On the Geometrical Gyro-Kinetic Theory

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Abstract - Considering a Hamiltonian Dynamical System describing the motion of charged particle in a Tokamak or a Stellarator, we build a change of coordinates to reduce its dimension. This change of coordinates is in fact an intricate succession of mappings that are built using Hyperbolic Partial Differential Equations, Differential Geometry, Hamiltonian Dynamical System Theory and Symplectic Geometry, Lie Transforms and a new tool which is here introduced : Partial Lie Sums.

Keywords - Tokamak; Stellarator; Gyro-Kinetic Approximation; Hamiltonian Dynamical System; Hyperbolic Partial Differential Equations; Differential Geometry; Hamiltonian Dynamical System Theory; Symplectic Geometry; Lie Transforms; Partial Lie Sums.

Contents

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1 Introduction

At the end of the 70', Littlejohn [25, 26, 27] shed new light on what is called *the Guiding Center Approximation.* His approach incorporated high level mathematical concepts from Hamiltonian Mechanics, Differential Geometry and Symplectic Geometry into a physical affordable theory in order to clarify what has been done for years in the domain (see Kruskal [24], Gardner [12], Northrop [28], Northrop & Rome [29]). This theory is a nice success. It has been beeing widely used by physicists to deduce related models (*Finite Larmor Radius Approximation, Drift-Kinetic Model, Quasi-Neutral Gyro-Kinetic Model, etc.*, see for instance Brizard [2], Dubin *et al.* [5], Frieman & Chen [10], Hahm [17], Hahm, Lee & Brizard [19], Parra & Catto [30, 31, 32]) making up the *Gyro-Kinetic Approximation Theory*, which is the basis of all kinetic codes used to simulate Plasma Turbulence emergence and evolution in Tokamak and Stellarators (see for instance Brizard [2], Quin *et al* [33, 34], Kawamura & Fukuyama [22], Hahm [18], Hahm, Wang & Madsen [20], Grandgirard *et al.* [15, 16], and the review of Garbet *et al.* [11]).
Yet, the resulting Geometrical Gyro-Kinetic Approximation Theory remains a physical theory which is formal from the mathematical point of view and not directly affordable for mathematicians. The present paper is a first step towards providing a mathematical affordable theory, particularly for the analysis, the applied mathematics and computer sciences communities.

Notice that, beside this Geometrical Gyro-Kinetic Approximation Theory, an alternative approach, based on Asymptotic Analysis and Homogenization Methods was developed by Frénod & Sonnendrücker [7, 8, 9], Frénod, Raviart & Sonnendrücker [6], Golse & Saint-Raymond [14] and Ghendrih, Hauray & Nouri [13].

Summarizing, the Geometrical Gyro-Kinetic Approximation Theory consists in building a change of coordinates in order to make two components of a dynamical system disappear. The method to achieve this goal involves, in a intricate way, elements of Hamiltonian Dynamical System Theory, Symplectic Geometry, Non-Linear Hyperbolic PDE, Hilbert Theory of Operators and Asymptotic Analysis.

In order to clarify the purpose and put things in context, notice that the domain of validity of the Gyro-Kinetic Approximation Theory is that of charged particles under the action of a strong magnetic field. Hence, we begin by considering a non-relativistic charged particle moving in a static magnetic field. It is well known that, in usual coordinates $(\underline{\mathbf{x}}, \underline{\mathbf{v}}) = (x_1, x_2, x_3, v_1, v_2, v_3)$, where $\underline{\mathbf{x}}$ stands for the position variable and $\underline{\mathbf{v}}$ for the velocity variable, the position $\underline{\mathbf{X}}(t; \underline{\mathbf{x}}, \underline{\mathbf{v}})$ and the velocity $\underline{\mathbf{V}}(t; \underline{\mathbf{x}}, \underline{\mathbf{v}})$ at time t of the particle which is in $\underline{\mathbf{x}}$ with velocity $\underline{\mathbf{v}}$ at time $t = 0$ are solutions to

$$\frac{\partial \underline{\mathbf{X}}}{\partial t} = \underline{\mathbf{V}}, \qquad \underline{\mathbf{X}}(0) = \underline{\mathbf{x}}, \tag{1.1}$$

$$\frac{\partial \underline{\mathbf{V}}}{\partial t} = \frac{q}{m} \underline{\mathbf{V}} \times \mathbf{B}(\underline{\mathbf{X}}), \qquad \underline{\mathbf{V}}(0) = \underline{\mathbf{v}}. \tag{1.2}$$

In this dynamical system, $\underline{\mathbf{X}}$ has three components $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$, as $\underline{\mathbf{V}}$: $(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$.

For the purpose of this paper, we restrict to magnetic fields of the form

$$\mathbf{B}\left(\underline{\mathbf{x}}\right) = (0, 0, B(x_1, x_2)), \tag{1.3}$$

and furthermore, we consider only the projection of the particle motion onto the x_1, x_2-plan. So writing $\mathbf{x} = (x_1, x_2)$ and $\mathbf{v} = (v_1, v_2)$, the bi-dimensional dynamical system for $\mathbf{X}(t; \mathbf{x}, \mathbf{v})$ $(\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2))$ and $\mathbf{V}(t; \mathbf{x}, \mathbf{v})$ $(\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2))$ resulting from (1.1) and (1.2) reads

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}, \qquad \mathbf{X}(0) = \mathbf{x}, \tag{1.4}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{q}{m} B(\mathbf{X})\, {}^{\perp}\mathbf{V}, \qquad \mathbf{V}(0) = \mathbf{v}, \tag{1.5}$$

where, for any $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$, the notation ${}^{\perp}\mathbf{u}$ stands for $(u_2, -u_1)$. Throughout the rest of this paper, we will be interested in trajectory $(\mathbf{X}, \mathbf{V})$ and in dynamical system (1.4) and (1.5).
Finally, we suppose that the sign of B remains constant and that B is nowhere close to 0.

Now we make precise the context of strong magnetic field we work within. It is well known that the norm $|\mathbf{V}|$ of the velocity $\mathbf{V}$ solution to (1.5) is constant and that if the magnetic field B is uniform, the trajectory $\mathbf{X}$ of the considered charged particle is a circle. The radius a_L of the circle, which is called the Larmor radius in the context of Tokamak and Stellarator plasma, equals the norm of the velocity times the particle charge divided by the particle mass times the norm of the magnetic field, i.e.

$$a_L = \frac{|\mathbf{V}|}{\omega_c} \quad \text{where} \quad \omega_c = \frac{q|B|}{m}, \tag{1.6}$$

is the cyclotron frequency. The center of the circle, which is called the guiding center, equals

$$\mathbf{C} = \mathbf{X}(t) - \boldsymbol{\rho}(t), \tag{1.7}$$

where

$$\boldsymbol{\rho}(t) = -a_L \frac{{}^{\perp}\mathbf{V}(t)}{|\mathbf{V}|}. \tag{1.8}$$

Here, we do not consider a uniform B. Nevertheless, at any time t_0 when the position of the particle is $\mathbf{X}(t_0)$, we can consider the local values of the Larmor radius, the cyclotron frequency and the guiding center

$$a_L(t_0) = \frac{|\mathbf{V}|}{\omega_c(t_0)}, \tag{1.9}$$

$$\omega_c(t_0) = \frac{q|B(\mathbf{X}(t_0)|}{m}, \tag{1.10}$$

$$\mathbf{C}(t_0) = \mathbf{X}(t_0) - \boldsymbol{\rho}(t_0), \tag{1.11}$$

$$\boldsymbol{\rho}(t_0) = -a_L(t_0) \frac{{}^{\perp}\mathbf{V}(t_0)}{|\mathbf{V}|}. \tag{1.12}$$

With the help of those quantities, we can say that considering a small Larmor Radius consists in considering that the magnetic field applied to the particle position $\mathbf{X}(t_0)$ is close to the magnetic field applied to the guiding center position $\mathbf{C}(t_0)$. Now, as the magnetic field is strong, the local value of the period of rotation

$$T_c(t_0) = (2\pi/\omega_c(t_0)) \approx (2\pi m/q|B(\mathbf{C}(t_0)|), \tag{1.13}$$

is small. Considering that $a_L(t)$ and $T_c(t)$ have the same magnitude and that they are both small implies that $(a_L(t))^2\omega_c(t)$ is small and hence that

$$\frac{\partial \mathbf{C}}{\partial t}(t) = (a_L(t))^2\omega_c(t) \frac{{}^{\perp}\mathbf{V}(t)}{|\mathbf{V}|} \frac{\nabla_{\mathbf{x}} B(\mathbf{X}(t)) \cdot (\mathbf{V}(t)/|\mathbf{V}|)}{B(\mathbf{X}(t))}$$

is small. As

$$a_L(t) \approx \frac{m|\mathbf{V}|}{q|B(\mathbf{C}(t)|}, \tag{1.14}$$

the fact that $\mathbf{C}(t)$ vary slowly has for consequence that the Larmor Radius vary slowly. To summarize, for any time t belonging to a time interval of length of order $(2\pi m/q|B(\mathbf{C}(t_0)|)$

the values of $a_L(t)$ and of $\mathbf{C}(t)$ remain almost constant, so that the particle trajectory is close to a circle of radius $m\left|\mathbf{V}\right|/qB\left(\mathbf{C}\left(t_0\right)\right)$ traveled in a time close to $(2\pi m/q|B(\mathbf{C}\left(t_0\right)|)$.

To reexplain this within a rigorous modeling procedure, called scaling, and to introduce rigorously the small and large quantities, we need to define some characteristic scales: $\bar{t}$ stands for a characteristic time, $\bar{L}$ for a characteristic length and $\bar{v}$ for a characteristic velocity. We now define new variables t', $\mathbf{x}'$ and $\mathbf{v}'$ by $t = \bar{t}t'$, $\mathbf{x} = \bar{L}\mathbf{x}'$, and $\mathbf{v} = \bar{v}\mathbf{v}'$, making the characteristic scales the units. In the same way, we define the scaling factor $\bar{B}$ for the magnetic field:

$$\mathbf{B}\left(\bar{L}\mathbf{x}'\right) = \bar{B}\mathcal{B}\left(\mathbf{x}'\right). \tag{1.15}$$

Using those dimensionless variables and magnetic field, we obtain from dynamical system (1.4)–(1.5) that dimensionless trajectory $(\mathbf{X}', \mathbf{V}')$ defined by

$$\mathbf{X}(\bar{t}t'; \bar{L}\mathbf{x}', \bar{v}\mathbf{v}') = \bar{L}\mathbf{X}'(t'; \mathbf{x}', \mathbf{v}'), \tag{1.16}$$
$$\mathbf{V}(\bar{t}t'; \bar{L}\mathbf{x}', \bar{v}\mathbf{v}') = \bar{v}\mathbf{V}'(t'; \mathbf{x}', \mathbf{v}'), \tag{1.17}$$

is solution to

$$\frac{\bar{L}}{\bar{t}}\frac{\partial \mathbf{X}'}{\partial t'} = \bar{v}\mathbf{V}', \qquad \mathbf{X}'(0) = \mathbf{x}', \tag{1.18}$$
$$\frac{\bar{v}}{\bar{t}}\frac{\partial \mathbf{V}'}{\partial t'} = \frac{q\bar{B}\bar{v}}{m}\mathcal{B}\left(\mathbf{X}'\right)\ {}^{\perp}\mathbf{V}', \qquad \mathbf{V}'(0) = \mathbf{v}'. \tag{1.19}$$

We introduce the characteristic cyclotron frequency and Larmor radius:

$$\bar{\omega}_c = \frac{q\bar{B}}{m} \quad \text{and} \quad \bar{a}_L = \frac{\bar{v}}{\bar{\omega}_c}. \tag{1.20}$$

Using those physical characteristic quantities, system (1.18)–(1.19) becomes

$$\frac{\partial \mathbf{X}'}{\partial t'} = \bar{t}\bar{\omega}_c\frac{\bar{a}_L}{\bar{L}}\mathbf{V}', \qquad \mathbf{X}'(0) = \mathbf{x}', \tag{1.21}$$
$$\frac{\partial \mathbf{V}'}{\partial t'} = \bar{t}\bar{\omega}_c\mathcal{B}\left(\mathbf{X}'\right)\ {}^{\perp}\mathbf{V}', \qquad \mathbf{V}'(0) = \mathbf{v}'. \tag{1.22}$$

Once this dimensionless system written, saying that the magnetic field is strong means that the characteristic time $\bar{t}$ is large when compared to the characteristic period $\bar{T}_c = 2\pi/\bar{\omega}_c$. Introducing a small parameter ε, this can be translated into:

$$\bar{t}\bar{\omega}_c = \frac{1}{\varepsilon}. \tag{1.23}$$

Now, saying that the Larmor radius is small when compared to the characteristic scale length consists in considering that

$$\frac{\bar{a}_L}{\bar{L}} = \varepsilon, \tag{1.24}$$

Hence, the dimensionless dynamical system is rewritten as:

$$\frac{\partial \mathbf{X}'}{\partial t'} = \mathbf{V}', \qquad \mathbf{X}'(0) = \mathbf{x}', \tag{1.25}$$

$$\frac{\partial \mathbf{V}'}{\partial t'} = \frac{1}{\varepsilon}\mathcal{B}\left(\mathbf{X}'\right){}^{\perp}\mathbf{V}', \qquad \mathbf{V}'(0) = \mathbf{v}'. \tag{1.26}$$

Yet, we have enough material to precise, within this framework, our intuition that, with accuracy of the order of ε, for any time t' belonging to a time interval $[t'_0, t'_1]$ of length of the order ε, $\mathbf{X}'$ draws a circle of radius $\varepsilon\,|\mathbf{v}'|\,/\mathcal{B}\left(\mathbf{X}'\left(t'_0\right)\right)$ over a time $2\pi\varepsilon/\mathcal{B}(\mathbf{X}'\left(t'_0\right))$. We let,

$$\mathbf{X}'\left(t'\right) = \mathbf{C}'\left(t'\right) + \boldsymbol{\rho}'\left(t'\right), \tag{1.27}$$

$$\boldsymbol{\rho}'\left(t'\right) = -\frac{\varepsilon}{\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right)}{}^{\perp}\mathbf{V}'\left(t'\right), \tag{1.28}$$

$$\boldsymbol{\rho}'\left(t'\right) = \left|\boldsymbol{\rho}'\left(t'\right)\right|\left(\cos\left(\Theta'\left(t'\right)\right), -\sin\left(\Theta'\left(t'\right)\right)\right), \tag{1.29}$$

where $\Theta'\left(t'\right)$ is the angle between the x_1-axis and $\boldsymbol{\rho}'\left(t'\right)$ mesured in the clockwise sense.

Using the usual mobile orthonormal frame $\left(\hat{\mathbf{c}}(\theta), \hat{\mathbf{a}}(\theta)\right)$, where $\hat{\mathbf{c}}(\theta) = (-\sin(\theta), -\cos(\theta))$ and $\hat{\mathbf{a}}\left(\theta\right) = \left(\cos\left(\theta\right), -\sin\left(\theta\right)\right)$, equations (1.28) and (1.29) yield the expression of $\mathbf{V}'$ in this frame and the ODE it satisfies:

$$\mathbf{V}'\left(t'\right) = \left|\mathbf{V}'\right|\hat{\mathbf{c}}\left(\Theta'\left(t'\right)\right), \tag{1.30}$$

$$\frac{\partial \mathbf{V}'}{\partial t'} = \left|\mathbf{V}'\right|\frac{\partial}{\partial t'}\hat{\mathbf{c}}\left(\Theta'\left(t'\right)\right) = -\left|\mathbf{V}'\right|\frac{\partial \Theta'}{\partial t'}\left(t'\right)\hat{\mathbf{a}}\left(\Theta'\left(t'\right)\right). \tag{1.31}$$

Injecting (1.31) in (1.26) and using the fact that, in frame $\left(\hat{\mathbf{c}}\left(\theta\right), \hat{\mathbf{a}}\left(\theta\right)\right)$, ${}^{\perp}\mathbf{V}'\left(t'\right)$ writes:

$${}^{\perp}\mathbf{V}'\left(t'\right) = -\left|\mathbf{V}'\right|\hat{\mathbf{a}}\left(\Theta'\left(t'\right)\right),$$

we obtain the equation satisfied by $\Theta'\left(t'\right)$.

In the case of a constant magnetic field, we have:

$$\frac{\partial \mathbf{C}'}{\partial t'}\left(t'\right) = 0, \tag{1.32}$$

$$\frac{\partial}{\partial t'}\left|\boldsymbol{\rho}'\left(t'\right)\right| = 0, \tag{1.33}$$

$$\frac{\partial \Theta'}{\partial t'}\left(t'\right) = \frac{1}{\varepsilon}\mathcal{B}, \tag{1.34}$$

and the particle draws a circle of center $\mathbf{C}'$, of radius $|\rho'|$ over a time $2\pi\varepsilon/\mathcal{B}$.
When the magnetic field is not constant the ODEs satisfied by $\mathbf{C}'$, $|\rho'|$, and Θ' are

$$\frac{\partial \mathbf{C}'}{\partial t'}\left(t'\right) = -\varepsilon\,{}^{\perp}\mathbf{V}'\left(t\right)\frac{\nabla_{\mathbf{x}'}\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right)\cdot\mathbf{V}'\left(t'\right)}{\left(\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right)\right)^2}, \tag{1.35}$$

$$\frac{\partial}{\partial t'}\left|\boldsymbol{\rho}'\left(t'\right)\right| = \varepsilon\left|\mathbf{V}'\right|\frac{\nabla_{\mathbf{x}'}\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right)\cdot\mathbf{V}'\left(t'\right)}{\left(\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right)\right)^2}, \tag{1.36}$$

$$\frac{\partial \Theta'}{\partial t'}\left(t'\right) = \frac{1}{\varepsilon}\mathcal{B}\left(\mathbf{X}'\left(t'\right)\right) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon}\mathcal{B}\left(\mathbf{C}'\left(t'\right)\right), \tag{1.37}$$

Hence, for any time t' belonging to a time interval $[t'_0, t'_1]$ of length of the order of ε, the usual Taylor inequality leads to $|\mathcal{B}(\mathbf{C}'(t')) - \mathcal{B}(\mathbf{C}'(t'_0))| = O(\varepsilon^2)$ and hence

$$\frac{\partial \Theta'}{\partial t'}(t') \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon}\mathcal{B}(\mathbf{C}'(t'_0)) \text{ and } \Theta'\left(t' + \frac{2\pi\varepsilon}{\mathcal{B}(\mathbf{C}'(t'_0))}\right) - \Theta'(t') \text{ is close to } 2\pi. \tag{1.38}$$

Moreover, for any time t belonging to $[t'_0, t'_1]$, from (1.35)–(1.37) we obtain

$$\left|\mathbf{C}'(t') - \mathbf{C}'(t'_0)\right| = O(\varepsilon^2), \left||\boldsymbol{\rho}'(t')| - |\boldsymbol{\rho}'(t'_0)|\right| = O(\varepsilon^2). \tag{1.39}$$

Writing $\mathbf{X}'(t') = \mathbf{C}'(t') + \rho'(t')$, (1.39) and (1.38) say nothing but that $\mathbf{X}'$ draws a circle of radius $\varepsilon|\mathbf{v}'|/\mathcal{B}(\mathbf{C}'(t'_0))$ around $\mathbf{C}'(t'_0)$ over a time $2\pi\varepsilon/\mathcal{B}(\mathbf{C}'(t'_0))$ with accuracy of the order of ε.

The regime in which dynamical system (1.25)–(1.26) is written is called the drift-kinetic regime.

Subsequently, we will work exclusively with dimensionless variables. Since no confusion is possible, to simplify the notations, we will remove the ' and replace $\mathcal{B}$ by B. Hence, for $\mathbf{x}_0 = (x_{10}, x_{20}) \in \mathbb{R}^2$ and $\mathbf{v} = (v_{10}, v_{20}) \in \mathbb{R}^2$ we consider $\mathbf{X}(t, \mathbf{x}_0, \mathbf{v}_0) = (\mathbf{X}_1(t, \mathbf{x}_0, \mathbf{v}_0), \mathbf{X}_2(t, \mathbf{x}_0, \mathbf{v}_0))$ and $\mathbf{V}(t, \mathbf{x}_0, \mathbf{v}_0) = (\mathbf{V}_1(t, \mathbf{x}_0, \mathbf{v}_0), \mathbf{V}_2(t, \mathbf{x}_0, \mathbf{v}_0))$ solutions to the following dynamical system

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}, \qquad \mathbf{X}(0) = \mathbf{x}_0, \tag{1.40}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon} B(\mathbf{X})\,{}^\perp\mathbf{V}, \qquad \mathbf{V}(0) = \mathbf{v}_0, \tag{1.41}$$

and we make the following assumptions on B:

$$B(\mathbf{x}) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}. \tag{1.42}$$

with $\mathbf{A} = (A_1, A_2)$ an analytic function on $\mathbb{R}^2$ and

$$\inf_{\mathbf{x}\in\mathbb{R}^2} B(\mathbf{x}) > 1. \tag{1.43}$$

Remark 1.1. *In his remark, we will evoke issues related to the numerical simulation of system* (1.40)–(1.41)*. What we learn from* (1.27) *is that the solution of* (1.40)–(1.41) *is made of two parts: a strongly oscillating one, related to the strong oscillations of* Θ*, and, the guiding center motion, i.e. the motion of* $\mathbf{C}$*. Because of the oscillation part, direct numerical simulation of* (1.40)–(1.41) *requires a very small time step, and then is not considered. One could think that a good option to compute an approximate solution of* (1.40)–(1.41) *would be to compute* $\mathbf{C}$*. Yet, the resolution of EDO* (1.35) $\mathbf{C}$ *is solution to, requires the knowledge* *of trajectory* $\mathbf{X}$ *and consequently the resolution of* (1.40)-(1.41). *This option seems then to be a dead end. The interest of the Gyro-Kinetic Approximation is that it yields a dynamical system for something which is close to* $\mathbf{X}$ *and* $\mathbf{C}$*, but which can be solved as a stand-alone system.*

Now, we give a detailed summarize of what the present paper contains. In the geometric formalism, in any system of coordinates on a manifold $\mathcal{M}$, a Hamiltonian dynamical system, which solution is $\mathbf{R} = \mathbf{R}(t, \mathbf{r}_0)$ in the considered system of coordinates, can be written in the following form

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R})\nabla_{\mathbf{r}} H(\mathbf{R}), \quad \mathbf{R}(0, \mathbf{r}_0) = \mathbf{r}_0, \tag{1.44}$$

where $\mathcal{P}(\mathbf{r})$ is a skew-symmetric matrix called the matrix of the Poisson Bracket (or Poisson Matrix in short), and $H(\mathbf{r})$ is called the Hamiltonian function. Moreover, a sufficient condition for a dynamical system to be Hamiltonian is that the dynamical system writes in the form (1.44) in one system of coordinates which is global on $\mathcal{M}$. Roughly speaking, the goal of the Geometrical Gyro-Kinetic Theory is to make a succession of change of coordinates in order to satisfy the assumptions of the following theorem.

Theorem 1.2. *If, in a given coordinate system* $\mathbf{r} = (r_1, r_2, r_3, r_4)$*, the Poisson Matrix has the following form:*

$$\mathcal{P}(\mathbf{r}) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathtt{M}(\mathbf{r})$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0\ 0 & & 0 & 1 \\ 0\ 0 & & -1 & 0 \end{array} \right), \tag{1.45}$$

and if the Hamiltonian function does not depend on the penultimate variable, i.e.

$$\frac{\partial H}{\partial r_3} = 0, \tag{1.46}$$

then, submatrix $\mathtt{M}$ *does not depend on the two last variables, i.e.*

$$\frac{\partial \mathtt{M}}{\partial r_3} = 0 \textit{ and } \frac{\partial \mathtt{M}}{\partial r_4} = 0. \tag{1.47}$$

Consequently, the time-evolution of the two first components $\mathbf{R}_1, \mathbf{R}_2$ *is independent of the penultimate component* $\mathbf{R}_3$*; and, the last component* $\mathbf{R}_4$ *of the trajectory in not time-evolving, i.e.*

$$\frac{\partial \mathbf{R}_4}{\partial t} = 0. \tag{1.48}$$

Proof. When the Poisson Matrix has the form given by (1.45), the last line of (1.44) reads

$$\frac{\partial \mathbf{R}_4}{\partial t} = -\frac{\partial H}{\partial r_3}(\mathbf{R}).$$

Hence, if the Hamiltonian function does not depend on the penultimate variable, then, the last component $\mathbf{R}_4$ of the trajectory is not time-evolving. Now, introducing the Poisson Bracket of two functions $f \equiv f(\mathbf{r})$ and $g \equiv g(\mathbf{r})$ defined by

$$\{f, g\}_{\mathbf{r}}(\mathbf{r}) = [\nabla_{\mathbf{r}} f(\mathbf{r})]^T \mathcal{P}(\mathbf{r}) \nabla_{\mathbf{r}} g(\mathbf{r}), \tag{1.49}$$

where $\mathcal{P}(\mathbf{r})$ is the Poisson Matrix, we have

$$\mathcal{P}_{i,j} = \{\mathbf{r}_i, \mathbf{r}_j\}_{\mathbf{r}} \text{ for } i, j = 1, 2, 3, 4, \tag{1.50}$$

where $\mathbf{r}_i$ is the i-th coordinate function $\mathbf{r} \mapsto r_i$ and a direct computation leads to

$$\{\{\mathbf{r}_1, \mathbf{r}_2\}_{\mathbf{r}}, \mathbf{r}_3\}_{\mathbf{r}}(\mathbf{r}) = -\frac{\partial \mathcal{P}_{1,2}}{\partial r_4}(\mathbf{r}) \text{ and } \{\{\mathbf{r}_1, \mathbf{r}_2\}_{\mathbf{r}}, \mathbf{r}_4\}_{\mathbf{r}}(\mathbf{r}) = \frac{\partial \mathcal{P}_{1,2}}{\partial r_3}(\mathbf{r}). \tag{1.51}$$

Using the Jacobi identity saying that for any regular function f, g, h,

$$\{\{f, g\}_{\mathbf{r}}, h\}_{\mathbf{r}} + \{\{h, f\}_{\mathbf{r}}, g\}_{\mathbf{r}} + \{\{g, h\}_{\mathbf{r}}, f\}_{\mathbf{r}} = 0, \tag{1.52}$$

and the facts that $\mathcal{P}_{3,1} = \mathcal{P}_{2,3} = \mathcal{P}_{4,1} = \mathcal{P}_{2,4} = 0$, we obtain

$$\{\{\mathbf{r}_1, \mathbf{r}_2\}_{\mathbf{r}}, \mathbf{r}_3\}_{\mathbf{r}} = -\{\{\mathbf{r}_3, \mathbf{r}_1\}_{\mathbf{r}}, \mathbf{r}_2\}_{\mathbf{r}} - \{\{\mathbf{r}_2, \mathbf{r}_3\}_{\mathbf{r}}, \mathbf{r}_1\}_{\mathbf{r}} = 0, \tag{1.53}$$

$$\{\{\mathbf{r}_1, \mathbf{r}_2\}_{\mathbf{r}}, \mathbf{r}_4\}_{\mathbf{r}} = -\{\{\mathbf{r}_4, \mathbf{r}_1\}_{\mathbf{r}}, \mathbf{r}_2\}_{\mathbf{r}} - \{\{\mathbf{r}_2, \mathbf{r}_4\}_{\mathbf{r}}, \mathbf{r}_1\}_{\mathbf{r}} = 0. \tag{1.54}$$

and consequently, since $\mathcal{P}_{1,1} = \mathcal{P}_{2,2} = 0$, (1.51) brings (1.47), ending the proof of the theorem. □

Theorem 1.2 is the Key Result that brings the understanding of the Geometrical Gyro-Kinetic Theory. Skipping most of the details, we can say that the Gyro-Kinetic Approximation of dynamical system (1.40)–(1.41) consists in writing it within a system of coordinates that satisfies the assumptions of Theorem 1.2 and which is close to the Historic Guiding-Center Coordinates which is such that:

$$y_1^{hgc} = x_1 - \varepsilon \frac{v}{B(\mathbf{x})} \cos(\theta), \tag{1.55}$$

$$y_2^{hgc} = x_2 + \varepsilon \frac{v}{B(\mathbf{x})} \sin(\theta), \tag{1.56}$$

$$\theta^{hgc} = \theta, \tag{1.57}$$

$$k^{hgc} = \frac{v^2}{2B(\mathbf{x})}, \tag{1.58}$$

where $v = |\mathbf{v}|$ and where θ is such that (v, θ) is a Polar Coordinate System for the velocity variable and is defined precisely by formula (2.21).

