 = 0$ means : $\cdots / \ldots$

Order 1 Two-Scale Numerical Methods - 3

solution:

$$\left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \left(NL(z,\zeta)\right)_1^* (\mathcal{R}(\zeta))^* \, d\zeta\right)^* V = 0$$

$$\left(\int_{[0,1]^n} (\mathcal{R}(\zeta))^* \left(NL(z,\zeta)\right)_1^* (\mathcal{R}(\zeta))^* \, d\zeta\right)^* V^1 = RHS$$

and compute:

$$U(z,\zeta) = (\mathcal{R}(\zeta))\left(V(z) + \varepsilon\left(V^1(z) + W(z,\zeta)\right)\right)$$

Order 1 Two-Scale Numerical Method - 4 : Algorithm

To compute u^{ε} solution to

$$O^{\varepsilon}\, u^{\varepsilon} = 0$$

for ε small:

Compute V solution to

$$\left(\int_{[0,1]^{\boldsymbol{n}}} (\mathcal{R}(\zeta))^{*} \left(NL(z,\zeta)\right)_{1}^{*} (\mathcal{R}(\zeta))^{*}\, d\zeta\right)^{*} V = 0$$

$$\left(\int_{[0,1]^{\boldsymbol{n}}} (\mathcal{R}(\zeta))^{*} \left(NL(z,\zeta)\right)_{1}^{*} (\mathcal{R}(\zeta))^{*}\, d\zeta\right)^{*} V^{1} = RHS$$

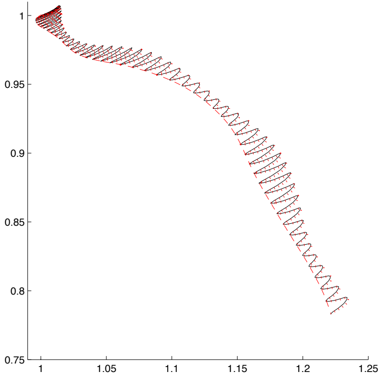
Build $W(z,\zeta)$
And use

$$u^{\varepsilon}(z) \sim U(z,\frac{z}{\varepsilon}) + \varepsilon U1(z,\frac{z}{\varepsilon}) = \left(\mathcal{R}(\frac{z}{\varepsilon})\right)\left(V(z) + \varepsilon\left(V^{1}(z) + W(z,\frac{z}{\varepsilon})\right)\right)$$

Order 1 Two-Scale simulation

First example of Two-Scale simulation: Drift in coastal ocean waters

In : Ailliot - Frénod - Monbet, *Multiscale Mod. Simul.*, 2006.



Objective of TSAPS

Recall : Strategy of AP Schemes

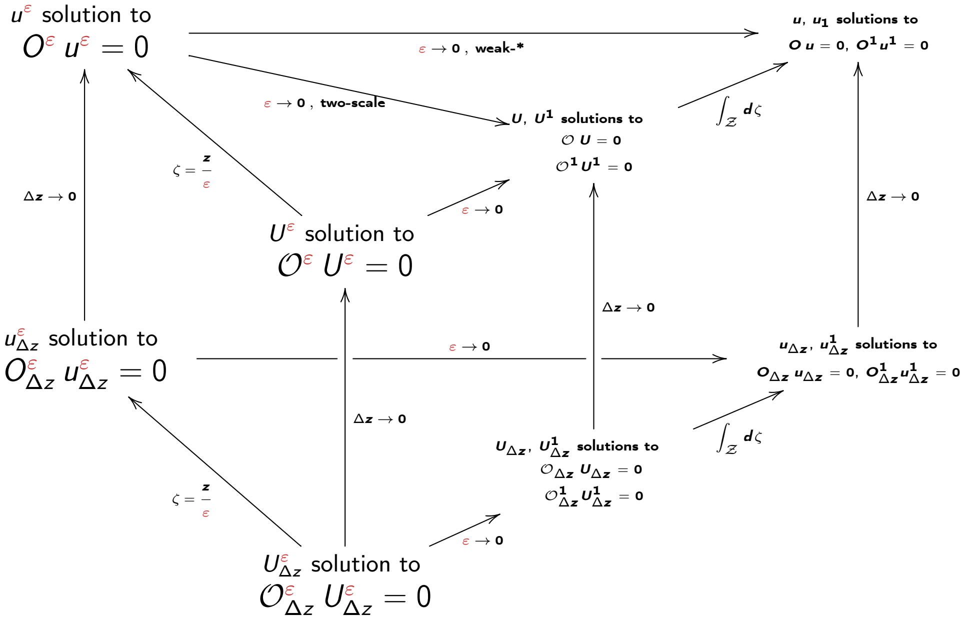
ε non uniformally small
$u^\varepsilon(z) \sim u(z)$ strongly for small ε

u^ε solution to $O^\varepsilon\, u^\varepsilon = 0$ $\xrightarrow{\varepsilon \to 0}$ u solution to $O\, u = 0$

$u^\varepsilon_{\Delta z}$ solution to $O^\varepsilon_{\Delta z}\, u^\varepsilon_{\Delta z} = 0$ $\xrightarrow{\varepsilon \to 0}$ $u_{\Delta z}$ solution to $O_{\Delta z}\, u_{\Delta z} = 0$

ε non uniformally small

If $u^\varepsilon(z) \sim U^\varepsilon(z, \frac{z}{\varepsilon}) = U_0^\varepsilon(z, \frac{z}{\varepsilon}) + \varepsilon U_1^\varepsilon(z, \frac{z}{\varepsilon})$



