Two-Scale Convergence and Two-Scale Numerical Methods

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Two-Scale Convergence

Two-Scale Convergence and Homogenization

Two-Scale Convergence first statements

G. Nguetseng.
A general convergence result for a functional related to the theory of homogenization.
SIAM Journal on Mathematical Analysis, 20(3):608–623, 1989.

G. Nguetseng.
Asymptotic analysis for a stiff variational problem arising in mechanics.
SIAM Journal on Mathematical Analysis, 21(6):1394–1414, 1990.

G. Allaire.
Homogenization and Two-scale Convergence.
SIAM Journal on Mathematical Analysis, 23(6):1482–1518, 1992.

The simplest example I know to introduce Homogenization

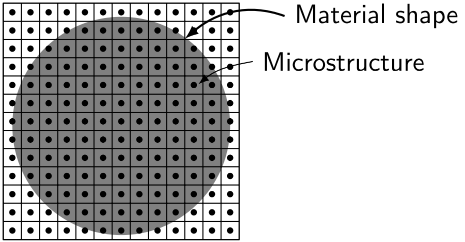


Figure : Composite material - macroscopic shape and a microstructure - Ratio size of the microstructure on the size of the material is ε.

u^ε : Temperature field

$$\nabla \cdot \left[a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon \right] = 0 \quad \text{within the material,}$$

u^ε given on the boundary of the material,

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 1

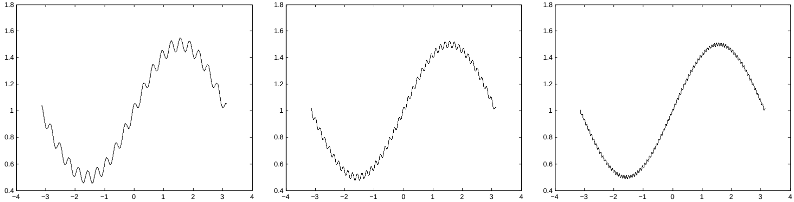


Figure : Graph of $\frac{1}{2}\sin(x) + 1 + \varepsilon\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 2

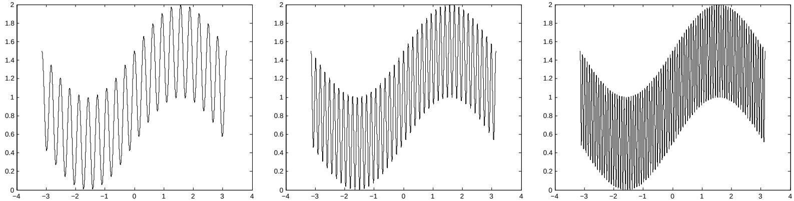


Figure : Graph of $\frac{1}{2}\sin(x) + 1 + \frac{1}{2}\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 3

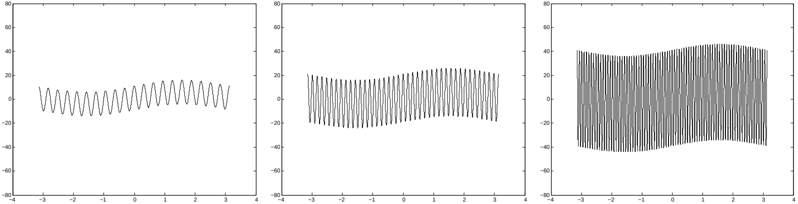


Figure : Graph of $5\sin(x) + 1 + \frac{1}{2\varepsilon}\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 4

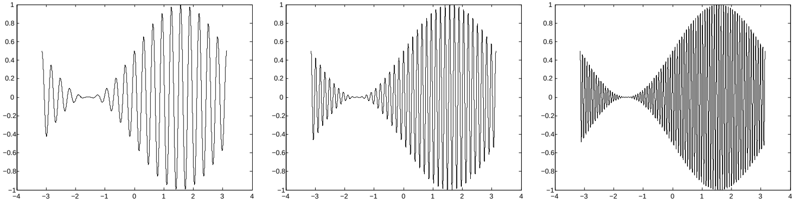


Figure : Graph of $\frac{1}{2}(\sin(x) + 1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 5

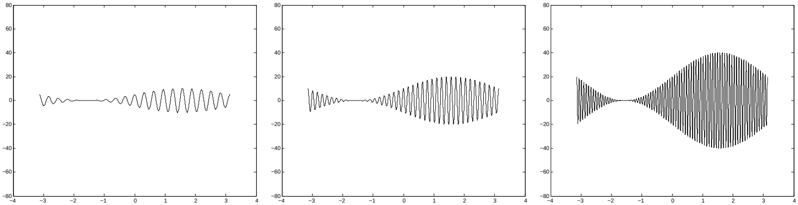


Figure : Graph of $\frac{1}{4\varepsilon}(\sin(x)+1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 6

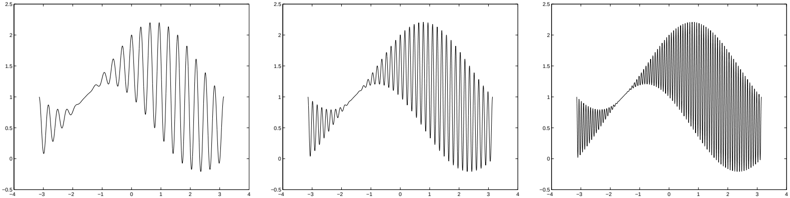


Figure : Graph of $\frac{1}{2}\cos(x) + 1 + \frac{1}{2}(\sin(x) + 1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 7

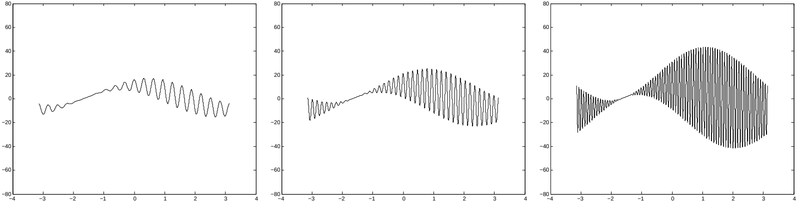


Figure : Graph of $10\cos(x) + 1 + \frac{1}{2\varepsilon}(\sin(x) + 1)\cos(\frac{x}{\varepsilon})$ for $\varepsilon = 1/20$ (left), $1/40$ (center) and $1/80$ (right) between $-\pi$ and π.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 8

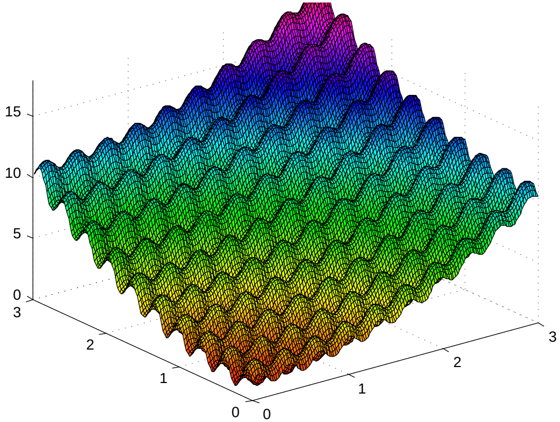


Figure : Graph of $x^2 + y^2 + \frac{1}{2}(\sin(\frac{y}{\varepsilon}) + 1) + (\sin(\frac{x}{\varepsilon}) + 1)$.

A slight digression to explain $a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 9

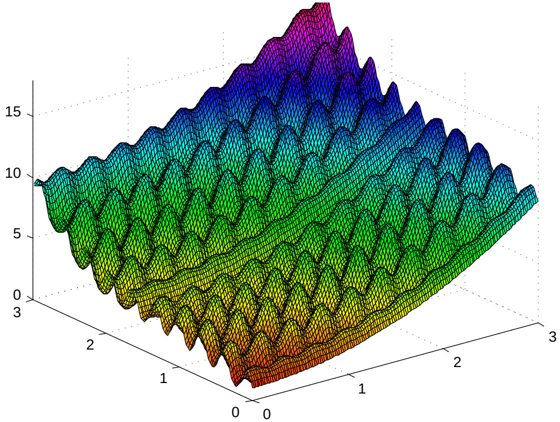


Figure : Graph of $x^2 + y^2 + \sin(2x)(\sin(\frac{y}{\varepsilon}) + 1) + (\sin(\frac{x}{\varepsilon}) + 1)$ for $\varepsilon = 1/20$ on $[0, 3]^2$.

A slight digression to explain $a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ (and even more) - 10

- $a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ can model a wide range of microscopic oscillations or heterogeneities.
- This is why we use it in the model.

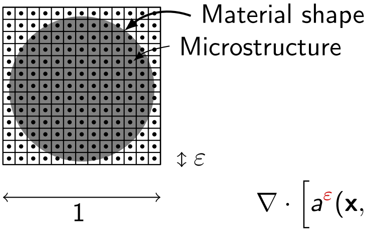
Remark

Two-Scale Convergence is based on this capability

Remark

- IF $\xi \mapsto a^\varepsilon(\mathbf{x}, \xi)$ THEN periodic microscopic scale variations are qualified of **high frequency periodic oscillations**.
- **Two-Scale Convergence is essentially designed for this context.**

Back to : the simplest example I know to introduce Homogenization



u^{ε} : Temperature field

$$\nabla \cdot \left[a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^{\varepsilon} \right] = 0 \quad \text{within the material,}$$

u^{ε} given on the boundary of the material,

IF Solved with a numerical method INDUCES : $\Delta x << \varepsilon$

- IF interested in the tiny variation of u^{ε}, WHY NOT (?)
- OTHERWISE: Clearly NOT REASONNABLE

Homogenization Goal

Find an operator $\mathcal{H}$ (that neither contains nor generates oscillations of size ε)
Such that u

$$\mathcal{H}u = 0 \quad \text{within the material,}$$
$$u = u_{\text{Given}} \quad \text{on the boundary of the material,}$$

close to u^ε (in some sense)

$$\nabla \cdot \left[a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon \right] = 0 \quad \text{within the material,}$$
$$u^\varepsilon = u_{\text{Given}} \quad \text{on the boundary of the material,}$$

INDEPENDENTLY of u_{Given}

This means

- $\mathcal{H}$ must induce average effect of oscillations in u
- In some sense: $\mathcal{H} = \lim_{\varepsilon \to 0} \nabla \cdot a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla$

Homogenization Theory

Homogenization Theory gathers a collection of methods that allow to build operators $\mathcal{H}$ satisfying the required constraint for every problem - containing or generating oscillations or heterogeneities - we can imagine.

Asymptotic Expansion: First Homogenization method set out by Engineers in the 1970s

In the case of $\nabla \cdot \left[a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon \right] = 0$:

$$u^\varepsilon(\mathbf{x}) = U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon U_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon^2 U_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \dots,$$

$U(\mathbf{x}, \xi)$, $U_1(\mathbf{x}, \xi)$, $U_2(\mathbf{x}, \xi)$, ... periodic with respect to ξ.

Gathering terms in factor of ε^{-2}, ε^{-1}, ε^0, ε, ε^2, ...:

$$H_{-2} U = 0, \; H_{-1} U_1 = \mathcal{I}(U), \; H_0 U_2 = \mathcal{I}'(U, U_1), \; \dots .$$

Get well-posed equations for U, U_1, U_2, ...

Mathematical justification of Asymptotic Expansion

Needed:

$$\left\| u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right\|_? \to 0,$$

or in a weaker sense:

$$\left(u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) \rightharpoonup 0.$$

For higher orders, needed:

$$\left(\frac{u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})}{\varepsilon} - U_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) \to 0,$$

$$\left(\frac{1}{\varepsilon} \left(\frac{1}{\varepsilon} \left(u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) - U_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) - U_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) \to 0,$$

and so on, in some sense.

Tools for mathematical justification of Asymptotic Expansion - 1

For Heat Equation with Dirichlet boundary conditions:

$$\nabla \cdot \left[a^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^{\varepsilon} \right] = 0 \quad \text{within the material,}$$

$$u^{\varepsilon} = u_{\text{Given}} \quad \text{on the boundary of the material,}$$

Maximum Principle and boundary estimates WORKS.
SEE

A. Bensoussan, J. L. Lions, and G. Papanicolaou.
Asymptotic analysis for periodic structures.
Studies in Mathematics and its Applications, Vol. 5. North Holland, 1978.

For any all problem: DOES NOT WORK.