Once this done, our attempts will be satisfied : the two first components of the resulting dynamical system will give raise to a good approximation of the evolution of the Guiding Center motion and hence of the evolution of the particle in the physical space, and, the dynamical system that needs to be solved will be easier. In particular, it will not involve the strong oscillations. Moreover, the last component of the solution will be constant and the evolution of the two firsts will not depend on the third component. As a consequence,

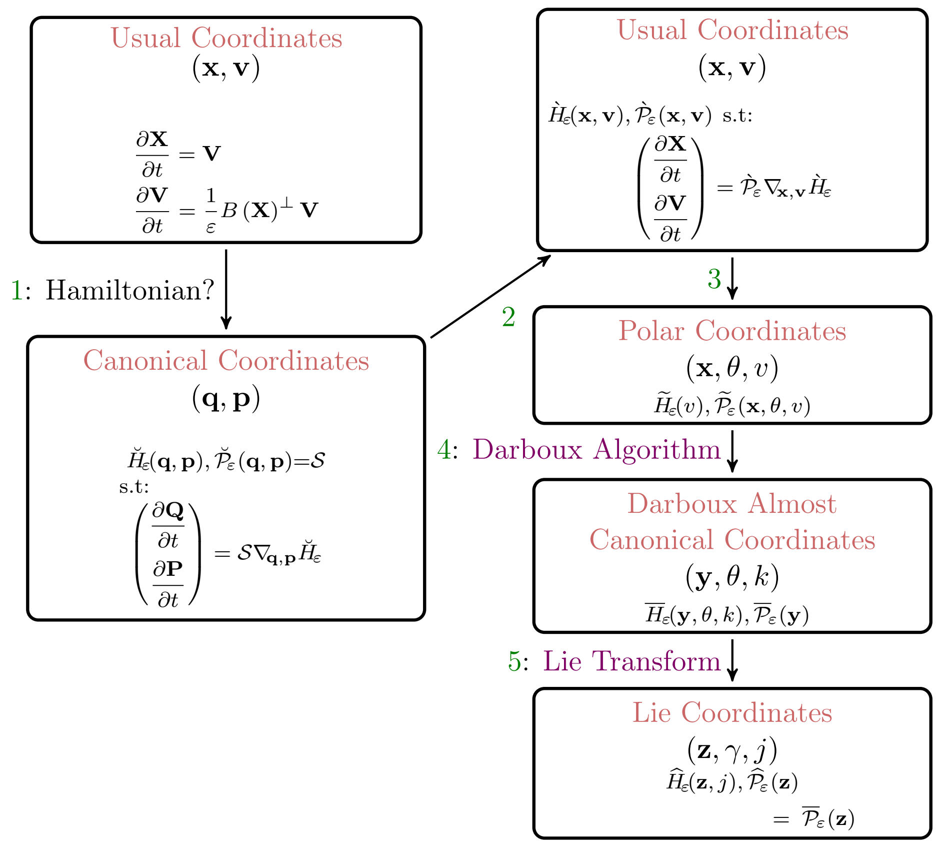

Figure 1: A schematic description of the method leading the Gyro-Kinetic Approximation.

if we are just interested in the motion of the particle in the physical space, i.e. just in the evolution of the two first components, solving the dynamical system in the new system of coordinates, reduces to find a trajectory in $\mathbb{R}^2$, in place of a trajectory in $\mathbb{R}^4$ when it is solved in the original system of coordinates.
To be more precise writing dynamical system (1.40)–(1.41) in a form satisfying the assumptions of Theorem 1.2 can only be done in a formal way using formal series expansion. Then, in fact, we will use a variant of Theorem 1.2 which result is the same, up to any order of ε. This variant is given in Theorem 4.1. From this variant, we can set dynamical system (1.40)– (1.41) in a form from which we can, for instance, prove the following theorem.

Theorem 1.3. *Assume that the magnetic field B satisfies assumptions* (1.42) *and* (1.43) *and that all its derivatives are bounded. Let N be a positive integer. Then, for any couple of open subsets $\left(\mathcal{O}\left(\mathbf{x}_0, R_{\mathbf{x}_0}; a, b\right), \mathcal{O}\left(\mathbf{x}_0, R'_{\mathbf{x}_0}; c, d\right)\right)$ (see formula* (1.69)*) such that $R'_{\mathbf{x}_0} > R_{\mathbf{x}_0}$ and $\left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right] \subset (c, d)$, there exists a positive real number η, such that for any $\varepsilon \in (-\eta, \eta)$, there exists a map $\mathcal{C}\!\mathcal{G}_\varepsilon$, one to one from $\mathcal{O}\left(\mathbf{x}_0, R'_{\mathbf{x}_0}; c, d\right)$ onto $\mathcal{C}\!\mathcal{G}_\varepsilon(\mathcal{O}\left(\mathbf{x}_0, R'_{\mathbf{x}_0}; c, d\right))$; a system of coordinates $(\mathbf{z}, \gamma, j) = \mathcal{C}\!\mathcal{G}_\varepsilon(\mathbf{x}, \mathbf{v})$ on $\mathcal{C}\!\mathcal{G}_\varepsilon(\mathcal{O}\left(\mathbf{x}_0, R'_{\mathbf{x}_0}; c, d\right))$; and for any $\varepsilon \in (0, \eta)$ a real t_e^ε, with $t_e^\varepsilon > \frac{\alpha_0}{\varepsilon}$, where α_0 is a positive real number depending only on $R'_{\mathbf{x}_0} - R_{\mathbf{x}_0}$ and B, such that the solution*

$$\left(\mathbf{Z}^T, \Gamma^T, \mathcal{J}^T\right) = \left(\mathbf{Z}^T(t, \mathbf{z}_0, \gamma_0, j_0), \Gamma^T(t, \mathbf{z}_0, \gamma_0, j_0), \mathcal{J}^T(t, \mathbf{z}_0, \gamma_0, j_0)\right) \tag{1.59}$$

of the following dynamical system, written within system of coordinates $(\mathbf{z}, \gamma, j)$,

$$\frac{\partial \mathbf{Z}^T}{\partial t} = -\frac{\varepsilon}{B\left(\mathbf{Z}^T\right)} \begin{pmatrix} \dfrac{\partial \hat{H}_{\varepsilon,T}}{\partial z_2}\left(\mathbf{Z}^T, j_0\right) \\ -\dfrac{\partial \hat{H}_{\varepsilon,T}}{\partial z_1}\left(\mathbf{Z}^T, j_0\right) \end{pmatrix}, \qquad \mathbf{Z}^T\left(0; \mathbf{z}_0, j_0\right) = \mathbf{z}_0, \tag{1.60}$$

$$\frac{\partial \Gamma^T}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial \hat{H}_{\varepsilon,T}}{\partial j}\left(\mathbf{Z}^T, j_0\right), \qquad \boldsymbol{\Gamma}^T\left(0; \mathbf{z}_0, j_0, \gamma_0\right) = \gamma_0, \tag{1.61}$$

$$\frac{\partial \mathcal{J}^T}{\partial t} = 0, \qquad \mathcal{J}^T\left(0; \mathbf{z}_0, j_0\right) = j_0, \tag{1.62}$$

where $H_{\varepsilon,T}$ is the function obtained by Algorithm 5.11 , satisfies

$$\left\|(\mathbf{Z}, \mathcal{J}) - (\mathbf{Z}^T, \mathcal{J}^T)\right\|_{\infty,\mathbf{init}} \leq C\varepsilon^{N-1} \tag{1.63}$$

for any $t \in (-t_e^\varepsilon, t_e^\varepsilon)$. In (1.63)*, $\|g\|_{\infty,\mathbf{init}}$ stands for*

$$\|g\|_{\infty,\mathbf{init}} = \sup_{(\mathbf{z}_0,\gamma_0,j_0)\in\mathcal{C}\!\mathcal{G}_\varepsilon\left(\mathcal{O}\left(\mathbf{x}_0,R_{\mathbf{x}_0};a,b\right)\right)} |g(\mathbf{z}_0, \gamma_0, j_0)|; \tag{1.64}$$

$$(\mathbf{Z}(t, \mathbf{z}_0, \gamma_0, j_0), \Gamma(t, \mathbf{z}_0, \gamma_0, j_0), \mathcal{J}(t, \mathbf{z}_0, \gamma_0, j_0)) = \mathcal{C}\!\mathcal{G}_\varepsilon\left(\mathbf{X}(t, \mathbf{x}_0, \mathbf{v}_0), \mathbf{V}(t, \mathbf{x}_0, \mathbf{v}_0)\right), \tag{1.65}$$

with $(\mathbf{X}, \mathbf{V})$ the solution of dynamical system (1.40)*–*(1.41) *and $(\mathbf{x}_0, \mathbf{v}_0) = \mathcal{C}\!\mathcal{G}_\varepsilon^{-1}(\mathbf{z}_0, \gamma_0, j_0)$; and, C only depends on B and $R'_{\mathbf{x}_0} - R_{\mathbf{x}_0}$ (and not on ε).*

The map $\mathscr{C}\!G_\varepsilon$ giving coordinate system $(\mathbf{z}, \gamma, j)$ *is a composition of several ones. A detailed summarize of these maps is given in the beginning of section 6 in Theorem 6.1.*

When $N = 2$*, system* (1.60)–(1.62) *reads*

$$\frac{\partial \mathbf{Z}^T}{\partial t} = -\frac{\varepsilon \mathcal{J}^T}{B\left(\mathbf{Z}^T\right)} \,{}^{\perp}\nabla B\left(\mathbf{Z}^T\right), \qquad \mathbf{Z}^T(0) = \mathbf{z}_0, \quad (1.66)$$

$$\frac{\partial \Gamma^T}{\partial t} = \frac{B\left(\mathbf{Z}^T\right)}{\varepsilon} + \varepsilon \frac{\mathcal{J}^T}{2B\left(\mathbf{Z}^T\right)^2}\Big(B\left(\mathbf{Z}^T\right)\nabla^2 B\left(\mathbf{Z}^T\right) - 3\left(\nabla B\left(\mathbf{Z}^T\right)\right)^2\Big), \qquad \Gamma^T(0) = \gamma_0, \quad (1.67)$$

$$\frac{\partial \mathcal{J}^T}{\partial t} = 0, \qquad \mathcal{J}^T(0) = j_0. \quad (1.68)$$

In this theorem and in the sequel, $\mathcal{O}\left(\mathbf{x}_0, R_{\mathbf{x}_0}; a, b\right)$ stands for the open subset defined by:

$$\mathcal{O}\left(\mathbf{x}_0, R_{\mathbf{x}_0}; a, b\right) = \mathfrak{b}^2(\mathbf{x}_0, R_{\mathbf{x}_0}) \times \mathfrak{C}\left(a; b\right), \quad (1.69)$$

where $\mathfrak{b}^2(\mathbf{x}_0, R_{\mathbf{x}_0}) \subset \mathbb{R}^2$ is the open Euclidian ball of radius $R_{\mathbf{x}_0}$ and of center $\mathbf{x}_0$, and $\mathfrak{C}\left(a; b\right)$ is the open crown of $\mathbb{R}^2$ defined by

$$\mathfrak{C}\left(a; b\right) = \left\{\mathbf{v} \in \mathbb{R}^2,\ |\mathbf{v}| \in (a, b)\right\}, \quad (1.70)$$

where $[a, b] \subset (0, +\infty)$.

If we are not interested in the evolution of Γ, since (1.62) is obviously solved, in this system of coordinates solving dynamical system (1.40)–(1.41) reduces to compute a bidimensionnal trajectory by solving (1.60).

Remark 1.4. *After having obtained the coordinate system given by Theorem 1.3, if, to make numerical simulations of the evolution of the particle, we compute* $\mathbf{Z}^T$ *in place of the particle trajectory* $\mathbf{X}$*, then the global error is given by:*

$$\textit{global error} \;=\; \textit{Series Truncation Error} \;+\; \textit{Numerical Error}\,. \quad (1.71)$$

The Series Truncation Error is made of two parts. As regard to the first part which is the error done when approximating $\mathbf{Z}$ *by* $\mathbf{Z}^T$*, inequality* (1.63) *of Theorem 1.3 claims that it can be pushed up to any order of* ε*. As regard to its second part which is related to the fact that the change of coordinates giving* $\mathbf{Z}$ *from* $\mathbf{X}$ *is not exactly known, the method for its construction allows also us to also push this expression to any order of* ε*. As a consequence, and because the numerical method (which is the one induced by the numerical scheme used to compute approximatted solution to* (1.60)*) can be as accurate as needed, our result claims that from the Geometrical Gyro-Kinetic Approximation numerical methods with any desired accuracy can be built to solve dynamical system* (1.40)–(1.41).

As illustrated in Figure 1, the method to build the desired change of coordinates is made of 5 steps. The first one consists in checking that dynamical system (1.40)–(1.41) is well Hamiltonian. This is symbolized by arrow 1 in the top of the figure. Once this is done, we can go back into the Usual Coordinate System but knowing that the dynamical system writes as in the square which is in the top-right of the figure, i.e., involving the

Poisson Matrix $\grave{\mathcal{P}}_\varepsilon$ and the gradient $\nabla_{\mathbf{x},\mathbf{v}}\grave{H}_\varepsilon$ of Hamiltonian function $\grave{H}_\varepsilon$. It may be written in that form in any coordinates system, and formulas give how to transform the Hamiltonian function and the Poisson Matrix while changing of coordinates. The goal of the third step is to introduce a Polar in velocity Coordinate System. In the fourth step, we make another change of coordinates, based on a Darboux Algorithm, in order to get the form of the Poisson Matrix allowing the application of the Key Result. This step consists essentially in solving a hyperbolic non-linear system of Partial Differential Equations (PDEs) for which a specific well adapted method of characteristics, in a spirit inspired by Abraham [1], is set out. We obtain a coordinate system close to the Historical Guiding-Center Coordinates of [24], [12], [28] and [29] involving the magnetic moment and the gyro-angle around the magnetic direction. In the fifth step, we make a last change of coordinates based on a Partial Lie Transform Method that we introduce in the present paper, leaving the form of the Poisson Matrix unchanged up to any desired order N of ε and bringing the Hamiltonian function independent of the last variable up to order $N+1$, allowing the proof of Theorem 1.3.

Remark 1.5. *The method in itself is complex and call upon concepts coming from several mathematical theories. Besides, as the method consists in building a succession of changes of coordinates, each of them being defined via a complex protocol, we have to check that what we build are well changes of coordinates, i.e. that they are well one to one, that they have the required regularity to be real changes of coordinates and that their inverse transformations have also the required regularity. Since moreover some of the domains on which those changes of coordinates are built are not straightforwardly comprehensible, those checks add a level of technical complexity.*

The motivation for introducing the Polar in velocity Coordinates $(\mathbf{x}, \theta, v)$ in the third step is not obvious. Indeed, whatever the system of coordinates chosen as point-of-departure, the resolution of the non-linear set of hyperbolic PDEs involved in the Darboux Algorithm is possible. Hence, we may wonder why taking this specific one. On another hand, the change of coordinates based on the Partial Lie Transform, which is a perturbation method, gives a system of coordinates close to the Historical Guiding Center-Coordinate System (1.55)–(1.58). Hence, we may also wonder if there is an a priori argument suggesting that method, starting from the Polar in velocity Coordinates and leading to a coordinate system close to the Historical Guiding-Center Coordinate System, generates a coordinate system satisfying the assumptions of Theorem 1.2. As a mater of fact, a few words are needed to attempt at providing some answers to those questions and, as a by-product, to explain this motivation.

Helping to provide some answers calls upon geometrical arguments put into the perspective of the regime under consideration. In a first place, the Darboux Algorithm needs to select a component of the coordinates which is left unchanged while doing the change of coordinates. This component plays the role of a pivot on which the change of coordinates leans to be defined.
If we look at what happens in the case of a constant magnetic field, we observe that the particle trajectory is a circle and obviously rotations of angle θ are its associated symmetries. Invoking the Noether's Theorem, which relates symmetries and invariants of the particle trajectory, yields that the magnetic moment, which is the product of the Larmor Radius by the norm of the velocity, is the invariant associated with those symmetries. Hence, a well adapted coordinate system to study the motion of a particle is such that its two first components

give the position of the center of the circle and its two lasts are related to the symmetries and the invariant linked by the Noether's Theorem. This coordinate system is exactly the Historical Guiding-Center Coordinate System. Notice that, in the mathematical literature, the coordinate related to the invariant is called "action" and the coordinate related to the symmetries is called "angle". Consequently, the best order 0 change of coordinate, before a method leading to a coordinate system close to the Historical Guiding-Center Coordinate System, is the one leading to the Polar in velocity Coordinates. With this point-of-departure, and with θ as pivot, the Darboux Algorithm gives the Historical Guiding-Center Coordinate System. Moreover the Hamiltonian function is independent of θ, bringing the conditions of application of the Key Result.

Going back to our concerns where the magnetic field $B = B(\mathbf{x})$ depends on the position, but under the assumptions of the drift-kinetic regime, in view of what we said while explaining this regime and in particular that the particle's trajectory is close to a circle, rotations of angles θ are close to symmetries, the Noether's Theorem allows us to hope that there exists a coordinate system close to the Historical Guiding-Center Coordinate System with a component related to an invariant close to the magnetic moment. Hence, the Polar in velocity Coordinate System is the best choice as point-of-departure. As we will see, with the Polar in velocity Coordinates System as point-of-departure, and with θ as pivot, the Darboux Algorithm gives a coordinate system close to the Historical Guiding-Center Coordinate System. This helps to provide answers to the two above questions: since the sought system of coordinates is, with accuracy of the order of ε, the Historical Guiding-Center Coordinates, and since the result of the Darboux algorithm is, with accuracy of the order of ε, the Historical Guiding-Center Coordinates, it can be obtained from the Darboux Algorithm result using a perturbation method like a Lie Transform Method.

Another remarkable fact is that in the Polar in velocity Coordinate System the Hamiltonian function does not depend on θ. As the Darboux algorithm gives a coordinate system close to the Historical Guiding-Center Coordinate System, and the Lie algorithm is close to the identity, this fact is indispensable. Indeed, this leads that the θ-dependency of the Hamiltonian function in the Darboux Coordinates System, appears only at the first order in ε. Consequently the Partial Lie Transform method has finally the ability to eliminate this θ-dependency up to any order N.

The context of the Gyro-Kinetic Approximation is Tokamak and Stellarator Plasma Physics. An artist vision of ITER, which is a Tokamak, is given in figure 1. The Vessel of a Tokamak is the interior of a torus with vertical axis of symmetry which, along the torus, electromagnets can generate a large magnetic field. At first sight, a Stellarator is similar but with a more complicated shape.

Clearly, dynamical system (1.25)–(1.26) is too simplified to be consider in a Tokamak or a Stellarator. Nevertheless, this model is simple enough to well understand the method and contains enough complexity to convince that it may be generalized to the genuine Dynamical System involved for particles in a Tokamak or a Stellarator.

The paper is organized as fallows: in section 2, we will construct a symplectic structure well adapted to the study of (1.40)–(1.41) and we will give the mathematical tools necessary

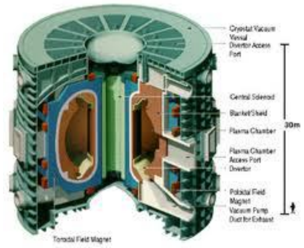

Figure 2: Artist vision of Iter.

for the comprehension of the Geometrical Gyro-Kinetic Theory we develop then. As a by-product of this section we obtain that the dynamical system we work with is a Hamiltonian one. In the third section, we will set out the method of characteristics and we will use it in order to derive the Darboux Coordinate System. In the fourth and fifth section, we will introduce the concept of Partial Lie Sum and develop the Partial Lie Transform Method in the case of a Poisson Matrix that depends on ε. This method is a mathematically rigorous version of the Lie Transform Method developed by Littlejohn [25, 26, 27]. In this new framework, we do not use formal Lie Series, we use Partial Lie Sums and we control the rests all along the method development. Then in section 6 we will derive the Gyro-Kinetic Coordinate System and prove Theorem 1.3.

2 Construction of the symplectic structure

We start by recalling the manifold that is used in order to study the motion of a charged particle. To avoid confusions, we will introduce the notation $\mathcal{M}$ for $\mathbb{R}^4$ endowed with its usual Cartesian coordinate system and with its usual Euclidian topology. As $\mathcal{M}$ is flat the first chart we choose is the global chart $(\mathcal{M}, \boldsymbol{\tau})$ where $\boldsymbol{\tau} : \mathcal{M} \to \mathbb{R}^4$; $\boldsymbol{\tau}(\mathbf{x}, \mathbf{v}) = (\mathbf{x}, \mathbf{v})$. Afterwards, we choose the maximal atlas containing this coordinate chart.

The first step of the construction of the symplectic structure consists in defining the Symplectic Two-Form. In the non-dimensionless case and within the general framework, the electromagnetic Lagrangian reads:

$$L\left(\mathbf{x}, \mathbf{v}, t\right) = \frac{\left|\mathbf{v}\right|^2}{2} - \frac{q}{m}\phi\left(\mathbf{x}, t\right) + \frac{q}{m}\mathbf{v}\cdot\mathbf{A}\left(\mathbf{x}, t\right), \tag{2.1}$$

where ϕ is the usual scalar potential and $\mathbf{A}$ is the usual vector potential. In the context of the present paper, where $\phi = 0$ and where the drift-kinetic regime is considered, the dimensionless electromagnetic Lagrangian reads

$$L_\varepsilon\left(\mathbf{x}, \mathbf{v}\right) = \frac{\left|\mathbf{v}\right|^2}{2} + \frac{1}{\varepsilon}\mathbf{v}\cdot\mathbf{A}\left(\mathbf{x}\right), \tag{2.2}$$

where $\mathbf{A}$ is given by (1.42). The Canonical Coordinates $(\mathbf{q}, \mathbf{p})$ are given by

$$\mathbf{q} = \mathbf{x} \text{ and } \mathbf{p} = \frac{\partial L_\varepsilon}{\partial \mathbf{v}}\left(\mathbf{x}, \mathbf{v}\right); \tag{2.3}$$

i.e. :

$$\mathbf{q} = \mathbf{x} \text{ and } \mathbf{p} = \mathbf{v} + \frac{1}{\varepsilon}\mathbf{A}(\mathbf{x}). \tag{2.4}$$

The Symplectic Two-Form $\mathbf{\Omega}_\varepsilon$ on $\mathcal{M}$ that is considered is the unique Two-Form which expression in the Canonical Coordinate chart is given by

$$\breve{\omega}_\varepsilon = d\mathbf{q} \wedge d\mathbf{p}. \tag{2.5}$$

Now, <u>*a*</u> Poisson Matrix $\mathcal{P}$ on an open subset is a skew-symmetric matrix satisfying:

$$\forall i, j, k \in \{1, \ldots, 4\}, \ \{\{\mathbf{r}_i, \mathbf{r}_j\}, \mathbf{r}_k\} + \{\{\mathbf{r}_k, \mathbf{r}_i\}, \mathbf{r}_j\} + \{\{\mathbf{r}_j, \mathbf{r}_k\}, \mathbf{r}_i\} = 0, \tag{2.6}$$

where for smooth functions f and g the Poisson Bracket $\{f, g\}$ is defined by (1.49), and in the case of a symplectic manifold, <u>*the*</u> Poisson Matrix in a given coordinate system is defined as follow: it is the inverse of the transpose of the matrix of the expression of the Symplectic Two-Form in this coordinate system. Notice that the Jacoby identities (2.6) are direct consequences of the closure of the Symplectic Two-Form.
For our purpose, in the Canonical Coordinates, the matrix associated with the Symplectic Two-Form is given by

$$\breve{\mathcal{K}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S} = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix}, \tag{2.7}$$

and then the Poisson Matrix is given by

$$\breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \left(\breve{\mathcal{K}}_\varepsilon(\mathbf{q}, \mathbf{p})\right)^{-T} = \mathcal{S} \tag{2.8}$$

We now turn to the change-of-coordinates rule for the Poisson Matrix. Firstly, if in a given coordinate chart $\overset{\star}{\mathbf{r}}$, the matrix associated with the Symplectic Two-Form $\mathbf{\Omega}_\varepsilon$ reads $\overset{\star}{\mathcal{K}}_\varepsilon\left(\overset{\star}{\mathbf{r}}\right)$, then, according to the previous definition, the Poisson Matrix is given by

$$\overset{\star}{\mathcal{P}}_\varepsilon\left(\overset{\star}{\mathbf{r}}\right) = \left(\overset{\star}{\mathcal{K}}_\varepsilon\left(\overset{\star}{\mathbf{r}}\right)\right)^{-T}. \tag{2.9}$$

If we make the change of coordinates $\sigma : \overset{\star}{\mathbf{r}} \mapsto \overset{\triangleright}{\mathbf{r}}$, then the usual change-of-coordinates rule for the expression of the Symplectic Two-Form leads to the following change of coordinates rule for the Poisson Matrix

$$\overset{\triangleright}{\mathcal{P}}_\varepsilon\left(\overset{\triangleright}{\mathbf{r}}\right) = \nabla_{\overset{\star}{\mathbf{r}}}\sigma\left(\sigma^{-1}\left(\overset{\triangleright}{\mathbf{r}}\right)\right) \overset{\star}{\mathcal{P}}_\varepsilon\left(\sigma^{-1}\left(\overset{\triangleright}{\mathbf{r}}\right)\right) \left[\nabla_{\overset{\star}{\mathbf{r}}}\sigma\left(\sigma^{-1}\left(\overset{\triangleright}{\mathbf{r}}\right)\right)\right]^T. \tag{2.10}$$

Using the Poisson Bracket defined in formula (1.49), the change-of-coordinates rule for the Poisson Matrix reads

$$\left(\overset{\triangleright}{\mathcal{P}}_\varepsilon\right)_{i,j}\left(\overset{\triangleright}{\mathbf{r}}\right) = \left\{\overset{\triangleright}{\mathbf{r}}_i, \overset{\triangleright}{\mathbf{r}}_j\right\}_{\overset{\star}{\mathbf{r}}}\left(\sigma^{-1}\left(\overset{\triangleright}{\mathbf{r}}\right)\right). \tag{2.11}$$

A Hamiltonian function is a smooth function on $\mathcal{M}$ and the Hamiltonian vector field associated with Hamiltonian function $\mathcal{G}$ is the unique vector field $\boldsymbol{\mathcal{X}}_{\mathcal{G}}^{\varepsilon}$ satisfying

$$i_{\boldsymbol{\mathcal{X}}_{\mathcal{G}}^{\varepsilon}} d\boldsymbol{\Omega}_{\varepsilon} = d\mathcal{G}, \tag{2.12}$$

where $i_{\boldsymbol{\mathcal{X}}_{\mathcal{G}}^{\varepsilon}} d\boldsymbol{\Omega}_{\varepsilon}$ is the interior product of differential two-form $d\boldsymbol{\Omega}_{\varepsilon}$ by vector field $\boldsymbol{\mathcal{X}}_{\mathcal{G}}^{\varepsilon}$. The expression of the Hamiltonian vector field associated with the Hamiltonian function $\mathcal{G}$, in the coordinate system $\overset{\star}{\mathbf{r}}$, is the vector field which reads:

$$\overset{\star}{\mathbf{X}}{}_{\overset{\star}{G}}^{\varepsilon}\left(\overset{\star}{\mathbf{r}}\right) = \overset{\star}{\mathcal{P}}_{\varepsilon}\left(\overset{\star}{\mathbf{r}}\right) \nabla_{\overset{\star}{\mathbf{r}}} \overset{\star}{G}\left(\overset{\star}{\mathbf{r}}\right), \tag{2.13}$$

where $\overset{\star}{G}$ is the representative of $\mathcal{G}$ in this coordinate system. In fact, we can consider Hamiltonian vector fields on $\mathcal{M}$, which requires that the Hamiltonian functions are smooth functions on $\mathcal{M}$, or just Hamiltonian vector fields on an open subset of $\mathcal{M}$, which requires that the Hamiltonian functions are defined on this open subset.

