Décomposition Micro-Macro 2 échelles et TSAPS pour un problème de physique des plasmas

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Introduction

Scheme based on an Approximated Equation

ϵ uniformally small
$u^\epsilon(z) \sim u(z)$ strongly for small ϵ

u^ϵ solution to

$$E^\epsilon\, u^\epsilon = 0$$

$\xrightarrow[\epsilon \to 0]{}$

u solution to

$$E\, u = 0$$

$\uparrow$ $\Delta z \to 0$

$u_{\Delta z}$ solution to

$$E_{\Delta z}\, u_{\Delta z} = 0$$

AP Schemes

ϵ non uniformally small
$u^\epsilon(z) \sim u(z)$ strongly for small ϵ

u^ϵ solution to $E^\epsilon\, u^\epsilon = 0$ $\xrightarrow[\epsilon \to 0]{}$ u solution to $E\, u = 0$

$\Delta z \to 0$ (↑) $\Delta z \to 0$ (↑)

$u^\epsilon_{\Delta z}$ solution to $E^\epsilon_{\Delta z}\, u^\epsilon_{\Delta z} = 0$ $\xrightarrow[\epsilon \to 0]{}$ $u_{\Delta z}$ solution to $E_{\Delta z}\, u_{\Delta z} = 0$

Two-Scale Numerical Methods

ϵ uniformally small
$u^\epsilon(z) \sim u(z)$ weakly and $u^\epsilon(z) \sim U(z, \zeta = \frac{z}{\epsilon})$ strongly for small ϵ

u^ϵ solution to
$$E^\epsilon\, u^\epsilon = 0$$

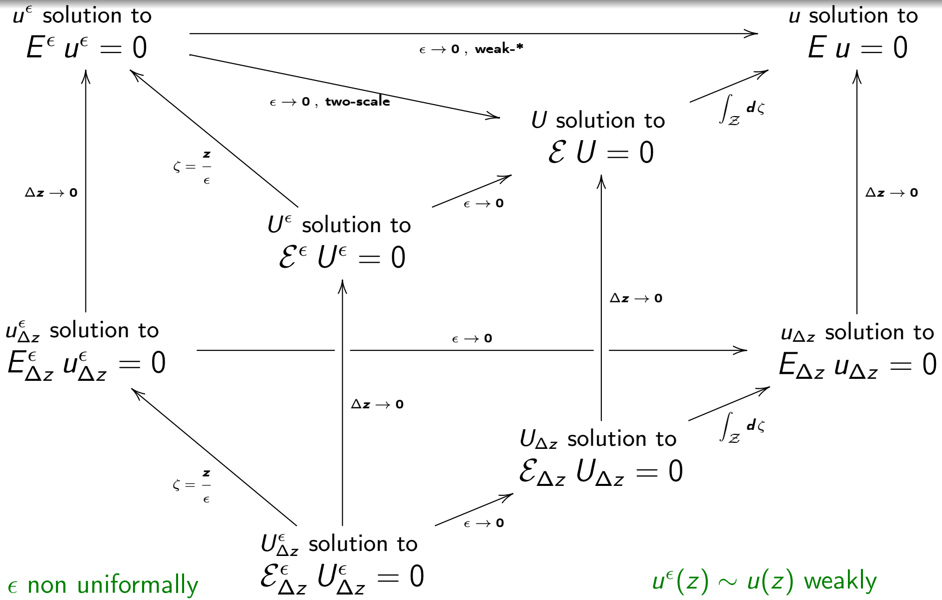
$\xrightarrow[\epsilon \to 0\ ,\ \text{two-scale}]{}$