Tools for mathematical justification of Asymptotic Expansion - 2 : "Oscillating Test Function Method"

L. Tartar.
Cours Peccot.
Collège de France, 1977.

F. Murat.
H-convergence.
Séminaire d'Analyse Fonctionnelle et Numérique d'Alger, 1977.

L. Tartar.
The General Theory of Homogenization. A Personalized Introduction.
Springer Verlag, dec 2009.

Brief overview of Oscillating Test Function Method

Weak Formulation with Oscillating Test Functions (WFWOTF).

$$\int_{\text{Material}} \nabla \cdot \left[a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon(\mathbf{x}) \right] \varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} = 0,$$

By the Stokes Formula:

$$\int_{\text{Material}} a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon(\mathbf{x}) \cdot \nabla \left[\varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right] d\mathbf{x} = \int_{\text{Boundary}} \text{Something},$$

or

$$\int_{\text{Material}} a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \nabla u^\varepsilon(\mathbf{x}) \cdot \left[\nabla_{\mathbf{x}} \varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_{\boldsymbol{\xi}} \varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right] d\mathbf{x} = \int_{\text{Boundary}} \text{Something}.$$

Difficulty: ∇u^ε, $a^\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, $\nabla_{\mathbf{x}} \varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ and $\nabla_{\boldsymbol{\xi}} \varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ converges in a weak sense only.
Passing to the limit involves relatively sophisticated analytical methods.

Two-Scale Convergence offers an efficient framework to pass to the limit in such terms, in the case when oscillations are periodic.

Link Homogenization - Two-Scale Convergence: Conclusion

- Two-Scale Convergence emerged as an efficient tools to justify Asymptotic Expansion
- Yet, it is more that this: It is a constructive Homogenization Method very well adapted to Singularly Perturbed Hyperbolic Equations.
- Well adapted for problems with oscillations at one frequency: $\frac{1}{\varepsilon}$.
- Can be improved to the case of oscillations with several frequencies, if scale separation, for instance : $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^2}$.
- Cannot be improved to the case of several frequencies if no scale separation.
- Cannot be improved to the case of a variable frequency.

Two proofs which are typical in Two-Scale Convergence

The Riemann-Lebesgue Lemma

The Lemma

If $\psi \in L^\infty_\#(\mathbb{R})$. Defining $[\psi]^\varepsilon$ by $[\psi]^\varepsilon(x) = \psi(\frac{x}{\varepsilon})$, then

$$[\psi]^\varepsilon \rightharpoonup \int_0^1 \psi(\xi)\, d\xi \text{ in } L^\infty(\mathbb{R}) \text{ weak-*}.$$

This means: for any function $\varphi \in L^1(\mathbb{R})$ (or $\in \mathcal{D}(\mathbb{R})$ by density)

$$\int_{\mathbb{R}} [\psi]^\varepsilon(x)\, \varphi(x)\, dx \to \int_0^1 \psi(\xi)\, d\xi \int_{\mathbb{R}} \varphi(x)\, dx.$$

The Riemann-Lebesgue Lemma proof - 1

- Fix $\varphi \in \mathcal{D}(\mathbb{R})$
- Choose M s.t. $\text{supp}(\varphi) \subset [-M, M]$
- Set $\{-M, -M+\varepsilon, \dots, -M+\mathbb{E}(2M/\varepsilon)\varepsilon, -M+(\mathbb{E}(2M/\varepsilon)+1)\varepsilon\}$ ($\mathbb{E}$: integer part)
- Split $\displaystyle\int_{\mathbb{R}} [\psi]^\varepsilon(x)\,\varphi(x)\,dx = \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,\varphi(x)\,dx$
- Use Taylor formula: $\forall x \in [-M+(i-1)\varepsilon, -M+i\varepsilon]$, $\exists c_i(x) \in [-M+(i-1)\varepsilon, x]$ such that $\varphi(x) = \varphi(-M+(i-1)\varepsilon) + (x+M-(i-1)\varepsilon)\varphi'(c_i(x))$
- $$\int_{\mathbb{R}} [\psi]^\varepsilon(x)\,\varphi(x)\,dx = \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,dx\;\varphi(-M(i-1)\varepsilon)$$ $$+ \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,(x+M-(i-1)\varepsilon)\varphi'(c_i(x))\,dx$$

The Riemann-Lebesgue Lemma proof - 2

- $$\int_{\mathbb{R}} [\psi]^\varepsilon(x)\,\varphi(x)\,dx = \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,dx\,\varphi(-M(i-1)\varepsilon)$$
$$+ \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,(x+M-(i-1)\varepsilon)\varphi'(c_i(x))\,dx$$
- $$\sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,dx\,\varphi(-M(i-1)\varepsilon) =$$
$$\int_0^1 \psi(\xi)\,d\xi \;\; \varepsilon \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \varphi(-M(i-1)\varepsilon) \xrightarrow{\varepsilon\to 0} \int_0^1 \psi(\xi)\,d\xi \;\int_{\mathbb{R}} \varphi(x)\,dx$$
- $$\left| \sum_{i=1}^{\mathbb{E}(2M/\varepsilon)+1} \int_{-M+(i-1)\varepsilon}^{-M+i\varepsilon} \psi(\frac{x}{\varepsilon})\,(x+M-(i-1)\varepsilon)\varphi'(c_i(x))\,dx \right|$$
$$\leq \int_0^1 |\psi(\xi)|\,\varepsilon d\xi \;\left(\frac{2M+1}{\varepsilon}\right)\varepsilon\|\varphi'\|_\infty \xrightarrow{\varepsilon\to 0} 0$$

The Riemann-Lebesgue Lemma generalization

The Lemma

If $\psi = \psi(x, \xi) \in \mathcal{C}^0(\mathbb{R}; \mathcal{C}^0_\#(\mathbb{R}))$ (or $\in L^\infty(\mathbb{R}; \mathcal{C}^0_\#(\mathbb{R}))$ but it is more technical). Defining $[\psi]^\varepsilon$ by $[\psi]^\varepsilon(x) = \psi(x, \frac{x}{\varepsilon})$, then

$$[\psi]^\varepsilon \rightharpoonup \int_0^1 \psi(x, \xi)\, d\xi \text{ in } L^\infty(\mathbb{R}) \text{ weak-*}.$$

This means: for any function $\varphi \in L^1(\mathbb{R})$ (or $\in \mathcal{D}(\mathbb{R})$ by density)

$$\int_{\mathbb{R}} [\psi]^\varepsilon(x)\, \varphi(x)\, dx \to \int_{\mathbb{R}} \left(\int_0^1 \psi(x, \xi)\, d\xi \right) \varphi(x)\, dx.$$

i.e.: $\forall \delta > 0$, $\exists \varepsilon_0 > 0$, s.t. $\forall \varepsilon \leq \varepsilon_0$,

$$\left| \int_{\mathbb{R}} [\psi]^\varepsilon(x)\, \varphi(x)\, dx - \int_{\mathbb{R}} \left(\int_0^1 \psi(x, \xi)\, d\xi \right) \varphi(x)\, dx \right| \leq \delta.$$

The Riemann-Lebesgue Lemma generalization proof - 1

step 1:

- $\forall m \in \mathbb{N}$: partition of $[0,1]$ with m intervals of length $1/m$
- χ_i^m : characteristic functions of i-th interval, for $i = 1 \ldots, m$ extended by periodicity over $\mathbb{R}$. ξ_i^m: center of the i-th interval
- $$\tilde{\psi}_m(x,\xi) = \sum_{i=1}^{m} \psi(x,\xi_i^m)\,\chi_i^m(\xi) \stackrel{m\to\infty}{\longrightarrow} \psi(x,\xi) \text{ uniformally}$$
- $$[\chi_i^m]^\varepsilon \stackrel{\varepsilon\to 0}{\rightharpoonup} \int_0^1 \chi_i^m(\xi)\,d\xi = \frac{1}{m} \text{ in } L^\infty(\mathbb{R}) \text{ weak-*}.$$

  Hence $$[\tilde{\psi}_m]^\varepsilon \stackrel{\varepsilon\to 0}{\rightharpoonup} \sum_{i=1}^{m} \psi(x,\xi_i^m)\,\frac{1}{m} = \int_0^1 \tilde{\psi}_m(x,\xi)\,d\xi$$

The Riemann-Lebesgue Lemma generalization proof - 2

step 2:

$$\left|\int_{\mathbb{R}} [\psi]^{\varepsilon}(x)\,\varphi(x)\,dx - \int_{\mathbb{R}} \left(\int_0^1 \psi(x,\xi)\,d\xi\right) \varphi(x)\,dx\right| \leq$$

$$\int_{\mathbb{R}} \left|[\psi]^{\varepsilon}(x) - [\tilde{\psi}_m]^{\varepsilon}(x)\right| |\varphi(x)|\,dx$$

$$+ \left|\int_{\mathbb{R}} \left([\tilde{\psi}_m]^{\varepsilon}(x) - \int_0^1 \tilde{\psi}_m(x,\xi)\,d\xi\right) \varphi(x)\,dx\right|$$

$$+ \int_{\mathbb{R}} \left(\int_0^1 \left|\tilde{\psi}_m(x,\xi) - \psi(x,\xi)\right| d\xi\right) |\varphi(x)|\,dx$$

The Riemann-Lebesgue Lemma generalization proof - 2

step 2:
Fix m s.t. :

$$\left|\int_{\mathbb{R}}[\psi]^{\varepsilon}(x)\,\varphi(x)\,dx-\int_{\mathbb{R}}\left(\int_0^1\psi(x,\xi)\,d\xi\right)\varphi(x)\,dx\right|\leq$$

$$\int_{\mathbb{R}}\left|[\psi]^{\varepsilon}(x)-[\tilde{\psi}_m]^{\varepsilon}(x)\right||\varphi(x)|\,dx\quad\leq\frac{\delta}{3},\forall\varepsilon>0$$

$$+\left|\int_{\mathbb{R}}\left([\tilde{\psi}_m]^{\varepsilon}(x)-\int_0^1\tilde{\psi}_m(x,\xi)\,d\xi\right)\varphi(x)\,dx\right|$$

$$+\int_{\mathbb{R}}\left(\int_0^1\left|\tilde{\psi}_m(x,\xi)-\psi(x,\xi)\right|d\xi\right)|\varphi(x)|\,dx\quad\leq\frac{\delta}{3}$$

The Riemann-Lebesgue Lemma generalization proof - 2

step 2:
Fix m and ε_0 s.t. :

$$\left|\int_{\mathbb{R}} [\psi]^\varepsilon(x)\,\varphi(x)\,dx - \int_{\mathbb{R}} \left(\int_0^1 \psi(x,\xi)\,d\xi\right)\varphi(x)\,dx\right| \leq$$

$$\int_{\mathbb{R}} \left|[\psi]^\varepsilon(x) - [\tilde{\psi}_m]^\varepsilon(x)\right| |\varphi(x)|\,dx \quad \leq \frac{\delta}{3}, \forall \varepsilon > 0$$

$$+ \left|\int_{\mathbb{R}} \left([\tilde{\psi}_m]^\varepsilon(x) - \int_0^1 \tilde{\psi}_m(x,\xi)\,d\xi\right)\varphi(x)\,dx\right| \quad \leq \frac{\delta}{3}, \forall \varepsilon \leq \varepsilon_0$$

$$+ \int_{\mathbb{R}} \left(\int_0^1 \left|\tilde{\psi}_m(x,\xi) - \psi(x,\xi)\right| d\xi\right) |\varphi(x)|\,dx \quad \leq \frac{\delta}{3}$$

$$\leq \delta, \forall \varepsilon \leq \varepsilon_0$$

Two-Scale Convergence: definitions and results

Key Points of the Theory - 1

- Several variants of the Two-Scale Convergence theory, for various targeted applications and involving various functional spaces.
- Very close to each other. All follow the same routine based :
  - A continuous injection Lemma
  - A compactness Theorem

See

G. Nguetseng.
A general convergence result for a functional related to the theory of homogenization.
SIAM Journal on Mathematical Analysis, 20(3):608–623, 1989.

G. Nguetseng.
Asymptotic analysis for a stiff variational problem arising in mechanics.
SIAM Journal on Mathematical Analysis, 21(6):1394–1414, 1990.

G. Allaire.
Homogenization and Two-scale Convergence.
SIAM Journal on Mathematical Analysis, 23(6):1482–1518, 1992.

Key Points of the Theory - 2

M. Amar.
Two-scale convergence and homogenization on BV(ω).
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J. Casado-Díaz and I. Gayte.
The two-scale convergence method applied to generalized Besicovitch spaces.
Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 458(2028):2925–2946, 2002.

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Asymptotic expansion of the Vlasov equation in a large external magnetic field.
J. Math. Pures et Appl., 80(8):815–843, 2001.

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$\Sigma-$convergence.
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$\Sigma-$Convergence of nonlinear parabolic operators.
Nonlinear Analysis: Theory, Methods & Applications, 66(4):968–1004, feb 2007.