The Hamiltonian dynamical system associated with Hamiltonian function $\mathcal{G}$ on $\mathcal{M}$ is the dynamical system which reads

$$\frac{\partial \boldsymbol{\mathcal{R}}}{\partial t}(t) = \boldsymbol{\mathcal{X}}_{\mathcal{G}}^{\varepsilon}\left(\boldsymbol{\mathcal{R}}(t)\right), \tag{2.14}$$

or equivalently as said in the introduction, the dynamical system whose expression in every coordinate system $\overset{\star}{\mathbf{r}}$ is given by

$$\frac{\partial \overset{\star}{\mathbf{R}}}{\partial t} = \overset{\star}{\mathcal{P}}_{\varepsilon}\left(\overset{\star}{\mathbf{R}}\right) \nabla_{\overset{\star}{\mathbf{r}}} \overset{\star}{G}\left(\overset{\star}{\mathbf{R}}\right). \tag{2.15}$$

In particular, if we check that on a global coordinate chart, a dynamical system is Hamiltonian, then the dynamical system is Hamiltonian on $\mathcal{M}$ and its expression in every coordinate chart $\overset{\star}{\mathbf{r}}$ is given by (2.15).

Going back to dynamical system (1.40)–(1.41), we can easily check that in coordinate system $(\mathbf{q}, \mathbf{p})$ defined by (2.4) trajectory $(\mathbf{Q}, \mathbf{P})$ defined by

$$\mathbf{Q}(s) = \mathbf{X}(s) \text{ and } \mathbf{P}(s) = \mathbf{V}(s) + \frac{1}{\varepsilon}\mathbf{A}\left(\mathbf{X}(s)\right), \tag{2.16}$$

is solution to

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\, \nabla_{\mathbf{q},\mathbf{p}} \breve{H}\left(\mathbf{Q}, \mathbf{P}\right), \tag{2.17}$$

where $\mathcal{S}$ is defined by (2.7) and $\breve{H}(\mathbf{q}, \mathbf{p}) = \left|\mathbf{p} - \frac{1}{\varepsilon}\mathbf{A}(\mathbf{q})\right|^2$, insuring that it is Hamiltonian. And, using (2.10) or (2.11), we obtain the expression of the Poisson Matrix in the Cartesian Coordinate System:

$$\grave{\mathcal{P}}_{\varepsilon}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \frac{B(\mathbf{x})}{\varepsilon} \\ 0 & -1 & -\frac{B(\mathbf{x})}{\varepsilon} & 0 \end{pmatrix}, \tag{2.18}$$

and of the Hamiltonian function:

$$\grave{H}_\varepsilon(\mathbf{x},\mathbf{v}) = \frac{|\mathbf{v}|^2}{2}. \tag{2.19}$$

Since the above coordinate charts are both global, dynamical system (1.40)–(1.41) is Hamiltonian on $\mathcal{M}$. We will denote by $\mathcal{H}_\varepsilon:\ \mathcal{M}\to\mathbb{R}$ the Hamiltonian function on the manifold.

Now, we will perform the third step (see Figure 1) which consists in setting the dynamical system (1.40)-(1.41) in a Polar in velocity Coordinate System $(\mathbf{x},\theta,v=|\mathbf{v}|)$. To be consistent with the physical literature, we define θ as the angle between the x_1-axis and the gyro-radius vector $\boldsymbol{\rho}_\varepsilon(\mathbf{x},\mathbf{v}) = -\frac{\varepsilon}{B(\mathbf{x})}{}^\perp\mathbf{v}$, measured in a clockwise sense; i.e.

$$\boldsymbol{\rho}_\varepsilon(\mathbf{x},\mathbf{v}) = |\boldsymbol{\rho}_\varepsilon(\mathbf{x},\mathbf{v})| \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}. \tag{2.20}$$

The change of coordinates leading to the Polar in velocity Coordinate System is then given

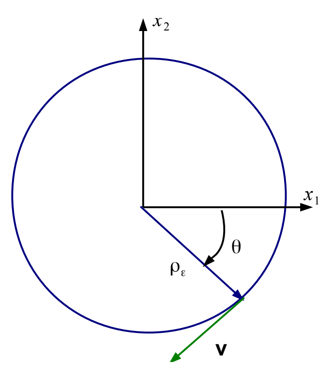

Figure 3: Polar coordinates for the velocity variable.

by:

$$\mathfrak{Pol}:\quad \begin{array}{ccc} \mathbb{R}^2\times\mathbb{R}^2 & \to & \mathbb{R}^2\times(\mathbb{R}/(2\pi\mathbb{Z}))\times(0,+\infty) \\ (\mathbf{x},\mathbf{v}) & \mapsto & (\mathbf{x},\theta,v) \end{array} \qquad \text{with } \mathbf{v} = v\begin{pmatrix} -\sin(\theta) \\ -\cos(\theta) \end{pmatrix}. \tag{2.21}$$

Using the change-of-coordinates formula we deduce from (2.19) that the expression of Hamiltonian function $\mathcal{H}_\varepsilon$ in the Polar in velocity Coordinate System is:

$$\tilde{H}_\varepsilon(\mathbf{x},\theta,v) = \frac{v^2}{2}, \tag{2.22}$$

and, using (2.11), the expression of the Poisson Matrix in this system is

$$\tilde{\mathcal{P}}_\varepsilon(\mathbf{x},\theta,v) = \begin{pmatrix} 0 & 0 & -\frac{\cos(\theta)}{v} & -\sin(\theta) \\ 0 & 0 & \frac{\sin(\theta)}{v} & -\cos(\theta) \\ \frac{\cos(\theta)}{v} & -\frac{\sin(\theta)}{v} & 0 & \frac{B(\mathbf{x})}{\varepsilon v} \\ \sin(\theta) & \cos(\theta) & -\frac{B(\mathbf{x})}{\varepsilon v} & 0 \end{pmatrix}. \tag{2.23}$$

According to formula (2.15), in this system, the characteristic $(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}, \Theta^{\varepsilon}, \mathcal{V}^{\varepsilon})(t; \mathbf{x}, \theta, v)$ laying on $\mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times (0, +\infty)$ and satisfying initial condition $(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}, \Theta^{\varepsilon}, \mathcal{V}^{\varepsilon})(0; \mathbf{x}, \theta, v) = (\mathbf{x}, \theta, v) \in \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times (0, +\infty)$ is defined by

$$\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}(t; \mathbf{x}, \theta, v) = \mathbf{X}^{\varepsilon}\left(t; \mathfrak{Pol}^{-1}(\mathbf{x}, \theta, v)\right), \tag{2.24}$$

$$\Theta^{\varepsilon}(t; \mathbf{x}, \theta, v) = \mathfrak{Pol}_3\left(\mathbf{X}^{\varepsilon}\left(t; \mathfrak{Pol}^{-1}(\mathbf{x}, \theta, v)\right), \mathbf{V}^{\varepsilon}\left(t; \mathfrak{Pol}^{-1}(\mathbf{x}, \theta, v)\right)\right), \tag{2.25}$$

$$\mathcal{V}^{\varepsilon}(t; \mathbf{x}, \theta, v) = \sqrt{\left(X^{\varepsilon}_1\left(t; \mathfrak{Pol}^{-1}(\mathbf{x}, \theta, v)\right)\right)^2 + \left(X^{\varepsilon}_2\left(t; \mathfrak{Pol}^{-1}(\mathbf{x}, \theta, v)\right)\right)^2}, \tag{2.26}$$

and is solution to

$$\frac{\partial X^{\varepsilon}_{\mathfrak{Pol},1}}{\partial t}(t; \mathbf{x}, \theta, v) = -\sin\left(\Theta^{\varepsilon}(t; \mathbf{x}, \theta, v)\right) \mathcal{V}^{\varepsilon}(t; \mathbf{x}, \theta, v), \tag{2.27}$$

$$\frac{\partial X^{\varepsilon}_{\mathfrak{Pol},2}}{\partial t}(t; \mathbf{x}, \theta, v) = -\cos\left(\Theta^{\varepsilon}(t; \mathbf{x}, \theta, v)\right) \mathcal{V}^{\varepsilon}(t; \mathbf{x}, \theta, v), \tag{2.28}$$

$$\frac{\partial \Theta^{\varepsilon}}{\partial t}(t; \mathbf{x}, \theta, v) = \frac{B\left(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}(t; \mathbf{x}, \theta, v)\right)}{\varepsilon}, \tag{2.29}$$

$$\frac{\partial \mathcal{V}^{\varepsilon}}{\partial t}(t; \mathbf{x}, \theta, v) = 0. \tag{2.30}$$

In particular, for any $t \in (0, +\infty)$ and for any $(\mathbf{x}, \theta, v)$, the characteristic $\mathcal{V}^{\varepsilon}$ satisfies $\mathcal{V}^{\varepsilon}(t; \mathbf{x}, \theta, v) = v$.

In the next subsection we will rather consider that the range of $\mathfrak{Pol}$ is $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ instead of $\mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times (0, +\infty)$. This is not a big issue, nevertheless, for purposes linked to what we will do in the fifth step, we need to define properly the periodic extensions of the characteristics expressed in the Polar in Velocity coordinate system as follows:

Definition 2.1. *Function $(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#}, \mathcal{V}^{\varepsilon,\#})$ ranging in $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, solution of dynamical system* (2.27)–(2.30) *and satisfying initial conditions*

$$\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(0, \mathbf{x}, \theta, v) = \mathbf{x}, \;\; \Theta^{\varepsilon,\#}(0, \mathbf{x}, \theta, v) = \theta, \;\; \mathcal{V}^{\varepsilon,\#}(0, \mathbf{x}, \theta, v) = v, \tag{2.31}$$

where $(\mathbf{x}, \theta, v) \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, is called the periodic extension of characteristic $(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}, \Theta^{\varepsilon}, \mathcal{V}^{\varepsilon})$.

Lemma 2.2. *Let $\mathfrak{p} : \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times (0, +\infty)$ be the canonical projection. Then, the periodic extension $(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#}, \mathcal{V}^{\varepsilon,\#})$ of $(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}, \Theta^{\varepsilon}, \mathcal{V}^{\varepsilon})$ satisfies*

$$\mathfrak{p} \circ \left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#}, \mathcal{V}^{\varepsilon,\#}\right)(t, \cdot) = \left(\mathbf{X}^{\varepsilon}_{\mathfrak{Pol}}, \Theta^{\varepsilon}, \mathcal{V}^{\varepsilon}\right)(t, \cdot) \circ \mathfrak{p}, \tag{2.32}$$

for any $t \in \mathbb{R}$.

Proof. Let $\left(\widetilde{\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}}(t; \mathbf{x}, \theta, v), \widetilde{\Theta^{\varepsilon,\#}}(t; \mathbf{x}, \theta, v), \widetilde{\mathcal{V}^{\varepsilon,\#}}(t; \mathbf{x}, \theta, v)\right)$ be the function defined by

$$\left(\widetilde{\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}}, \widetilde{\Theta^{\varepsilon,\#}}, \widetilde{\mathcal{V}^{\varepsilon,\#}}\right)(t; \mathbf{x}, \theta, v) = \left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#} - 2\pi, \mathcal{V}^{\varepsilon,\#}\right)(t; \mathbf{x}, \theta + 2\pi, v). \tag{2.33}$$

Then, $\left(\widetilde{\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}}, \widetilde{\Theta^{\varepsilon,\#}}, \widetilde{\mathcal{V}^{\varepsilon,\#}}\right)(t;\mathbf{x},\theta,v)$ satisfies (2.27)–(2.30) and satisfies

$$\left(\widetilde{\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}}, \widetilde{\Theta^{\varepsilon,\#}}, \widetilde{\mathcal{V}^{\varepsilon,\#}}\right)(0;\mathbf{x},\theta,v) = (\mathbf{x},\theta,v)\,. \tag{2.34}$$

Consequently, the Cauchy-Lipschitz Theorem yields that

$$\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#}, \mathcal{V}^{\varepsilon,\#}\right)(t;\mathbf{x},\theta+2\pi,v) = \left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}, \Theta^{\varepsilon,\#}+2\pi, \mathcal{V}^{\varepsilon,\#}\right)(t;\mathbf{x},\theta,v)\,. \tag{2.35}$$

This ends the proof of Lemma 2.2. □

Having this geometrical material on hand, we end this section by clarifying our comments about the Noether's Theorem done in the introduction. For this we first make precise the Noether's Theorem statement. Consider the following definitions:

Definition 2.3. *Let F be a Hamiltonian function on $\mathbb{R}^4$ and ψ_t be the flow associated with the Hamiltonian vector field of F. Let H be another Hamiltonian function, we say that ψ_t is a symplectic symmetry for H if*

$$\forall t\in\mathbb{R}, \forall \mathbf{r}\in\mathbb{R}^4,\;\; H(\mathbf{r}) = H(\psi_t(\mathbf{r}))\,. \tag{2.36}$$

Definition 2.4. *Let φ_t be the flow of the Hamiltonian system with Hamiltonian H. We say that a function G is an invariant of the Hamiltonian system with Hamiltonian H if*

$$\forall t\in\mathbb{R}, \forall \mathbf{r}\in\mathbb{R}^4,\;\; G(\mathbf{r}) = G(\varphi_t(\mathbf{r}))\,. \tag{2.37}$$

The Noether's Theorem links symmetry and invariants in the following way:

Theorem 2.5. *If the flow ψ_t associated with Hamiltonian function F is a symplectic symmetry of H, then F is an invariant of the Hamiltonian system of Hamiltonian H.*

Now, turning to dynamical system (1.40)–(1.41) assuming that the magnetic field is constant, if we make the Historical Guiding-Center change of coordinates; i.e. if we apply (1.55)– (1.58), using (2.11), we obtain the following expression of the Poisson Matrix in this Historical Guiding-Center Coordinate System

$$\bar{\mathcal{P}}_\varepsilon\left(\mathbf{y}^{hgc},\theta^{hgc},k^{hgc}\right) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B} & 0 & 0 \\ \frac{\varepsilon}{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix} \tag{2.38}$$

and the Hamiltonian function (2.22) becomes $\bar{H}_\varepsilon\left(\mathbf{y}^{hgc},\theta^{hgc},k^{hgc}\right) = k^{hgc}B$. The vector field $\frac{\partial}{\partial\theta^{hgc}}$ is clearly hamiltonian. Indeed, its related Hamiltonian function is εk^{hgc}. As $\bar{H}_\varepsilon$ does not depend on θ^{hgc}, the flow of $\frac{\partial}{\partial\theta^{hgc}}$ is a symplectic symmetry. The Noether's Theorem ensures us that εk^{hgc} is an invariant for the Hamiltonian system with Hamiltonian $\bar{H}_\varepsilon$. Notice also that this coordinate system satisfies the assumptions of Theorem 1.2. Now, if the magnetic field is not constant, the trajectory of a particle is close to a circle. Hence, the flow of $\frac{\partial}{\partial\theta^{hgc}}$ is close to a symmetry and εk^{hgc} defined by (1.58) is close to an invariant.

3 The Darboux algorithm

3.1 Objectives

The fourth step (see Figure 1) on the way to build the Gyro-Kinetic Approximation is the application of the mathematical algorithm, so called the Darboux Algorithm, to build a global Coordinate System (y_1, y_2, θ, k) close to the Historic Guiding-Center Coordinate System (1.55)–(1.58), and in which the Poisson Matrix has the required form (1.45) to apply the Key Result (Theorem 1.2). In fact, in order to manage the small parameter ε, we will build the Coordinate System (y_1, y_2, θ, k) in order to have $\bar{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k)$ with the following form:

$$\bar{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) = \left(\begin{array}{c|cc} \mathbb{M}_\varepsilon(\mathbf{y}) & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0\ 0\\0\ 0\end{matrix} & \begin{matrix}0\\-\frac{1}{\varepsilon}\end{matrix} & \begin{matrix}\frac{1}{\varepsilon}\\0\end{matrix} \end{array}\right), \tag{3.1}$$

An important and constitutive fact in the Darboux Algorithm is that the θ-variable is left unchanged.

We first introduce the following notations: we will denote by

$$\boldsymbol{\Upsilon}(\mathbf{x}, \theta, v) = \left(\boldsymbol{\Upsilon}_1(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_2(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_3(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_4(\mathbf{x}, \theta, v)\right), \tag{3.2}$$

the mapping, defined on a subset of $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, (which will be built) giving the coordinates $(\mathbf{y}, \theta, k)$ and by

$$\boldsymbol{\kappa} = \boldsymbol{\Upsilon}^{-1}, \tag{3.3}$$

its inverse map (which existence will be set out). Then, the change of coordinates write $(\mathbf{y}, \theta, k) = \boldsymbol{\Upsilon}(\mathbf{x}, \theta, v)$ and $(\mathbf{x}, \theta, v) = \boldsymbol{\kappa}(\mathbf{y}, \theta, k)$.

Applying formula (2.11), the matrix entries of (3.1) can be rewritten for $i = 1, \dots, 4$ and $j = 1, \dots, 4$ as

$$(\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k))_{ij} = \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\}_{\mathbf{x},\theta,v}(\boldsymbol{\kappa}(\mathbf{y}, \theta, k)), \quad \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\}_{\mathbf{x},\theta,v} = (\nabla \boldsymbol{\Upsilon}_i) \cdot (\widetilde{\mathcal{P}}_\varepsilon(\nabla \boldsymbol{\Upsilon}_j)), \tag{3.4}$$

with $\tilde{\mathcal{P}}_\varepsilon$ given by (2.23). Hence, the bottom-right of matrix form given in (3.1), results from:

$$\{\boldsymbol{\Upsilon}_4, \boldsymbol{\Upsilon}_3\} = -\frac{1}{\varepsilon}. \tag{3.5}$$

In the sequel, we will use the following notations:

$$\boldsymbol{\Upsilon}_1 = \boldsymbol{\Upsilon}_{y_1}, \ \boldsymbol{\Upsilon}_2 = \boldsymbol{\Upsilon}_{y_2}, \ \boldsymbol{\Upsilon}_3 = \boldsymbol{\Upsilon}_\theta \text{ and } \boldsymbol{\Upsilon}_4 = \boldsymbol{\Upsilon}_k. \tag{3.6}$$

Remark 3.1. *If we use those notations,* (3.5) *may be also read* $\{\boldsymbol{\Upsilon}_k, \boldsymbol{\Upsilon}_\theta\} = -\frac{1}{\varepsilon}$. *In papers of physicists, this last equation reads* $\{k, \theta\} = -\frac{1}{\varepsilon}$.

In the same way, the fact that the two last lines (or columns) contain only zeros results from:

$$\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_3\} = 0, \tag{3.7}$$

$$\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_4\} = 0, \tag{3.8}$$

$$\{\boldsymbol{\Upsilon}_2, \boldsymbol{\Upsilon}_3\} = 0, \tag{3.9}$$

$$\{\boldsymbol{\Upsilon}_2, \boldsymbol{\Upsilon}_4\} = 0. \tag{3.10}$$

Remark 3.2. *Using notations* (3.6)*,* (3.7)–(3.10) *may also read*

$$\begin{aligned} &\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_k\} = 0, (\mathit{or}\ \{y_1, k\} = 0), && \{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_\theta\} = 0, (\mathit{or}\ \{y_1, \theta\} = 0), \\ &\{\boldsymbol{\Upsilon}_{y_2}, \boldsymbol{\Upsilon}_k\} = 0, (\mathit{or}\ \{y_2, k\} = 0), && \{\boldsymbol{\Upsilon}_{y_2}, \boldsymbol{\Upsilon}_\theta\} = 0, (\mathit{or}\ \{y_2, \theta\} = 0). \end{aligned} \tag{3.11}$$

Equations (3.5) and (3.7)–(3.10) make a non linear hyperbolic system of PDEs that needs to be solved to build mapping $\boldsymbol{\Upsilon}$ and consequently the desired change of coordinates. In the perspective of the fifth step (see Figure 1), mapping $\boldsymbol{\Upsilon}$ needs to be close to the Historic Guiding-Center Coordinate System (1.55)–(1.58).

The non-linear nature of (3.5) and (3.7)–(3.10) is balanced by the fact that θ is left unchanged by $\boldsymbol{\Upsilon}$ and then that $\boldsymbol{\Upsilon}_3 = \boldsymbol{\Upsilon}_\theta$ is known. As a consequence, (3.5), (3.7) and (3.9) may be solved by an ad-hoc method of characteristics.

Besides, the choice of the boundary conditions, that need to be set to make system (3.5) and (3.7)–(3.10) to be well-posed, makes that (3.8) and (3.10) are consequences of (3.7) and (3.9). This choice is compatible with the constraint to get a system of coordinates close to the Historic Guiding-Center System which requires, among others, k to be close to the magnetic moment. As the magnetic moment is 0 when velocity v is 0, we choose to set the boundary conditions in $v = 0$, and for $\boldsymbol{\Upsilon}_k = \boldsymbol{\Upsilon}_4$ we set

$$\boldsymbol{\Upsilon}_k (\mathbf{x}, \theta, 0) = 0. \tag{3.12}$$

This choice to set the boundary condition in $v = 0$ generates a small difficulty because $\tilde{\mathcal{P}}_\varepsilon (\mathbf{x}, \theta, v)$ has a singularity in $v = 0$. To overcome this difficulty, we use that entry $(3, 4)$ of matrix $\tilde{\mathcal{P}}_\varepsilon$ satisfies:

$$(\tilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v))_{3,4} = (\tilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v))_{\theta,v} = \frac{B(\mathbf{x})}{\varepsilon v} > 0, \tag{3.13}$$

for all $(\mathbf{x}, \theta, v)$ belonging to $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ because of (1.43). We will denote:

$$\frac{B(\mathbf{x})}{\varepsilon v} = \omega_\varepsilon(\mathbf{x}, v). \tag{3.14}$$

That allows us to introduce the matrix $\tilde{\mathcal{Q}}_\varepsilon (\mathbf{x}, \theta, v)$ defined by

$$\tilde{\mathcal{P}}_\varepsilon (\mathbf{x}, \theta, v) = \omega_\varepsilon (\mathbf{x}, v)\, \tilde{\mathcal{Q}}_\varepsilon (\mathbf{x}, \theta, v) \tag{3.15}$$

Using this matrix, (3.5) and (3.7)–(3.10) are equivalent, for $v \neq 0$, to equations involving $\tilde{\mathcal{Q}}_\varepsilon$:

$$(\nabla \boldsymbol{\Upsilon}_k) \cdot (\tilde{\mathcal{Q}}_\varepsilon(\nabla \boldsymbol{\Upsilon}_\theta)) = -\frac{v}{B\,(\mathbf{x})}, \tag{3.16}$$

and

$$(\nabla\boldsymbol{\Upsilon}_{y_1})\cdot(\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_\theta))=0, \tag{3.17}$$
$$(\nabla\boldsymbol{\Upsilon}_{y_1})\cdot(\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_k))=0, \tag{3.18}$$
$$(\nabla\boldsymbol{\Upsilon}_{y_2})\cdot(\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_\theta))=0, \tag{3.19}$$
$$(\nabla\boldsymbol{\Upsilon}_{y_2})\cdot(\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_k))=0. \tag{3.20}$$

that have no singularity in $v=0$. Consequently in place of solving (3.5) and (3.7)–(3.10) we will solve (3.16)–(3.20) which is provided with boundary condition (3.12) and, with regard to (3.17)–(3.20), with:

$$\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,0)=x_1, \tag{3.21}$$
$$\boldsymbol{\Upsilon}_{y_2}(\mathbf{x},\theta,0)=x_2. \tag{3.22}$$

Obviously in these PDEs the variable v belongs to $(0,+\infty)$. Nevertheless, in subsections 3.2 to 3.5 we will solve PDEs (3.17)–(3.20) on $\mathbb{R}^4$. Afterwards, in subsection 3.6, we will take the restriction of $\boldsymbol{\Upsilon}$ to $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ and we will show that this restriction is a diffeomorphisms from $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ onto itself.