U solution to
$$\mathcal{E}\, U = 0$$

$\uparrow$ $\Delta z \to 0$

$u_{\Delta z}$ solution to
$$\mathcal{E}_{\Delta z}\, U_{\Delta z} = 0$$

TSAPS



ϵ non uniformly small

$u^\epsilon(z) \sim u(z)$ weakly

$u^\epsilon(z) \sim U(z, \zeta = \frac{z}{\epsilon})$ strongly for small ϵ

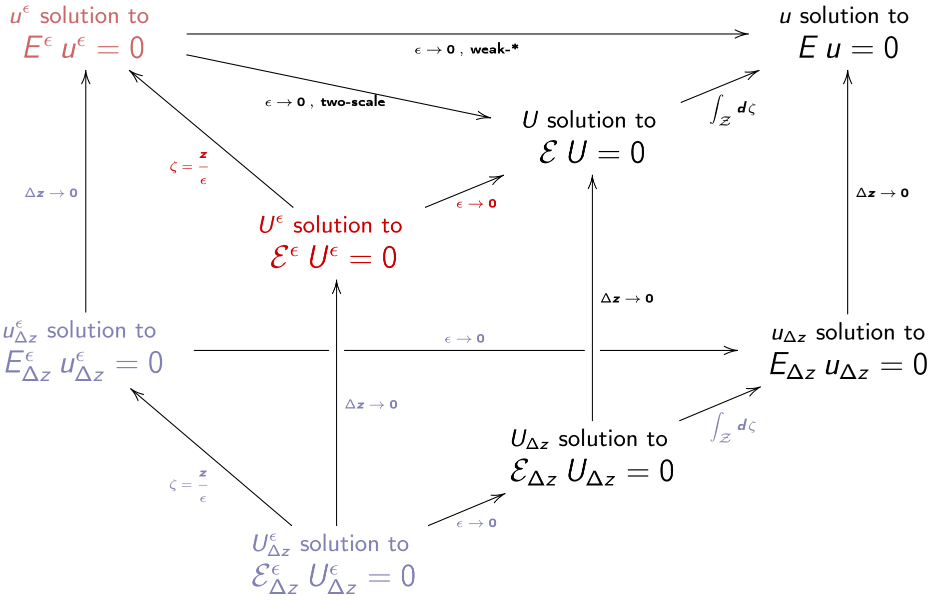
Equation of Interest

Vlasov Equation with Strong Magnetic Field

$$\begin{cases} \dfrac{\partial f^\epsilon}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\epsilon + \left(\mathbf{E} + \mathbf{v} \times \dfrac{\mathcal{M}}{\epsilon}\right) \cdot \nabla_{\mathbf{v}} f^\epsilon = 0\,, \\ f^\epsilon(t=0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v})\,, \end{cases}$$

$t \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{v} \in \mathbb{R}^3$
Distribution function: $f^\epsilon = f^\epsilon(t, \mathbf{x}, \mathbf{v})$
Electric field: $\mathbf{E}^\epsilon = \mathbf{E}^\epsilon(t, \mathbf{x})$
Magnetic field: $\mathcal{M} = \mathbf{e}_1$

Déjà fait - Fait - Reste à faire



Convergence as $\epsilon \to 0$ for f^ϵ

As $\epsilon \to 0$:

$f^\epsilon(t, \mathbf{x}, \mathbf{v}) = f^\epsilon$ Two-Scale converges to $F = F(t, \tau, \mathbf{x}, \mathbf{v})$

i.e: $\boxed{f^\epsilon(t, \mathbf{x}, \mathbf{v}) \sim F(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v})}$ when $\epsilon \sim 0$

or: $\boxed{\int f^\epsilon \, (\psi)^\epsilon \, dt d\mathbf{x} d\mathbf{v} \to \int \int_0^{2\pi} F \, \psi \, d\tau dt d\mathbf{x} d\mathbf{v}}$ as $\epsilon \to 0$

$(\psi)^\epsilon(t, \mathbf{x}, \mathbf{v}) = \psi(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v})$
$\psi(t, \tau, \mathbf{x}, \mathbf{v})$ regular, compactly supported in t, $\mathbf{x}$ and $\mathbf{v}$ and periodic in τ

$$f^\epsilon \rightharpoonup f = \int_0^{2\pi} F \, d\tau \quad \text{weak} - *$$

Properties of F

Weak formulation with oscillating test functions :

$$\int f^\epsilon \Big((\frac{\partial\psi}{\partial t})^\epsilon + \frac{1}{\epsilon}(\frac{\partial\psi}{\partial\tau})^\epsilon + \mathbf{v}\cdot(\nabla_{\mathbf{x}}\psi)^\epsilon + \Big(\mathbf{E}+\mathbf{v}\times\frac{\mathcal{M}}{\epsilon}\Big)\cdot(\nabla_{\mathbf{v}}\psi)^\epsilon \Big)\, dtd\mathbf{x}d\mathbf{v}$$
$$= \int f_0\psi(0,0,.,.)\, d\mathbf{x}d\mathbf{v}$$