E. Frénod.
Two-Scale Convergence.
ESAIM: Proceedings, 38:1–35, December 2012.

Key Points of the Theory - 2

H. E. Pak.
Geometric two-scale convergence on forms and its applications to maxwell's equations.
Proceedings of the Royal Society of Edinburgh, 135A:133–147, 2005.

Definitions

- Ω: a regular domain in $\mathbb{R}^n$
- $\mathcal{L}$ a usual functional Banach space: $\mathcal{L}'$ its topological dual space. ${}_{\mathcal{L}'}\langle .,. \rangle_{\mathcal{L}}$: duality bracket. $|.|_{\mathcal{L}}$, $|.|_{\mathcal{L}'}$: norms
- $q \in [1,+\infty)$ and $p \in (1,+\infty]$ s.t. $1/q + 1/p = 1$
- $\mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L})$: continuous functions $\mathbb{R}^n \to \mathcal{L}$, periodic of period 1 with respect to every variable
- $L^p(\Omega,\mathcal{L}')$: functions $f:\Omega \to \mathcal{L}'$
  - s.t. $|f|^p_{\mathcal{L}'}$ is integrable if $p<\infty$
  - s.t. $|f|_{\mathcal{L}'}$ is essentially bounded if $p=\infty$
- $L^p_\#(\mathbb{R}^n;\mathcal{L}')$: functions $f:\mathbb{R}^n \to \mathcal{L}'$
  - s.t. $|f|^p_{\mathcal{L}'}$ is locally integrable if $p<\infty$
  - s.t. $|f|_{\mathcal{L}'}$ is locally essentially bounded if $p=\infty$

  and periodic of period 1.
- $L^p_\#(\mathbb{R}^n;\mathcal{L}') = (L^q_\#(\mathbb{R}^n;\mathcal{L}))'$ (because of the separability of $\mathcal{L}$)
- $L^q(\Omega; L^q_\#(\mathbb{R}^n,\mathcal{L}))$, $L^q(\Omega;\mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L}))$ and $L^p(\Omega; L^p_\#(\mathbb{R}^n,\mathcal{L}'))$

Definition

$(u^\varepsilon) = (u^\varepsilon(\mathbf{x})) \subset L^p(\Omega; \mathcal{L}')$ Two-Scale converges to

$$U = U(\mathbf{x}, \boldsymbol{\xi}) \in L^p(\Omega; L^p_\#(\mathbb{R}^n, \mathcal{L}'))$$

if, for any function $\phi = \phi(\mathbf{x}, \boldsymbol{\xi}) \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$,

$$\lim_{\varepsilon \to 0} \int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\rangle_{\mathcal{L}}\, d\mathbf{x} = \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}'}\langle U(\mathbf{x}, \boldsymbol{\xi}), \phi(\mathbf{x}, \boldsymbol{\xi})\rangle_{\mathcal{L}}\, d\mathbf{x} d\boldsymbol{\xi},$$

Strong Two-Scale Convergence definition

Definition

IF $p = q = 2$, $\mathcal{L}$ is a Hilbert space,
IF

$$(u^\varepsilon) = (u^\varepsilon(\mathbf{x})) \subset L^2(\Omega; \mathcal{L}') \text{ Two-Scale converges to } U = U(\mathbf{x}, \boldsymbol{\xi})$$

and IF $U \in L^2(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}'))$.

THEN we say

$$(u^\varepsilon) = (u^\varepsilon(\mathbf{x})) \text{ Strongly Two-Scale converges to } U = U(\mathbf{x}, \boldsymbol{\xi})$$

if

$$\lim_{\varepsilon \to 0} \int_\Omega \left| u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right|^2_{\mathcal{L}'} d\mathbf{x} = 0$$

Link with weak-$*$ convergence

Link with weak-∗ convergence

Proposition

If $(u^\varepsilon) \subset L^p(\Omega; \mathcal{L}')$ Two-Scale converges to $U \in L^p(\Omega; L^p_\#(\mathbb{R}^n; \mathcal{L}'))$, then

$$u^\varepsilon \rightharpoonup \int_{[0,1]^n} U(.,\xi)\, d\xi \text{ weak-* in } L^p(\Omega; \mathcal{L}').$$

In the definition of Two-Scale Convergence: $\phi(\mathbf{x}, \xi) = \phi(\mathbf{x})$.

$$\lim_{\varepsilon\to 0} \int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x})\rangle_{\mathcal{L}}\, d\mathbf{x} = \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}'}\langle U(\mathbf{x},\xi), \phi(\mathbf{x})\rangle_{\mathcal{L}}\, d\mathbf{x} d\xi = \int_\Omega {}_{\mathcal{L}'}\left\langle \left(\int_{[0,1]^n} U(\mathbf{x},\xi) d\xi \right), \phi(\mathbf{x})\right\rangle_{\mathcal{L}}\, d\mathbf{x}.$$

Two-Scale Convergence criterion

Injection Lemma

If $\phi \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$, then for all $\varepsilon > 0$, function $[\phi]^\varepsilon : \Omega \to \mathcal{L}$ defined by

$$[\phi]^\varepsilon(\mathbf{x}) = \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$$

satisfies

$$\|[\phi]^\varepsilon\|_{L^q(\Omega;\mathcal{L})} \leq \|\phi\|_{L^q(\Omega;\mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L}))}$$

$$\|\phi\|^q_{L^q(\Omega;\mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L}))} = \int_\Omega \left(\sup_{\boldsymbol{\xi}\in[0,1]^n} |\phi(\mathbf{x}, \boldsymbol{\xi})|_\mathcal{L} \right)^q d\mathbf{x}$$

$$\|[\phi]^\varepsilon\|_{L^q(\Omega;\mathcal{L})} = \int_\Omega \left|\phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\right|^q_\mathcal{L} d\mathbf{x} \leq \int_\Omega \left(\sup_{\boldsymbol{\xi}\in[0,1]^n} |\phi(\mathbf{x}, \boldsymbol{\xi})|_\mathcal{L} \right)^q d\mathbf{x}$$

Injection Lemma - 2: Supplementary Proposition

Supplementary Proposition

If $\phi \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$, then

$$\lim_{\varepsilon \to 0} \|[\phi]^\varepsilon\|^q_{L^q(\Omega;\mathcal{L})} = \lim_{\varepsilon \to 0} \int_\Omega \left|\phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x}$$

$$= \int_\Omega \int_{[0,1]^n} |\phi(\mathbf{x}, \boldsymbol{\xi})|^q_{\mathcal{L}} \, d\mathbf{x} d\boldsymbol{\xi} = \|\phi\|^q_{L^q(\Omega; L^q_\#(\mathbb{R}^n;\mathcal{L}))}$$

Two-Scale Convergence criterion
Injection Lemma - 3: Suppl. Proposition proof

step 1:

- $\forall m \in \mathbb{N}$: partition of $[0,1]^n$ with m hypercubes of measure $1/m$
- ξ_i^m: center of the i-th hypercube χ_i^m: characteristic function of the i-th hypercube extended by periodicity to $\mathbb{R}^n$
- $[\chi_i]^\varepsilon \rightharpoonup \displaystyle\int_{[0,1]^n} \chi_i(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{1}{m}$ in $L^\infty(\Omega;\mathbb{R})$ weak-*
- $([\chi_i]^\varepsilon)^q \rightharpoonup \displaystyle\int_{[0,1]^n} \chi_i^q(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{1}{m}$ in $L^\infty(\Omega;\mathbb{R})$ weak-*
- $\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi}) = \displaystyle\sum_{i=1}^{m} \phi(\mathbf{x},\boldsymbol{\xi}_i)\,\chi_i(\boldsymbol{\xi})$ s.t.

$$\lim_{\varepsilon\to 0}\int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|_{\mathcal{L}}^q d\mathbf{x} = \lim_{\varepsilon\to 0}\int_\Omega \sum_{i=1}^{m} |\phi(\mathbf{x},\boldsymbol{\xi}_i)|_{\mathcal{L}}^q\,([\chi_i]^\varepsilon)^q d\mathbf{x} =$$

$$\sum_{i=1}^{m}\frac{1}{m}\int_\Omega |\phi(\mathbf{x},\boldsymbol{\xi}_i)|_{\mathcal{L}}^q\, d\mathbf{x} = \int_\Omega\int_{[0,1]^n}\left|\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|_{\mathcal{L}}^q d\mathbf{x}d\boldsymbol{\xi}$$

step 2: We have

$$\|[\phi]^{\varepsilon}\|^q_{L^q(\Omega;\mathcal{L})} = \int_\Omega \left|\phi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x} = \left(\int_\Omega \left|\phi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x} - \int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x}\right) +$$
$$\left(\int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x} - \int_\Omega\int_{[0,1]^n} \left|\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|^q_{\mathcal{L}} d\mathbf{x}d\boldsymbol{\xi}\right) + \left(\int_\Omega\int_{[0,1]^n} \left|\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|^q_{\mathcal{L}} d\mathbf{x}d\boldsymbol{\xi}\right)$$

And

$$\left(\int_\Omega\int_{[0,1]^n} \left|\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|^q_{\mathcal{L}} d\mathbf{x}d\boldsymbol{\xi}\right) \to \left(\int_\Omega\int_{[0,1]^n} |\phi(\mathbf{x},\boldsymbol{\xi})|^q_{\mathcal{L}} d\mathbf{x}d\boldsymbol{\xi}\right) \quad \text{as } m \to +\infty$$

$$\left(\int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x} - \int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|^q_{\mathcal{L}} d\mathbf{x}d\boldsymbol{\xi}\right) \to 0 \quad \text{as } \varepsilon \to 0$$

$$\left|\int_\Omega \left|\phi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x} - \int_\Omega \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} d\mathbf{x}\right| \leq \int_\Omega \left|\left|\phi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}} - \left|\tilde{\phi}_m(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\right|^q_{\mathcal{L}}\right| d\mathbf{x}$$
$$\leq \int_\Omega \sup_{\boldsymbol{\xi}\in[0,1]^n} \left|\phi(\mathbf{x},\boldsymbol{\xi}) - \tilde{\phi}_m(\mathbf{x},\boldsymbol{\xi})\right|^q_{\mathcal{L}} d\mathbf{x} \to 0 \quad \text{as } m \to +\infty$$

Theorem

If a sequence (u^ε) is bounded in $L^p(\Omega;\mathcal{L}')$, i.e. if

$$\|u^\varepsilon\|_{L^p(\Omega;\mathcal{L}')} = \left(\int_\Omega |u^\varepsilon(\mathbf{x})|^p_{\mathcal{L}'}\, d\mathbf{x}\right)^{\frac{1}{p}} \leq c,$$

for a constant c independent of ε, then, there exists a profile $U \in L^p(\Omega; L^p_\#(\mathbb{R}^n;\mathcal{L}'))$ such that, up to a subsequence,

$$(u^\varepsilon) \text{ Two-Scale converges to } U.$$

Two ingredients for the proof

- the sequential Banach-Alaoglu Theorem
- the Riesz Representation Theorem.