3.2 First equation processing

In this subsection we deduce $\boldsymbol{\Upsilon}_k$ from equation (3.16). Since the θ-variable is left unchanged by $\boldsymbol{\Upsilon}$,

$$\nabla\boldsymbol{\Upsilon}_\theta=(0,0,1,0)^T, \tag{3.23}$$

we deduce $\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_\theta)$ is the penultimate column of $\tilde{\mathcal{Q}}_\varepsilon$. Hence, in view of (3.16), (3.15) and (2.23), we have the following lemma

Lemma 3.3. *The last component $\boldsymbol{\Upsilon}_4=\boldsymbol{\Upsilon}_k$ of mapping $\boldsymbol{\Upsilon}$ that gives the Darboux Coordinates in terms of the Polar in velocity Coordinates is the unique solution to*

$$-\varepsilon\frac{\cos(\theta)}{B(\mathbf{x})}\frac{\partial\boldsymbol{\Upsilon}_k}{\partial x_1}+\varepsilon\frac{\sin(\theta)}{B(\mathbf{x})}\frac{\partial\boldsymbol{\Upsilon}_k}{\partial x_2}-\frac{\partial\boldsymbol{\Upsilon}_k}{\partial v}=-\frac{v}{B(\mathbf{x})}, \tag{3.24}$$
$$\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,0)=0. \tag{3.25}$$

Moreover, when $\varepsilon=0$, $\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v)=v^2/(2B(\mathbf{x}))$ is the rescaled magnetic moment associated with the trajectory solution to (1.40)–(1.41).

Proof. As already said the boundary value problem $\boldsymbol{\Upsilon}_k$ is solution to is a consequence of (3.16) and of the choice concerning the boundary condition. Problem (3.24)–(3.25) is clearly well-posed as a consequence of the linearity of (3.24) and of the regularity of its coefficients. Uniqueness of the solution is obvious. The remark concerning the case when $\varepsilon=0$ is easily obtained by solving (3.24)–(3.25) with $\varepsilon=0$. □

We now turn to the resolution of (3.24)–(3.25). And, as already mentioned, we will solve this equation on $\mathbb{R}^4$. Defining vector field $\mathbf{\Lambda}$ of $\mathbb{R}^3$ as

$$\mathbf{\Lambda}(\mathbf{x},\theta) = \frac{\cos(\theta)}{B(\mathbf{x})}\frac{\partial}{\partial x_1} - \frac{\sin(\theta)}{B(\mathbf{x})}\frac{\partial}{\partial x_2}, \tag{3.26}$$

$\mathbf{\Lambda}^n\cdot$ its iterated application acting on regular functions f as

$$\mathbf{\Lambda}^0\cdot f = f, \qquad \mathbf{\Lambda}^1\cdot f = \frac{\cos(\theta)}{B(\mathbf{x})}\frac{\partial f}{\partial x_1} - \frac{\sin(\theta)}{B(\mathbf{x})}\frac{\partial f}{\partial x_2}, \tag{3.27}$$

$$\mathbf{\Lambda}^n\cdot f = \mathbf{\Lambda}\cdot\left(\mathbf{\Lambda}^{n-1}\cdot f\right) \qquad \forall n\geq 2. \tag{3.28}$$

and $\mathcal{G}_\lambda = \mathcal{G}_\lambda(\mathbf{x},\theta)$ its flow, we obtain the following expansion of $\mathbf{\Upsilon}_k$.

Theorem 3.4. *Under assumptions* (1.42) *and* (1.43)*, for any $n\geq 0$, for any $\varepsilon\in\mathbb{R}$, and for any $(\mathbf{x},\theta,v)\in\mathbb{R}^4$,*

$$\begin{aligned}\mathbf{\Upsilon}_k(\mathbf{x},\theta,v) = &\sum_{l=0}^{n}\frac{(-\varepsilon)^l v^{l+2}}{(l+2)!}\left(\mathbf{\Lambda}^l\cdot\frac{1}{B}\right)(\mathbf{x},\theta)\\ &+\frac{(-\varepsilon)^{n+1}}{(n+2)!}\int_0^v (v-u)^{n+2}\left(\mathbf{\Lambda}^{n+1}\cdot\frac{1}{B}\right)\circ\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\,du,\end{aligned} \tag{3.29}$$

with $\mathbf{\Lambda}^n\cdot$ defined by (3.28)*. Moreover, for any $l\in\mathbb{N}$, $\left(\mathbf{\Lambda}^l\cdot\frac{1}{B}\right)$ is in $\mathcal{C}^\infty_\#\left(\mathbb{R}^4\right)$; for any $n\in\mathbb{N}$, $(\varepsilon,\mathbf{x},\theta,v)\mapsto\int_0^v(v-u)^{n+2}\left(\mathbf{\Lambda}^{n+1}\cdot\frac{1}{B}\right)\circ\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\,du$ is in $\mathcal{C}^\infty_\#\left(\mathbb{R}^5\right)$; for any $l\in\mathbb{N}$, $\left(\mathbf{\Lambda}^l\cdot\frac{1}{B}\right)$ is in $\mathcal{C}^\infty_b\left(\mathbb{R}^3\right)$; and, for any $v\in\mathbb{R}$ and for any $n\in\mathbb{N}$, $(\varepsilon,\mathbf{x},\theta)\mapsto\int_0^v(v-u)^{n+2}\left(\mathbf{\Lambda}^{n+1}\cdot\frac{1}{B}\right)\circ\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\,du$ is bounded by $C_n^{\mathbf{\Upsilon}_k}(v) = \frac{|v|^{n+3}}{n+3}\left\|\mathbf{\Lambda}^{n+1}\cdot\frac{1}{B}\right\|_\infty$.*

Here and hereafter, $\mathcal{C}^\infty_b(\mathbb{R}^m)$ (where $m\in\mathbb{N}$) stands of the space of functions being in $\mathcal{C}^\infty(\mathbb{R}^m)$ and with their derivatives at any order which are bounded. For a 2π-periodic set $I^\#$ included in $\mathbb{R}$, $\mathcal{C}^\infty_{\text{per}}\big(I^\#\big)$ stands of the space of functions being in $\mathcal{C}^\infty(I^\#)$ and 2π-periodic. For a set $\mathfrak{M}^\#$ included in $\mathbb{R}^m$ (where $m\in\mathbb{N}$ and $m\geq 2$) which is 2π-periodic with respect to the l-th variable ($l\leq m$); meaning that there exists a set $\mathfrak{I}^\flat\subset\{(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)\in\mathbb{R}^{m-1}\}$ and, for any $(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)\in\mathfrak{I}^\flat$, another 2π-periodic set $\mathfrak{I}^\sharp(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)$ included in $\mathbb{R}$, such that $\mathfrak{M}^\# = \{\mathbf{r},(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)\in\mathfrak{I}^\flat, r_l\in\mathfrak{I}^\sharp(r_1,\ldots,r_{l-1},\, r_{l+1},\ldots,r_m)\}$; we denote

$$\begin{aligned}\mathcal{C}^\infty_{\#,l}\left(\mathfrak{M}^\#\right) = \Big\{f\in\mathcal{C}^\infty\left(\mathfrak{M}^\#\right)\ &\text{such that } r_l\mapsto f(\mathbf{r})\\ &\in\mathcal{C}^\infty_{per}(\mathfrak{I}^\sharp(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)),\ \forall(r_1,\ldots,r_{l-1},r_{l+1},\ldots,r_m)\in\mathfrak{I}^\flat\Big\}.\end{aligned} \tag{3.30}$$

Since the case when the variable with respect to which periodicity occurs is the penultimate happens very often in the following, we define

$$\mathcal{C}^\infty_\#\left(\mathfrak{M}^\#\right) = \mathcal{C}^\infty_{\#,(m-1)}\left(\mathfrak{M}^\#\right) \tag{3.31}$$

The proof of Theorem 3.4 is based on a method of characteristics that we will develop now.

3.3 The method of Characteristics

In this subsection we will set out a method of characteristics which brings the capability of building solution to a PDE, related to (3.24), (3.17) and (3.19), on which the resolution of (3.24), (3.17) and (3.19) themselves will be based in the next subsection. This method of characteristics is in the same spirit as the method developed in Abraham [1, page 233].
In a first place, we give the regularity property of the flow $\mathcal{G}_\lambda$ of vector field $\mathbf{\Lambda}$ defined by (3.26).

Lemma 3.5. *Flow $\mathcal{G}_\lambda = \mathcal{G}_\lambda(\mathbf{x},\theta)$ of vector field $\mathbf{\Lambda}$ defined by (3.26) is complete, in $\mathcal{C}^\infty(\mathbb{R}^3)$, $\left(\mathcal{G}^1_\lambda, \mathcal{G}^2_\lambda\right)$ is in $\mathcal{C}^\infty_{\#,3}(\mathbb{R}^3)$ and $\mathcal{G}^3_\lambda(\mathbf{x},\theta) = \theta$.*

Proof. The fact that $\mathcal{G}_\lambda$ is complete is obvious. The regularity of $\mathcal{G}_\lambda$ comes from the assumed regularity of B (see (1.42) and (1.43)). Integrating $\frac{\partial \mathcal{G}^3_\lambda}{\partial \lambda}(\mathbf{x},\theta) = 0$ with initial condition $\mathcal{G}^3_0(\mathbf{x},\theta) = \theta$ yields $\mathcal{G}^3_\lambda(\mathbf{x},\theta) = \theta$. Finally, let $\tilde{\mathcal{G}}_\lambda$ be defined by $\tilde{\mathcal{G}}_\lambda(\mathbf{x},\theta) = \mathcal{G}_\lambda(\mathbf{x},\theta+2\pi)$. Then, the two first components of $\mathcal{G}_\lambda$ and $\tilde{\mathcal{G}}_\lambda$ satisfy the same dynamical system. Furthermore, since

$$\left(\tilde{\mathcal{G}}^1_0(\mathbf{x},\theta), \tilde{\mathcal{G}}^2_0(\mathbf{x},\theta)\right) = \mathbf{x} = \left(\mathcal{G}^1_0(\mathbf{x},\theta), \mathcal{G}^2_0(\mathbf{x},\theta)\right),$$

the Cauchy-Lipschitz Theorem allows us to conclude that

$$\left(\tilde{\mathcal{G}}^1_\lambda(\mathbf{x},\theta), \tilde{\mathcal{G}}^2_\lambda(\mathbf{x},\theta)\right) = \left(\mathcal{G}^1_\lambda(\mathbf{x},\theta), \mathcal{G}^2_\lambda(\mathbf{x},\theta)\right)$$

and consequently that the two first components of $\mathcal{G}_\lambda$ are 2π-periodic. □

Secondly, we define vector field $\boldsymbol{\xi}$ on $\mathbb{R}^5$ by:

$$\boldsymbol{\xi}\left(\mathbf{x},\theta,v,x_5\right) = -\varepsilon\mathbf{\Lambda}\left(\mathbf{x},\theta\right) - \frac{\partial}{\partial v}, \tag{3.32}$$

with $\mathbf{\Lambda}$ given by (3.26), and its flow $\mathcal{F}_\lambda \equiv \mathcal{F}_\lambda\left(\mathbf{x},\theta,v,x_5\right)$ which is such that

$$\frac{d\mathcal{F}_\lambda\left(\mathbf{x},\theta,v,x_5\right)}{d\lambda} = \boldsymbol{\xi}\left(\mathcal{F}_\lambda\left(\mathbf{x},\theta,v,x_5\right)\right), \tag{3.33}$$

$$\mathcal{F}_0\left(\mathbf{x},\theta,v,x_5\right) = \left(\mathbf{x},\theta,v,x_5\right). \tag{3.34}$$

Since the function

$$\left(\mathbf{x},\theta,v,x_5\right) \mapsto \left(-\varepsilon\frac{\cos\left(\theta\right)}{B\left(\mathbf{x}\right)}, \varepsilon\frac{\sin\left(\theta\right)}{B\left(\mathbf{x}\right)}, 0, -1, 0\right),$$

is Lipschitz continuous on $\mathbb{R}^5$, the flow is complete and we can consider the manifolds

$$\Gamma = \left\{\left(\mathbf{x},\theta,v,x_5\right) \in \mathbb{R}^5,\ v=0,\ x_5 = \frac{1}{B\left(\mathbf{x}\right)}\right\},$$

and $P = \underset{\lambda\in\mathbb{R}}{\cup} \mathcal{F}_\lambda\left(\Gamma\right)$.
The following lemma holds true.

Lemma 3.6. *If there exists a function $\varphi \equiv \varphi(\mathbf{x}, \theta, v)$, from $\mathbb{R}^4$ to $\mathbb{R}$, which is in $\mathcal{C}^1(\mathbb{R}^4)$ such that P writes*

$$P = \{(\mathbf{x}, \theta, v, x_5),\ x_5 = \varphi(\mathbf{x}, \theta, v)\}, \tag{3.35}$$

then φ is solution to the following PDE

$$-\varepsilon \mathbf{\Lambda}^1 \cdot \varphi - \frac{\partial \varphi}{\partial v} = 0, \tag{3.36}$$

$$\varphi(\mathbf{x}, \theta, 0) = \frac{1}{B(\mathbf{x})}. \tag{3.37}$$

Proof. By construction, $\boldsymbol{\xi}$ is a vector field tangent to manifold P. On another hand, if P writes as in formula (3.35), then the vector field

$$\mathbf{n}(\mathbf{x}, \theta, v) = \frac{\partial \varphi}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial \varphi}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial \varphi}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial v}\frac{\partial}{\partial v} - \frac{\partial}{\partial x_5} \tag{3.38}$$

is orthogonal to P in every point of P. Then, using $\boldsymbol{\xi}(\mathbf{x}, \theta, v, x_5) \cdot \mathbf{n}(\mathbf{x}, \theta, v) = 0$ yields (3.36). Moreover the boundary condition of (3.37) is obviously satisfied by φ. □

Using this lemma we obtain the following theorem.

Theorem 3.7. *The unique solution φ to* (3.36)–(3.37) *is given by*

$$\varphi(\mathbf{x}, \theta, v) = \frac{1}{B\left(\mathcal{G}^1_{-\varepsilon v}(\mathbf{x}, \theta), \mathcal{G}^2_{-\varepsilon v}(\mathbf{x}, \theta)\right)}, \tag{3.39}$$

where $\mathcal{G}_\lambda = \mathcal{G}_\lambda(\mathbf{x}, \theta)$ is the flow of $\mathbf{\Lambda}$. Moreover, φ is in $\mathcal{C}^\infty_\#(\mathbb{R}^4)$ and is bounded (see (3.31) *for notation).*

Proof. By definition, the flow $\mathcal{G}_\lambda$ of $\mathbf{\Lambda}$ is solution to:

$$\begin{aligned} &\frac{d\mathcal{G}_\lambda(\mathbf{x}, \theta)}{d\lambda} = \mathbf{\Lambda}(\mathcal{G}_\lambda(\mathbf{x}, \theta)), \\ &\mathcal{G}_0(\mathbf{x}, \theta) = (\mathbf{x}, \theta). \end{aligned} \tag{3.40}$$

As

$$\left[\frac{d\mathcal{G}}{d\lambda}\right]_{-\varepsilon\lambda}(\mathbf{x}, \theta) = -\varepsilon \frac{d(\mathcal{G}_{-\varepsilon\lambda})}{d\lambda}(\mathbf{x}, \theta) = -\varepsilon \mathbf{\Lambda}(\mathcal{G}_{-\varepsilon\lambda}(\mathbf{x}, \theta)), \tag{3.41}$$

we deduce that the flow of $\boldsymbol{\xi}$ writes:

$$\mathcal{F}_\lambda(\mathbf{x}, \theta, v, x_5) = (\mathcal{G}_{-\varepsilon\lambda}(\mathbf{x}, \theta), -\lambda + v, x_5). \tag{3.42}$$

Now, using the following parametric representation of Γ:

$$\Gamma = \left\{x_1 = t_1,\ x_2 = t_2,\ \theta = t_3,\ v = 0,\ x_5 = \frac{1}{B(t_1, t_2)};\ (t_1, t_2, t_3) \in \mathbb{R}^3\right\}, \tag{3.43}$$

for any $\mathbf{z}$ $(= (z_1, z_2, z_3, z_4, z_5)) \in P$, there exists $\lambda \in \mathbb{R}$ and $\mathbf{m}$ $\left(= \left(t_1, t_2, t_3, 0, \frac{1}{B(t_1,t_2)}\right)\right) \in \Gamma$ such that

$$\mathbf{z} = \mathcal{F}_\lambda (\mathbf{m}) \text{ or } \mathbf{m} = \mathcal{F}_{-\lambda} (\mathbf{z}) . \tag{3.44}$$

This last equality reads also

$$(t_1, t_2, t_3) = \mathcal{G}_{\varepsilon\lambda} (z_1, z_2, z_3) , \tag{3.45}$$

$$\lambda + z_4 = 0, \tag{3.46}$$

$$z_5 = \frac{1}{B (t_1, t_2)}. \tag{3.47}$$

Hence,

$$\lambda = -z_4, \tag{3.48}$$

$$(t_1, t_2) = \left(\mathcal{G}^1_{-\varepsilon z_4} (z_1, z_2, z_3) , \mathcal{G}^2_{-\varepsilon z_4} (z_1, z_2, z_3)\right) , \tag{3.49}$$

$$z_5 = \frac{1}{B \left(\mathcal{G}^1_{-\varepsilon z_4} (z_1, z_2, z_3) , \mathcal{G}^2_{-\varepsilon z_4} (z_1, z_2, z_3)\right)}. \tag{3.50}$$

In view of the expression of z_5 in this formula, applying Lemma 3.6, the solution $\varphi = \varphi (\mathbf{x}, \theta, v)$ of (3.36)–(3.37) is given by (3.39). Uniqueness of the solution of problem (3.36)–(3.37) is obvious. Since the regularity and periodicity of φ is a direct consequence of Lemma 3.5 and of assumptions (1.42) and (1.43), this ends the proof of Theorem 3.7. □

Now, we will look for an asymptotic expansion, with respect to ε, of the solution φ to (3.36)–(3.37). This will be based on a Lie expansion of the flow $\mathcal{G}_\lambda$.

Definition 3.8. *If $\mathbf{\Lambda}$ is a vector field of $\mathbb{R}^3$ with coefficients which are in $\mathcal{C}^\infty_b(\mathbb{R}^3)$, then we define the Lie Series $S^\infty_L (\mathbf{\Lambda}) \cdot$ associated with $\mathbf{\Lambda}$ by*

$$S^\infty_L (\mathbf{\Lambda}) \cdot = \sum_{l\geq 0} \frac{(\mathbf{\Lambda})^l \cdot}{l!}, \tag{3.51}$$

where $(\mathbf{\Lambda})^l$ is defined by (3.27) and (3.28), and the partial Lie Sum of order n:

$$S^n_L (\mathbf{\Lambda}) \cdot = \sum_{l=0}^{n} \frac{(\mathbf{\Lambda})^l \cdot}{l!}. \tag{3.52}$$

It is known that, formally, the flow $\mathcal{G}_\lambda$ associated with $\mathbf{\Lambda}$ may be expressed in terms of the Lie Series of $\mathbf{\Lambda}$:

$$\mathcal{G}_\lambda = S^\infty_L (\lambda\mathbf{\Lambda}) \cdot = \sum_{l\geq 0} \frac{(\lambda\mathbf{\Lambda})^l \cdot}{l!}. \tag{3.53}$$

More rigorously, as the flow is complete, using its partial Lie Sum we have

$$f \circ \mathcal{G}_\lambda = \sum_{l=0}^{n} \frac{\lambda^l (\mathbf{\Lambda})^l \cdot}{l!} f + \int_0^\lambda \frac{(\lambda - u)^n}{n!} \left(\mathbf{\Lambda}^{n+1} \cdot f\right) \circ \mathcal{G}_u du, \tag{3.54}$$

for any function $f: \mathbb{R}^3 \to \mathbb{R}$ being $\mathcal{C}^\infty(\mathbb{R}^3)$. Taking now $\frac{1}{B}$ as function f in (3.54), we obtain

$$\varphi(\mathbf{x},\theta,v) = \sum_{l=0}^{n} \frac{(-\varepsilon v)^l}{l!} \left(\mathbf{\Lambda}^l \cdot \frac{1}{B}\right)(\mathbf{x},\theta) + \int_0^{-\varepsilon v} \frac{(-\varepsilon v - u)^n}{n!} \left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_u(\mathbf{x},\theta)\, du. \tag{3.55}$$

Hence we have proven the following lemma

Lemma 3.9. *Function φ, solution to* (3.36)–(3.37)*, admits for any $n \in \mathbb{N}$, for any $\varepsilon \in \mathbb{R}$ and for any $(\mathbf{x},\theta,v) \in \mathbb{R}^4$ the following expansion in power of ε*

$$\begin{aligned} \varphi(\mathbf{x},\theta,v) = & \sum_{l=0}^{n} \frac{(-\varepsilon v)^l}{l!} \left(\mathbf{\Lambda}^l \cdot \frac{1}{B}\right)(\mathbf{x},\theta) \\ & + \frac{(-\varepsilon)^{n+1}}{n!} \int_0^{v} \frac{(v-u)^n}{n!} \left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\, du. \end{aligned} \tag{3.56}$$

Moreover, for any $l \in \mathbb{N}$, $\left(\mathbf{\Lambda}^l \cdot \frac{1}{B}\right)$ is in $\mathcal{C}^\infty_{\#,3}(\mathbb{R}^3) \cap \mathcal{C}^\infty_b(\mathbb{R}^3)$; for any $n \in \mathbb{N}$, $(\varepsilon,\mathbf{x},\theta,v) \mapsto \int_0^v \frac{(v-u)^n}{n!} \left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\, du$ is in $\mathcal{C}^\infty_\#(\mathbb{R}^5)$; and for any $v \in \mathbb{R}$ and any $n \in \mathbb{N}$, $(\varepsilon,\mathbf{x},\theta) \mapsto \int_0^v \frac{(v-u)^n}{n!} \left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\, du$ is bounded by $C^\varphi_n(v) = \frac{|v|^{n+1}}{(n+1)!} \left\| \mathbf{\Lambda}^{n+1} \cdot \frac{1}{B} \right\|_\infty$.

This ends this subsection. We will now use these results to solve (3.24)–(3.25) and consequently prove Theorem 3.4.

3.4 Proof of Theorem 3.4

For φ being given by (3.39), let

$$\psi(\mathbf{x},\theta,v) = \int_0^v \varphi(\mathbf{x},\theta,s)\, ds. \tag{3.57}$$

Integrating (3.36) between 0 and v, using (3.37) and expression (3.27) of $\mathbf{\Lambda}^1\cdot$, we obtain that ψ satisfies $-\varepsilon \mathbf{\Lambda}^1 \cdot \psi(\mathbf{x},\theta,v) - \varphi(\mathbf{x},\theta,v) = -\varphi(\mathbf{x},\theta,0)$ and then is the unique solution to

$$-\varepsilon \frac{\cos(\theta)}{B(\mathbf{x})} \frac{\partial \psi}{\partial x_1} + \varepsilon \frac{\sin(\theta)}{B(\mathbf{x})} \frac{\partial \psi}{\partial x_2} - \frac{\partial \psi}{\partial v} = -\frac{1}{B(\mathbf{x})}, \tag{3.58}$$

$$\psi(\mathbf{x},\theta,0) = 0. \tag{3.59}$$

Integrating now (3.58) between 0 and v and using (3.59) we obtain that $\int_0^v \psi(\mathbf{x},\theta,s)\, ds$ is the unique solution of (3.24)–(3.25), from which we deduce that

$$\mathbf{\Upsilon}_k(\mathbf{x},\theta,v) = \int_0^v \psi(\mathbf{x},\theta,s)\, ds. \tag{3.60}$$

Moreover, integrating between 0 and v the expansion of φ given by Lemma 3.9, we obtain

$$\begin{aligned} \psi(\mathbf{x},\theta,v) = & \sum_{l=0}^{n} \frac{(-\varepsilon)^l v^{l+1}}{(l+1)!} \left(\mathbf{\Lambda}^l \cdot \frac{1}{B}\right)(\mathbf{x},\theta) \\ & + \frac{(-\varepsilon)^{n+1}}{(n+1)!} \int_0^{v} (v-u)^{n+1} \left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta)\, du, \end{aligned} \tag{3.61}$$

and integrating the expansion of ψ given by (3.61) we obtain the expansion (3.29). This ends the proof of Theorem 3.4. $\square$

3.5 The other equations

Equation (3.16) was processed and gave expression of $\boldsymbol{\Upsilon}_k$. Equations (3.17), (3.18), (3.19) and (3.20) will be processed using results of the previous sections. Here, there is an additional difficulty which is that $\boldsymbol{\Upsilon}_{y_1}$ and $\boldsymbol{\Upsilon}_{y_2}$ are simultaneously solutions of two PDEs; one involving $\boldsymbol{\Upsilon}_\theta$ and another one involving $\boldsymbol{\Upsilon}_k$.

We have the following theorem.

Theorem 3.10. *The first component $\boldsymbol{\Upsilon}_1 = \boldsymbol{\Upsilon}_{y_1}$ of mapping $\boldsymbol{\Upsilon}$ which is solution to* (3.18) *and* (3.17) *and which satisfies* (3.21) *is given by*

$$\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v) = x_1 - \varepsilon\cos(\theta)\,\psi(\mathbf{x},\theta,v), \tag{3.62}$$

where ψ is defined by formula (3.57)*. For any $n \geq 1$, for any $\varepsilon \in \mathbb{R}$, and for any $(\mathbf{x},\theta,v) \in \mathbb{R}^4$ we have the following expansion for $\boldsymbol{\Upsilon}_{y_1}$*

$$\begin{aligned}\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v) = x_1 + \cos(\theta)\sum_{l=1}^{n}\frac{v^l(-\varepsilon)^l}{l!}\left(\boldsymbol{\Lambda}^{l-1}\cdot\frac{1}{B}\right)(\mathbf{x},\theta) + \\ \cos(\theta)\frac{(-\varepsilon)^{n+1}}{n!}\int_0^v (v-u)^n\left(\boldsymbol{\Lambda}^n\cdot\frac{1}{B}\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,du.\end{aligned} \tag{3.63}$$

Moreover, $(\varepsilon,\mathbf{x},\theta,v) \mapsto \boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v)$ and $(\varepsilon,\mathbf{x},\theta,v) \mapsto \psi(\mathbf{x},\theta,v)$ are in $\mathcal{C}^\infty_\#(\mathbb{R}^5)$; ψ is bounded by $C^\psi(v) = |v|^2\left\|\frac{1}{B}\right\|_\infty$; for any $l \in \mathbb{N}$, $\left(\boldsymbol{\Lambda}^l\cdot\frac{1}{B}\right)$ is in $\mathcal{C}^\infty_b(\mathbb{R}^3)\cap\mathcal{C}^\infty_\#(\mathbb{R}^4)$; $(\varepsilon,\mathbf{x},\theta,v) \mapsto \int_0^v (v-u)^n\left(\boldsymbol{\Lambda}^n\cdot\frac{1}{B}\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,du$ is in $\mathcal{C}^\infty_\#(\mathbb{R}^5)$, and for any $v \in \mathbb{R}$ and any $n \in \mathbb{N}$, it is bounded by $C_n^{\boldsymbol{\Upsilon}_{y_1}}(v) = \frac{|v|^{n+1}}{n+1}\left\|\boldsymbol{\Lambda}^n\cdot\frac{1}{B}\right\|_\infty$.