$F \in \mathrm{Ker}(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}})$ i.e.:

$$\boxed{F(t,\tau,\mathbf{x},\mathbf{v}) = G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big)} \quad G\big(t,\mathbf{x},\mathbf{u}\big),\ r(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial G}{\partial t} + \mathbf{u}_{\|}\cdot\nabla_{\mathbf{x}_{\|}} G + \mathbf{E}_{\|}\cdot\nabla_{\mathbf{u}_{\|}} G = 0\,, \\ G(t=0,\mathbf{x},\mathbf{u}) = \dfrac{1}{2\pi} f_0(\mathbf{x},\mathbf{u})\,, \end{cases}$$

Properties of F - Resuming

$$\frac{1}{\epsilon}\Big(f^{\epsilon} - \big(F\big)^{\epsilon}\Big) \text{ Two-Scale converges to } F_1$$

$$\boxed{F_1(t,\tau,\mathbf{x},\mathbf{v}) = G_1\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) + l(t,\tau,\mathbf{x},\mathbf{v})}$$

Equation for G_1

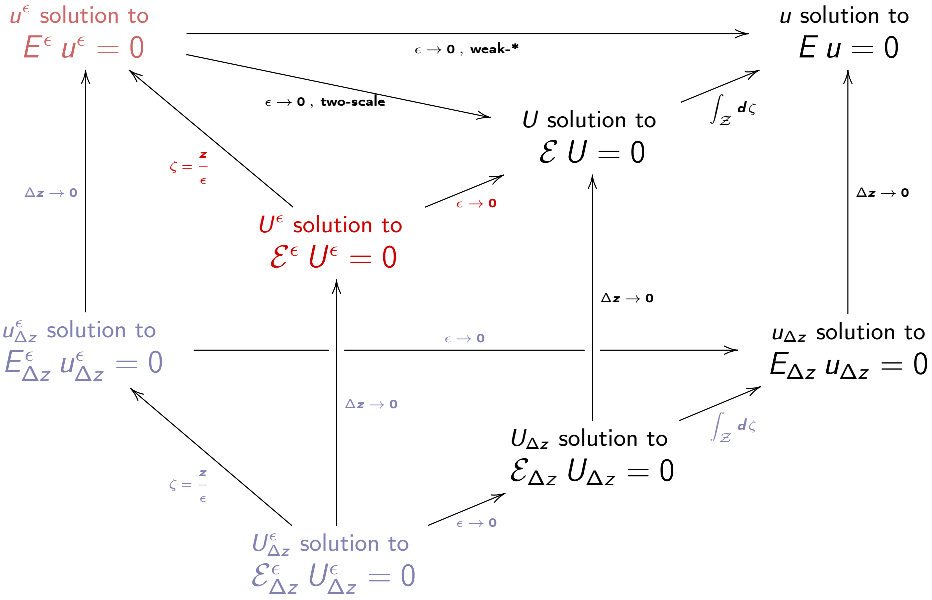
$$\begin{aligned} l(t,\tau,\mathbf{x},\mathbf{v}) = (r(\tau+\tfrac{\pi}{2}) - r(\tfrac{\pi}{2}))\mathbf{v}_\perp \cdot \nabla_{\mathbf{x}_\perp} G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) \\ +(r(\tau+\tfrac{\pi}{2}) - r(\tfrac{\pi}{2}))\mathbf{E}_\perp \cdot \nabla_{\mathbf{u}_\perp} G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) \end{aligned}$$

Equation for the weak $-*$ limit f

$$\int_0^{2\pi} (\text{Eq. for } F \text{ or } G)\, d\tau \Longrightarrow$$

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} + \mathbf{v}_{\|} \cdot \nabla_{\mathbf{x}_{\|}} f + \mathbf{E}_{\|} \cdot \nabla_{\mathbf{v}_{\|}} f = 0\,, \\ f(t=0,\mathbf{x},\mathbf{v}) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_0\big(\mathbf{x}, \mathbf{u}(\tau,\mathbf{v})\big)\, d\tau\,. \end{array} \right.$$

Déjà fait


u^ϵ solution to
$E^\epsilon\, u^\epsilon = 0$
$\epsilon \to 0$, weak-*
u solution to
$E\, u = 0$
$\epsilon \to 0$, two-scale
U solution to
$\mathcal{E}\, U = 0$
$\int_{\mathcal{Z}} d\zeta$
$\zeta = \frac{z}{\epsilon}$
$\Delta z \to 0$
$\Delta z \to 0$
U^ϵ solution to
$\mathcal{E}^\epsilon\, U^\epsilon = 0$
$\epsilon \to 0$
$\Delta z \to 0$
$u^\epsilon_{\Delta z}$ solution to
$E^\epsilon_{\Delta z}\, u^\epsilon_{\Delta z} = 0$
$\epsilon \to 0$
$u_{\Delta z}$ solution to
$E_{\Delta z}\, u_{\Delta z} = 0$
$\Delta z \to 0$
$\int_{\mathcal{Z}} d\zeta$
$U_{\Delta z}$ solution to
$\mathcal{E}_{\Delta z}\, U_{\Delta z} = 0$
$\zeta = \frac{z}{\epsilon}$
$\epsilon \to 0$
$U^\epsilon_{\Delta z}$ solution to
$\mathcal{E}^\epsilon_{\Delta z}\, U^\epsilon_{\Delta z} = 0$

