Synthetic introduction to homogenization based numerical methods

Emmanuel Frénod

LMBA, Université de Bretagne-Sud, Vannes, France
and
EP Inria Calvi and Irma, Strasbourg, France

AIMS conference
Special Session 79:
Numerical Methods based on Homogenization and on Two-Scale Convergence
Orlando - July 1st, 2012

Introduction

General framework

$$O^{\varepsilon}\, u^{\varepsilon} = 0,$$

Essentially : $\varepsilon \sim 0$ (but not uniformally)

O^{ε} induces oscillations of period ε (possibly with non-small amplitude) in $u^{\varepsilon} = u^{\varepsilon}(z)$.

Strong and Weak-$*$ Convergence based Numerical Methods

Convergence Results

Weak-$*$ Convergence definition

$$u^\varepsilon(z) \rightharpoonup u(z) \quad \text{weak-}* \text{ if } \forall \phi : \int (u^\varepsilon(z) - u(z))\phi(z)\, dz \xrightarrow{\varepsilon\to 0} 0$$

Weak-$*$ Convergence result

If $\|u^\varepsilon\|$ in bounded, then $u^\varepsilon(z) \rightharpoonup u(z)$ weak-$*$

High-frequency-non-small-amplitude oscillations

Strong Convergence

If $\|u^\varepsilon\| \xrightarrow{\varepsilon\to 0} \|u\|$, then $\|u^\varepsilon(z) - u(z)\| \xrightarrow{\varepsilon\to 0} 0$

High-frequency-small-amplitude oscillations

Set of Methods

$$O^\varepsilon\, u^\varepsilon = 0 \quad \text{-->} \quad O\, u = 0$$

Strong Convergence based Numerical Methods - Diagram

Where ε is small
If $u^\varepsilon(z) \sim u(z)$ strongly for small ε $\quad (\|u^\varepsilon(z) - u(z)\| \xrightarrow{\varepsilon\to 0} 0)$

$$\begin{tikzcd}
\begin{array}{c} u^\varepsilon \text{ solution to} \\ O^\varepsilon\, u^\varepsilon = 0 \end{array} \arrow[r, "\varepsilon \to 0"'] & \begin{array}{c} u \text{ solution to} \\ O\, u = 0 \end{array} \\
 & \begin{array}{c} u_{\Delta z} \text{ solution to} \\ O_{\Delta z}\, u_{\Delta z} = 0 \end{array} \arrow[u, "\Delta z \to 0"]
\end{tikzcd}$$

$u^\varepsilon(z) \sim u(z)$ strongly —> high-frequency-small-amplitude oscillations.
u good approximation of u^ε.

Weak-$*$ Convergence based Numerical Methods - Diagram

Where ε is small
If $u^\varepsilon(z) \sim u(z)$ weakly for small ε
$\left(\int (u^\varepsilon(z) - u(z))\varphi(z)\,dz \xrightarrow{\varepsilon\to 0} 0\right.$ for any test function $\left.\varphi\right)$

$$\begin{tikzcd}
\begin{array}{c} u^\varepsilon \text{ solution to} \\ O^\varepsilon\, u^\varepsilon = 0 \end{array} \arrow[r, "\varepsilon\to 0"'] & \begin{array}{c} u \text{ solution to} \\ O\, u = 0 \end{array} \\
 & \begin{array}{c} u_{\Delta z} \text{ solution to} \\ O_{\Delta z}\, u_{\Delta z} = 0 \end{array} \arrow[u, "\Delta z \to 0"]
\end{tikzcd}$$

$u^\varepsilon(z) \sim u(z)$ weakly $\longrightarrow$ high-frequency-non-small-amplitude oscillations.
u approximation of u^ε in average only.