Proof. In a first place, by definition, $\psi(\mathbf{x},\theta,0) = 0$ and hence $\boldsymbol{\Upsilon}_{y_1}$ given by (3.62) satisfies $\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,0) = x_1$. Moreover ψ satisfies $\left(-\varepsilon\boldsymbol{\Lambda} - \frac{\partial}{\partial v}\right)\cdot\psi = -\frac{1}{B}$ and hence by linearity $\boldsymbol{\Upsilon}_{y_1}$ given by (3.62) is solution of (3.17). On another hand from expansion (3.61) of ψ we deduce expansion (3.63) of $\boldsymbol{\Upsilon}_{y_1}$.

Secondly we show that $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}$, which is defined (with worth 0) for $v \neq 0$ because of the singularity of $\tilde{\mathcal{P}}_\varepsilon$, can be extended smoothly by 0 in $v = 0$. Using expansion (3.63) of $\boldsymbol{\Upsilon}_{y_1}$ for $n = 1$ we obtain

$$\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v) = x_1 - \frac{\varepsilon\cos(\theta)\,v}{B(\mathbf{x})} + \varepsilon^2\cos(\theta)\int_0^v (v-u)\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,du. \tag{3.64}$$

In the same way, applying formula (3.29) with $n = 0$ we obtain:

$$\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v) = \frac{v^2}{2B(\mathbf{x})} - \frac{\varepsilon}{2}\int_0^v (v-u)\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,du. \tag{3.65}$$

Differentiating (3.64) with respect to x_1 yields

$$\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial x_1}(\mathbf{x},\theta,v) = 1 - \varepsilon\cos(\theta)\, v \left(\frac{\partial}{\partial x_1}\left(\frac{1}{B}\right)\right)(\mathbf{x}) + \varepsilon^2 \cos(\theta) \int_0^v (v-u) \left[\left(\frac{\partial}{\partial x_1}\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,\frac{\partial \mathcal{G}^1_{-\varepsilon u}}{\partial x_1}(\mathbf{x},\theta) + \left(\frac{\partial}{\partial x_2}\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,\frac{\partial \mathcal{G}^2_{-\varepsilon u}}{\partial x_1}(\mathbf{x},\theta)\right] du. \quad (3.66)$$

As $\frac{1}{B}$ and all its derivatives are bounded and as $\frac{\partial \mathcal{G}^1_\lambda}{\partial x_1}$ and $\frac{\partial \mathcal{G}^2_\lambda}{\partial x_1}$ are continuous with respect to λ we obtain the following estimate:

$$\left|\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial x_1}(\mathbf{x},\theta,v)\right| \leq 1 + \varepsilon|v| \left\|\frac{\partial}{\partial x_1}\frac{1}{B}\right\|_\infty + \frac{\varepsilon^2|v|^2}{2}\left[\left\|\frac{\partial}{\partial x_1}\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)\right\|_\infty \sup_{u\in[-|v|,|v|]}\left|\frac{\partial \mathcal{G}^1_{-\varepsilon u}}{\partial x_1}\right|(\mathbf{x},\theta) + \left\|\frac{\partial}{\partial x_2}\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)\right\|_\infty \sup_{u\in[-|v|,|v|]}\left|\frac{\partial \mathcal{G}^2_{-\varepsilon u}}{\partial x_1}\right|(\mathbf{x},\theta)\right].$$

Hence $\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial x_1}(\mathbf{x},\theta,v) = \epsilon^{x_1}_{y_1}(\mathbf{x},\theta,v)$ with $\epsilon^{x_1}_{y_1}(\mathbf{x},\theta,v)$ such that for any $(\mathbf{x},\theta)$, $v \mapsto \epsilon^{x_1}_{y_1}(\mathbf{x},\theta,v)$ is smooth, and is bounded in the neighborhood of $v=0$. In the same way we can show that

$$\begin{aligned}
&\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial x_2}(\mathbf{x},\theta,v) = v\epsilon^{x_2}_{y_1}(\mathbf{x},\theta,v), && \frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial \theta}(\mathbf{x},\theta,v) = v\epsilon^{\theta}_{y_1}(\mathbf{x},\theta,v),\\
&\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial v}(\mathbf{x},\theta,v) = \epsilon^{v}_{y_1}(\mathbf{x},\theta,v), && \frac{\partial \boldsymbol{\Upsilon}_{k}}{\partial x_1}(\mathbf{x},\theta,v) = v^2\epsilon^{x_1}_{k}(\mathbf{x},\theta,v),\\
&\frac{\partial \boldsymbol{\Upsilon}_{k}}{\partial x_2}(\mathbf{x},\theta,v) = v^2\epsilon^{x_2}_{k}(\mathbf{x},\theta,v), && \frac{\partial \boldsymbol{\Upsilon}_{k}}{\partial \theta}(\mathbf{x},\theta,v) = v^3\epsilon^{\theta}_{k}(\mathbf{x},\theta,v),\\
&\frac{\partial \boldsymbol{\Upsilon}_{k}}{\partial v}(\mathbf{x},\theta,v) = v\epsilon^{v}_{k}(\mathbf{x},\theta,v).
\end{aligned} \quad (3.67)$$

with $\epsilon^{x_2}_{y_1}(\mathbf{x},\theta,v)$, $\epsilon^{\theta}_{y_1}(\mathbf{x},\theta,v)$, $\epsilon^{v}_{y_1}(\mathbf{x},\theta,v)$, $\epsilon^{x_1}_{k}(\mathbf{x},\theta,v)$, $\epsilon^{x_2}_{k}(\mathbf{x},\theta,v)$, $\epsilon^{\theta}_{k}(\mathbf{x},\theta,v)$, $\epsilon^{v}_{k}(\mathbf{x},\theta,v)$ such that for any $(\mathbf{x},\theta)$, the functions $v \mapsto \epsilon^{\bullet}_{\bullet}(\mathbf{x},\theta,v)$ are smooth, and are bounded in the neighborhood of $v=0$. Injecting these expressions in $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}(\mathbf{x},\theta,v) = (\nabla\boldsymbol{\Upsilon}_{y_1})\cdot\left(\tilde{\mathcal{P}}_\varepsilon\nabla\boldsymbol{\Upsilon}_k\right)$ we obtain $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}(\mathbf{x},\theta,v) = v\epsilon_{y_1,k}(\mathbf{x},\theta,v)$ with $\epsilon_{y_1,k}(\mathbf{x},\theta,v)$ such that $v\mapsto \epsilon_{y_1,k}(\mathbf{x},\theta,v)$ is smooth, and is bounded in the neighborhood of $v=0$ leading that $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}$ can be smoothly extended by 0 in $v=0$.

As the last step of this proof, because of the Jacobi identity we have

$$\forall v\neq 0,\quad \{\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\},\boldsymbol{\Upsilon}_\theta\} + \{\{\boldsymbol{\Upsilon}_\theta,\boldsymbol{\Upsilon}_{y_1}\},\boldsymbol{\Upsilon}_k\} + \{\{\boldsymbol{\Upsilon}_k,\boldsymbol{\Upsilon}_\theta\},\boldsymbol{\Upsilon}_{y_1}\} = 0, \quad (3.68)$$

which reads, because the gradient of a constant is zero, because, according to (3.5), $\{\boldsymbol{\Upsilon}_k,\boldsymbol{\Upsilon}_\theta\} = \frac{1}{\varepsilon}$ and, as we just saw, because $\boldsymbol{\Upsilon}_{y_1}$ given by (3.62) satisfies $\{\boldsymbol{\Upsilon}_\theta,\boldsymbol{\Upsilon}_{y_1}\}=0$,

$$\{\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\},\boldsymbol{\Upsilon}_\theta\} = 0. \quad (3.69)$$

Dividing (3.69) by $\omega_\varepsilon(\mathbf{x},\theta)$ defined by (3.14), we obtain that for $v\neq 0$, $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}$ is solution to

$$(\nabla\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\})\cdot\left(\tilde{\mathcal{Q}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_\theta)\right) = 0. \quad (3.70)$$

By continuity of the left hand side of (3.70) on $\mathbb{R}^4$, we deduce that equality (3.70) is valid on $\mathbb{R}^4$. As $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_k\}$ may be smoothly extended by 0 in $v = 0$, and as (3.70) admits an unique solution satisfying the boundary condition $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_k\}(\mathbf{x}, \theta, 0) = 0$, we deduce that $\boldsymbol{\Upsilon}_{y_1}$ given by (3.62) satisfies $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_k\} = 0$ for all $(\mathbf{x}, \theta, v)$. Hence (3.18) follows and we deduce that $\boldsymbol{\Upsilon}_{y_1}$ given by (3.62) is well $\boldsymbol{\Upsilon}_{y_1}$. This ends the proof of Theorem 3.10. □

In the same way we can prove:

Theorem 3.11. *The second component* $\boldsymbol{\Upsilon}_2 = \boldsymbol{\Upsilon}_{y_2}$ *of mapping* $\boldsymbol{\Upsilon}$ *that gives the Darboux Coordinates in terms of the Polar in velocity Coordinates is given by*

$$\boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, v) = x_2 + \varepsilon \sin(\theta)\, \psi(\mathbf{x}, \theta, v), \tag{3.71}$$

where ψ *is defined by formula* (3.57)*. For any* $n \geq 1$*, for any* $\varepsilon \in \mathbb{R}$*, and for any* $(\mathbf{x}, \theta, v) \in \mathbb{R}^4$ *we have the following expansion for* $\boldsymbol{\Upsilon}_{y_2}$

$$\begin{aligned}\boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, v) = x_2 - \sin(\theta) \sum_{l=1}^{n} \frac{v^l (-\varepsilon)^l}{l!} \left(\boldsymbol{\Lambda}^{l-1} \cdot \frac{1}{B}\right)(\mathbf{x}, \theta) \\ - \sin(\theta) \frac{(-\varepsilon)^{n+1}}{n!} \int_0^v (v-u)^n \left(\boldsymbol{\Lambda}^n \cdot \frac{1}{B}\right) \left(\mathcal{G}_{-\varepsilon u}(\mathbf{x}, \theta)\right) du.\end{aligned} \tag{3.72}$$

Moreover, $(\varepsilon, \mathbf{x}, \theta, v) \mapsto \boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, v)$ *is* $\mathcal{C}^\infty_\#\left(\mathbb{R}^5\right)$.

3.6 The Darboux Coordinates System

In subsection 3.2 and 3.5 we solved equations (3.5) and (3.7)–(3.10) on $\mathbb{R}^4$. Now, we need to check that the restriction of $\boldsymbol{\Upsilon}$ to $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, also denoted by $\boldsymbol{\Upsilon}$, is a diffeomorphism (onto $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$) and hence that $(\mathbf{y}, \theta, k)$ makes a true coordinate system on $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. We will also prove that function $\boldsymbol{\kappa}$ defined by $\boldsymbol{\kappa} = \boldsymbol{\Upsilon}^{-1}$ is smooth with respect to the small parameter ε and we will give its expansion in power of ε.

Firstly, using expressions (3.62) and (3.71) of $\boldsymbol{\Upsilon}_{y_1}$ and $\boldsymbol{\Upsilon}_{y_2}$, formula (3.39) that gives the expression of $\varphi = \partial_v \psi$, expression (3.26) of $\boldsymbol{\Lambda}$ and the definition (3.40) of its flow $\mathcal{G}_\lambda$, we deduce that

$$\frac{\partial \boldsymbol{\Upsilon}_{y_1}}{\partial v}(\mathbf{x}, \theta, v) = \frac{\partial}{\partial v} \mathcal{G}^1_{-\varepsilon v}(\mathbf{x}, \theta), \tag{3.73}$$

$$\frac{\partial \boldsymbol{\Upsilon}_{y_2}}{\partial v}(\mathbf{x}, \theta, v) = \frac{\partial}{\partial v} \mathcal{G}^2_{-\varepsilon v}(\mathbf{x}, \theta), \tag{3.74}$$

$$\frac{\partial \boldsymbol{\Upsilon}_{\theta}}{\partial v}(\mathbf{x}, \theta, v) = \frac{\partial}{\partial v} \mathcal{G}^3_{-\varepsilon v}(\mathbf{x}, \theta). \tag{3.75}$$

Hence, since $\boldsymbol{\Upsilon}_{y_1}(\mathbf{x}, \theta, 0) = x_1$, $\boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, 0) = x_2$ and $\boldsymbol{\Upsilon}_\theta(\mathbf{x}, \theta, 0) = \theta$ we obtain that

$$\left(\boldsymbol{\Upsilon}_{y_1}(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_\theta(\mathbf{x}, \theta, v)\right) = \mathcal{G}_{-\varepsilon v}(\mathbf{x}, \theta). \tag{3.76}$$

From this, it is clear that $(\mathbf{y}, \theta, v)$ makes a coordinate system and that the reciprocal change of coordinates is given by $(\mathbf{x}, \theta, v) = \left(\mathcal{G}_{\varepsilon v}(\mathbf{y}, \theta), v\right)$.

In order to show that $(\mathbf{y}, \theta, k)$ makes also a coordinate system we will proceed as follows: we will express $\boldsymbol{\Upsilon}_k$ in the $(\mathbf{y}, \theta, v)$-coordinate system and using this expression, we will express v in terms of $\mathbf{y}$ and θ and the yielding expression of $\boldsymbol{\Upsilon}_k$ in the $(\mathbf{y}, \theta, v)$-coordinate system.

Lemma 3.12. *The representative of $\boldsymbol{\Upsilon}_k$ in the $(\mathbf{y}, \theta, v)$-coordinate system is given by*

$$\tilde{\boldsymbol{\Upsilon}}_k(\mathbf{y}, \theta, v) = \int_0^v \frac{u}{B\left(\mathcal{G}^1_{\varepsilon u}(\mathbf{y}, \theta), \mathcal{G}^2_{\varepsilon u}(\mathbf{y}, \theta)\right)} du. \tag{3.77}$$

Proof. Using function φ involved in the expression of $\boldsymbol{\Upsilon}_k$ (see (3.60) and (3.57)), we obtain:

$$\begin{aligned}
\boldsymbol{\Upsilon}_k(\mathbf{x}, \theta, v) &= \int_0^v \left(\int_0^s \varphi(\mathbf{x}, \theta, u)\, du\right) ds \\
&= \int_0^v \left(\int_u^v \varphi(\mathbf{x}, \theta, u)\, ds\right) du \\
&= \int_0^v (v-u)\, \varphi(\mathbf{x}, \theta, u)\, du.
\end{aligned} \tag{3.78}$$

Now, using expressions (3.39) of φ and (3.76) of $(\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_{y_2}, \boldsymbol{\Upsilon}_\theta)$, we obtain

$$\begin{aligned}
\boldsymbol{\Upsilon}_k(\mathbf{x}, \theta, v) &= \int_0^v \frac{(v-u)}{B\left(\mathcal{G}^1_{-\varepsilon u}(\mathbf{x}, \theta), \mathcal{G}^2_{-\varepsilon u}(\mathbf{x}, \theta)\right)} du \\
&= \int_0^v \frac{(v-u)}{B\left(\mathcal{G}^1_{\varepsilon(v-u)}\left(\mathcal{G}_{-\varepsilon v}(\mathbf{x}, \theta)\right), \mathcal{G}^2_{\varepsilon(v-u)}\left(\mathcal{G}_{-\varepsilon v}(\mathbf{x}, \theta)\right)\right)} du \\
&= \int_0^v (v-u)\ \varphi(\mathcal{G}_{-\varepsilon v}(\mathbf{x}, \theta), u-v)\, du, \\
&= \int_0^v u\varphi\left(\boldsymbol{\Upsilon}_{y_1}(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_{y_2}(\mathbf{x}, \theta, v), \boldsymbol{\Upsilon}_\theta(\mathbf{x}, \theta, v), -u\right) du,
\end{aligned} \tag{3.79}$$

implying (3.77) and consequently proving the lemma. □

Having expression (3.77) of $\tilde{\boldsymbol{\Upsilon}}_k$ on hand, for all $(\mathbf{y}, \theta) \in \mathbb{R}^3$ we can define the parametrized smooth function $\eta = [\eta(\mathbf{y}, \theta)]$ of v by

$$[\eta(\mathbf{y}, \theta)](v) = \tilde{\boldsymbol{\Upsilon}}_k(\mathbf{y}, \theta, v). \tag{3.80}$$

Lemma 3.13. *For any $(\mathbf{y}, \theta) \in \mathbb{R}^2 \times \mathbb{R}$, function $[\eta(\mathbf{y}, \theta)]$ is a $\mathcal{C}^\infty$-diffeomorphism from $(0, +\infty)$ onto itself and function $\tilde{\eta} = \tilde{\eta}(\mathbf{y}, \theta, k)$ defined by:*

$$\tilde{\eta}(\mathbf{y}, \theta, k) = [\eta(\mathbf{y}, \theta)]^{-1}(k) \tag{3.81}$$

which gives the expression of v, is $\mathcal{C}^\infty_\#(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty))$.

Proof. As

$$\left[\frac{d[\eta(\mathbf{y}, \theta)]}{dv}\right](v) = \frac{v}{B\left(\mathcal{G}^1_{\varepsilon v}(\mathbf{y}, \theta), \mathcal{G}^2_{\varepsilon v}(\mathbf{y}, \theta)\right)} > 0,$$

$[\eta(\mathbf{y},\theta)]$ is a $\mathcal{C}^\infty$-diffeomorphism from $(0,+\infty)$ onto

$$\left(\lim_{v\to 0}[\eta(\mathbf{y},\theta)](v)\,,\lim_{v\to+\infty}[\eta(\mathbf{y},\theta)](v)\right) \tag{3.82}$$

for all $(\mathbf{y},\theta)$. Moreover, according to formula (3.77) we have for any $v>0$ the following estimates:

$$\frac{v^2}{2\|B\|_\infty}\le[\eta(\mathbf{y},\theta)](v)\le\frac{v^2}{2}, \tag{3.83}$$

and consequently for any $(\mathbf{y},\theta)\in\mathbb{R}^3$

$$[\eta(\mathbf{y},\theta)]((0,+\infty))=(0,+\infty)\,. \tag{3.84}$$

Particularly, for any $v\in(0,+\infty)$ there exists $k\in(0,+\infty)$ such that

$$v=[\eta(\mathbf{y},\theta)]^{-1}(k)\,. \tag{3.85}$$

The regularity of $\tilde{\eta}$ with respect to k is easily obtained from the fact that $[\eta(\mathbf{y},\theta)]$ is a $\mathcal{C}^\infty$-diffeomorphism. The $\mathcal{C}^\infty$-nature of $\tilde{\eta}$ with respect to $\mathbf{y}$ and θ is obtained by computing the successive derivatives of $[\eta(\mathbf{y},\theta)]\circ[\eta(\mathbf{y},\theta)]^{-1}=id$ and using the regularity of η that comes from the regularity of $\boldsymbol{\Upsilon}_k$ and then from the regularity of B and flow $\mathcal{G}_\lambda$. Moreover, the periodicity of $\tilde{\eta}$ with respect to θ comes from the fact that $\theta\mapsto\left(\mathcal{G}^1_\lambda(\mathbf{x},\theta),\mathcal{G}^2_\lambda(\mathbf{x},\theta)\right)$ is in $\mathcal{C}^\infty_{\text{per}}(\mathbb{R})$ for any $\mathbf{x}\in\mathbb{R}^2$ as set out in Lemma 3.5. □

Hence we have proven the following theorem.

Theorem 3.14. $(\mathbf{y},\theta,k)$ *makes a coordinate system on* $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ *and function* $\boldsymbol{\kappa}$ *defined by* (3.3) *is given by*

$$\boldsymbol{\kappa}(\mathbf{y},\theta,k)=\left(\mathcal{G}_{\varepsilon\tilde{\eta}(\mathbf{y},\theta,k)}(\mathbf{y},\theta)\,,\tilde{\eta}(\mathbf{y},\theta,k)\right), \tag{3.86}$$

where $\tilde{\eta}$ *is defined by* (3.81).

Now, we will focus on the ε-dependency of $\boldsymbol{\kappa}$. For this purpose, we will introduce the functions $\alpha=[\alpha(\mathbf{y},\theta)](v)$, which is defined for $v\in\mathbb{R}_+$, $\beta=[\beta(\mathbf{y},\theta,k)](\varepsilon)$, which is defined for $\varepsilon\in(0,+\infty)$, and $\gamma=[\gamma(\mathbf{y},\theta,k)](\varepsilon)$, which is defined for $\varepsilon\in\mathbb{R}_+$ by

$$[\alpha(\mathbf{y},\theta)](v)=\int_0^v\frac{s}{B\left(\mathcal{G}^1_s(\mathbf{y},\theta)\,,\mathcal{G}^2_s(\mathbf{y},\theta)\right)}ds, \tag{3.87}$$

$$[\beta(\mathbf{y},\theta,k)](\varepsilon)=[\alpha(\mathbf{y},\theta)]^{-1}\left(\varepsilon^2k\right), \tag{3.88}$$

$$[\gamma(\mathbf{y},\theta,k)](\varepsilon)=\sqrt{\frac{[\alpha(\mathbf{y},\theta)](\varepsilon)}{k}}. \tag{3.89}$$

With their help, we can state the following lemma.

Lemma 3.15. *Function β defined by formula* (3.88) *admits a smooth continuation to $\mathbb{R}_+$ such that*

$$[\beta(\mathbf{y},\theta,k)](0)=0, \tag{3.90}$$

Moreover, for any $\varepsilon>0$ we have

$$[\beta(\mathbf{y},\theta,k)]'(\varepsilon)=\frac{1}{[\gamma(\mathbf{y},\theta,k)]'([\beta(\mathbf{y},\theta,k)](\varepsilon))}, \tag{3.91}$$

where γ is defined by (3.89)*.*

Proof. By definition, function $\varepsilon\mapsto[\beta(\mathbf{y},\theta,k)](\varepsilon)$ is in $\mathcal{C}^\infty(\mathbb{R}_+^\star)$ for every $(\mathbf{y},\theta,k)\in\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. Moreover function γ is such that

$$\forall\varepsilon>0,\ [\gamma(\mathbf{y},\theta,k)](\varepsilon)=[\beta(\mathbf{y},\theta,k)]^{-1}(\varepsilon). \tag{3.92}$$

Hence, in order to show that β admits a smooth continuation on $\mathbb{R}_+$ we just have to show that γ admits a smooth inverse function in the neighborhood of 0 in $\mathbb{R}_+$. And yet, for all $\varepsilon\geq 0$, we have

$$[\gamma(\mathbf{y},\theta,k)](\varepsilon)=\varepsilon\sqrt{\frac{1}{k}\int_0^1\frac{u}{B\left(\mathcal{G}^1_{\varepsilon u}(\mathbf{y},\theta),\mathcal{G}^2_{\varepsilon u}(\mathbf{y},\theta)\right)}du}. \tag{3.93}$$

This function is in $\mathcal{C}^\infty(\mathbb{R}_+)$ and

$$\left[\frac{d\gamma(\mathbf{y},\theta,k)}{d\varepsilon}\right](0)=\frac{1}{\sqrt{2kB(\mathbf{y})}}\neq 0. \tag{3.94}$$

Hence, there exists a neighborhood I of 0 and a smooth function $\delta=[\delta(\mathbf{y},\theta,k)](\varepsilon)$ defined on $J=[\gamma(\mathbf{y},\theta,k)](I\cap\mathbb{R}_+)$ such that $[\gamma(\mathbf{y},\theta,k)]\circ[\delta(\mathbf{y},\theta,k)]=id$. Hence we have shown that the smooth function β defined on $\mathbb{R}_+^\star$ admits a smooth continuation to $\mathbb{R}_+$. Then, since (3.91) follows directly (3.92), Lemma 3.15 is proven □

Lemma 3.16. *Function*

$$(\mathbf{y},\theta,k,\varepsilon)\mapsto[\beta(\mathbf{y},\theta,k)](\varepsilon) \tag{3.95}$$

is in $\mathcal{C}^\infty_{\#,3}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\times\mathbb{R}_+\big)$ (see (3.30) *for the definition of this space).*

The proof of the periodicity with respect to the third variable is similar to the one of Lemma 3.13.

We will now use Lemmas 3.15 and 3.16 to deduce an expression of the expansion with respect to ε of the v-component of $\boldsymbol{\kappa}=\boldsymbol{\Upsilon}^{-1}$.

Lemma 3.17. *For any $n\in\mathbb{N}^\star$, there exists $P_n\in\mathbb{R}_{n-1}[X_1,\dots,X_n]$ (where $\mathbb{R}_{n-1}[X_1,\dots,X_n]$ stands for the space of the homogeneous polynomial of degree $n-1$ in n variables) such that*

$$[\beta(\mathbf{y},\theta,k)]^{(n)}(\varepsilon)=\frac{P_n\left([\gamma(\mathbf{y},\theta,k)]^{(1)}(\beta(\varepsilon)),\dots,[\gamma(\mathbf{y},\theta,k)]^{(n)}(\beta(\varepsilon))\right)}{\left([\gamma(\mathbf{y},\theta,k)]^{(1)}(\beta(\varepsilon))\right)^{2n-1}}. \tag{3.96}$$

Proof. For $n = 1$ it is just formula (3.91). For $n \geq 2$, we will prove formula (3.96) by induction.

Differentiating (3.91) we obtain

$$[\beta(\mathbf{y}, \theta, k)]^{(2)}(\varepsilon) = \frac{P_2\left([\gamma(\mathbf{y}, \theta, k)]^{(1)}(\beta(\varepsilon)), [\gamma(\mathbf{y}, \theta, k)]^{(2)}(\beta(\varepsilon))\right)}{\left([\gamma(\mathbf{y}, \theta, k)]^{(1)}(\beta(\varepsilon))\right)^3}, \tag{3.97}$$

where $P_2(X_1, X_2) = -X_2$.

Now, assume that formula (3.96) is true for some given $n \geq 2$. Differentiating (3.96) yields:

$$[\beta(\mathbf{y}, \theta, k)]^{(n+1)}(\varepsilon) = \frac{P_{n+1}\left([\gamma(\mathbf{y}, \theta, k)]^{(1)}(\beta(\varepsilon)), \ldots, [\gamma(\mathbf{y}, \theta, k)]^{(n+1)}(\beta(\varepsilon))\right)}{\left([\gamma(\mathbf{y}, \theta, k)]^{(1)}(\beta(\varepsilon))\right)^{2n+1}},$$

where

$$P_{n+1}(X_1, \ldots, X_{n+1}) = -(2n-1)X_2 P_n(X_1, \ldots, X_n) + \sum_{k=1}^{n} X_1 X_{k+1} \frac{\partial P_n}{\partial X_k}(X_1, \ldots, X_n).$$

This ends the proof of Lemma 3.17. □

Lemma 3.18. *For any $l \in \mathbb{N}^\star$, there exists $a_l \in \mathcal{O}^\infty_{T,b}$ such that*

$$[\beta(\mathbf{y}, \theta, k)]^{(l)}(0) = \sqrt{k}^{\,l} a_l(\mathbf{y}, \theta). \tag{3.98}$$

Here and hereafter, $\mathcal{O}^\infty_{T,b}$ stands of the algebra of functions spanned by the functions of the form

$$(\mathbf{y}, \theta) \mapsto f_1(\mathbf{y})\cos(\theta) + f_2(\mathbf{y})\sin(\theta), \tag{3.99}$$

where $f_1, f_2 \in \mathcal{C}^\infty_b(\mathbb{R}^2)$.