Decompositon

Sought shape for f^ϵ - Work program

Use:

$$\text{Ker}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big) \oplus \text{Im}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big)$$

$$\in \qquad\qquad\qquad \in$$

$$\alpha^\epsilon(t,\tau,\mathbf{x},\mathbf{v}) \qquad\qquad \beta^\epsilon(t,\tau,\mathbf{x},\mathbf{v})$$

$$f^\epsilon(t,\mathbf{x},\mathbf{v}) = \alpha^\epsilon(t,\frac{t}{\epsilon},\mathbf{x},\mathbf{v}) + \beta^\epsilon(t,\frac{t}{\epsilon},\mathbf{x},\mathbf{v})$$

Project Eq. for f^ϵ on

$$\text{Ker}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big) \text{ and } \text{Im}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big)$$

and find Eq. for α^ϵ and β^ϵ

which is the Two-Scale-Micro-Macro decomposition

Sought shape for f^ϵ - Choices

$$f^\epsilon(t,\mathbf{x},\mathbf{v}) = \overbrace{G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) + \epsilon G_1^\epsilon\big(t,\mathbf{x},r(\tau)\mathbf{v}\big)}^{\in \operatorname{Ker}\left(\frac{\partial}{\partial \tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\right)}$$

$$+ \underbrace{\epsilon l(t,\tau,\mathbf{x},\mathbf{v}) + \frac{\partial k^\epsilon}{\partial \tau}(t,\tau,\mathbf{x},\mathbf{v}) + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}} k^\epsilon(t,\tau,\mathbf{x},\mathbf{v})}_{\in \operatorname{Im}\left(\frac{\partial}{\partial \tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\right)}$$

The Two-Scale Macro Equation

$$\int \Big((f^\epsilon =) \big(G \circ r\big)^\epsilon + \epsilon \big(G_1^\epsilon \circ r\big)^\epsilon + \epsilon (l)^\epsilon + \big(\frac{\partial k^\epsilon}{\partial \tau}\big)^\epsilon + (\mathbf{v} \times \mathcal{M}) \cdot \big(\nabla_{\mathbf{v}} k^\epsilon\big)^\epsilon \Big)$$
$$\Big(\big(\frac{\partial \psi}{\partial t}\big)^\epsilon + \frac{1}{\epsilon} \big(\frac{\partial \psi}{\partial \tau}\big)^\epsilon + \mathbf{v} \cdot \big(\nabla_{\mathbf{x}} \psi\big)^\epsilon + \Big(\mathbf{E} + \mathbf{v} \times \frac{\mathcal{M}}{\epsilon}\Big) \cdot \big(\nabla_{\mathbf{v}} \psi\big)^\epsilon \Big)\, dt d\mathbf{x} d\mathbf{v}$$
$$= \int f_0 \psi(0, 0, ., .)\, d\mathbf{x} d\mathbf{v}$$

$$\psi(t, \tau, \mathbf{x}, \mathbf{v}) = \big(\gamma \circ r\big)\big(t, \tau, \mathbf{x}, \mathbf{v}\big) = \gamma\big(t, \mathbf{x}, r(\tau)\mathbf{v}\big)$$