Two-Scale Macro-Micro Decomposition

Ideas for the Macro-Micro Decomposition - 1

We want: $u^\varepsilon(z) \sim U^\varepsilon(z, \frac{z}{\varepsilon}) = U_0^\varepsilon(z, \frac{z}{\varepsilon}) + \varepsilon U_1^\varepsilon(z, \frac{z}{\varepsilon})$

We have, for ε small:

$$u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon}) + \varepsilon U^1(z, \frac{z}{\varepsilon}) = \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(V(z) + \varepsilon\Big(V^1(z) + W(z, \frac{z}{\varepsilon})\Big)$$

$$(\mathcal{R}(\zeta))(V(z)) \text{ and } (\mathcal{R}(\zeta))(V^1(z)) \in \operatorname{Ker}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big)$$

$$(\mathcal{R}(\zeta))(W(z,\zeta)) \in \operatorname{Im}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big)$$

$$(\operatorname{Ker}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big) \oplus \operatorname{Im}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big))$$

So, we write:

$$u^\varepsilon(z) = \underbrace{\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z)) + \varepsilon\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V_1^\varepsilon(z))}_{\text{Macro Part}}$$

$$+ \underbrace{\varepsilon\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(W(z, \frac{z}{\varepsilon})\Big) + \varepsilon K^\varepsilon((z, \frac{z}{\varepsilon}))}_{\text{Micro Part}}$$

Ideas for the Macro-Micro Decomposition - 2

So, we wrote:

$$u^\varepsilon(z) = \underbrace{\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z)) + \varepsilon\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V_1^\varepsilon(z))}_{\text{Macro Part}}$$

$$+ \underbrace{\varepsilon\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(W(z,\frac{z}{\varepsilon})\Big) + \varepsilon K^\varepsilon((z,\frac{z}{\varepsilon}))}_{\text{Micro Part}}$$

$$(\mathcal{R}(\zeta))(V(z)) \text{ and } (\mathcal{R}(\zeta))(V_1^\varepsilon(z)) \in \text{Ker}\left(\Big(NL(z,\zeta)\Big)_{\mathcal{P}} + L_1\right)$$

$$(\mathcal{R}(\zeta))(W(z,\zeta)) \text{ and } K^\varepsilon(z,\zeta) \in \text{Im}\left(\Big(NL(z,\zeta)\Big)_{\mathcal{P}} + L_1\right)$$

$$V_1^\varepsilon \xrightarrow{\varepsilon\to 0} V^1$$

$$K^\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

Projection

vs

Ad-hoc Oscillating Test Functions

Projection ??

Plug

$$u^\varepsilon(z) = \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z)) + \varepsilon\,\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V_1^\varepsilon(z)) + \varepsilon\,\Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(W(z,\frac{z}{\varepsilon})\Big) + \varepsilon K^\varepsilon((z,\frac{z}{\varepsilon}))$$

in $O^\varepsilon u^\varepsilon(z) = 0$ i.e. $\left(\Big(NL(z,\frac{z}{\varepsilon})\Big) + \frac{1}{\varepsilon}L\right)u^\varepsilon(z) = 0$

Then project on $\mathrm{Ker}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big)$ to get the Macro part of the Macro-Micro System

And project on $\mathrm{Im}\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big)$ to get the Micro part of the Macro-Micro System

Troubles:

As soon as ζ is replaced by $\frac{z}{\varepsilon}$, $\Big(\big(NL(z,\zeta)\big)_{\mathcal{P}} + L_1\Big)$ does not exist.

Ad-hoc Oscillating Test Functions

Take WFWOTF: $\int \left(\left(\left(NL(z,\frac{z}{\varepsilon})\right) + \frac{1}{\varepsilon}L\right)u^\varepsilon(z)\right)[\psi]^\varepsilon(z)\,dz = 0$

Plug

$$u^\varepsilon(z) = \left(\mathcal{R}(\frac{z}{\varepsilon})\right)(V(z)) + \varepsilon\left(\mathcal{R}(\frac{z}{\varepsilon})\right)(V_1^\varepsilon(z)) + \varepsilon\left(\mathcal{R}(\frac{z}{\varepsilon})\right)\left(W(z,\frac{z}{\varepsilon})\right) + \varepsilon K^\varepsilon((z,\frac{z}{\varepsilon}))$$

in it.

- when $\psi \in \mathrm{Ker}\left(\left(NL(z,\zeta)\right)_{\mathcal{P}} + L_1\right)$ $\Rightarrow$ Two-Scale Macro Piece
- when $\psi \in \mathrm{Im}\left(\left(NL(z,\zeta)\right)_{\mathcal{P}} + L_1\right)$ $\Rightarrow$ Two-Scale Micro Piece

Model Hierarchy

TSAPS:

$$u^\varepsilon(z) = \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z)) + \varepsilon \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V_1^\varepsilon(z)) + \varepsilon \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(W(z, \frac{z}{\varepsilon})\Big) + \varepsilon K^\varepsilon((z, \frac{z}{\varepsilon}))$$

Order 1 Two-Scale Numerical Methods:

$$u^\varepsilon(z) \sim \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z)) + \varepsilon \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V^1(z)) + \varepsilon \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)\Big(W(z, \frac{z}{\varepsilon})\Big)$$

Order 0 Two-Scale Numerical Methods:

$$u^\varepsilon(z) \sim \Big(\mathcal{R}(\frac{z}{\varepsilon})\Big)(V(z))$$