Proof. On the one hand, for any $(\mathbf{y}, \theta, k)$ and for any $n \in \mathbb{N}$, $[\gamma(\mathbf{y}, \theta, k)]$ admits a Taylor-MacLaurin expansion of order n.

$$\begin{aligned}[\gamma(\mathbf{y}, \theta, k)](\varepsilon) &= [\gamma(\mathbf{y}, \theta, k)](0) + \varepsilon[\gamma(\mathbf{y}, \theta, k)]^{(1)}(0) + \ldots + \frac{\varepsilon^n}{n!}[\gamma(\mathbf{y}, \theta, k)]^{(n)}(0) \\ &+ \int_0^\varepsilon \frac{[\gamma(\mathbf{y}, \theta, k)]^{(n+1)}(t)}{n!}(t-\varepsilon)^n \, dt.\end{aligned} \tag{3.100}$$

On the other hand, applying formula (3.54) with $\frac{1}{B}$, multiplying by λ and integrating between 0 and ε yields:

$$\begin{aligned}[\alpha(\mathbf{y}, \theta)](\varepsilon) = \varepsilon^2 \Big(&\sum_{l=0}^{n} \frac{\varepsilon^l}{(l+2)\, l!}\left(\mathbf{\Lambda}^l \cdot \frac{1}{B}\right)(\mathbf{y}, \theta) \\ &+ \frac{\varepsilon^{n+1}}{(n+1)!}\int_0^1 (1-u)^{n+1}(n+1+u)\left(\mathbf{\Lambda}^{n+1} \cdot \frac{1}{B}\right) \circ \mathcal{G}_{\varepsilon u} du\Big).\end{aligned} \tag{3.101}$$

Injecting formula (3.101) in (3.89) yields:

$$[\gamma(\mathbf{y},\theta,k)](\varepsilon) =$$

$$\frac{\varepsilon}{\sqrt{k}}\sqrt{\left(\sum_{l=0}^{n}\frac{\varepsilon^l}{(l+2)\,l!}\left(\boldsymbol{\Lambda}^l\cdot\frac{1}{B}\right)(\mathbf{y},\theta)+\frac{\varepsilon^{n+1}}{(n+1)!}\int_0^1(1-u)^{n+1}(n+1+u)\left(\boldsymbol{\Lambda}^{n+1}\cdot\frac{1}{B}\right)\circ\mathcal{G}_{\varepsilon u}du\right)}. \tag{3.102}$$

Expanding formula (3.102) with respect to ε, up to order n, by using the usual expansion of $s\mapsto\sqrt{1+s}$, and identifying with formula (3.100) yields that for any $l\in\{0,\dots,n\}$, $\sqrt{k}\,[\gamma(\mathbf{y},\theta,k)]^{(l)}(0)\in\mathcal{O}^{\infty}_{T,b}$.

Finally, using formula (3.96) we obtain formula (3.98). This ends the proof of Lemma 3.18. □

Theorem 3.19. *The v-component $\boldsymbol{\kappa}_v$ of $\boldsymbol{\kappa}=\boldsymbol{\Upsilon}^{-1}$ admits the following expansion in power of ε:*

$$\boldsymbol{\kappa}_v(\mathbf{y},\theta,k)=\sum_{i=0}^{n}\sqrt{k}^{i+1}a_{i+1}(\mathbf{y},\theta)\frac{\varepsilon^i}{(i+1)!} \\ +\frac{\varepsilon^{n+1}}{(n+1)!}\int_0^1(1-u)^{n+1}[\beta(\mathbf{y},\theta,k)]^{(n+2)}(\varepsilon u)\,du, \tag{3.103}$$

where the terms a_i of the expansion are defined in Lemma 3.18 and are easily obtained by using formulas (3.90) *and* (3.91). *Moreover,*

$$(\mathbf{y},\theta,k,\varepsilon)\mapsto\int_0^1(1-u)^{n+1}[\beta(\mathbf{y},\theta,k)]^{(n+2)}(\varepsilon u)\,du\in\mathcal{C}^{\infty}_{\#,3}(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\times\mathbb{R}_+), \tag{3.104}$$

(see (3.30) *for the definition of this space).*

Proof. We just have to notice that, because of expression (3.79) of $\boldsymbol{\Upsilon}_k$, or more precisey of expression (3.77) of $\tilde{\boldsymbol{\Upsilon}}_k$ and of (3.87), we have

$$\tilde{\boldsymbol{\Upsilon}}_k(\mathbf{y},\theta,v)=\frac{1}{\varepsilon^2}[\alpha(\mathbf{y},\theta)](\varepsilon v)\ \text{ or }\varepsilon v=[\alpha(\mathbf{y},\theta)]^{-1}(\varepsilon^2\tilde{\boldsymbol{\Upsilon}}_k(\mathbf{y},\theta,v)), \tag{3.105}$$

and consequently, in view of (3.88), function $\boldsymbol{\kappa}_v$ expresses as

$$\forall\varepsilon>0,\ \boldsymbol{\kappa}_v(\mathbf{y},\theta,k)=\frac{1}{\varepsilon}[\beta(\mathbf{y},\theta,k)](\varepsilon). \tag{3.106}$$

Hence equality (3.103) follows directly from a Taylor expansion and using (3.98) and (3.90). Property (3.104) is a direct consequence of the regularity of function β. Hence, the theorem is proven. □

Applying Theorem 3.19, up to order 2, we obtain

$$\begin{aligned}\boldsymbol{\kappa}_v(\mathbf{y},\theta,k) = &\sqrt{2kB(\mathbf{y})} + \varepsilon\frac{2kB(\mathbf{y})}{3}\hat{a}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{y}) \\ &-\varepsilon^2 k\sqrt{\frac{kB(\mathbf{y})}{2}}\left[\frac{7}{18B(\mathbf{y})^3}\left(\hat{a}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{y})\right)^2 - \frac{\hat{a}(\theta)^T\mathcal{H}_B(\mathbf{y})\hat{a}(\theta)}{2B(\mathbf{y})^2}\right] + \\ &\frac{\varepsilon^3}{3!}\int_0^1(1-u)^3\left[\beta(\mathbf{y},\theta,k)\right]^{(4)}(\varepsilon u)\,du\end{aligned} \tag{3.107}$$

where $\hat{\mathbf{a}} = \hat{\mathbf{a}}(\theta)$, already used in the introduction, is defined by

$$\hat{\mathbf{a}}(\theta) = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{3.108}$$

and where $\mathcal{H}_B$ is the Hessian Matrix of B.

3.7 Expression of the Poisson Matrix

We have solved Equations (3.5) and (3.7)–(3.10) and obtained the change-of-coordinates mapping $\boldsymbol{\Upsilon}$. Furthermore, by construction, from formula (1.50), we know all the Poisson Matrix entries, except its entry number $(1,2)$: $\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_{y_2}\}_{\mathbf{x},\theta,v}(\mathbf{x},\theta,v)$. Its expression is given by the following theorem.

Theorem 3.20. *The Poisson Bracket between the two first components $\boldsymbol{\Upsilon}_{y_1}$ and $\boldsymbol{\Upsilon}_{y_2}$ of mapping $\boldsymbol{\Upsilon}$ is given by*

$$\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_{y_2}\}_{\mathbf{x},\theta,v}(\mathbf{x},\theta,v) = -\frac{\varepsilon}{B\left(\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v),\boldsymbol{\Upsilon}_{y_2}(\mathbf{x},\theta,v)\right)}. \tag{3.109}$$

Proof. The proof consists in identifying the Poisson Bracket between $\boldsymbol{\Upsilon}_{y_1}$ and $\boldsymbol{\Upsilon}_{y_2}$ as the unique solution of the PDE of unknown u

$$-\varepsilon\boldsymbol{\Lambda}^1\cdot u - \frac{\partial u}{\partial v} = 0, \tag{3.110}$$

$$u(\mathbf{x},\theta,0) = \frac{-\varepsilon}{B(\mathbf{x})}. \tag{3.111}$$

In a first place, as function φ defined by (3.39) is the unique solution of (3.36)–(3.37), the unique solution of (3.110)–(3.111) is given by

$$u(\mathbf{x},\theta,v) = -\varepsilon\varphi(\mathbf{x},\theta,v);$$

i.e. by (3.109).

On an another hand as for any $v\neq 0$, $\{\boldsymbol{\Upsilon}_\theta,\boldsymbol{\Upsilon}_{y_1}\}_{\mathbf{x},\theta,v} = 0$ and $\{\boldsymbol{\Upsilon}_{y_2},\boldsymbol{\Upsilon}_\theta\}_{\mathbf{x},\theta,v} = 0$, the Jacobi identity ensures that

$$\forall v\neq 0,\ \left\{\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_{y_2}\},\boldsymbol{\Upsilon}_\theta\right\}_{\mathbf{x},\theta,v} = 0. \tag{3.112}$$

Hence, dividing (3.112) by $\omega_\varepsilon(\mathbf{x}, v)$, we obtain that for $v \neq 0$, $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_{y_2}\}$ is solution of (3.110). Using now the same method as when proving Theorem 3.10, we obtain

$$\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_{y_2}\}_{\mathbf{x},\theta,v}(\mathbf{x}, \theta, v) = -\frac{\varepsilon}{B(\mathbf{x})} + v\epsilon_{y_1,y_2}(\mathbf{x}, \theta, v), \tag{3.113}$$

with $\epsilon_{y_1,y_2}(\mathbf{x}, \theta, v)$ such that for any $(\mathbf{x}, \theta)$, $v \mapsto \epsilon_{y_1,y_2}(\mathbf{x}, \theta, v)$ is bounded in the neighborhood of $v = 0$ and consequently that $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_{y_2}\}_{\mathbf{x},\theta,v}(\mathbf{x}, \theta, 0) = \frac{-\varepsilon}{B(\mathbf{x})}$.

As a conclusion, $\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_{y_2}\}_{\mathbf{x},\theta,v} = u$, and u is given by (3.109). Hence the theorem is proven. □

Since the entries $\left(\bar{\mathcal{P}}_\varepsilon\right)_{i,j}$ of $\bar{\mathcal{P}}_\varepsilon$ are given by $\left(\bar{\mathcal{P}}_\varepsilon\right)_{i,j} = \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\}$ and since we used the convention (3.6) we have enough information to state the following Corollary.

Corollary 3.21. *The Poisson Matrix in the Darboux Coordinate System is given by*

$$\bar{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 \\ \frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix}. \tag{3.114}$$

3.8 Expression of the Hamiltonian in the Darboux Coordinate System

In the Darboux Coordinate System, the Hamiltonian is given by $\bar{H}_\varepsilon(\mathbf{y}, \theta, k) = \tilde{H}_\varepsilon(\boldsymbol{\kappa}(\mathbf{y}, \theta, k))$. Since $\tilde{H}_\varepsilon(\mathbf{x}, \theta, v) = \frac{v^2}{2}$, we have

$$\bar{H}_\varepsilon(\mathbf{y}, \theta, k) = \frac{\boldsymbol{\kappa}_v^2(\mathbf{y}, \theta, k)}{2}. \tag{3.115}$$

Hence, according to Theorem 3.19, Hamiltonian function $\bar{H}_\varepsilon$ is regular with respect to ε on $\mathbb{R}_+$ and it admits an expansion in power of ε. More precisely, using expansion (3.103), we obtain the following corollaries.

Corollary 3.22. *The Hamiltonian function in the Darboux Coordinate System admits the following expansion in power of ε:*

$$\bar{H}_\varepsilon(\mathbf{y}, \theta, k) = \bar{H}_0(\mathbf{y}, k) + \sum_{n=1}^{N} \varepsilon^n \bar{H}_n(\mathbf{y}, \theta, k) + \varepsilon^{N+1}\iota_{N+1}(\varepsilon, \mathbf{y}, \theta, k), \tag{3.116}$$

where function ι_{N+1} is in $\mathcal{C}^\infty_\#(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty))$. Moreover, for any $n \in \{1, \ldots, N\}$ there exists a function $b_n \in \mathcal{O}^\infty_{T,b}$ such that

$$\bar{H}_n(\mathbf{y}, \theta, k) = \sqrt{k}^{n+2} b_n(\mathbf{y}, \theta), \tag{3.117}$$

with $\mathcal{O}^\infty_{T,b}$ defined by (3.99).

Here and hereafter, $\mathcal{Q}^{\infty}_{T,b}$ stands of the space of functions

$$\mathcal{Q}^{\infty}_{T,b} = \Bigg\{ f \in \mathcal{C}^{\infty}\left(\mathbb{R}^3 \times (0,+\infty)\right),\ f(\mathbf{y},\theta,k) = \sum_{n\in\mathbb{I}_f} c_n(\mathbf{y},\theta)\sqrt{k}^n \\ \text{where } \mathbb{I}_f \subset \mathbb{Z} \text{ is finite and } \forall n \in \mathbb{I}_f,\ c_n \in \mathcal{O}^{\infty}_{T,b} \Bigg\}. \tag{3.118}$$

Corollary 3.23. *The Hamiltonian function in the Darboux Coordinate System admits, up to order two, the following expansion in power of* ε*:*

$$\begin{aligned} \bar{H}_{\varepsilon}(\mathbf{y},\theta,k) = {} & B(\mathbf{y})\,k + \varepsilon \frac{\hat{\mathbf{a}}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{y})}{3B(\mathbf{y})^2}\left(2B(\mathbf{y})\,k\right)^{\frac{3}{2}} + \\ & \varepsilon^2 \frac{\left(2B(\mathbf{y})\,k\right)^2}{24B(\mathbf{y})^2}\Big[-\hat{\mathbf{a}}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{y}) + 3B(\mathbf{y})\,\hat{\mathbf{a}}(\theta)^T\,\mathcal{H}_B(\mathbf{y})\,\hat{\mathbf{a}}(\theta)\Big] + \\ & \varepsilon^3 \iota_3(\mathbf{y},\theta,k,\varepsilon), \end{aligned} \tag{3.119}$$

where $\hat{\mathbf{a}}$ *is defined by* (3.108)*, function* ι_3 *is in* $\mathcal{C}^{\infty}_{\#,3}(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\times\mathbb{R}_+)$*, and where* $\mathcal{H}_B$ *stands for the Hessian matrix associated with* B*.*

In expression (3.116), there is an important fact for the setting out of the to come Lie Transform based Method: the first term is independent of θ.

3.9 Characteristics in the Darboux Coordinate System

We denote by $(\mathbf{Y}^{\varepsilon}(t;\mathbf{y},\theta,k), \Theta^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k), \boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k))$ the characteristics expressed in the Darboux coordinate system. According to formula (2.15), these characteristics satisfy

$$\frac{\partial \begin{pmatrix} \mathbf{Y}^{\varepsilon} \\ \Theta^{\varepsilon}_{\mathfrak{Dar}} \\ \boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}} \end{pmatrix}}{\partial t} = \bar{\mathcal{P}}_{\varepsilon}(\mathbf{Y}^{\varepsilon})\,\nabla\bar{H}_{\varepsilon}(\mathbf{Y}^{\varepsilon},\Theta^{\varepsilon}_{\mathfrak{Dar}},\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}), \tag{3.120}$$

equipped with $(\mathbf{Y}^{\varepsilon}(0;\mathbf{y},\theta,k), \Theta^{\varepsilon}_{\mathfrak{Dar}}(0;\mathbf{y},\theta,k), \boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(0;\mathbf{y},\theta,k)) = (\mathbf{y},\theta,k)$, where $\bar{\mathcal{P}}_{\varepsilon}$ is the Poisson matrix expressed in the Darboux coordinate system and given by formula (3.114), and $\bar{H}_{\varepsilon}$ is the Hamiltonian function expressed in the Darboux Coordinate System and given by (3.115). The characteristic expressed in the Darboux Coordinate System is related with the periodic extension of the characteristic expressed in the Polar in velocity Coordinate System (see Definition 2.1 and Lemma 2.2) by

$$Y_1^{\varepsilon}(t;\mathbf{y},\theta,k) = \boldsymbol{\Upsilon}_{y_1}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \Theta^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \mathcal{V}^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k))\Big), \tag{3.121}$$

$$Y_2^{\varepsilon}(t;\mathbf{y},\theta,k) = \boldsymbol{\Upsilon}_{y_2}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \Theta^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \mathcal{V}^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k))\Big), \tag{3.122}$$

$$\Theta^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k) = \Theta^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \tag{3.123}$$

$$\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k) = \boldsymbol{\Upsilon}_{k}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \Theta^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \mathcal{V}^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k))\Big). \tag{3.124}$$

The purpose of this subsection is the two following theorems:

Theorem 3.24. *Let $[a,b]$ be an interval such that $[a,b] \subset (0,+\infty)$. Then, for any $(\mathbf{y},\theta) \in \mathbb{R}^3$ and for any $v \in [a,b]$, $\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v) \in \left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right]$. And, for any $(\mathbf{y},\theta,k) \in \boldsymbol{\Upsilon}\left(\mathbb{R}^3 \times [a,b]\right)$, for any $\varepsilon \in (0,+\infty)$, and for any $t \in \mathbb{R}$, $\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k) \in \left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right]$.*

Theorem 3.25. *Let $[a,b]$ be an interval such that $[a,b] \subset (0,+\infty)$, $\mathbf{x}_0 \in \mathbb{R}^2$, and $R_{\mathbf{x}_0}$ and $R'_{\mathbf{x}_0}$ be two real numbers satisfying $0 < R_{\mathbf{x}_0} < R'_{\mathbf{x}_0}$. Then, for any $\varepsilon \in \left(-\frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{b}, \frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{b}\right)$,*

$$\boldsymbol{\Upsilon}\left(\overline{\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})} \times \mathbb{R} \times [a,b]\right) \subset \mathfrak{b}^2(\mathbf{x}_0,R'_{\mathbf{x}_0}) \times \mathbb{R} \times \left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right]. \tag{3.125}$$

Moreover, there exists two positive real numbers α_0 and η, such that for any $\varepsilon \in (0,\eta)$, there exists a real number $t^\varepsilon_e > \frac{\alpha_0}{\varepsilon}$, such that for any $t \in (-t^\varepsilon_e, t^\varepsilon_e)$ and for any $(\mathbf{y},\theta,k) \in \boldsymbol{\Upsilon}\left(\overline{\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})} \times \mathbb{R} \times [a,b]\right)$,

$$\mathbf{Y}^\varepsilon(t;\mathbf{y},\theta,k) \in \mathfrak{b}^2(\mathbf{x}_0,R'_{\mathbf{x}_0}). \tag{3.126}$$

We will prove Theorems 3.24 and 3.25 in subsection 3.10.

3.10 Proof of Theorems 3.24 and 3.25

By definition $\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v) = \int_0^v \psi(\mathbf{x},\theta,s)\,ds$, and $\psi(\mathbf{x},\theta,s) = \int_0^s \varphi(\mathbf{x},\theta,u)\,du$ where ψ is defined by (3.57) and where φ is given by Theorem 3.7. Hence,

$$\psi(\mathbf{x},\theta,s) = \int_0^s \varphi(\mathbf{x},\theta,u)\,du = \int_0^s \frac{1}{B\left(\mathcal{G}^1_{-\varepsilon u}(\mathbf{x},\theta), \mathcal{G}^2_{-\varepsilon u}(\mathbf{x},\theta)\right)}du \geq \frac{s}{\|B\|_\infty}, \tag{3.127}$$

and consequently, for any $v \in [a,b]$ and for any $(\mathbf{x},\theta) \in \mathbb{R}^2 \times \mathbb{R}$, we obtain

$$\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v) = \int_0^v \psi(\mathbf{x},\theta,s)\,ds \geq \frac{v^2}{2\|B\|_\infty} \geq \frac{a^2}{2\|B\|_\infty}. \tag{3.128}$$

On another hand, since $\inf_{\mathbf{x}\in\mathbb{R}^2} B(\mathbf{x}) \geq 1$, we obtain $\psi(\mathbf{x},\theta,s) \leq s$, and consequently for any $v \in [a,b]$ and for any $(\mathbf{x},\theta)$, we obtain

$$\boldsymbol{\Upsilon}_k(\mathbf{x},\theta,v) \leq \frac{v^2}{2} \leq \frac{b^2}{2}. \tag{3.129}$$

According to Lemma 2.2 and formula (2.30), for any $(\mathbf{x},\theta,v) \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$ and for any $t \in \mathbb{R}$, $\mathcal{V}^{\varepsilon,\#}(t;\mathbf{x},\theta,v) = v$, and consequently for any $(\mathbf{y},\theta,k) \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$ and for any $t \in \mathbb{R}$, formula (3.124) can be rewritten:

$$\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k) = \boldsymbol{\Upsilon}_k\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \Theta^{\varepsilon,\#}(t,\boldsymbol{\kappa}(\mathbf{y},\theta,k)), \boldsymbol{\kappa}_v(\mathbf{y},\theta,k)\right). \tag{3.130}$$

Now, for any $(\mathbf{y},\theta,k) \in \boldsymbol{\Upsilon}\left(\mathbb{R}^2 \times \mathbb{R} \times [a,b]\right)$, $\boldsymbol{\kappa}_v(\mathbf{y},\theta,k) \in [a,b]$ and estimates (3.128) and (3.129) yield that $\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}(t;\mathbf{y},\theta,k) \in \left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right]$. This ends the proof of Theorem 3.24. $\square$

Concerning Theorem 3.25, firstly, for any $(\mathbf{x},\theta)\in\mathbb{R}^2\times\mathbb{R}$ and for any $v\in[a,b]$, function ψ satisfies $|\psi(\mathbf{x},\theta,v)|\leq b$. Consequently, for any $(\mathbf{x},\theta,v)\in\overline{\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})}\times\mathbb{R}\times[a,b]$ and for any $\varepsilon\in\mathbb{R}$ we have:

$$|(\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v),\boldsymbol{\Upsilon}_{y_2}(\mathbf{x},\theta,v))-\mathbf{x}_0|\leq R_{\mathbf{x}_0}+|\varepsilon|\,b. \tag{3.131}$$

Eventually, since for any $\varepsilon\in\mathbb{R}$, $\boldsymbol{\Upsilon}_k$ satisfies (3.128) and (3.129), and since $(\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_{y_2})$ satisfies (3.131), we obtain (3.125).

Applying formula (3.63) with $n=1$ yields:

$$\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v)=\boldsymbol{\Upsilon}^s_{y_1}(\mathbf{x},\theta,v)+\boldsymbol{\Upsilon}^b_{y_1}(\mathbf{x},\theta,v), \tag{3.132}$$

where

$$\begin{aligned}\boldsymbol{\Upsilon}^s_{y_1}(\mathbf{x},\theta,v)&=x_1-\varepsilon v\frac{\cos(\theta)}{B(\mathbf{x})},\\ \boldsymbol{\Upsilon}^b_{y_1}(\mathbf{x},\theta,v)&=-\varepsilon^2\cos(\theta)\int_0^v(v-u)\left(\boldsymbol{\Lambda}\cdot\frac{1}{B}\right)(\mathcal{G}_{-\varepsilon u}(\mathbf{x},\theta))\,du.\end{aligned} \tag{3.133}$$

For any $(\mathbf{x},\theta)\in\mathbb{R}^2\times\mathbb{R}$, for any $v\in[a,b]$ and for any $\varepsilon\in\mathbb{R}$ we have:

$$\left|\boldsymbol{\Upsilon}^b_{y_1}(\mathbf{x},\theta,v)\right|\leq\frac{\varepsilon^2b^2}{2}\left\|\boldsymbol{\Lambda}\cdot\frac{1}{B}\right\|_\infty, \tag{3.134}$$

and consequently for any $(\mathbf{x},\theta)\in\mathbb{R}^2\times\mathbb{R}$, for any $v\in[a,b]$, for any $\varepsilon\in\mathbb{R}^\star$ and for any $t\in\mathbb{R}$

$$\left|\boldsymbol{\Upsilon}^b_{y_1}\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\mathbf{x},\theta,v),\Theta^{\varepsilon,\#}(t,\mathbf{x},\theta,v),v\right)\right|\leq\frac{\varepsilon^2b^2}{2}\left\|\boldsymbol{\Lambda}\cdot\frac{1}{B}\right\|_\infty. \tag{3.135}$$

On another hand, evaluating $\boldsymbol{\Upsilon}^s_{y_1}$ in $\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\mathbf{x},\theta,v),\Theta^{\varepsilon,\#}(t,\mathbf{x},\theta,v),v\right)$ and differentiating with respect to t yields:

$$\frac{\partial}{\partial t}\left(\boldsymbol{\Upsilon}^s_{y_1}\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}},\Theta^{\varepsilon,\#},v\right)\right)=\varepsilon v^2\cos\left(\Theta^{\varepsilon,\#}\right)\frac{\hat{\mathbf{c}}\left(\Theta^{\varepsilon,\#}\right)\cdot\nabla_{\mathbf{x}}B\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}\right)}{B\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}\right)^2}, \tag{3.136}$$

where

$$\hat{\mathbf{c}}(\theta)=\begin{pmatrix}-\sin(\theta)\\-\cos(\theta)\end{pmatrix}, \tag{3.137}$$

was already used in the introduction, and consequently

$$\left|\frac{\partial}{\partial t}\left(\boldsymbol{\Upsilon}^s_{y_1}\left(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}},\Theta^{\varepsilon,\#},v\right)\right)\right|\leq|\varepsilon|\,b^2\sup_{(\mathbf{x},\theta)\in\mathbb{R}^3}\left|\frac{\hat{\mathbf{c}}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{x})}{B(\mathbf{x})^2}\right|. \tag{3.138}$$

Combining estimates (3.134), (3.135) and (3.138) yields that for any $(\mathbf{x},\theta)\in\mathbb{R}^2\times\mathbb{R}$, for any $v\in[a,b]$, for any $\varepsilon\in\mathbb{R}^\star$ and for any $t\in\mathbb{R}$

$$\begin{aligned}\Big|\boldsymbol{\Upsilon}_{y_1}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\mathbf{x},\theta,v),\Theta^{\varepsilon,\#}(t,\mathbf{x},\theta,v),v\Big)-\boldsymbol{\Upsilon}_{y_1}(\mathbf{x},\theta,v)\Big|\\ \leq |t|\,|\varepsilon|\,b^2\sup_{(\mathbf{x},\theta)\in\mathbb{R}^3}\left|\frac{\hat{\mathbf{c}}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{x})}{B(\mathbf{x})^2}\right|+\varepsilon^2 b^2\left\|\boldsymbol{\Lambda}\cdot\frac{1}{B}\right\|_\infty.\end{aligned}\tag{3.139}$$

The same estimate holds true by replacing $\boldsymbol{\Upsilon}_{y_1}$ by $\boldsymbol{\Upsilon}_{y_2}$.