The Two-Scale Macro Equation - 2

$$
\begin{aligned}
\int G_1^\epsilon \Big[\frac{\partial \gamma}{\partial t} + \big[r(-\frac{t}{\epsilon})\mathbf{u} \big] \cdot \nabla_{\mathbf{x}} \gamma + \big[r(\frac{t}{\epsilon})\mathbf{E} \big] \cdot \nabla_{\mathbf{u}} \gamma \Big] \, dtd\mathbf{x}d\mathbf{v} \\
- \int \Big(\frac{\partial (l \circ r^-)}{\partial t} \Big)^\epsilon \gamma \, dtd\mathbf{x}d\mathbf{v} \\
+ \frac{1}{\epsilon} \int \Big[\Big(\frac{\partial k^\epsilon}{\partial \tau} \circ r^- \Big)^\epsilon + \big(\big[r(-\frac{t}{\epsilon})\mathbf{u} \big] \times \mathcal{M} \big) \cdot \big(\nabla_{\mathbf{v}} k^\epsilon \circ r^- \big)^\epsilon \Big] \frac{\partial \gamma}{\partial t} \, dtd\mathbf{x}d\mathbf{v} \\
+ \frac{1}{\epsilon} \int \Big[\epsilon (l \circ r^-)^\epsilon + \Big(\frac{\partial k^\epsilon}{\partial \tau} \circ r^- \Big)^\epsilon + \big(\big[r(-\frac{t}{\epsilon})\mathbf{u} \big] \times \mathcal{M} \big) \cdot \big(\nabla_{\mathbf{v}} k^\epsilon \circ r^- \big)^\epsilon \Big] \\
\Big[\big[r(-\frac{t}{\epsilon})\mathbf{u} \big] \cdot \nabla_{\mathbf{x}} \gamma + \big[r(\frac{t}{\epsilon})\mathbf{E} \big] \cdot \nabla_{\mathbf{u}} \gamma \Big] \, dtd\mathbf{x}d\mathbf{v} = 0
\end{aligned}
$$

The Two-Scale Micro Equation

$$\int \Big((f^\epsilon =) \big(G \circ r\big)^\epsilon + \epsilon \big(G_1^\epsilon \circ r\big)^\epsilon + \epsilon (l)^\epsilon + \big(\frac{\partial k^\epsilon}{\partial \tau}\big)^\epsilon + (\mathbf{v} \times \mathcal{M}) \cdot \big(\nabla_{\mathbf{v}} k^\epsilon\big)^\epsilon \Big)$$

$$\Big(\big(\frac{\partial \psi}{\partial t}\big)^\epsilon + \frac{1}{\epsilon}\big(\frac{\partial \psi}{\partial \tau}\big)^\epsilon + \mathbf{v} \cdot \big(\nabla_{\mathbf{x}} \psi\big)^\epsilon + \Big(\mathbf{E} + \mathbf{v} \times \frac{\mathcal{M}}{\epsilon}\Big) \cdot \big(\nabla_{\mathbf{v}} \psi\big)^\epsilon \Big)\, dt d\mathbf{x} d\mathbf{v}$$

$$= \int f_0 \psi(0, 0, ., .)\, d\mathbf{x} d\mathbf{v}$$

$$\psi(t, \tau, \mathbf{x}, \mathbf{v}) = \frac{\partial \kappa}{\partial \tau}(t, \tau, \mathbf{x}, \mathbf{v}) + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_{\mathbf{v}} \kappa(t, \tau, \mathbf{x}, \mathbf{v})$$

The Two-Scale Micro Equation - 2

$$-\int\Bigg[\left(\frac{\partial^2 k^\epsilon}{\partial t\partial\tau}\right)^\epsilon+(\mathbf{v}\times\mathcal{M})\cdot\left(\frac{\partial\nabla_\mathbf{v} k^\epsilon}{\partial t}\right)^\epsilon+\frac{1}{\epsilon}\left(\frac{\partial^2 k^\epsilon}{\partial\tau^2}\right)^\epsilon+\frac{1}{\epsilon}(\mathbf{v}\times\mathcal{M})\cdot\left(\frac{\partial\nabla_\mathbf{v} k^\epsilon}{\partial\tau}\right)^\epsilon\Bigg]$$
$$\left[\left(\frac{\partial\kappa}{\partial\tau}\right)^\epsilon+(\mathbf{v}\times\mathcal{M})\cdot(\nabla_\mathbf{v}\kappa)^\epsilon\right] dtd\mathbf{x}d\mathbf{v}$$
$$+\int\left[\left(\frac{\partial k^\epsilon}{\partial\tau}\right)^\epsilon+(\mathbf{v}\times\mathcal{M})\cdot(\nabla_\mathbf{v} k^\epsilon)^\epsilon\right]\Bigg[\mathbf{v}\cdot\left(\frac{\partial\nabla_\mathbf{x}\kappa}{\partial\tau}\right)^\epsilon+(\mathbf{v}\times\mathcal{M})\cdot(\nabla_\mathbf{x}\nabla_\mathbf{v}\kappa)^\epsilon\mathbf{v}+\mathbf{E}\cdot\left(\frac{\partial\nabla_\mathbf{x}\kappa}{\partial\tau}\right)^\epsilon$$
$$+(\mathbf{E}\times\mathcal{M})\cdot(\nabla_\mathbf{v}\kappa)^\epsilon+\mathbf{E}\cdot({\nabla_\mathbf{v}}^2\kappa)^\epsilon(\mathbf{v}\times\mathcal{M})+(\mathbf{v}\times\mathcal{M})\cdot\left(\frac{1}{\epsilon}\frac{\partial\nabla_\mathbf{x}\kappa}{\partial\tau}\right)^\epsilon$$
$$-\mathbf{v}\cdot(\nabla_\mathbf{v}\kappa)^\epsilon+(\mathbf{v}\times\mathcal{M})\cdot({\nabla_\mathbf{v}}^2\kappa)^\epsilon(\mathbf{v}\times\mathcal{M})\Bigg] dtd\mathbf{x}d\mathbf{v}$$