Order 1 Weak-$*$ Convergence based Numerical Methods

Order 1 Weak-$*$ Convergence based Numerical Methods

$$u^{\varepsilon}(z) \sim u(z) + \varepsilon u^1(z).$$

$$u^{\varepsilon}(z) \to u(z,\zeta)$$
$$\frac{1}{\varepsilon}\big(u^{\varepsilon}(z) - u(z)\big) \rightharpoonup u^1(z)$$

Set of Methods

$$O^{\varepsilon}\, u^{\varepsilon} = 0 \quad \longrightarrow \quad O\, u = 0 \text{ and } O^1\, u^1 = 0$$

Order 1 Weak-$*$ Convergence based Numerical Methods - Diagram

Where ε is small
If $u^\varepsilon(z) \sim u(z) + \varepsilon u^1(z)$ for small ε

$$\begin{tikzcd}
\begin{array}{c} u^\varepsilon \text{ solution to} \\ O^\varepsilon\, u^\varepsilon = 0 \end{array} \arrow[r, "\varepsilon \to 0"'] & \begin{array}{c} u,\ u_1 \text{ solutions to} \\ O\, u = 0 \\ O^1\, u^1 = 0 \end{array} \\
 & \begin{array}{c} u_{\Delta z},\ u_{\Delta z} \text{ solution to} \\ O_{\Delta z}\, u_{\Delta z} = 0 \\ O^1_{\Delta z}\, u^1_{\Delta z} = 0 \end{array} \arrow[u, "\Delta z \to 0"]
\end{tikzcd}$$

Two-Scale Convergence

Approx. of high-frequency-non-small-amplitude oscillating functions

$u^{\varepsilon}(z) \longrightarrow$ high-frequency-non-small-amplitude oscillations.

$$u^{\varepsilon}(z) \sim U(z, \tfrac{z}{\varepsilon}) \text{ with } U(z,\zeta) \text{ periodic in } \zeta.$$

Illustration $\cdots / \ldots$

function with large scale variation and high-frequency-non-small-amplitude oscillations

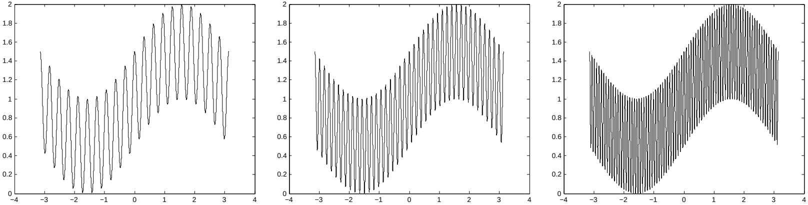

Figure: Graph of $\frac{1}{2}\sin(x) + 1 + \frac{1}{2}\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

function with high-frequency-modulated-amplitude oscillations

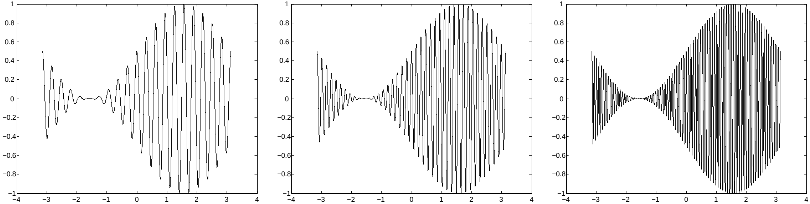

Figure: Graph of $\frac{1}{2}(\sin(x)+1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

function with large scale variation and high-frequency-modulated-amplitude oscillations

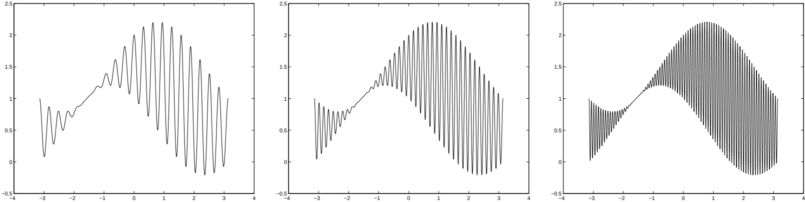

Figure: Graph of $\dfrac{1}{2}\cos(x) + 1 + \dfrac{1}{2}(\sin(x) + 1)\cos(\dfrac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

Two-Scale Convergence Definition

$(z \in \mathbb{R}^n)$

$(u^\varepsilon(z))$ Two-Scale Converges to $U(z,\zeta)$ periodic of period $[0,1]^n$ in ζ

if

$\forall \psi(z,\zeta)$ regular, compactly supported in z and periodic of period $[0,1]^n$ in ζ

$$\int u^\varepsilon(z)\psi(z,\frac{z}{\varepsilon})\,dz \xrightarrow{\varepsilon\to 0} \int\int_{\zeta\in[0,1]^n} U(z,\zeta)\psi(z,\zeta)\,d\zeta dz$$