Using estimates (3.139) and (3.131), we obtain for any $(\mathbf{x},\theta,v)\in\overline{\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})}\times\mathbb{R}\times[a,b]$, for any $t\in\mathbb{R}$ and for any $\varepsilon\in\mathbb{R}^\star$

$$\begin{aligned}&\Big|\Big(\boldsymbol{\Upsilon}_{y_1}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\mathbf{x},\theta,v),\Theta^{\varepsilon,\#}(t,\mathbf{x},\theta,v),v\Big),\boldsymbol{\Upsilon}_{y_2}\Big(\mathbf{X}^{\varepsilon,\#}_{\mathfrak{Pol}}(t,\mathbf{x},\theta,v),\Theta^{\varepsilon,\#}(t,\mathbf{x},\theta,v),v\Big)\Big)-\mathbf{x}_0\Big|\\ &\leq\sqrt{2}\left(|t|\,|\varepsilon|\,b^2\alpha_1+\varepsilon^2b^2\alpha_2\right)+\left(R_{\mathbf{x}_0}+|\varepsilon|\,b\right),\end{aligned}$$

where $\alpha_1=\sup_{(\mathbf{x},\theta)\in\mathbb{R}^3}\left|\frac{\hat{\mathbf{c}}(\theta)\cdot\nabla_{\mathbf{x}}B(\mathbf{x})}{B(\mathbf{x})^2}\right|$ and $\alpha_2=\left\|\boldsymbol{\Lambda}\cdot\frac{1}{B}\right\|_\infty$. The right hand side of the above estimate is smaller than $R'_{\mathbf{x}_0}$ if and only if t satisfies

$$|t|<\frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{\sqrt{2}\,|\varepsilon|\,b^2\alpha_1}-\frac{1}{\sqrt{2}b\alpha_1}-|\varepsilon|\,\frac{\alpha_2}{\alpha_1}.\tag{3.140}$$

Let $\eta\in(0,+\infty)$ be such that

$$\frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{\sqrt{2}b^2\alpha_1}-\frac{\eta}{\sqrt{2}b\alpha_1}-\eta^2\frac{\alpha_2}{\alpha_1}>0.\tag{3.141}$$

Hence, setting

$$\alpha_0=\frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{\sqrt{2}b^2\alpha_1}-\frac{\eta}{\sqrt{2}b\alpha_1}-\eta^2\frac{\alpha_2}{\alpha_1}\tag{3.142}$$

yields that for any ε in $(-\eta,\eta)\setminus\{0\}$,

$$\frac{\alpha_0}{|\varepsilon|}\leq\frac{R'_{\mathbf{x}_0}-R_{\mathbf{x}_0}}{\sqrt{2}\,|\varepsilon|\,b^2\alpha_1}-\frac{1}{\sqrt{2}b\alpha_1}-|\varepsilon|\,\frac{\alpha_2}{\alpha_1},\tag{3.143}$$

and consequently formulas (3.121) and (3.122) yield (3.126). This ends the proof of Theorem 3.25. □

3.11 Consistency with the Torus

The change of coordinate map $\boldsymbol{\Upsilon}=(\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_\theta,\boldsymbol{\Upsilon}_k)$ is such that components 1, 2 and 4 are 2π-periodic with respect to θ, and the penultimate component is given by $\boldsymbol{\Upsilon}_\theta(\mathbf{x},\theta,v)=\theta$. Hence the map $\mathfrak{p}\circ\boldsymbol{\Upsilon}$, where $\mathfrak{p}$ is the canonical projection on the torus, induces a $\mathcal{C}^\infty$-diffeomorphism

$$\boldsymbol{\Upsilon}^\circ:\ \mathbb{R}^2\times(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}\to\mathbb{R}^2\times(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R},\tag{3.144}$$

such that $\mathfrak{p}\circ\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^{\circ}\circ\mathfrak{p}$. We denote by $\mathbf{Y}^{\varepsilon,\circ}$, $\Theta^{\varepsilon,\circ}_{\mathfrak{Dar}}$ and $\boldsymbol{\mathcal{K}}^{\varepsilon,\circ}_{\mathfrak{Dar}}$ the expression of the characteristics solution to (1.40)–(1.41) and expressed in the coordinate system $(\mathbf{y},\theta,v) = \boldsymbol{\Upsilon}^{\circ}(\mathbf{x},\theta,v)$. Then, these characteristics are solutions to the dynamical system (3.121)–(3.124) (viewed as a dynamical system on $\mathbb{R}^2 \times \mathbb{R}\,|_{2\pi\mathbb{Z}} \times (0,+\infty)$), and they satisfy

$$\mathfrak{p}\circ\left(\mathbf{Y}^{\varepsilon},\Theta^{\varepsilon}_{\mathfrak{Dar}},\boldsymbol{\mathcal{K}}^{\varepsilon}_{\mathfrak{Dar}}\right)(t,\cdot) = \left(\mathbf{Y}^{\varepsilon,\circ},\Theta^{\varepsilon,\circ}_{\mathfrak{Dar}},\boldsymbol{\mathcal{K}}^{\varepsilon,\circ}_{\mathfrak{Dar}}\right)(t,\cdot)\circ\mathfrak{p}. \tag{3.145}$$

4 The Partial Lie Sums

4.1 Objectives

As a result of the Darboux Algorithm, we obtained a Poisson Matrix $\bar{\mathcal{P}}_{\varepsilon}(\mathbf{y},\theta,k)$ (see (3.114)) with the required form to apply the Key Result (Theorem 1.2), but the resulting Hamiltonian Function given by (3.116) depends on θ. In order to be under the assumptions of the Key Result, we would need to make this dependency to vanish. Since $\bar{H}_0$ does not depend on θ, it seems to be possible to make the penultimate coordinate vanish using a mapping parametrized by ε and close to the identity map for small ε.

Moreover, we would like to build this mapping in such a way that it does not change the Poisson Matrix expression. This means that, as regarded as functions, $\bar{\mathcal{P}}_{\varepsilon}$ and the expression $\hat{\mathcal{P}}_{\varepsilon}$ of the Poisson Matrix in the sought coordinate system would need to be the same; i.e.:

$$\bar{\mathcal{P}}_{\varepsilon}(\mathbf{z},\gamma,j) = \hat{\mathcal{P}}_{\varepsilon}(\mathbf{z},\gamma,j) \;\text{ or }\; \bar{\mathcal{P}}_{\varepsilon}(\mathbf{y},\theta,k) = \hat{\mathcal{P}}_{\varepsilon}(\mathbf{y},\theta,k). \tag{4.1}$$

for any $(\mathbf{z},\gamma,j)$ or $(\mathbf{y},\theta,k)$. Changes of variables having this property are called symplectic. It is well-known that, in the case of a Poisson Matrix that does not depend on ε, flows of Hamiltonian vector fields, parametrized by ε, are symplectic and moreover close to the identity map for small values of their parameter.
In the case we are dealing with, the Poisson Matrix depends on ε. In general, as illustrated by the example of Appendix A, flows of Hamiltonian vector fields are no longer symplectic.

In order to avoid this problem, in Littlejohn [25, 26, 27], $\bar{H}_{\varepsilon}$ is formally expanded as a series. Then a Lie Transform method based on the use of Lie Series

$$S_L^{\infty}\left(\varepsilon\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)\cdot = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\cdot, \tag{4.2}$$

where $\bar{f} = \bar{f}(\mathbf{y},\theta,k)$ is a smooth function and where $\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}$ is the smooth Hamiltonian vector field defined by

$$\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}} = \varepsilon\bar{\mathcal{P}}_{\varepsilon}\nabla\bar{f}, \tag{4.3}$$

is developed. Notice that $\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}$ is the Hamiltonian vector field associated with Hamiltonian Function $\varepsilon\bar{f}$. Formally, i.e. if convergence of the series are not considered, the map $(\mathbf{y},\theta,k) \mapsto S_L^{\infty}\left(\varepsilon\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)\cdot(\mathbf{y},\theta,k)$ is symplectic and from this Lie Series, a symplectic and close-to-identity change-of-coordinates mapping may be built such that, in the resulting coordinate system, the Hamiltonian function, which is expressed as a series, does not depend on the penultimate variable. The drawback of using such a formal Lie Series method is that its convergence is neither ensured nor controlled.

Unfortunately, building a coordinate system that satisfies the assumptions of the Key Theorem can only be led in a formal way and not in a mathematical rigorous way. Hence, in order to insure its existence, we will rather build a coordinate system satisfying the assumption of the following variant of Theorem 1.2:

Theorem 4.1. *Let $\mathbf{z}_0 \in \mathbb{R}^2$ and $\mathfrak{b}^2(\mathbf{z}_0, R_{\mathbf{z}_0}) \subset \mathbb{R}^2$ be the open ball of radius $R_{\mathbf{z}_0}$; $\mathcal{O}$ be the open subset of $\mathbb{R}^4$ defined by $\mathcal{O} = \mathfrak{b}^2(\mathbf{z}_0, R_{\mathbf{z}_0}) \times \mathbb{R} \times (a,b)$, where $[a,b] \subset (0,+\infty)$; $\mathcal{O}'$ be an open subset such that $\overline{\mathcal{O}'} \subset \mathfrak{b}^2(\mathbf{z}_0, R'_{\mathbf{z}_0}) \times \mathbb{R} \times (c,d)$, where $0 < R'_{\mathbf{z}_0} < R_{\mathbf{z}_0}$ and $[c,d] \subset (a,b)$; and $\mathbf{r} = (r_1, r_2, r_3, r_4)$ be a coordinate system on $\mathcal{O}$. If, there exists a real number η such that for any $\varepsilon \in (0,\eta)$ the Poisson Matrix, expressed in the coordinate system $\mathbf{r}$, has the following form for $N \in \mathbb{N}^*$:*

$$\mathcal{P}_\varepsilon(\mathbf{r}) = \bar{\mathcal{P}}_\varepsilon(\mathbf{r}) + \varepsilon^N \boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon, \mathbf{r}), \tag{4.4}$$

where

$$\bar{\mathcal{P}}_\varepsilon(\mathbf{r}) = \left(\begin{array}{cc|cc} 0 & -\dfrac{\varepsilon}{B(r_1,r_2)} & 0 & 0 \\ \dfrac{\varepsilon}{B(r_1,r_2)} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dfrac{1}{\varepsilon} \\ 0 & 0 & -\dfrac{1}{\varepsilon} & 0 \end{array}\right), \tag{4.5}$$

and where $\boldsymbol{\rho}^N_{\mathcal{P}}$ is in $\mathcal{C}^\infty_\#([0,\eta)\times\mathcal{O})$ (see (3.31)); if for any $\varepsilon \in [0,\eta)$ the Hamiltonian function H_ε, expressed in the coordinate system $\mathbf{r}$, writes

$$H_\varepsilon(\mathbf{r}) = H_{\varepsilon,T}(r_1, r_2, r_4) + \varepsilon^{N+1} \rho_H(\varepsilon, \mathbf{r}), \tag{4.6}$$

where ρ_H is in $\mathcal{C}^\infty_\#([0,\eta) \times \mathcal{O})$; if

$$H_{\varepsilon,T}(r_1, r_2, r_4) = H_0(r_1, r_2, r_4) + \ldots + \varepsilon^N H_N(r_1, r_2, r_4), \tag{4.7}$$

where H_i, $i = 0 \ldots N-1$ are in $\mathcal{Q}^\infty_{T,b}$ (see (3.118)); if

$$R_{\mathbf{z}_0} > 1 + R'_{\mathbf{z}_0} + \sqrt{2} \sup_{\varepsilon \in [-\eta,\eta]} \frac{\|\nabla H_{\varepsilon,T}\|_\infty}{\|B\|_\infty}; \tag{4.8}$$

and if for any $\varepsilon \in (0,\eta)$ there exists a time $t^\varepsilon_e > \frac{\alpha_0}{\varepsilon}$, where $\alpha_0 \in (0,+\infty)$ depends only on the difference between $R_{\mathbf{z}_0}$ and $R'_{\mathbf{z}_0}$, such that for any $t \in (-t^\varepsilon_e, t^\varepsilon_e)$ and for any $\mathbf{r} \in \overline{\mathcal{O}'}$, the trajectory $\mathbf{R}^\varepsilon(t;\mathbf{r})$ solution to dynamical system:

$$\frac{\partial \mathbf{R}^\varepsilon}{\partial t} = \mathcal{P}_\varepsilon(\mathbf{R}^\varepsilon) \nabla H_\varepsilon(\mathbf{R}^\varepsilon), \quad \mathbf{R}^\varepsilon(0) = \mathbf{r}, \tag{4.9}$$

belongs to $\mathcal{O}$; then, the following truncated dynamical system

$$\frac{\partial \mathbf{R}^\varepsilon_T}{\partial t} = \bar{\mathcal{P}}_\varepsilon(\mathbf{R}^\varepsilon_T) \nabla H_{\varepsilon,T}(\mathbf{R}^\varepsilon_T), \quad \mathbf{R}^\varepsilon_T(0) = \mathbf{r}, \tag{4.10}$$

is Hamiltonian of Hamiltonian function $H_{\varepsilon,T}(r_1, r_2, r_4)$ *and satisfies the assumptions of Theorem 1.2, so that* $\mathbf{R}^\varepsilon_T$ *satisfies the conclusions of this same Theorem 1.2.*

Moreover, $\underline{\mathbf{R}^\varepsilon} = (\mathbf{R}^\varepsilon_1, \mathbf{R}^\varepsilon_2, \mathbf{R}^\varepsilon_4)$ *a priori defined for* $\varepsilon \in (0,\eta)$ *is extensible to* $[0,\eta)$ *and this extension, also denoted by* $\underline{\mathbf{R}^\varepsilon}$*, belongs to* $\mathcal{C}^{N-1}([0,\eta))$ *for any* $\mathbf{r} \in \mathcal{O}'$*; and, for any* $\varepsilon \in (0,\eta)$*, for any* $t \in \left(-\frac{\min(1,\alpha_0)}{\varepsilon}, \frac{\min(1,\alpha_0)}{\varepsilon}\right)$ *and for any* $\mathbf{r} \in \overline{\mathcal{O}'}$*,* $\underline{\mathbf{L}^\varepsilon} = (\mathbf{L}^\varepsilon_1, \mathbf{L}^\varepsilon_2, \mathbf{L}^\varepsilon_4)$ *defined by*

$$\underline{\mathbf{L}^\varepsilon} = \frac{\underline{\mathbf{R}^\varepsilon} - \underline{\mathbf{R}^\varepsilon_T}}{\varepsilon^{N-1}} \tag{4.11}$$

is extensible to $[0,\eta)$ *and this extension, also denoted by* $\underline{\mathbf{L}^\varepsilon}$*, is smooth and continuous with respect to* ε*. Eventually, for any* $\alpha \in (0,\eta)$*, we have, for any* $\varepsilon \in [0,\alpha]$ *and for any* $t \in \left(-\frac{\min(1,\alpha_0)}{\varepsilon}, \frac{\min(1,\alpha_0)}{\varepsilon}\right)$*,*

$$\left\| \underline{\mathbf{R}^\varepsilon} - \underline{\mathbf{R}^\varepsilon_T} \right\|_{\infty,\mathcal{O}'} \leq |\varepsilon|^{N-1} \sup_{\varepsilon\in[0,\alpha]} \|\underline{\mathbf{L}^\varepsilon}\|_{\infty,\overline{\mathcal{O}'}}, \tag{4.12}$$

where $\underline{\mathbf{R}^\varepsilon_T} = ((\mathbf{R}^\varepsilon_T)_1, (\mathbf{R}^\varepsilon_T)_2, (\mathbf{R}^\varepsilon_T)_4)$ *and where* $\|g\|_{\infty,\mathcal{O}'}$ *stands for*

$$\|g\|_{\infty,\mathcal{O}'} = \sup_{\mathbf{r}\in\mathcal{O}'} |g(\mathbf{r})|. \tag{4.13}$$

.

Proof. In a first place, since $\bar{\mathcal{P}}_\varepsilon$ is a Poisson Matrix and since $H_{\varepsilon,T}$ is smooth, dynamical system (4.10) is Hamiltonian of Hamiltonian function $H_{\varepsilon,T}$.

Setting

$$\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r}) = \begin{pmatrix} \left(\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r})\right)^{\mathrm{TL}} & \left(\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r})\right)^{\mathrm{TR}} \\ \left(\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r})\right)^{\mathrm{BL}} & \left(\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r})\right)^{\mathrm{BR}} \end{pmatrix} = \left(\left(\boldsymbol{\rho}^N_{\mathcal{P}}(\varepsilon,\mathbf{r})\right)^{i,j}\right)_{i,j=1,\dots 4}, \tag{4.14}$$

and using the skew-symmetry of $\mathcal{P}_\varepsilon$ in (4.4) yields:

$$\mathcal{P}_\varepsilon(\mathbf{r}) =$$
$$\begin{pmatrix} 0 & \bar{\mathcal{P}}^{1,2}_\varepsilon + \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,2} & \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,3} & \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,4} \\ -\bar{\mathcal{P}}^{1,2}_\varepsilon - \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,2} & 0 & \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{2,3} & \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{2,4} \\ -\varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,3} & -\varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{2,3} & 0 & \frac{1}{\varepsilon} + \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{3,4} \\ -\varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{1,4} & -\varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{2,4} & -\frac{1}{\varepsilon} - \varepsilon^N\left(\boldsymbol{\rho}^N_{\mathcal{P}}\right)^{3,4} & 0 \end{pmatrix}. \tag{4.15}$$

The second part of the proof consists essentially in checking that $\underline{\mathbf{R}^\varepsilon}$ is in $\mathcal{C}^{N-1}([0,\eta))$. In order to check this, we define for any $\varepsilon \in (0,\eta)$, for any $t \in \left(-\frac{t^\varepsilon_e}{\varepsilon}, \frac{t^\varepsilon_e}{\varepsilon}\right)$, and for any $\mathbf{r} \in \overline{\mathcal{O}'}$, $\widetilde{\mathbf{R}^\varepsilon} = \widetilde{\mathbf{R}^\varepsilon}(t,\mathbf{r})$ by

$$\widetilde{\mathbf{R}^\varepsilon}(t,\mathbf{r}) = \mathbf{R}^\varepsilon(\varepsilon t,\mathbf{r}). \tag{4.16}$$

It satisfies

$$\frac{\partial \widetilde{\mathbf{R}^{\varepsilon}}}{\partial t}(t) = \varepsilon \mathcal{P}_{\varepsilon}(\widetilde{\mathbf{R}^{\varepsilon}}(t)) \nabla H_{\varepsilon}\Big(\widetilde{\mathbf{R}^{\varepsilon}}(t)\Big), \tag{4.17}$$

Since $\varepsilon \mapsto \varepsilon \mathcal{P}_{\varepsilon}(\mathbf{r})$ is in $\mathcal{C}^{\infty}([0,\eta))$ for any $\mathbf{r} \in \mathcal{O}$, the solution of (4.17) depends smoothly on the parameter ε. In particular function $\widetilde{\mathbf{R}^{\varepsilon}}$, defined by (4.16), is smoothly extensible at $\varepsilon = 0$. Notice that for $\varepsilon = 0$, we obtain

$$\widetilde{\mathbf{R}^{0}}(t, \mathbf{r}) = \left(r_1, r_2, r_3 + t \frac{\partial H_0}{\partial r_4}(r_1, r_2, r_4), r_4 \right). \tag{4.18}$$

On another hand, for any $\varepsilon \in (0, \eta)$, for any $\mathbf{r} \in \overline{\mathcal{O}'}$, and for any $t \in (-t_e^{\varepsilon}, t_e^{\varepsilon})$, $\underline{\mathbf{R}}^{\varepsilon}(t, \mathbf{r}) = (\mathbf{R}_1^{\varepsilon}, \mathbf{R}_2^{\varepsilon}, \mathbf{R}_4^{\varepsilon})(t, \mathbf{r})$ is solution to

$$\begin{aligned}
&\frac{\partial \begin{pmatrix} \mathbf{R}_1^{\varepsilon} \\ \mathbf{R}_2^{\varepsilon} \end{pmatrix}}{\partial t} = \mathtt{M}_{\varepsilon}(\mathbf{R}_1^{\varepsilon}, \mathbf{R}_2^{\varepsilon}) \begin{pmatrix} \dfrac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \dfrac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} (\underline{\mathbf{R}}^{\varepsilon}) + \\
&\varepsilon^N \left[\varepsilon \mathtt{M}_{\varepsilon} \begin{pmatrix} \dfrac{\partial \rho_H}{\partial r_1} \\ \dfrac{\partial \rho_H}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{\mathrm{TL}} \begin{pmatrix} \dfrac{\partial H_{\varepsilon}}{\partial r_1} \\ \dfrac{\partial H_{\varepsilon}}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{\mathrm{TR}} \begin{pmatrix} \dfrac{\partial H_{\varepsilon}}{\partial r_3} \\ \dfrac{\partial H_{\varepsilon}}{\partial r_4} \end{pmatrix} \right] \left(\mathbf{R}_1^{\varepsilon}, \mathbf{R}_2^{\varepsilon}, \widetilde{\mathbf{R}_3^{\varepsilon}}\left(\frac{t}{\varepsilon}, \mathbf{r}\right), \mathbf{R}_4^{\varepsilon} \right), \\
&\frac{\partial \mathbf{R}_4^{\varepsilon}}{\partial t} = -\varepsilon^N \left[\big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{1,4} \frac{\partial H_{\varepsilon}}{\partial r_1} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{2,4} \frac{\partial H_{\varepsilon}}{\partial r_2} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{3,4} \frac{\partial H_{\varepsilon}}{\partial r_3} + \frac{\partial \rho_H}{\partial r_3}(\varepsilon, .) \right] \\
&\qquad\qquad \left(\mathbf{R}_1^{\varepsilon}, \mathbf{R}_2^{\varepsilon}, \widetilde{\mathbf{R}_3^{\varepsilon}}\left(\frac{t}{\varepsilon}, \mathbf{r}\right), \mathbf{R}_4^{\varepsilon} \right),
\end{aligned} \tag{4.19}$$

where

$$\mathtt{M}_{\varepsilon}(r_1, r_2) = \begin{pmatrix} 0 & -\dfrac{\varepsilon}{B(r_1, r_2)} \\ \dfrac{\varepsilon}{B(r_1, r_2)} & 0 \end{pmatrix}. \tag{4.20}$$

Notice that, in this system, $\widetilde{\mathbf{R}_3^{\varepsilon}}$ is known and then considered as given. Beside this,

$$\varepsilon \mathtt{M}_{\varepsilon} \begin{pmatrix} \dfrac{\partial \rho_H}{\partial r_1} \\ \dfrac{\partial \rho_H}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{\mathrm{TL}} \begin{pmatrix} \dfrac{\partial H_{\varepsilon}}{\partial r_1} \\ \dfrac{\partial H_{\varepsilon}}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{\mathrm{TR}} \begin{pmatrix} \dfrac{\partial H_{\varepsilon}}{\partial r_3} \\ \dfrac{\partial H_{\varepsilon}}{\partial r_4} \end{pmatrix}, \tag{4.21}$$

and

$$\big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{1,4} \frac{\partial H_{\varepsilon}}{\partial r_1} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{2,4} \frac{\partial H_{\varepsilon}}{\partial r_2} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon, .)\big)^{3,4} \frac{\partial H_{\varepsilon}}{\partial r_3} + \frac{\partial \rho_H}{\partial r_3}(\varepsilon, .) \tag{4.22}$$

are 2π-periodic and smooth, and consequently $\mathcal{C}_b^{\infty}(\mathbb{R})$ with respect to the third variable r_3. Hence, computing the successive derivatives of (4.19) with respect to ε, we obtain that

$\varepsilon \mapsto \underline{\mathbf{R}}^\varepsilon(t)$ is $\mathcal{C}^{N-1}$ in the neighborhood of $\varepsilon = 0$.
Moreover, as $\underline{\mathbf{R}_T^\varepsilon} = ((\mathbf{R}_T^\varepsilon)_1, (\mathbf{R}_T^\varepsilon)_2, (\mathbf{R}_T^\varepsilon)_4)$ is solution to

$$\begin{aligned} &\frac{\partial \begin{pmatrix} (\mathbf{R}_T^\varepsilon)_1 \\ (\mathbf{R}_T^\varepsilon)_2 \end{pmatrix}}{\partial t} = \mathtt{M}_\varepsilon((\mathbf{R}_T^\varepsilon)_1, (\mathbf{R}_T^\varepsilon)_2) \begin{pmatrix} \dfrac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \dfrac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} \left(\underline{\mathbf{R}_T^\varepsilon}\right), \\ &\frac{\partial (\mathbf{R}_T^\varepsilon)_4}{\partial t} = 0, \end{aligned} \tag{4.23}$$

$\underline{\mathbf{R}_T^\varepsilon}$ is smooth with respect to ε, for any $t \in \mathbb{R}$ and for any $\mathbf{r} \in \mathbb{R}^3 \times (0, +\infty)$. Moreover, for any $\mathbf{r} = (\mathbf{z}, r_3, r_4)$, we have

$$\left| \begin{pmatrix} (\mathbf{R}_T^\varepsilon)_1 \\ (\mathbf{R}_T^\varepsilon)_2 \end{pmatrix} (t, \mathbf{r}) - \mathbf{z} \right| \leq \sqrt{2}\, |\varepsilon|\, |t| \, \frac{\|\nabla H_{\varepsilon,T}\|_\infty}{\|B\|_\infty}. \tag{4.24}$$

Consequently, assumption (4.8) yields that for any $\varepsilon \in (-\eta, \eta)$, for any $t \in \left(-\frac{1}{|\varepsilon|}, \frac{1}{|\varepsilon|}\right)$ (By convention if $\varepsilon = 0$, $\left(-\frac{1}{|\varepsilon|}, \frac{1}{|\varepsilon|}\right) = \mathbb{R}$) and for any $\mathbf{r} \in \overline{\mathcal{O}'}$,

$$\begin{pmatrix} (\mathbf{R}_T^\varepsilon)_1 \\ (\mathbf{R}_T^\varepsilon)_2 \end{pmatrix} (t; \mathbf{r}) \in \mathfrak{b}^2(\mathbf{z}_0, R_{\mathbf{z}_0}), \tag{4.25}$$

and consequently, $\mathbf{R}_T^\varepsilon$ remains in $\mathcal{O}$.