Two-Scale Convergence criterion
Proof of the Theorem - 1

Injection Lemma and assumption of the Theorem $\rightarrow$
$\forall \phi = \phi(\mathbf{x}, \boldsymbol{\xi}) \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$ $((1/p) + (1/q) = 1)$

$$\left| \int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \rangle_{\mathcal{L}} \, d\mathbf{x} \right| \leq c \, \| [\phi]^\varepsilon \|_{L^q(\Omega, \mathcal{L})}$$
$$\leq c \|\phi\|_{L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))}$$

Hence (thanks to the second inequality)

$$\begin{array}{llll} \mu^\varepsilon : & L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L})) & \rightarrow & \mathbb{R} \\ & \phi & \mapsto & \displaystyle\int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \rangle_{\mathcal{L}} \, d\mathbf{x} \end{array}$$

bounded in $(L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L})))'$
As $(L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L})))'$ dual of separable space $L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$

$$\mu^\varepsilon \rightharpoonup \mu \quad \text{in } (L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L})))' \text{ weak-* (up to a subsequence)}$$

In particular: $\langle \mu^\varepsilon, \phi \rangle \rightarrow \langle \mu, \phi \rangle, \forall \phi \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$

Two-Scale Convergence criterion
Proof of the Theorem - 2

We have: $\forall \phi = \phi(\mathbf{x}, \boldsymbol{\xi}) \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$ $((1/p) + (1/q) = 1)$

$$\left| \int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \rangle_{\mathcal{L}} \, d\mathbf{x} \right| \leq c \, \|[\phi]^\varepsilon\|_{L^q(\Omega,\mathcal{L})} \leq c \|\phi\|_{L^q(\Omega;\mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L}))}$$

Making $\varepsilon \to 0 \to$

$$|\langle \mu, \phi \rangle| \leq c \, \|\phi\|_{L^q(\Omega; L^q_\#(\mathbb{R}^n;\mathcal{L}))} \quad \forall \phi \in L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$$

Since $L^q(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$ is dense in $L^q(\Omega; L^q_\#(\mathbb{R}^n; \mathcal{L}))$
(whose dual is $L^p(\Omega; L^p_\#(\mathbb{R}^n; \mathcal{L}')))$
Riez Representation Theorem $\to \exists U \in L^p(\Omega; L^p_\#(\mathbb{R}^n; \mathcal{L}'))$ s.t.

$$\langle \mu, \phi \rangle = \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}'}\langle U(\mathbf{x}, \boldsymbol{\xi}), \phi(\mathbf{x}, \boldsymbol{\xi}) \rangle_{\mathcal{L}} \, d\mathbf{x} d\boldsymbol{\xi},$$

$$\int_\Omega {}_{\mathcal{L}'}\langle u^\varepsilon(\mathbf{x}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \rangle_{\mathcal{L}} \, d\mathbf{x} \to \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}'}\langle U(\mathbf{x}, \boldsymbol{\xi}), \phi(\mathbf{x}, \boldsymbol{\xi}) \rangle_{\mathcal{L}} \, d\mathbf{x} d\boldsymbol{\xi}$$

as $\varepsilon \to 0$

Strong Two-Scale Convergence criterion

Lemma

IF $\psi = \psi(\mathbf{x}, \boldsymbol{\xi}) \in L^2(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$

$$([\psi]^\varepsilon) \text{ Strongly Two-Scale converges to } \psi$$

(recall: $[\psi]^\varepsilon(\mathbf{x}) = \psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}))$

step 1: Two-Scale convergence
Consequence of the Riemann-Lebesgue generalization

$$\int_\Omega {}_{\mathcal{L}}\langle \psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), \phi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \rangle_{\mathcal{L}} \, d\mathbf{x} \to \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}}\langle \psi(\mathbf{x}, \boldsymbol{\xi}), \phi(\mathbf{x}, \boldsymbol{\xi}) \rangle_{\mathcal{L}} \, d\mathbf{x} d\boldsymbol{\xi}$$

$\forall \phi \in L^2(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$, i.e.

$$([\psi]^\varepsilon) \text{ Two-Scale converges to } \psi$$

Strong Two-Scale Convergence criterion Preliminary results - 2

step 2: Strong Two-Scale convergence

$$\int_\Omega \Big| [\psi]^\varepsilon(\mathbf{x}) \; - \psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \Big|^2_{\mathcal{L}'} \, d\mathbf{x} \to 0,$$

Completely obvious: $[\psi]^\varepsilon(\mathbf{x}) = \psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$

Hence:

$$([\psi]^\varepsilon) \text{ Strongly Two-Scale converges to } \psi$$

Strong Two-Scale Convergence criterion Preliminary results - 3

Also easy to prove:

Lemma

IF $\psi = \psi(\mathbf{x}, \xi) \in L^2(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n; \mathcal{L}))$

$$\|[\psi]^\varepsilon\|_{L^2(\Omega;\mathcal{L})} = \left(\int_\Omega \left|\psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\right|^2_{\mathcal{L}} \, d\mathbf{x}\right)^{\frac{1}{2}} =$$

$$\left(\int_\Omega {}_{\mathcal{L}}\langle\psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), \psi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\rangle_{\mathcal{L}} \, d\mathbf{x}\right)^{\frac{1}{2}} \to \left(\int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}}\langle\psi(\mathbf{x}, \xi), \psi(\mathbf{x}, \xi)\rangle_{\mathcal{L}} \, d\mathbf{x} d\xi\right)^{\frac{1}{2}}$$

$$= \left(\int_\Omega \int_{[0,1]^n} |\psi(\mathbf{x}, \xi)|^2_{\mathcal{L}} \, d\mathbf{x}\right)^{\frac{1}{2}} = \|\psi\|_{L^2(\Omega; L^2_\#(\mathbb{R}^n;\mathcal{L}))}.$$

Theorem

IF$(u^\varepsilon) \subset L^2(\Omega;\mathcal{L})$ Two-Scale converges to U
IF $U \in L^2(\Omega; \mathcal{C}^0_\#(\mathbb{R}^n;\mathcal{L}))$
IF

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(\Omega;\mathcal{L})} = \|U\|_{L^2(\Omega;L^2([0,1]^n;\mathcal{L})},$$

THEN

$$(u^\varepsilon) \text{ Strongly Two-Scale converges to } U,$$

and, $\forall (v^\varepsilon) \subset L^2(\Omega;\mathcal{L})$ Two-Scale converging towards V,

$$_{\mathcal{L}}\langle u^\varepsilon, v^\varepsilon\rangle_{\mathcal{L}} \rightharpoonup \int_{[0,1]^n} {_{\mathcal{L}}\langle U(.,\xi), V(.,\xi)\rangle_{\mathcal{L}}}\, d\xi, \quad \text{in } \mathcal{D}'(\Omega).$$

First part of the Theorem

$$\int_\Omega \left| u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\right|^2_{\mathcal{L}} \, d\mathbf{x} =$$

$$\int_\Omega |u^\varepsilon(\mathbf{x})|^2_{\mathcal{L}} \, d\mathbf{x} - 2\int_\Omega {}_{\mathcal{L}}\langle u^\varepsilon(\mathbf{x}), U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\rangle_{\mathcal{L}} \, d\mathbf{x} + \int_\Omega \left|U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\right|^2_{\mathcal{L}} \, d\mathbf{x}$$

$$\stackrel{\varepsilon \to 0}{\longrightarrow}$$

$$\lim_{\varepsilon\to 0}\int_\Omega |u^\varepsilon(\mathbf{x})|^2_{\mathcal{L}} \, d\mathbf{x} - 2\int_\Omega\int_{[0,1]^n} {}_{\mathcal{L}}\langle U(\mathbf{x},\xi), U(\mathbf{x},\xi)\rangle_{\mathcal{L}} \, d\mathbf{x}d\xi$$
$$+ \int_\Omega\int_{[0,1]^n} |U(\mathbf{x},\xi)|^2_{\mathcal{L}} \, d\mathbf{x}d\xi =$$

$$\int_\Omega\int_{[0,1]^n} |U(\mathbf{x},\xi)|^2_{\mathcal{L}} \, d\mathbf{x}d\xi - 2\int_\Omega\int_{[0,1]^n} |U(\mathbf{x},\xi)|^2_{\mathcal{L}} \, d\mathbf{x}d\xi$$
$$+ \int_\Omega\int_{[0,1]^n} |U(\mathbf{x},\xi)|^2_{\mathcal{L}} \, d\mathbf{x}d\xi = 0.$$

Second part of the Theorem - $\forall \varphi \in \mathcal{D}(\Omega)$

$$_{\mathcal{D}'}\langle {}_{\mathcal{L}}\langle\, u^\varepsilon, v^\varepsilon \rangle_{\mathcal{L}}, \varphi \rangle_{\mathcal{D}} = \int_\Omega {}_{\mathcal{L}}\langle u^\varepsilon(\mathbf{x}), v^\varepsilon(\mathbf{x})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} =$$

$$\int_\Omega {}_{\mathcal{L}}\langle U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), v^\varepsilon(\mathbf{x})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} + \int_\Omega {}_{\mathcal{L}}\langle u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), v^\varepsilon(\mathbf{x})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x}.$$

- $u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \to 0 \quad \to \int_\Omega {}_{\mathcal{L}}\langle\, u^\varepsilon(\mathbf{x}) - U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), v^\varepsilon(\mathbf{x})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} \to 0$
- $\int_\Omega {}_{\mathcal{L}}\langle\, U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), v^\varepsilon(\mathbf{x})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} = \int_\Omega {}_{\mathcal{L}}\langle\, v^\varepsilon(\mathbf{x}), U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} =$

$$\int_\Omega {}_{\mathcal{L}}\langle\, v^\varepsilon(\mathbf{x}), \varphi(\mathbf{x}) U(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\rangle_{\mathcal{L}}\; d\mathbf{x} \to \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}}\langle\, V(\mathbf{x}, \xi), \varphi(\mathbf{x}) U(\mathbf{x}, \xi)\rangle_{\mathcal{L}}\; d\mathbf{x} d\xi$$

$$= \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}}\langle\, V(\mathbf{x}, \xi), U(\mathbf{x}, \xi)\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} d\xi$$

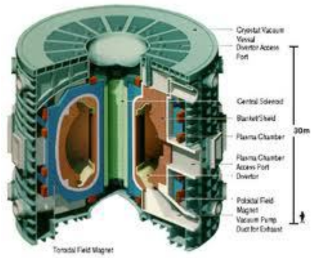
$$= \int_\Omega \int_{[0,1]^n} {}_{\mathcal{L}}\langle\, U(\mathbf{x}, \xi), V(\mathbf{x}, \xi)\rangle_{\mathcal{L}}\; \varphi(\mathbf{x})\; d\mathbf{x} d\xi$$

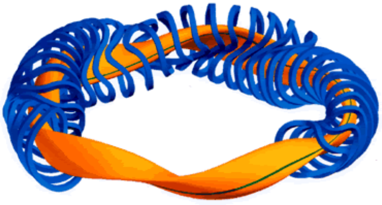
Homogenization of singularly perturbed Hyperbolic Partial Differential Equations

Motivation : Tokamaks and Stellarators

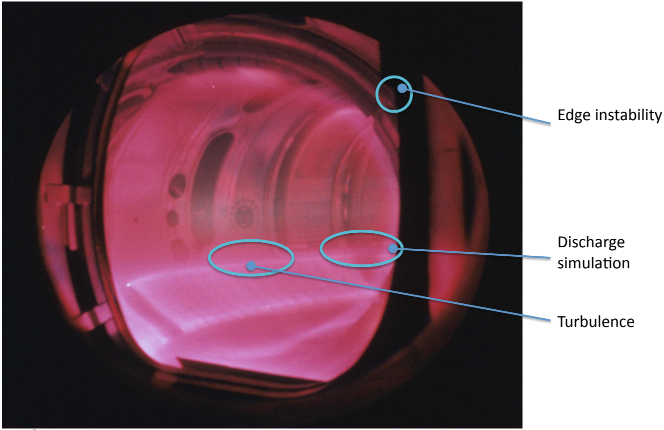
Equation of interest

Some words on Tokamaks and Stellarators - 1





Some words on Tokamaks and Stellarators - 2



$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \left(\mathbf{E}^\varepsilon + \mathbf{v}\times\left(\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon}\right)\right)\cdot\nabla_v f^\varepsilon = 0$$

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon + \frac{\mathbf{v}_\perp}{\varepsilon}\cdot\nabla_x f^\varepsilon + \left(\mathbf{E}^\varepsilon + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon}\right)\cdot\nabla_v f^\varepsilon = 0$$

Equation of interest and setting

Equation of interest and setting

$$\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon = 0$$
$$u^\varepsilon{}_{|t=0} = u_0$$

$u^\varepsilon = u^\varepsilon(t, \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, $t \in [0, T)$, for $T > 0$

Assumptions:

- $\mathbf{a}$ is regular
- $\nabla \cdot \mathbf{a} = 0$
- $\tau \mapsto \mathbf{a}(t, \tau, \mathbf{x})$ periodic of period 1
- $\mathbf{b} = \mathbf{b}(\mathbf{x}) = M\mathbf{x}$, M matrix s.t.
  - $\mathrm{tr} M = 0$
  - $\tau \mapsto e^{\tau M}$ periodic of period 1

  $\Rightarrow \nabla \cdot \mathbf{b} = 0$ and $\tau \mapsto \mathbf{X}(\tau) = e^{\tau M}\mathbf{x}$ periodic of period 1
  $\left(\frac{\partial \mathbf{X}}{\partial \tau} = M\mathbf{X} = \mathbf{b}(\mathbf{X}),\ \mathbf{X}(0) = \mathbf{x}\right)$
- $u_0 \in L^2(\mathbb{R}^d)$

A priori estimate

A priori estimate

$$\left(\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon = 0\right) \times u^\varepsilon, \int_{\mathbb{R}^d} d\mathbf{x} \rightarrow$$

$$\int_{\mathbb{R}^d} \frac{\partial u^\varepsilon}{\partial t} u^\varepsilon \, d\mathbf{x} + \int_{\mathbb{R}^d} \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot \nabla u^\varepsilon u^\varepsilon \, d\mathbf{x} + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \mathbf{b} \cdot \nabla u^\varepsilon u^\varepsilon \, d\mathbf{x} = 0$$