MEANS: $u^\varepsilon(z) \sim U(z,\frac{z}{\varepsilon})$

Two-Scale Convergence Results

Two-Scale Convergence

If $\|u^{\varepsilon}\|$ in bounded, then

$$(u^{\varepsilon}(z)) \text{ Two-Scale Converges to } U(z,\zeta)$$

$$u^{\varepsilon}(z) \rightharpoonup u(z) = \int_{\zeta\in[0,1]^{n}} U(z,\zeta)\,d\zeta \quad \text{weak-}*$$

Strong Two-Scale Convergence

If $\|u^{\varepsilon}\| \xrightarrow{\varepsilon\to 0} \|U\|$, then $\|u^{\varepsilon}(z) - U(z,\frac{z}{\varepsilon})\| \xrightarrow{\varepsilon\to 0} 0$

Set of Methods

$$O^{\varepsilon}\, u^{\varepsilon} = 0 \;\longrightarrow\; \mathcal{O}\, U = 0$$

Order 0 Two-Scale Numerical Methods

Order 0 Two-Scale Numerical Methods - Diagram

Where ε is small

If $u^\varepsilon(z) \sim u(z)$ weakly and $u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon})$ more strongly for small ε

$$\begin{tikzcd}
\begin{array}{c} u^\varepsilon \text{ solution to} \\ O^\varepsilon\, u^\varepsilon = 0 \end{array} \arrow[rr, "{\varepsilon \to 0\,,\ \text{weak-*}}"'] \arrow[dr, "{\varepsilon \to 0\,,\ \text{two-scale}}"'] & & \begin{array}{c} u \text{ solution to} \\ O\, u = 0 \end{array} \\
& \begin{array}{c} U \text{ solution to} \\ \mathcal{O}\, U = 0 \end{array} \arrow[ur, "\int_{\mathcal{Z}} d\zeta"'] & \\
& & \begin{array}{c} u_{\Delta z} \text{ solution to} \\ O_{\Delta z}\, u_{\Delta z} = 0 \end{array} \arrow[uu, "\Delta z \to 0"'] \\
& \begin{array}{c} U_{\Delta z} \text{ solution to} \\ \mathcal{O}_{\Delta z}\, U_{\Delta z} = 0 \end{array} \arrow[uu, "\Delta z \to 0"'] \arrow[ur, "\int_{\mathcal{Z}}^{\boxed{\text{Num}}} d\zeta"'] &
\end{tikzcd}$$

Order 1 Two-Scale Numerical Methods

Order 1 Two-Scale Numerical Methods

$u^\varepsilon(z) \sim U(z, \frac{z}{\varepsilon}) + \varepsilon U^1(z, \frac{z}{\varepsilon})$ with $U(z,\zeta)$ and $U^1(z,\zeta)$ periodic in ζ.

$(u^\varepsilon(z))$ Two-Scale Converges to $U(z,\zeta)$

$\left(\frac{1}{\varepsilon}\left(u^\varepsilon(z) - U(z, \frac{z}{\varepsilon})\right)\right)$ Two-Scale Converges to $U^1(z,\zeta)$

Set of Methods

$$O^\varepsilon\, u^\varepsilon = 0 \quad \longrightarrow \quad \mathcal{O}\, U = 0 \text{ and } \mathcal{O}^1\, U^1 = 0$$

Where ε is small

If $u^{\varepsilon}(z) \sim U(z, \frac{z}{\varepsilon}) + \varepsilon U^1(z, \frac{z}{\varepsilon})$ for small ε $(u^{\varepsilon}(z) \sim u(z) + \varepsilon u^1(z)$ weakly)