$$
\begin{aligned}
&- \int \epsilon \left(\frac{\partial (G_1^\epsilon \circ r)}{\partial t} \right)^\epsilon \left[\left(\frac{\partial \kappa}{\partial \tau} \right)^\epsilon + (\mathbf{v} \times \mathcal{M}) \cdot \left(\nabla_{\mathbf{v}} \kappa \right)^\epsilon \right] dtd\mathbf{x}d\mathbf{v} \\
&+ \int \epsilon \left(G_1^\epsilon \circ r \right)^\epsilon \Bigg[\mathbf{v} \cdot \left(\frac{\partial \nabla_{\mathbf{x}} \kappa}{\partial \tau} \right)^\epsilon + (\mathbf{v} \times \mathcal{M}) \cdot \left(\nabla_{\mathbf{x}} \nabla_{\mathbf{v}} \kappa \right)^\epsilon \mathbf{v} + \mathbf{E} \cdot \left(\frac{\partial \nabla_{\mathbf{v}} \kappa}{\partial \tau} \right)^\epsilon \\
&\qquad + (\mathbf{E} \times \mathcal{M}) \cdot \left(\nabla_{\mathbf{v}} \kappa \right)^\epsilon + \mathbf{E} \cdot \left({\nabla_{\mathbf{v}}}^2 \kappa \right)^\epsilon (\mathbf{v} \times \mathcal{M}) \Bigg] dtd\mathbf{x}d\mathbf{v} \\
&- \int \epsilon \left(\frac{\partial (I \circ r^-)}{\partial t} \right)^\epsilon \left[\left(\frac{\partial \kappa}{\partial \tau} \circ r^- \right)^\epsilon + \left(\left[r(-\frac{t}{\epsilon}) \mathbf{u} \right] \times \mathcal{M} \right) \cdot \left(\nabla_{\mathbf{v}} \kappa \circ r^- \right)^\epsilon \right] dtd\mathbf{x}d\mathbf{u} \\
&+ \int \epsilon (I)^\epsilon \Bigg[\mathbf{v} \cdot \left(\frac{\partial \nabla_{\mathbf{x}} \kappa}{\partial \tau} \right)^\epsilon + (\mathbf{v} \times \mathcal{M}) \cdot \left(\nabla_{\mathbf{x}} \nabla_{\mathbf{v}} \kappa \right)^\epsilon \mathbf{v} + \mathbf{E} \cdot \left(\frac{\partial \nabla_{\mathbf{v}} \kappa}{\partial \tau} \right)^\epsilon + (\mathbf{E} \times \mathcal{M}) \left(\nabla_{\mathbf{v}} \kappa \right)^\epsilon \\
&\qquad + \mathbf{E} \cdot \left({\nabla_{\mathbf{v}}}^2 \kappa \right)^\epsilon (\mathbf{v} \times \mathcal{M}) \Bigg] dtd\mathbf{x}d\mathbf{v} \\
&+ \int \left(G_1^\epsilon (0, \mathbf{x}, \mathbf{v}) + k^\epsilon (0, 0, \mathbf{x}, \mathbf{v}) \right) \\
&\qquad \left[\left(\frac{\partial \kappa}{\partial \tau} \right) (0, 0, \mathbf{x}, \mathbf{v})) + (\mathbf{v} \times \mathcal{M}) \left(\nabla_{\mathbf{v}} \kappa \right) (0, 0, \mathbf{x}, \mathbf{v})) \right] d\mathbf{x}d\mathbf{v} = 0
\end{aligned}
$$