- $\displaystyle \int_{\mathbb{R}^d} \frac{\partial u^\varepsilon}{\partial t} u^\varepsilon \, d\mathbf{x} = \frac{1}{2} \frac{d\left(\int_{\mathbb{R}^d} |u^\varepsilon|^2 \, d\mathbf{x}\right)}{dt} = \frac{1}{2} \frac{d\left(\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}\right)}{dt}$
- $\displaystyle \int_{\mathbb{R}^d} \mathbf{a} \cdot \nabla u^\varepsilon u^\varepsilon \; dx = -\int_{\mathbb{R}^d} \mathbf{a} \cdot \nabla u^\varepsilon u^\varepsilon \; d\mathbf{x} - \int_{\mathbb{R}^d} \nabla \cdot \mathbf{a} \; u^\varepsilon u^\varepsilon \; d\mathbf{x} = -\int_{\mathbb{R}^d} \mathbf{a} \cdot \nabla u^\varepsilon u^\varepsilon \; d\mathbf{x} = 0$
- Same thing for last term

$$\frac{d\left(\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}\right)}{dt} = 0 \rightarrow \|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \text{ constant} \rightarrow \|u^\varepsilon\|_{L^2([0,T);L^2(\mathbb{R}^d))} \text{ bounded}$$

(u^ε) Two-Scale Converges to $U = U(t, \tau, \mathbf{x}) \in L^2([0, T); L^2_\#((\mathbb{R}; L^2(\mathbb{R}^d)))$

up to a subsequence

Order 0 Homogenization

Weak Formulation with Oscillating Test Functions

Order 0 Homogenization
Weak Formulation With Oscillating Test Functions

For $\phi = \phi(t, \tau, \mathbf{x})$ regular: $[\phi]^\varepsilon(t, \mathbf{x}) = \phi(t, \frac{t}{\varepsilon}, \mathbf{x})$

$$\frac{\partial [\phi]^\varepsilon}{\partial t} = \left[\frac{\partial \phi}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \phi}{\partial \tau}\right]^\varepsilon$$

$$[\phi]^\varepsilon \times \left(\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon\right), \int, \text{ IBP } \Rightarrow$$

$$\int_0^T \int_{\mathbb{R}^d} u^\varepsilon \left(\left[\frac{\partial \phi}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \phi}{\partial \tau}\right]^\varepsilon + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot [\nabla \phi]^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot [\nabla \phi]^\varepsilon\right) d\mathbf{x}dt$$

$$+ \int_{\mathbb{R}^d} u_0\, \phi(0, 0, .)\, d\mathbf{x} = 0$$

Order 0 Homogenization - Constraint

Order 0 Homogenization Constraint

WFOTF:

$$\int_0^T \int_{\mathbb{R}^d} u^\varepsilon \left(\left[\frac{\partial \phi}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \phi}{\partial \tau}\right]^\varepsilon + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot [\nabla \phi]^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot [\nabla \phi]^\varepsilon \right) d\mathbf{x}dt + \int_{\mathbb{R}^d} u_0 \, \phi(0,0,.) \, d\mathbf{x} = 0$$

$\times \varepsilon$, $\varepsilon \to 0$ $\to$

$$\frac{\partial U}{\partial \tau} + \mathbf{b} \cdot \nabla U = 0$$

$\to$

$$\exists V(t, \mathbf{y}) \in L^2([0, T); L^2(\mathbb{R}^d)) \text{ s.t. } U(t, \tau, \mathbf{x}) = V(t, e^{-\tau M}\mathbf{x})$$

(Recall: $\dfrac{\partial (e^{\tau M}\mathbf{x})}{\partial \tau} = M(e^{\tau M}\mathbf{x}) = \mathbf{b}(e^{\tau M}\mathbf{x})$

$$\frac{\partial (V(t, e^{-\tau M}\mathbf{x}))}{\partial \tau} + \mathbf{b} \cdot \nabla (V(t, e^{-\tau M}\mathbf{x})) =$$

$$\nabla V(t, e^{-\tau M}\mathbf{x}))\cdot((-e^{-\tau M})M\mathbf{x}) + ((e^{-\tau M})M\mathbf{x})\cdot\nabla V(t, e^{-\tau M}\mathbf{x})) = 0)$$

Order 0 Homogenization - Equation for V

Order 0 Homogenization Equation for V - 1

For $\gamma = \gamma(t, \mathbf{y})$ regular: $\phi(t, \tau, \mathbf{x}) = \gamma(t, e^{-\tau M}\mathbf{x})$ s.t. $\dfrac{\partial \phi}{\partial \tau} + \mathbf{b} \cdot \nabla \phi = 0$

In WFOTF $\rightarrow$

$$\int_0^T \int_{\mathbb{R}^d} u^\varepsilon \left(\left[\frac{\partial \phi}{\partial t} \right]^\varepsilon + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot [\nabla \phi]^\varepsilon \right) d\mathbf{x}dt + \int_{\mathbb{R}^d} u_0 \, \phi(0,0,.) \, d\mathbf{x} = 0$$

$\varepsilon \to 0 \quad \rightarrow$

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} U(t,\tau,\mathbf{x}) \left(\frac{\partial \phi}{\partial t}(t,\tau,\mathbf{x}) + \mathbf{a}(t,\tau,\mathbf{x}) \cdot \nabla \phi(t,\tau,\mathbf{x}) \right) d\mathbf{x}d\tau dt$$

$$+ \int_{\mathbb{R}^d} u_0 \, \phi(0,0,.) \, d\mathbf{x} = 0$$

Order 0 Homogenization Equation for V - 2

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} U(t,\tau,\mathbf{x}) \left(\frac{\partial \phi}{\partial t}(t,\tau,\mathbf{x}) + \mathbf{a}(t,\tau,\mathbf{x}) \cdot \nabla \phi(t,\tau,\mathbf{x}) \right) d\mathbf{x} d\tau dt + \int_{\mathbb{R}^d} u_0 \, \phi(0,0,.) \, d\mathbf{x} = 0$$

U in terms of V; ϕ in terms of γ

$$\frac{\partial \phi}{\partial t}(t,\tau,\mathbf{x}) = \frac{\partial \gamma}{\partial t}(t, e^{-\tau M}\mathbf{x}) \text{ and } \nabla \phi(t,\tau,\mathbf{x}) = (e^{-\tau M})^T \nabla \gamma(t, e^{-\tau M}\mathbf{x})$$

$\rightarrow$

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} V(t,\mathbf{y}) \left(\frac{\partial \gamma}{\partial t}(t,\mathbf{y}) + e^{-\tau M}\mathbf{a}(t,\tau,e^{\tau M}\mathbf{y}) \cdot \nabla \gamma(t,\mathbf{y}) \right) d\mathbf{y} d\tau dt + \int_{\mathbb{R}^d} u_0(\mathbf{y}) \, \gamma(0,\mathbf{y}) \, d\mathbf{y} = 0$$

$$\frac{\partial V}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(t,\sigma,e^{\sigma M}\mathbf{y}) \, d\sigma \right) \cdot \nabla V = 0 \qquad V_{|t=0} = u_0$$

Order 1 Homogenization

To simplify computations :

From now: $\mathbf{a}(t, \tau, \mathbf{x}) = \mathbf{a}(\mathbf{x})$

Order 1 Homogenization - Preparations: Equation for U and u

Order 1 Homogenization
Equation for U and u - 1

Linearity $\rightarrow$ Equation for U $\rightarrow$ Equation for u (w-$*$ limit of (u^ε)):
WRITE

$$\frac{\partial V}{\partial t} + \left(\int_0^1 e^{-\sigma M} \mathbf{a}(e^{\sigma M}\mathbf{y})\, d\sigma \right) \cdot \nabla V = 0 \text{ in } \mathbf{y} = e^{-\tau M}\mathbf{x}$$

USE: $U(t,\tau,\mathbf{x}) = V(t, e^{-\tau M}\mathbf{x})$
$\nabla U(t,\tau,\mathbf{x}) = (e^{-\tau M})^T \nabla V(t, e^{-\tau M}\mathbf{x})$ i.e.
$\nabla V(t, e^{-\tau M}\mathbf{x}) = (e^{\tau M})^T \nabla U(t,\tau,\mathbf{x}) \rightarrow$

$$0 = \frac{\partial \left(V(t, e^{-\tau M}\mathbf{x}) \right)}{\partial t} + \left(\int_0^1 e^{-\sigma M} \mathbf{a}(e^{\sigma M} e^{-\tau M}\mathbf{x}) d\sigma \right) \cdot \nabla V(t, e^{-\tau M}\mathbf{x})$$

$$= \frac{\partial U}{\partial t} + \left(e^{\tau M} \int_0^1 e^{-\sigma M} \mathbf{a}(e^{(\sigma-\tau)M}\mathbf{x}) d\sigma \right) \cdot \nabla U$$

$$= \frac{\partial U}{\partial t} + \left(\int_0^1 e^{(\tau-\sigma)M} \mathbf{a}(e^{(\sigma-\tau)M}\mathbf{x}) d\sigma \right) \cdot \nabla U$$

$$= \frac{\partial U}{\partial t} + \left(\int_0^1 e^{-\sigma M} \mathbf{a}(e^{\sigma M}\mathbf{x}) d\sigma \right) \cdot \nabla U,$$

Order 1 Homogenization Equation for U and u - 2

$$0 = \frac{\partial \left(V(t, e^{-\tau M}\mathbf{x})\right)}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}e^{-\tau M}\mathbf{x})d\sigma\right)\cdot\nabla V(t, e^{-\tau M}\mathbf{x})$$

$$= \frac{\partial U}{\partial t} + \left(\int_0^1 e^{(\tau-\sigma)M}\mathbf{a}(e^{(\sigma-\tau)M}\mathbf{x})d\sigma\right)\cdot\nabla U$$

$$= \frac{\partial U}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla U,$$

$\rightarrow$

$$\frac{\partial U}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla U = 0, \quad U_{|t=0} = u_0(e^{-\tau M}\mathbf{x})$$

$$u = \int_0^1 U(.,\tau,.)d\tau \rightarrow$$

$$\frac{\partial u}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla u = 0, \; u_{|t=0} = \int_0^1 u_0(e^{-\tau M}\mathbf{x})\; d\tau$$

Order 1 Homogenization - Strong Two-Scale convergence of u^{ε}

Order 1 Homogenization
Strong Two-Scale convergence of U - 1

$$\frac{\partial (u^\varepsilon)^2}{\partial t} = 2u^\varepsilon \frac{\partial u^\varepsilon}{\partial t} \quad \text{and} \quad \nabla (u^\varepsilon)^2 = 2u^\varepsilon \nabla u^\varepsilon$$

multiplying equation for u^ε by $2u^\varepsilon$ $\rightarrow$

$$\frac{\partial (u^\varepsilon)^2}{\partial t} + \mathbf{a} \cdot \nabla (u^\varepsilon)^2 + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla (u^\varepsilon)^2 = 0 \quad (u^\varepsilon)^2_{|t=0} = u_0^2$$

IF $u_0^2 \in L^2(\mathbb{R}^d)$, *i.e.* if $u_0 \in L^4(\mathbb{R}^d)$, doing the same $\rightarrow$

$(u^\varepsilon)^2$ Two-Scale converges to Z solution to

$$\frac{\partial Z}{\partial t} + \left(\int_0^1 e^{-\sigma M} \mathbf{a}(e^{\sigma M}\mathbf{x}) d\sigma \right) \cdot \nabla Z = 0$$
$$Z_{|t=0} = u_0^2(e^{-\tau M}\mathbf{x})$$

$\rightarrow Z = U^2$

$((u^\varepsilon)^2)$ Two-Scale Converges to U^2

Order 1 Homogenization
Strong Two-Scale convergence of U - 2

$$((u^\varepsilon)^2) \text{ Two-Scale Converges to } U^2$$

$\rightarrow$

$$\|u^\varepsilon\|_{L^2([0,T);L^2(\mathbb{R}^d))} \rightarrow \|U\|_{L^2([0,T);L^2_\#((\mathbb{R};L^2(\mathbb{R}^d)))}$$

Moreover: IF $u_0 \in \mathcal{C}^0(\mathbb{R}^d)$ $\rightarrow$
$u^\varepsilon \in \mathcal{C}^0([0,T);\mathcal{C}^0(\mathbb{R}^d))$, $U \in \mathcal{C}^0([0,T);\mathcal{C}^0_\#((\mathbb{R};\mathcal{C}^0(\mathbb{R}^d)))$,
$V \in \mathcal{C}^0([0,T);\mathcal{C}^0(\mathbb{R}^d))$