$$\begin{tikzcd}
\begin{array}{c} u^{\varepsilon} \text{ solution to} \\ O^{\varepsilon}\, u^{\varepsilon} = 0 \end{array} \arrow[rr, "{\varepsilon \to 0\,,\ \text{weak-*}}"'] \arrow[dr, "{\varepsilon \to 0\,,\ \text{two-scale}}"'] & & \begin{array}{c} u,\ u_1 \text{ solutions to} \\ O\, u = 0 \\ O^1 u^1 = 0 \end{array} \\
 & \begin{array}{c} U,\ U^1 \text{ solutions to} \\ \mathcal{O}\, U = 0 \\ \mathcal{O}^1 U^1 = 0 \end{array} \arrow[ur, "{\int_{\mathcal{Z}} d\zeta}"'] & \\
 & & \begin{array}{c} u_{\Delta z},\ u^1_{\Delta z} \text{ solutions to} \\ O_{\Delta z}\, u_{\Delta z} = 0 \\ O^1_{\Delta z} u^1_{\Delta z} = 0 \end{array} \arrow[uu, "{\Delta z \to 0}"'] \\
 & \begin{array}{c} U_{\Delta z},\ U^1_{\Delta z} \text{ solutions to} \\ \mathcal{O}_{\Delta z}\, U_{\Delta z} = 0 \\ \mathcal{O}^1_{\Delta z} U^1_{\Delta z} = 0 \end{array} \arrow[uu, "{\Delta z \to 0}"'] \arrow[ur, "{\int_{\mathcal{Z}}^{\boxed{\textbf{Num}}} d\zeta}"'] &
\end{tikzcd}$$

TSAPS

ε non uniformally small

If $u^\varepsilon(z) \sim U^\varepsilon(z, \frac{z}{\varepsilon}) = U_0^\varepsilon(z, \frac{z}{\varepsilon}) + \varepsilon U_1^\varepsilon(z, \frac{z}{\varepsilon})$

$$\begin{tikzcd}
\begin{array}{c} u^\varepsilon \text{ solution to} \\ O^\varepsilon\, u^\varepsilon = 0 \end{array} \arrow[rrr, "\varepsilon \to 0\text{ , weak-*}"'] \arrow[drr, "\varepsilon \to 0\text{ , two-scale}"'] & & & \begin{array}{c} u,\ u_1 \text{ solutions to} \\ O\, u = 0,\ O^1 u^1 = 0 \end{array} \\
& & \begin{array}{c} U,\ U^1 \text{ solutions to} \\ \mathcal{O}\, U = 0 \\ \mathcal{O}^1 U^1 = 0 \end{array} \arrow[ur, "\int_{\mathcal{Z}} d\zeta"'] & \\
& \begin{array}{c} U^\varepsilon \text{ solution to} \\ \mathcal{O}^\varepsilon\, U^\varepsilon = 0 \end{array} \arrow[uul, "\zeta = \frac{z}{\varepsilon}"] \arrow[ur, "\varepsilon \to 0"'] & & \\
\begin{array}{c} u_{\Delta z}^\varepsilon \text{ solution to} \\ O_{\Delta z}^\varepsilon\, u_{\Delta z}^\varepsilon = 0 \end{array} \arrow[uuu, "\Delta z \to 0"] \arrow[rrr, "\varepsilon \to 0"] & & & \begin{array}{c} u_{\Delta z},\ u_{\Delta z}^1 \text{ solutions to} \\ O_{\Delta z}\, u_{\Delta z} = 0,\ O_{\Delta z}^1 u_{\Delta z}^1 = 0 \end{array} \arrow[uuu, "\Delta z \to 0"'] \\
& & \begin{array}{c} U_{\Delta z},\ U_{\Delta z}^1 \text{ solutions to} \\ \mathcal{O}_{\Delta z}\, U_{\Delta z} = 0 \\ \mathcal{O}_{\Delta z}^1 U_{\Delta z}^1 = 0 \end{array} \arrow[uuu, "\Delta z \to 0"'] \arrow[ur, "\int_{\mathcal{Z}} d\zeta"'] & \\
& \begin{array}{c} U_{\Delta z}^\varepsilon \text{ solution to} \\ \mathcal{O}_{\Delta z}^\varepsilon\, U_{\Delta z}^\varepsilon = 0 \end{array} \arrow[uuu, "\Delta z \to 0"'] \arrow[uul, "\zeta = \frac{z}{\varepsilon}"] \arrow[ur, "\varepsilon \to 0"'] & &
\end{tikzcd}$$