HENCE: IF $u_0 \in (L^2 \cap L^4 \cap \mathcal{C}^0)(\mathbb{R}^d)$, THEN in addition to every already stated results

$$(u^\varepsilon) \text{ Strongly Two-Scale Converges to } U$$

(We have: $(u^\varepsilon - [U]^\varepsilon) \rightarrow 0$
Now: Get more: $((u^\varepsilon - [U]^\varepsilon)/\varepsilon)$ Two-Scale Converges)

Order 1 Homogenization - Function W_1

Order 1 Homogenization - Function W_1 - 1

Step 1:

$$\frac{\partial U}{\partial \tau} + \mathbf{b}\cdot\nabla U = 0 \rightarrow \frac{\partial [U]^\varepsilon}{\partial t} = \left[\frac{\partial U}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial U}{\partial \tau}\right]^\varepsilon = \left[\frac{\partial U}{\partial t}\right]^\varepsilon - \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla[U]^\varepsilon$$

$$\frac{\partial U}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla U = 0, \quad U_{|t=0} = u_0(e^{-\tau M}\mathbf{x})$$

$$\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(\mathbf{x})\cdot\nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon = 0, \quad {u^\varepsilon}_{|t=0} = u_0$$

$\rightarrow$

$$\frac{\partial\left(\dfrac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right)}{\partial t} + \mathbf{a}\cdot\nabla\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right) = -\frac{1}{\varepsilon}\left(\mathbf{a} - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla[U]^\varepsilon$$

$$\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right)_{|t=0} = 0$$

Order 1 Homogenization - Function W_1 - 2

Step 2: DEFINE: $W_1 = W_1(t, \tau, \mathbf{y})$ s.t

$$\tilde{W}_1 = \tilde{W}_1(t, \tau, \mathbf{x}) = W_1(t, \tau, e^{-\tau M}\mathbf{x}) \text{ solution to}$$

$$\frac{\partial \tilde{W}_1}{\partial \tau} + \mathbf{b} \cdot \nabla \tilde{W}_1 = -\left(\mathbf{a} - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right) \cdot \nabla U$$

THEN: $[\tilde{W}_1]^\varepsilon = [\tilde{W}_1]^\varepsilon(t, \mathbf{x}) = \tilde{W}_1(t, t/\varepsilon, \mathbf{x})$:

$$\frac{\partial [\tilde{W}_1]^\varepsilon}{\partial t} + \mathbf{a} \cdot \nabla [\tilde{W}_1]^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla [\tilde{W}_1]^\varepsilon$$

$$= \left[\frac{\partial \tilde{W}_1}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \tilde{W}_1}{\partial \tau}\right]^\varepsilon + \mathbf{a} \cdot \nabla [\tilde{W}_1]^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla [\tilde{W}_1]^\varepsilon$$

$$= \left[\frac{\partial \tilde{W}_1}{\partial t}\right]^\varepsilon + \mathbf{a} \cdot \nabla [\tilde{W}_1]^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{a} - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right) \cdot \nabla [U]^\varepsilon$$

Order 1 Homogenization - Function W_1 - 3

$$\frac{\partial\left(\dfrac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}\right)}{\partial t}+\mathbf{a}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}\right)+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}\right)=-\frac{1}{\varepsilon}\left(\mathbf{a}-\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla[U]^\varepsilon$$

$$\frac{\partial[\tilde{W}_1]^\varepsilon}{\partial t}+\mathbf{a}\cdot\nabla[\tilde{W}_1]^\varepsilon+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla[\tilde{W}_1]^\varepsilon=\left[\frac{\partial\tilde{W}_1}{\partial t}\right]^\varepsilon+\mathbf{a}\cdot\nabla[\tilde{W}_1]^\varepsilon-\frac{1}{\varepsilon}\left(\mathbf{a}-\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla[U]^\varepsilon$$

$$\frac{\partial\left(\dfrac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde{W}_1]^\varepsilon\right)}{\partial t}+\mathbf{a}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde{W}_1]^\varepsilon\right)+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde{W}_1]^\varepsilon\right)=-\left[\frac{\partial\tilde{W}_1}{\partial t}\right]^\varepsilon-\mathbf{a}\cdot\nabla[\tilde{W}_1]^\varepsilon$$

$$\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde{W}_1]^\varepsilon\right)_{|t=0}=-[\tilde{W}_1]^\varepsilon{}_{|t=0}$$

Order 1 Homogenization - Function W_1 - 4

Step 3: expression of the function W_1:

$$\tilde{W}_1(t,\tau,\mathbf{x}) = W_1(t,\tau,e^{-\tau M}\mathbf{x})$$

$$\frac{\partial \tilde{W}_1}{\partial \tau} + \mathbf{b}\cdot\nabla\tilde{W}_1 = -\left(\mathbf{a} - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{x})d\sigma\right)\cdot\nabla U$$

$\rightarrow$

$$\frac{\partial W_1}{\partial \tau} = -\left(\mathbf{a}(e^{\tau M}\mathbf{y}) - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{(\sigma+\tau)M}\mathbf{y})d\sigma\right)\cdot\nabla U(t,\tau,e^{\tau M}\mathbf{y})$$

$$\nabla U(t,\tau,e^{\tau M}\mathbf{y}) = (e^{-\tau M})^T\ \nabla\big(U(t,\tau,e^{\tau M}\mathbf{y})\big) = (e^{-\tau M})^T\nabla V(t,\mathbf{y})$$

$\rightarrow$

$$\frac{\partial W_1}{\partial \tau} = -\left(e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) - \int_0^1 e^{-(\sigma+\tau)M}\mathbf{a}(e^{(\sigma+\tau)M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$

$$= -\left(e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$

Order 1 Homogenization - Function W_1 - 5

$$\frac{\partial W_1}{\partial \tau} = -\left(e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) - \int_0^1 e^{-(\sigma+\tau)M}\mathbf{a}(e^{(\sigma+\tau)M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$
$$= -\left(e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) - \int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$

$\rightarrow$

$$W_1(t,\tau,\mathbf{y}) =$$
$$-\left(\int_0^\tau e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma - \tau\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$

By-product: $[\tilde W_1]^\varepsilon{}_{|t=0} = 0$

$$\left\| -\left[\frac{\partial \tilde W_1}{\partial t}\right]^\varepsilon - \mathbf{a}\cdot\nabla[\tilde W_1]^\varepsilon \right\|_{L^\infty([0,T);L^2(\mathbb{R}^d))} \le C_1$$

Order 1 Homogenization - A priori estimate and convergence

Order 1 Homogenization
A priori estimate and convergence - 1

$$\frac{\partial\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right)}{\partial t}+\mathbf{a}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right)$$

$$+\frac{1}{\varepsilon}\mathbf{b}\cdot\nabla\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right)=-\left[\frac{\partial \tilde W_1}{\partial t}\right]^\varepsilon-\mathbf{a}\cdot\nabla[\tilde W_1]^\varepsilon$$

$$\left(\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right)_{|t=0}=-[\tilde W_1]^\varepsilon{}_{|t=0}=0$$

$\times((u^\varepsilon-[U]^\varepsilon)/\varepsilon-[\tilde W_1]^\varepsilon)$, $\int_{\mathbb{R}^d}d\mathbf{x}$, IBP $\rightarrow$

$$\frac{d\left(\int_{\mathbb{R}^d}\left|\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right|^2d\mathbf{x}\right)}{dt}\leq C_1\left(\int_{\mathbb{R}^d}\left|\frac{u^\varepsilon-[U]^\varepsilon}{\varepsilon}-[\tilde W_1]^\varepsilon\right|^2d\mathbf{x}\right)^{\frac12}$$

Order 1 Homogenization
A priori estimate and convergence - 2

$$\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon} - [\tilde{W}_1]^\varepsilon\right) \text{ and consequently } \left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right)$$

bounded in $L^2([0,T); L^2(\mathbb{R}^d))$. Then, up to subsequences,

$$\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon}\right) \text{ Two-Scale Converges to } U_1 = U_1(t, \tau, \mathbf{x})$$

$$\left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon} - [\tilde{W}_1]^\varepsilon\right) \text{ Two-Scale Converges to } U_1 - \tilde{W}_1$$

Order 1 Homogenization - Constraint

Order 1 Homogenization
Constraint

$$\text{WFOTF}: \phi = \phi(t,\tau,\mathbf{x}) \in \mathcal{C}^1([0,T);\mathcal{C}^1_\#((\mathbb{R};\mathcal{C}^1(\mathbb{R}^d)))$$

$$\int_0^T \int_{\mathbb{R}^d} \left(\frac{u^\varepsilon - [U]^\varepsilon}{\varepsilon} - [\tilde{W}_1]^\varepsilon \right) \left(\left[\frac{\partial \phi}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \phi}{\partial \tau}\right]^\varepsilon + \mathbf{a}\cdot[\nabla\phi]^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot[\nabla\phi]^\varepsilon \right) dxdt$$
$$= \int_0^T \int_{\mathbb{R}^d} \left(-\left[\frac{\partial \tilde{W}_1}{\partial t}\right]^\varepsilon - \mathbf{a}\cdot\nabla[\tilde{W}_1]^\varepsilon \right) [\phi]^\varepsilon dxdt$$

$\times\varepsilon$, $\varepsilon \to 0 \quad \to$

$$\frac{\partial (U_1 - \tilde{W}_1)}{\partial \tau} + \mathbf{b}\cdot\nabla(U_1 - \tilde{W}_1) = 0$$

$\exists V_1 = V_1(t,\mathbf{y}) \in L^2([0,T);L^2(\mathbb{R}^d))$ s.t.

$$U_1(t,\tau,\mathbf{x}) - \tilde{W}_1(t,\tau,\mathbf{x}) = V_1(t, e^{-\tau M}\mathbf{x}) \text{ i.e.}$$
$$U_1(t,\tau,\mathbf{x}) = V_1(t, e^{-\tau M}\mathbf{x}) + W_1(t,\tau,e^{-\tau M}\mathbf{x})$$

Order 1 Homogenization - Equation for V_1

Order 1 Homogenization Equation for V_1 - 1

For $\gamma = \gamma(t, \mathbf{y})$ regular: $\phi(t, \tau, \mathbf{x}) = \gamma(t, e^{-\tau M}\mathbf{x})$ s.t. $\dfrac{\partial \phi}{\partial \tau} + \mathbf{b} \cdot \nabla \phi = 0$

USE $\phi(t, \tau, \mathbf{x})$ in WFOTF, $\varepsilon \to 0 \to$

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} V_1(t, e^{-\tau M}\mathbf{x}) \left(\frac{\partial \gamma}{\partial t}(t, e^{-\tau M}\mathbf{x}) + e^{-\tau M}\mathbf{a}(\mathbf{x}) \cdot \nabla \gamma(t, e^{-\tau M}\mathbf{x}) \right) d\mathbf{x} d\tau dt$$

$$= \int_0^T \int_0^1 \int_{\mathbb{R}^d} \left(-\frac{\partial \tilde{W}_1}{\partial t} - \mathbf{a}(\mathbf{x}) \cdot \nabla \tilde{W}_1 \right) \gamma(t, e^{-\tau M}\mathbf{x}) d\mathbf{x} d\tau dt$$

change of variables $(t, \tau, \mathbf{x}) \mapsto (t, \tau, \mathbf{y} = e^{-\tau M}\mathbf{x})$ gives

$$\int_0^T \int_0^1 \int_{\mathbb{R}^d} V_1(t, \mathbf{y}) \left(\frac{\partial \gamma}{\partial t}(t, \mathbf{y}) + e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) \cdot \nabla \gamma(t, \mathbf{y}) \right) d\mathbf{y} d\tau dt$$

$$= \int_0^T \int_0^1 \int_{\mathbb{R}^d} \left(-\frac{\partial W_1}{\partial t} - e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) \cdot \nabla W_1 \right) \gamma(t, \mathbf{y}) d\mathbf{y} d\tau dt$$

Order 1 Homogenization Equation for V_1 - 2

$$\int_0^T\int_0^1\int_{\mathbb{R}^d} V_1(t,\mathbf{y})\left(\frac{\partial\gamma}{\partial t}(t,\mathbf{y}) + e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y})\cdot\nabla\gamma(t,\mathbf{y})\right)d\mathbf{y}\,d\tau\,dt$$

$$= \int_0^T\int_0^1\int_{\mathbb{R}^d}\left(-\frac{\partial W_1}{\partial t} - e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y})\cdot\nabla W_1\right)\gamma(t,\mathbf{y})d\mathbf{y}\,d\tau\,dt$$

$\rightarrow$

$$\frac{\partial V_1}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})d\sigma\right)\cdot\nabla V_1 =$$

$$\int_0^1\left(-\frac{\partial W_1}{\partial t} - e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y})\cdot\nabla W_1\right)d\tau$$

$$V_{1|t=0} = 0$$

Order 1 Homogenization
Equation for V_1 - 3

Heavy computation to get:

$$\int_0^1 \left(-\frac{\partial W_1}{\partial t} - e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) \cdot \nabla W_1 \right) d\tau$$

$$\frac{\partial V_1}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y}) d\sigma \right) \cdot \nabla V_1 =$$

$$\left(\int_0^1 \left(\left[\nabla \left[e^{-\tau M}\mathbf{a}(e^{\tau M}\mathbf{y}) \right] \right] \left(\int_0^\tau e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y}) \right) d\tau \right.\right.$$

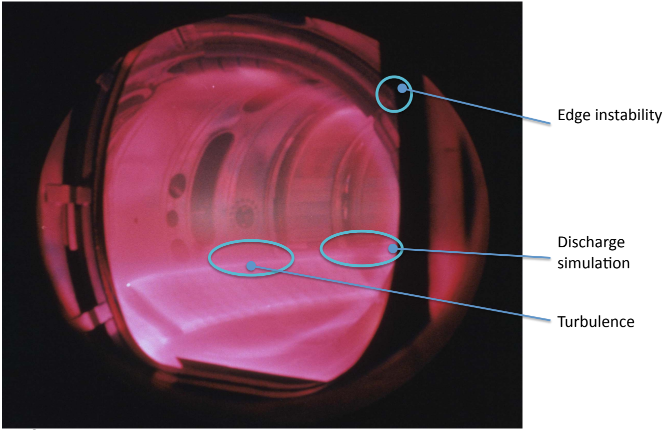
$$\left.\left. + \frac{1}{2} \left[\nabla \left[\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\, d\sigma \right] \right] \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\, d\sigma \right) \right) \right) \cdot (\nabla V)$$

$$V_{1|t=0} = 0.$$

Two-Scale Numerical Methods

Motivation : Tokamaks and Stellarators

Long term target : 10 ms of a Tokamak working



$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v}\times(\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon}))\cdot\nabla_v f^\varepsilon = 0$$

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_{\|}\cdot\nabla_x f^\varepsilon + \frac{\mathbf{v}_\perp}{\varepsilon}\cdot\nabla_x f^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon})\cdot\nabla_v f^\varepsilon = 0$$

Two-Scale Numerical Method Algorithms

Algorithm for order 0 Two-Scale Numerical Method

To compute u^ε solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(t, \frac{t}{\varepsilon}, \mathbf{x}) \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon = 0 \qquad {u^\varepsilon}_{|t=0} = u_0.$$

for ε small:

Compute V solution to

$$\frac{\partial V}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(t, \sigma, e^{\sigma M}\mathbf{y})\, d\sigma\right) \cdot \nabla V = 0 \quad V_{|t=0} = u_0$$

And use

$$u^\varepsilon(t, \mathbf{x}) \sim U(t, \frac{t}{\varepsilon}, \mathbf{x}) \qquad U(t, \frac{t}{\varepsilon}, \mathbf{x}) = V(t, e^{-\frac{t}{\varepsilon}M}\mathbf{x})$$

Algorithm for order 1 Two-Scale Numerical Method

For ε small, to compute u^ε solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \mathbf{a}(\mathbf{x}) \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon = 0 \qquad u^\varepsilon{}_{|t=0} = u_0.$$

Compute: $W_1(t,\tau,\mathbf{y}) =$

$$-\left(\int_0^\tau e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma - \tau\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma\right)\cdot\nabla V(t,\mathbf{y})$$

Compute: V and V_1 solution to

$$\frac{\partial V}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})\,d\sigma\right)\cdot\nabla V = 0 \quad V_{|t=0} = u_0$$

$$\frac{\partial V_1}{\partial t} + \left(\int_0^1 e^{-\sigma M}\mathbf{a}(e^{\sigma M}\mathbf{y})d\sigma\right)\cdot\nabla V_1 = RHS(V)$$

And use

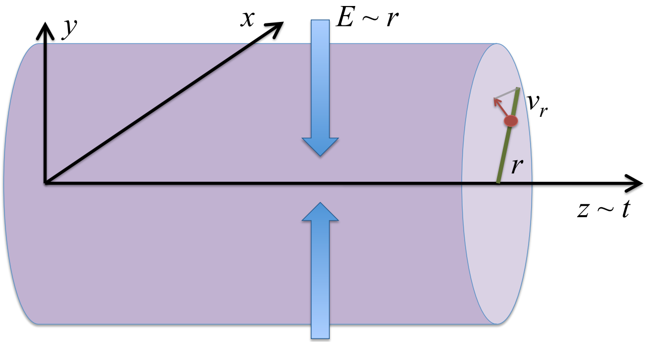
$$u^\varepsilon(t,\mathbf{x}) \sim U(t,\frac{t}{\varepsilon},\mathbf{x}) + \varepsilon U_1(t,\frac{t}{\varepsilon},\mathbf{x})$$

$$= V(t,e^{-\frac{t}{\varepsilon}M}\mathbf{x}) + \varepsilon\big(V_1(t,e^{-\frac{t}{\varepsilon}M}\mathbf{x}) + W_1(t,\frac{t}{\varepsilon},e^{-\frac{t}{\varepsilon}M}\mathbf{x})\big)$$

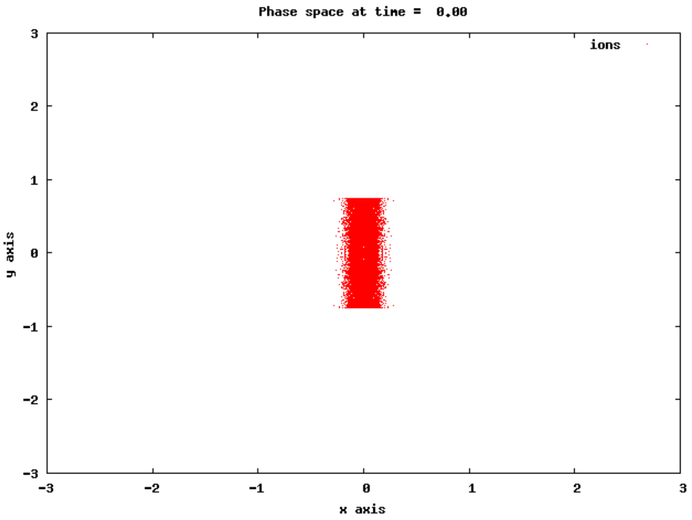
Two-Scale Numerical Method implementation for beam simulation

A beam in a focusing channel

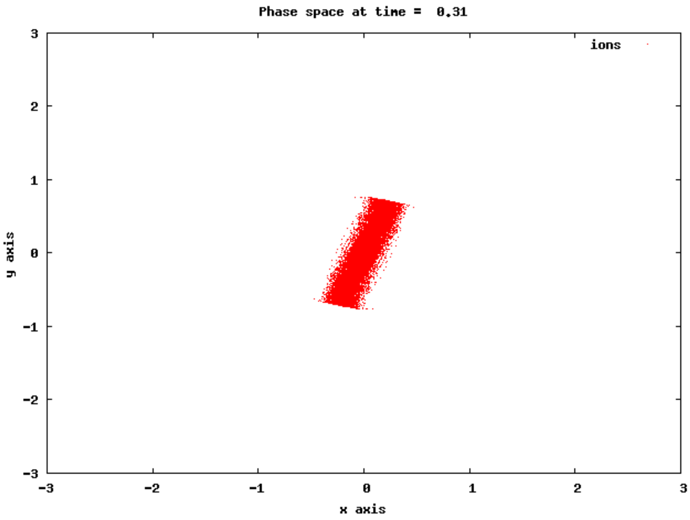
A beam in a focusing channel



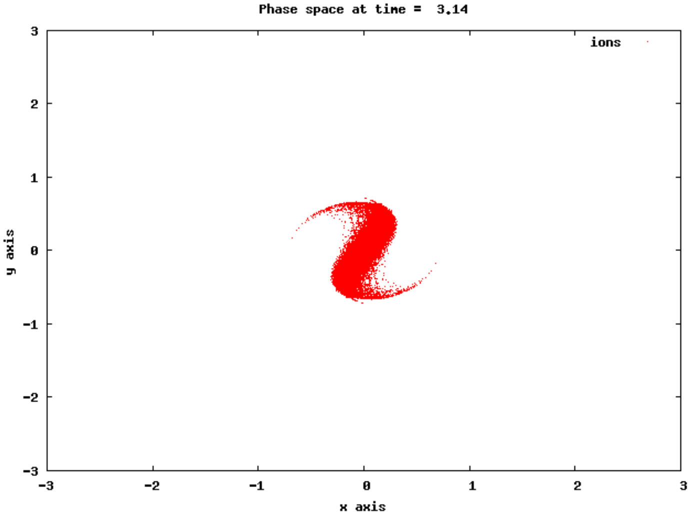
A beam in a focusing channel - Simulation



A beam in a focusing channel - Simulation

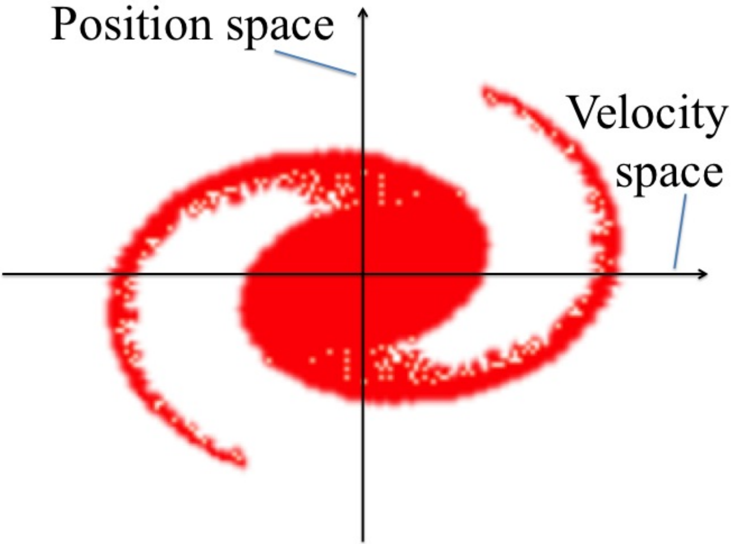


A beam in a focusing channel - Simulation

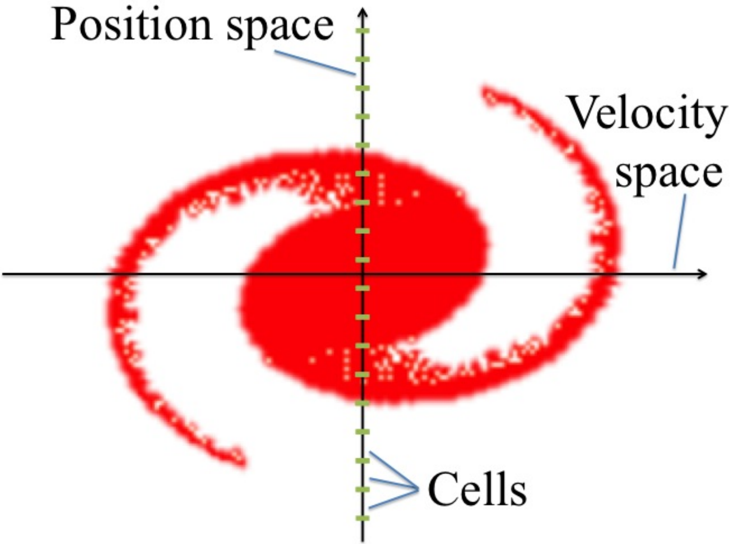


Digression on Pic Methods

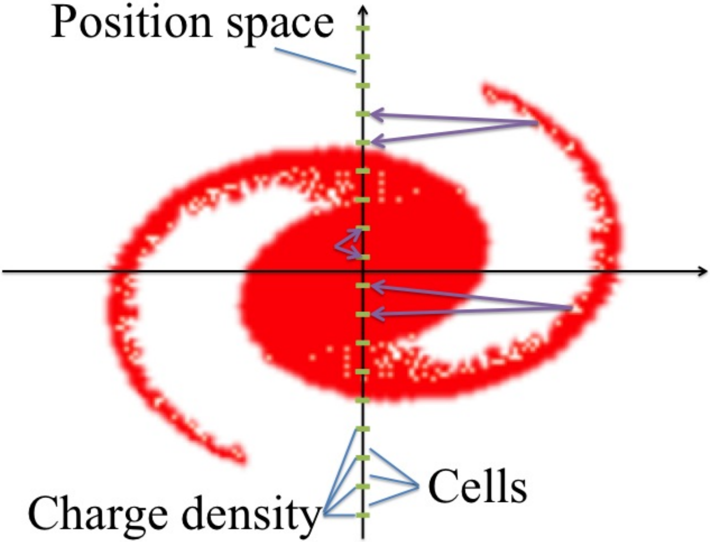
Pic Method explained - 1



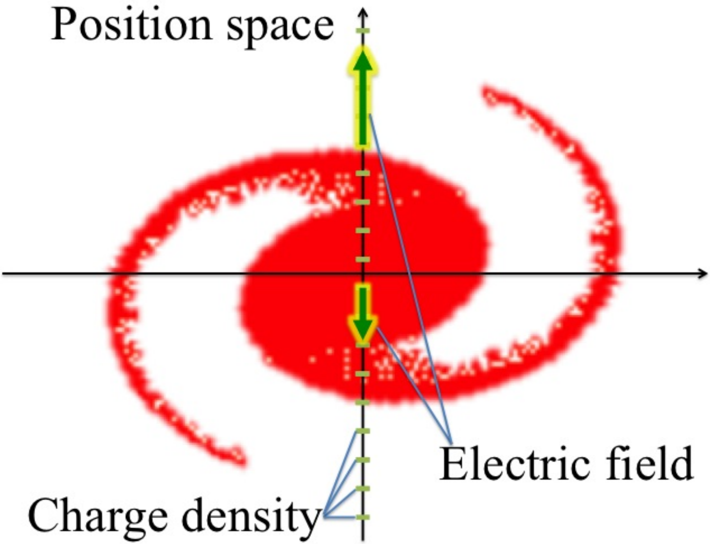
Pic Method explained - 2



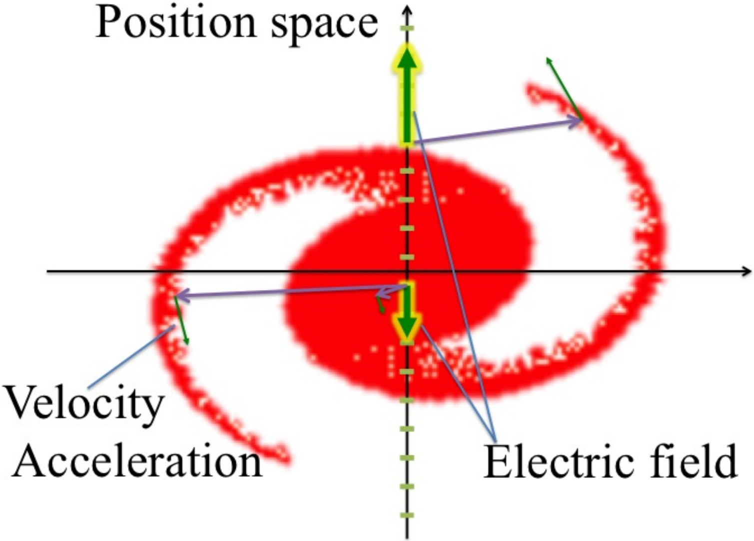
Pic Method explained - 3



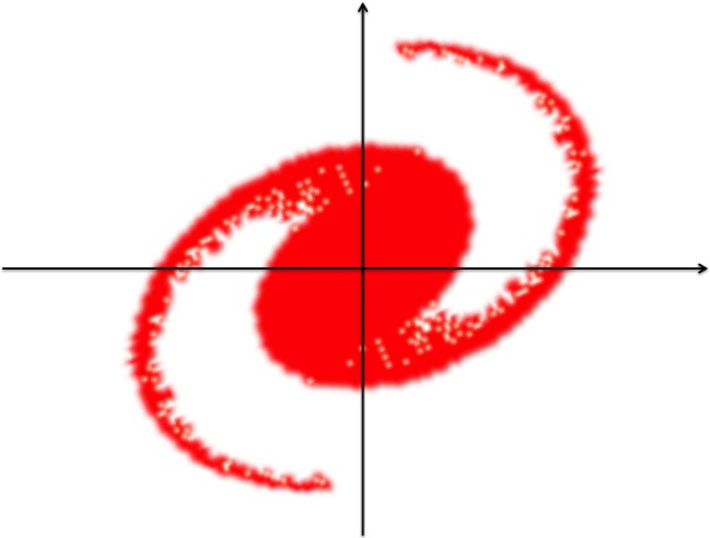
Pic Method explained - 4



Pic Method explained - 5



Pic Method explained - 6



Beam in a focusing channel : PDE Model

PDE Model

$f_\varepsilon = f_\varepsilon(t,r,v_r)$, $t \in [0,T)$, $r \in \mathbb{R}^+$ and $v_r \in \mathbb{R}$:

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t} + \dfrac{4\pi^2}{\varepsilon} v_r \dfrac{\partial f_\varepsilon}{\partial r} + \left(\mathbf{E}_{r\varepsilon} - \dfrac{4\pi^2}{\varepsilon} r\right) \dfrac{\partial f_\varepsilon}{\partial v_r} = 0 \\ \dfrac{1}{r}\dfrac{\partial(r\mathbf{E}_{r\varepsilon})}{\partial r} = \rho_\varepsilon(t,r), \qquad \rho_\varepsilon(t,r) = \displaystyle\int_{\mathbb{R}} f_\varepsilon(t,r,v_r)\, dv_r \\ f_\varepsilon(t=0,r,v_r) = f_0 \end{cases}$$

$\dfrac{\partial u^\varepsilon}{\partial t} + \mathbf{a}^\varepsilon \cdot \nabla u^\varepsilon + \dfrac{1}{\varepsilon}\mathbf{b} \cdot \nabla u^\varepsilon = 0$ with $\mathbf{x}$ replaced by (r, v_r) and

$$\mathbf{a}^\varepsilon = \begin{pmatrix} 0 \\ \mathbf{E}_{r\varepsilon}(t,r) \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4\pi^2 v_r \\ -4\pi^2 r \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix} e^{\tau M} = \begin{pmatrix} \cos(2\pi\tau) & \sin(2\pi\tau) \\ -\sin(2\pi\tau) & \cos(2\pi\tau) \end{pmatrix}$$

Order 0 Homogenization

Assumptions: $f_0 \geq 0,\, f_0 \in (L^1 \cap L^p)(\mathbb{R}^2; rdrdv_r)$ for $p \geq 2$

$$\int_{\mathbb{R}^2} (r^2 + v_r^2) f_0 \, rdrdv_r < +\infty$$

Then:

f_ε Two-Scale Converges to $F \in L^\infty([0,T); L^\infty_\#(\mathbb{R}; L^2(\mathbb{R}^2; rdrdv_r)))$

$\mathbf{E}_{r\varepsilon}$ Two-Scale Converges to $\mathcal{E}_r \in L^\infty([0,T); L^\infty_\#(\mathbb{R}; W^{1,3/2}(\mathbb{R}; rdr)))$

$\exists G = G(t,q,u_r) \in L^\infty([0,T); L^2(\mathbb{R}^2; qdqdu_r))$:

$$F(t,\tau,r,v_r) = G(t, \cos(2\pi\tau)r - \sin(2\pi\tau)v_r, \sin(2\pi\tau)r + \cos(2\pi\tau)v_r)$$

$$\begin{cases} \dfrac{\partial G}{\partial t} + \displaystyle\int_0^1 -\sin(2\pi\sigma)\,\mathcal{E}_r(t,\sigma,\cos(2\pi\sigma)q + \sin(2\pi\sigma)u_r)\,d\sigma\,\dfrac{\partial G}{\partial q} \\ \qquad\qquad + \displaystyle\int_0^1 \cos(2\pi\sigma)\,\mathcal{E}_r(t,\sigma,\cos(2\pi\sigma)q + \sin(2\pi\sigma)u_r)\,d\sigma\,\dfrac{\partial G}{\partial u_r} = 0 \\ G(t=0) = f_0 \end{cases}$$

$\mathcal{E}_r = \mathcal{E}_r(t,\tau,r,v_r)$:

$$\frac{1}{r}\frac{\partial(r\mathcal{E}_r)}{\partial r} = \int_{\mathbb{R}} G\big(t, \cos(2\pi\tau)r - \sin(2\pi\tau)v_r, \sin(2\pi\tau)r + \cos(2\pi\tau)v_r\big)\,dv_r$$

Two-Scale Pic Method for a beam in a focusing channel

G approximated by $G_N(q,u,t) = \sum_{k=1}^{N} w_k \delta(q - Q_k(t))\delta(u - U_k(t))$

From (Q_k^l, U_k^l) at time t_l, compute (Q_k^{l+1}, U_k^{l+1}) as an approximated solution to

$$\frac{dQ_k}{dt} = -\int_0^1 \sin(2\pi\sigma)\,\mathcal{E}_r(t,\sigma,\cos(2\pi\sigma)Q_k + \sin(2\pi\sigma)U_k)\,d\sigma, \quad Q_k(t_l) = Q_k^l$$

$$\frac{dU_k}{dt} = \int_0^1 \cos(2\pi\sigma)\,\mathcal{E}_r(t,\sigma,\cos(2\pi\sigma)Q_k + \sin(2\pi\sigma)U_k)\,d\sigma, \qquad U_k(t_l) = U_k^l$$

at time $t_{l+1} = t_l + \Delta t$

Two-Scale Pic Method to compute G - 1
Recall Runge-Kutta 4 Method

$$t_{l,1} = t_l, \quad y^{l,1} = y^l$$

$$t_{l,2} = t_l + \frac{\Delta t}{2}, \quad y^{l,2} = y^l + \frac{1}{2} l^1 \text{ with } l^1 = \Delta t\, K(t_{l,1}, y^{l,1}),$$

$$t_{l,3} = t_l + \frac{\Delta t}{2}, \quad y^{l,3} = y^l + \frac{1}{2} l^2 \text{ with } l^2 = \Delta t\, K(t_{l,2}, y^{l,2}),$$

$$t_{l,4} = t_l + \Delta t, \quad y^{l,4} = y^l + l^3, \text{ with } l^3 = \Delta t\, K(t_{l,3}, y^{l,3})$$

$$y^{l+1} = y^l + \frac{1}{6} l^1 + \frac{1}{3} l^2 + \frac{1}{3} l^3 + \frac{1}{6} l^4 \text{ with } l^4 = \Delta t\, K(t_{l,4}, y^{l,4})$$

In other words, we have to compute $Q_k^{l,2}$ as follows:

$$Q_k^{l,2} = Q_k^l + \frac{1}{2} I^1 \text{ with}$$

$$I^1 = \Delta t \Big(- \sum_{m=1}^{p} \gamma_m \sin(2\pi\sigma_m)\, \mathcal{E}_r(t_l, \sigma_m, \cos(2\pi\sigma_m) Q_k^l + \sin(2\pi\sigma_m) U_k^l) \Big)$$

$$Q_k^{l,3} = Q_k^l + \frac{1}{2} I^2 \text{ with}$$

$$I^2 = \Delta t \Big(- \sum_{m=1}^{p} \gamma_m \sin(2\pi\sigma_m) \\ \mathcal{E}_r^2(t_l + \frac{\Delta t}{2}, \sigma_m, \cos(2\pi\sigma_m) Q_k^{l,2} + \sin(2\pi\sigma_m) U_k^{l,2}) \Big)$$

Two-Scale Pic Method to compute G - 1 Implementation - 2

$$Q_k^{l,4} = Q_k^l + I^3, \text{ with}$$

$$I^3 = \Delta t\Big(-\sum_{m=1}^{p} \gamma_m \sin(2\pi\sigma_m) \mathcal{E}_r^3\big(t_l + \frac{\Delta t}{2}, \sigma_m, \cos(2\pi\sigma_m) Q_k^{l,3} + \sin(2\pi\sigma_m) U_k^{l,3}\big)\Big)$$

$$Q_k^{l+1} = Q_k^l + \frac{1}{6} I^1 + \frac{1}{3} I^2 + \frac{1}{3} I^3 + \frac{1}{6} I^4, \text{ with}$$

$$I^4 = \Delta t\Big(-\sum_{m=1}^{p} \gamma_m \sin(2\pi\sigma_m) \mathcal{E}_r^4\big(t_l + \Delta t, \sigma_m, \cos(2\pi\sigma_m) Q_k^{l,4} + \sin(2\pi\sigma_m) U_k^{l,4}\big)\Big)$$