Now, we will show that $\underline{\mathbf{L}}^\varepsilon$ defined for $\varepsilon \in (0, \eta)$ by (4.11) is extensible to $[0, \eta)$ and that the yielding extension is continuous with respect to ε. By definition for any $\varepsilon \in (0, \eta)$, for any $t \in (-t_e^\varepsilon, t_e^\varepsilon)$, and for any $\mathbf{r} \in \overline{\mathcal{O}'}$, $\mathbf{L}^\varepsilon$ is smooth on $\overline{\mathcal{O}'}$ and $\mathcal{C}^{N-1}((0, \eta))$ with respect to ε. So, we just have to show that $\varepsilon \mapsto \underline{\mathbf{L}}^\varepsilon$ is extensible as a continuous function on $[0, \eta)$, and particularly, that $\varepsilon = 0$ is not a singularity.
In a first place, for any $\varepsilon \in (0, \eta)$, we will explicit the dynamical system $\underline{\mathbf{L}}^\varepsilon$ satisfies. Injecting

$$\underline{\mathbf{R}}^\varepsilon (t, \mathbf{r}) = \underline{\mathbf{R}_T}^\varepsilon (t, \mathbf{r}) + \varepsilon^{N-1} \underline{\mathbf{L}}^\varepsilon (t, \mathbf{r}), \tag{4.26}$$

in (4.19) gives

$$\frac{\partial \begin{pmatrix} (\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon \\ (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon \end{pmatrix}}{\partial t}$$

$$= \mathtt{M}_\varepsilon\big((\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon, (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon\big) \begin{pmatrix} \frac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \frac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} \Big(\underline{\mathbf{R}_T^\varepsilon} + \varepsilon^{N-1}\underline{\mathbf{L}^\varepsilon}\Big)$$

$$+ \varepsilon^N \left[\varepsilon \mathtt{M}_\varepsilon \begin{pmatrix} \frac{\partial \rho_H}{\partial r_1} \\ \frac{\partial \rho_H}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon,.)\big)^{\mathrm{TL}} \begin{pmatrix} \frac{\partial H_\varepsilon}{\partial r_1} \\ \frac{\partial H_\varepsilon}{\partial r_2} \end{pmatrix} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon,.)\big)^{\mathrm{TR}} \begin{pmatrix} \frac{\partial H_\varepsilon}{\partial r_3} \\ \frac{\partial H_\varepsilon}{\partial r_4} \end{pmatrix}\right]$$

$$\left((\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon, (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon, \widetilde{\mathbf{R}_3^\varepsilon}\left(\frac{t}{\varepsilon}, \mathbf{r}\right), (\mathbf{R}_T^\varepsilon)_4 + \varepsilon^{N-1}\mathbf{L}_4^\varepsilon\right),$$

$$\frac{\partial (\mathbf{R}_T^\varepsilon)_4}{\partial t} + \varepsilon^{N-1}\frac{\partial \mathbf{L}_4^\varepsilon}{\partial t}$$

$$= -\varepsilon^N \left[\big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon,.)\big)^{1,4}\frac{\partial H_\varepsilon}{\partial r_1} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon,.)\big)^{2,4}\frac{\partial H_\varepsilon}{\partial r_2} + \big(\boldsymbol{\rho}_{\mathcal{P}}^N(\varepsilon,.)\big)^{3,4}\frac{\partial H_\varepsilon}{\partial r_3} + \frac{\partial \rho_H}{\partial r_3}(\varepsilon,.)\right]$$

$$\left((\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon, (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon, \widetilde{\mathbf{R}_3^\varepsilon}\left(\frac{t}{\varepsilon}, \mathbf{r}\right), (\mathbf{R}_T^\varepsilon)_4 + \varepsilon^{N-1}\mathbf{L}_4^\varepsilon\right). \tag{4.27}$$

According to formula (4.25), and by assumption, for any $\varepsilon \in (0,\eta)$, for any $t \in \Big(-\frac{\min(1,\alpha_0)}{\varepsilon}, \frac{\min(1,\alpha_0)}{\varepsilon}\Big)$ and for any $\mathbf{r} \in \overline{\mathcal{O}'}$, $\mathbf{R}_T^\varepsilon$ and $\mathbf{R}^\varepsilon$ are both in $\mathcal{O}$, which is a convex set. This allows us to use a Taylor expansion in

$$\mathtt{M}_\varepsilon\big((\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon, (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon\big) \begin{pmatrix} \frac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \frac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} \Big(\underline{\mathbf{R}_T^\varepsilon} + \varepsilon^{N-1}\underline{\mathbf{L}^\varepsilon}\Big)$$

leading to

$$\begin{aligned} &\mathtt{M}_\varepsilon\big((\mathbf{R}_T^\varepsilon)_1 + \varepsilon^{N-1}\mathbf{L}_1^\varepsilon, (\mathbf{R}_T^\varepsilon)_2 + \varepsilon^{N-1}\mathbf{L}_2^\varepsilon\big) \begin{pmatrix} \frac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \frac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} \Big(\underline{\mathbf{R}_T^\varepsilon} + \varepsilon^{N-1}\underline{\mathbf{L}^\varepsilon}\Big) \\ &\quad = \mathtt{M}_\varepsilon((\mathbf{R}_T^\varepsilon)_1, (\mathbf{R}_T^\varepsilon)_2) \begin{pmatrix} \frac{\partial H_{\varepsilon,T}}{\partial r_1} \\ \frac{\partial H_{\varepsilon,T}}{\partial r_2} \end{pmatrix} \Big(\underline{\mathbf{R}_T^\varepsilon}\Big) + \beta_1\left(\varepsilon, \underline{\mathbf{R}_T^\varepsilon}, \underline{\mathbf{L}^\varepsilon}\right), \end{aligned} \tag{4.28}$$

where $\beta_1 = \beta_1(\varepsilon, \underline{\mathbf{r}}, \underline{\mathbf{l}})$ belongs to $\mathcal{C}^\infty(\mathcal{U}_\eta^1)$, with

$$\begin{aligned} \mathcal{U}_\eta^1 = \Big\{ &(\varepsilon, \underline{\mathbf{r}}, \underline{\mathbf{l}}) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ &\text{s.t. } \varepsilon \in [0,\eta) \text{ and } \underline{\mathbf{r}} + \varepsilon^{N-1}\underline{\mathbf{l}} \in \mathbf{b}(\mathbf{z}_0, R_{\mathbf{z}_0}) \times (a,b)\Big\}. \end{aligned} \tag{4.29}$$

Injecting (4.28) in (4.27) and using (4.23) yields

$$\frac{\partial \begin{pmatrix} \mathbf{L}_1^\varepsilon \\ \mathbf{L}_2^\varepsilon \end{pmatrix}}{\partial t} = \beta_1\Big(\varepsilon, \underline{\mathbf{L}^\varepsilon}, \underline{\mathbf{R}_T^\varepsilon}\Big) + \varepsilon\beta_2\bigg(\varepsilon, \underline{\mathbf{L}^\varepsilon}, \underline{\mathbf{R}_T^\varepsilon}, \widetilde{\mathbf{R}_3^\varepsilon}\Big(\frac{t}{\varepsilon}, \mathbf{r}\Big)\bigg), \tag{4.30}$$

and

$$\frac{\partial \mathbf{L}_4^\varepsilon}{\partial t} = \varepsilon\beta_3\bigg(\varepsilon, \underline{\mathbf{L}^\varepsilon}, \underline{\mathbf{R}_T^\varepsilon}, \widetilde{\mathbf{R}_3^\varepsilon}\Big(\frac{t}{\varepsilon}, \mathbf{r}\Big)\bigg), \tag{4.31}$$

where β_2 and β_3 are in $\mathcal{C}^\infty(\mathcal{U}_\eta^2)$, with

$$\begin{aligned}\mathcal{U}_\eta^2 = \Big\{ &(\varepsilon, \underline{\mathbf{r}}, \underline{\mathbf{l}}, r_3) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \\ &\text{s.t. } \varepsilon \in [0, \eta)\,, \text{ and } \underline{\mathbf{r}} + \varepsilon^{N-1}\underline{\mathbf{l}} \in \mathbf{b}\left(\mathbf{z}_0, R_{\mathbf{z}_0}\right) \times (a, b) \Big\}.\end{aligned} \tag{4.32}$$

Moreover, β_2 and β_3 are smooth and 2π-periodic with respect to r_3. Besides, the solutions of this dynamical system are continuous with respect to ε. Clearly the initial data for $\underline{\mathbf{L}^\varepsilon}$ is $\underline{\mathbf{L}^\varepsilon}(0) = 0$. Hence, $\underline{\mathbf{L}^\varepsilon}$ is continuous with respect to ε. Since $\underline{\mathbf{R}^\varepsilon} - \underline{\mathbf{R}_T^\varepsilon} = \varepsilon^{N-1}\underline{\mathbf{L}^\varepsilon}$, estimate (4.12) follows. This ends the proof of Theorem 4.1. □

Theorem 4.1 means that we can approximate with accuracy ε^{N-1} dynamical system (4.9) by dynamical system (4.10).

Let us fix

$$N \in \mathbb{N}^\star. \tag{4.33}$$

In order to find a coordinate system satisfying the assumptions of Theorem 4.1, with this given N, we will introduce and develop a Partial Lie Transform Method of order N based on the use of the partial Lie Sums of order (i, j)

$$S_L^i\left(\varepsilon^j \bar{\mathbf{X}}_{\varepsilon\bar{f}}^\varepsilon\right) \cdot = \sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}_{\varepsilon\bar{f}}^\varepsilon\right)^n \cdot, \tag{4.34}$$

where i and j are non-negative integers and where $\bar{\mathbf{X}}_{\varepsilon\bar{f}}^\varepsilon$ is defined by (4.3). For any positive integer N, the partial Lie Transform of order N will allow us to build a coordinate system satisfying at this order the assumptions of Theorem 4.1. The resulting coordinate system will be the Gyro-Kinetic Coordinate System $(\mathbf{z}, \gamma, j)$.
In the forthcoming subsections, we will introduce the Partial Lie Sums of order (i, j) and set out their properties. In the next section we will develop for any $N \in \mathbb{N}^\star$ the Partial Lie Transform Method of order N. This method will then be applied in section 6 to deduce Theorem 1.3 from Theorem 4.1.

4.2 The partial Lie Sums: definitions and properties

Firstly, to simplify notations to come, we introduce

$$\bar{\mathbf{r}} = (\bar{r}_1, \ldots, \bar{r}_4) \text{ defined on } \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty), \tag{4.35}$$

as the Darboux Coordinates and the domain on which they are defined (notice that with this notation $\bar{r}_3 = \theta$) and we also introduce the set $\mathfrak{b}^{\#}(\bar{\mathbf{r}}_0, R_{\bar{\mathbf{r}}_0}) \subset \mathbb{R}^4$ defined by:

$$\mathfrak{b}^{\#}(\bar{\mathbf{r}}_0, R_{\bar{\mathbf{r}}_0}) = \left\{ \bar{\mathbf{r}} \in \mathbb{R}^4, \ |\bar{\mathbf{r}} - \bar{\mathbf{r}}_0|_{1,2,4} < R_{\bar{\mathbf{r}}_0} \right\}. \tag{4.36}$$

Here and hereafter, $|\bar{\mathbf{r}}|_{1,2,4}$ stands for $|(\bar{r}_1, \bar{r}_2, \bar{r}_4)|$, where $|\cdot|$ corresponds to the Euclidian norm on $\mathbb{R}^3$. We call such sets: open periodic balls. We then consider $\bar{\mathbf{r}}_0^*$ in $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and $R_0^* > 0$ such that

$$\overline{\mathfrak{b}^{\#}(\bar{\mathbf{r}}_0^*, R_0^*)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty). \tag{4.37}$$

We also recall properties of real analytic functions. For a positive integer p, let

$$S(\mathbf{Z}) = \sum_{\mathbf{l} \in \mathbb{N}^p} a_{\mathbf{l}} \mathbf{Z}^{\mathbf{l}}, \tag{4.38}$$

where $\mathbf{l} = (l_1, \ldots, l_p) \in \mathbb{N}^p$, $\mathbf{Z} = (Z_1, \ldots, Z_p) \in \mathbb{R}^p$ and $\mathbf{Z}^{\mathbf{l}} = Z_1^{l_1} \ldots Z_p^{l_p}$, be a formal power series of p variables. We can associate with this formal power series the numerical series

$$\sum_{\mathbf{l} \in \mathbb{N}^p} |a_{\mathbf{l}}| \, \mathbf{r}^{\mathbf{l}}, \tag{4.39}$$

where $\mathbf{r} = (r_1, \ldots, r_p) \in \mathbb{R}_+^p$. We denote by $\mathbf{\Gamma}_S$ the set

$$\mathbf{\Gamma}_S = \left\{ \mathbf{r} \in \mathbb{R}_+^p \text{ such that } \sum_{\mathbf{l} \in \mathbb{N}^p} |a_{\mathbf{l}}| \, \mathbf{r}^{\mathbf{l}} < +\infty \right\}, \tag{4.40}$$

and the set $\mathring{\mathbf{\Gamma}}_S$ is called the "set of convergence" of this series. By abuse of language,

$$\mathbf{\Sigma}_S = \left\{ \mathbf{x} \in \mathbb{R}^p \text{ such that } \sum_{\mathbf{l} \in \mathbb{N}^p} |a_{\mathbf{l}}| \left| \mathbf{x}^{\mathbf{l}} \right| < \infty \right\}. \tag{4.41}$$

is also called "set of convergence". We recall that on $\mathbf{\Sigma}_S$, the series $S(\mathbf{r})$ is infinitely differentiable, that the set of convergence of the partial derivatives is the same as of S and that the derivatives are obtained by permuting sum and derivatives. Notice also that if the closure of a non-empty open ball is included in the set of convergence, then, the series converges normally on this ball.

Definition 4.2. *We say that a function f is a real analytic function on the open subset $U \subset \mathbb{R}^p$ if, for all $\mathbf{r}_0 \in U$, there exists a formal power series $S_{\mathbf{r}_0}$ and a real number $R_{\mathbf{r}_0} > 0$ such that the n-dimensional open ball $\mathfrak{b}^n(\mathbf{r}_0, R_{\mathbf{r}_0})$ is included in U, such that $\mathfrak{b}^n(0, R_{\mathbf{r}_0})$ is included in $\mathbf{\Sigma}_S$ and such that*

$$\forall \mathbf{r} \in \mathfrak{b}^n(\mathbf{r}_0, R_{\mathbf{r}_0}), \ f(\mathbf{r}) = S(\mathbf{r} - \mathbf{r}_0). \tag{4.42}$$

We denote by $\mathcal{A}(U)$ the space of real analytic functions on U.

We recall the two following theorems.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^p$ and $\Omega' \subset \mathbb{R}^q$ be two open subsets. Let $f: \Omega \to \mathbb{R}^q$ in $(\mathcal{A}(\Omega))^q$ and $g: \Omega' \to \mathbb{R}^p$ in $(\mathcal{A}(\Omega'))^p$ be such that $g(\Omega') \subset \Omega$. Then, $f \circ g$ is in $(\mathcal{A}(\Omega'))^q$.*

Theorem 4.4. *Let f be a real analytic function in a neighborhood of $a = (a_1, \ldots, a_p)$. If the differential $(df)_a$ of f in a is non-singular, then f^{-1} is defined and real analytic in a neighborhood of $f(a)$.*

For more details about the real analytic functions, or for the proofs of the two previous theorems, see Cartan [4] or Krantz & Park [23].

We also recall the following version of the global inversion Theorem:

Theorem 4.5. *Let E be a Banach space and F a normed vector space. Let $A: E \to F$ be a linear, continuous and invertible map such that A^{-1} is continuous. Let $\varphi: E \to F$ be a Lipshitz continuous map such that its Lipshitz constant $Lip(\varphi)$ satisfies $Lip(\varphi) < \left\|A^{-1}\right\|^{-1}$. Then,*

- *$h = A + \varphi$ is invertible.*
- *h^{-1} is Lipschitz continuous.*
- *If $U \subset E$ is an open subset, if $h \in \mathcal{C}^1(U)$ and if for any $x \in U$, $(dh)_x$ is an isomorphism from E onto F, then h^{-1} is $\mathcal{C}^1(h(U))$, and for any $x \in U$, $\left(dh^{-1}\right)_{h(x)} = (dh)_x^{-1}$.*

We will use the results just recalled. The Poisson Matrix $\bar{\mathcal{P}}_\varepsilon = \bar{\mathcal{P}}_\varepsilon(\bar{\mathbf{r}})$ in the coordinate system $\bar{\mathbf{r}}$ satisfies:

Lemma 4.6. *The Poisson matrix $\bar{\mathcal{P}}_\varepsilon = \bar{\mathcal{P}}_\varepsilon(\bar{\mathbf{r}})$ given by formula (3.114) and defined by construction on $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, is extensible, with the same expression, as a Poisson Matrix to $\mathbb{R}^4$, the extension being also denoted $\bar{\mathcal{P}}_\varepsilon$. Moreover, all entries of $\bar{\mathcal{P}}_\varepsilon$ are independent of $\bar{r}_3$ and $\bar{r}_4$ and they can be viewed as functions in $\mathcal{A}\left(\mathbb{R}^2\right) \cap \mathcal{C}_b^\infty\left(\mathbb{R}^2\right)$.*

Proof. The symplectic Two-Form associated with $\bar{\mathcal{P}}_\varepsilon$

$$\bar{\omega}_\varepsilon(\bar{\mathbf{r}}) = \frac{\varepsilon}{B(\bar{r}_1, \bar{r}_2)} d\bar{r}_1 \wedge d\bar{r}_2 - \frac{1}{\varepsilon} d\bar{r}_3 \wedge d\bar{r}_4, \tag{4.43}$$

on the Darboux Coordinate chart is clearly extensible with the same expression to $\mathbb{R}^4$ and the yielding extension is clearly closed and non-degenerate. Hence the extension is a Poisson Matrix on $\mathbb{R}^4$. Moreover, as the magnetic field is defined on $\mathbb{R}^2$ and since it is by assumption bounded, larger than 1, and analytic on $\mathbb{R}^2$ so is $\frac{1}{B}$. Hence, all entries of $\bar{\mathcal{P}}_\varepsilon$ are analytic and in $\mathcal{C}_b^\infty\left(\mathbb{R}^2\right)$. This ends the proof of Lemma 4.6. ∎

In addition, $\bar{\mathcal{P}}_\varepsilon$ can be rewritten as

$$\bar{\mathcal{P}}_\varepsilon(\bar{\mathbf{r}}) = \frac{1}{\varepsilon}\bar{\mathcal{T}}_\varepsilon(\bar{\mathbf{r}}), \tag{4.44}$$

which define matrix $\bar{\mathcal{T}}_\varepsilon(\bar{\mathbf{r}})$.

We will now introduce the Partial Lie Sum of order (i,j) and the Partial Lie Sum Function as follows:

Definition 4.7. *Let $\mathfrak{C}$ be a convex subset of $\mathbb{R}^4$. For any $\bar{f} = \bar{f}(\bar{\mathbf{r}})$ in $\mathcal{C}^\infty(\mathfrak{C})$, let $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ be the differential operator acting on functions $\bar{g} = \bar{g}(\bar{\mathbf{r}})$ of $\mathcal{C}^\infty(\mathfrak{C})$ in the following way:*

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}} \cdot \bar{g} = S^i_L\left(\varepsilon^j \bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{f}}\right) \cdot \bar{g}, \tag{4.45}$$

where S^i_L is defined by (4.34) and $\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{f}}$ by (4.3). From operator $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ we define, with the same notation, function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}} = \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\bar{\mathbf{r}})$ from $\mathfrak{C}$ to $\mathbb{R}$ by

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}} = \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\bar{\mathbf{r}}) = \left(\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}} \cdot \bar{\mathbf{r}}_1\right)(\bar{\mathbf{r}}), \ldots, \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}} \cdot \bar{\mathbf{r}}_4\right)(\bar{\mathbf{r}})\right), \tag{4.46}$$

where $\bar{\mathbf{r}}_i$ stands for $\bar{\mathbf{r}} \mapsto \bar{r}_i$.

Definition 4.8. *Viewed as a differential operator, $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ is called the Partial Lie Sum of order (i,j) generated by f, and view as a function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ is called the Partial Lie Sum map generated by f.*

The first property that we will prove is the following theorem.

Theorem 4.9. *Let i,j be two positive integers, $\mathfrak{C}$ be a convex subset of $\mathbb{R}^3$ and $\mathfrak{C}^\#$ defined by:*

$$\mathfrak{C}^\# = \left\{\bar{\mathbf{r}} \in \mathbb{R}^4,\ (\bar{r}_1, \bar{r}_2, \bar{r}_4) \in \mathfrak{C} \text{ and } \bar{r}_3 \in \mathbb{R}\right\}, \tag{4.47}$$

and let $\bar{f} = \bar{f}(\bar{\mathbf{r}})$ be in $\mathcal{C}^\infty_\#\left(\mathfrak{C}^\#\right) \cap \mathcal{C}^\infty_b\left(\mathfrak{C}^\#\right)$. Then, there exists a real number $\bar{\eta}_1 > 0$ such that for any $\varepsilon \in [-\bar{\eta}_1, \bar{\eta}_1]$, function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ defined by (4.46) is a diffeomorphism from $\mathfrak{C}^\#$ onto its range. We denote by $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ the inverse function of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$.

Remark 4.10. *Thanks to Theorem 4.9, we will be able to consider change of coordinates $\hat{\mathbf{r}} = \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\bar{\mathbf{r}})$ for $\bar{\mathbf{r}}$ in $\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$ and $\bar{f}$ defined on $\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$, where $\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$ is defined by (4.36). We will denote by $\bar{\mathbf{r}} = \boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\hat{\mathbf{r}})$ the reciprocal change of coordinates. In view of the Partial Lie Transform method, that we will construct in the next section, we need to express $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ in terms of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}$. The problem to reach this goal, is that function $\bar{f}$ is not necessarily defined on $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)\right)$. In order to overcome this difficulty, we can choose among two options. The first option consists in restricting the method to an open subset $\mathfrak{V} \subset \mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$ such that $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\mathfrak{V}) \subset \mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$. But this option implies a restriction of the domain, and consequently, we do not choose it. The second option, which is the one we opt for, consists in taking an open subset $\mathfrak{U}$ containing $\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)$ and $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^\#(\bar{\mathbf{r}}^*_0, R^*_0)\right)$ and defining an extension of function $\bar{f}$ to this set. Of course this option requires the Poisson Matrix to be extensible to $\mathfrak{U}$, which is the case because of Lemma 4.6.*

Remark 4.11. *Another restriction regarding the class of functions $\bar{f}$ generating the Partial Lie Sum of order N (see Definition 4.8) required by the Partial Lie Transform Method, is that their restriction to $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right)$ will depend on the N first terms of the expansion in power of ε of the Hamiltonian function (see* (3.116)*). This implies that those N first terms need to be extensible to set $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right)$. Hence, we could think that the best choice for $\mathfrak{U}$ is an open subset on which both the Poisson Matrix and the N first terms of the expansion in power of ε of the Hamiltonian function can be extended and such that $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right) \subset \mathfrak{U}$.*

Remark 4.12. *In the context of the application of the Partial Lie Transform Method to the Gyro-Kinetic Coordinates, $\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)$ (see* (4.36)*) is a set on which function $\boldsymbol{\kappa}$ is defined (see* (3.3), (3.86) *and* (3.103)*), the Hamiltonian function is function $\bar{H}_\varepsilon$ defined by* (3.115) *and expanded in* (3.116) *and the Poisson Matrix is the matrix $\bar{\mathcal{P}}_\varepsilon$ defined by* (3.114)*. As already noticed, the Poisson matrix is clearly extensible to $\mathbb{R}^4$. The maximal open subset on which the N first terms of the Hamiltonian expansion in power of ε (see* (3.116)*) can be smoothly extended is $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. Hence, one could think that a good choice for $\mathfrak{U}$ is $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. Notice that $\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0) \subset \mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. Yet with this choice, there is no reason for the requirement $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right) \subset \mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ to be satisfied.*
Nevertheless, according to the expression of the N first terms of the expansion in power of ε of the Hamiltonian function (see (3.116)*) we will be able to proceed as follows. We will choose functions $\bar{f}$ in $\mathcal{C}^\infty_{\#}\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$. We will show that if $\overline{\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)} \subset \mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$, then, for any $\mathfrak{r}_0$ in $\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ and any $R_0>0$ such that $\overline{\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)} \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ and $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)} \subset \mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$, there exists a real number $\eta>0$ such that for any $\varepsilon\in[-\eta,\eta]$, $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$. Consequently, we will choose for $\mathfrak{U}$: $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ with $\mathfrak{r}_0$ and R_0 properly chosen.*

Theorem 4.13. *Let i,j be two positive integers, $f\in\mathcal{C}^\infty_{\#}(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty))$, $\mathfrak{r}_0\in\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$, R_0 be such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)} \subset \mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ and R_0' be such that $0<R_0'<R_0$. Then, there exists a real number $\eta>0$ such, that for any $\varepsilon\in[-\eta,\eta]$, function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ is a diffeomorphism from $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0')$ onto its range and such that*

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0')\right) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0). \tag{4.48}$$

The two following theorems are consequences of the previous one.

Theorem 4.14. *Let $\mathfrak{r}_0\in\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$ and $R_0>0$ be such that $\overline{\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)} \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$, where $\bar{\mathbf{r}}^*_0$, R^*_0 and $\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)$ are set by* (4.37)*. Let $\bar{f}\in\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$ (see definition* (3.118) *of $\mathcal{Q}^\infty_{T,b}$) and $i,j\in\mathbb{N}$ be such that $ij\geq N$, where N is fixed by* (4.33)*. Then, there exists a real number $\bar{\eta}_2>0$ such that for any $\varepsilon\in[-\bar{\eta}_2;\bar{\eta}_2]$*

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^*_0, R^*_0)\right) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0), \tag{4.49}$$

$\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ *is well defined and analytic on* $\mathfrak{b}^{\#}(\mathbf{r}_0, R_0)$*, the components* 1*,* 2 *and* 4 *of* $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ *are in* $\mathcal{C}^{\infty}_{\#}\Big(\mathfrak{b}^{\#}(\mathbf{r}_0, R_0)\Big)$ *and its penultimate component satisfies for any* $\bar{\mathbf{r}} \in \mathfrak{b}^{\#}(\mathbf{r}_0, R_0)$*:*

$$\Big(\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}\Big)_3(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2, \bar{\mathbf{r}}_3 + 2\pi, \bar{\mathbf{r}}_4) = \Big(\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}\Big)_3(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2, \bar{\mathbf{r}}_3, \bar{\mathbf{r}}_4) + 2\pi. \tag{4.50}$$

Moreover, for any $\mathbf{r} \in \mathfrak{b}^{\#}(\mathbf{r}_0, R_0)$ *the following equality holds true:*

$$\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\mathbf{r}) = \boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathbf{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^{N}_{\boldsymbol{\Xi}^{i,j}}(\varepsilon, \mathbf{r}), \tag{4.51}$$

where $\boldsymbol{\rho}^{N}_{\boldsymbol{\Xi}^{i,j}}$ *is in* $\mathcal{C}^{\infty}_{\#}\Big([-\bar{\eta}_2; \bar{\eta}_2] \times \mathfrak{b}^{\#}(\mathbf{r}_0, R_0)\Big)$ *(see* (3.31)*).*

Theorem 4.15. *Let* $\mathbf{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ *and* $R_0 > 0$ *be such that* $\overline{\mathfrak{b}^{\#}(\mathbf{r}_0, R_0)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$*. Let* $\bar{f} \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^{\infty}_{T,b}$*,* $i, j \in \mathbb{N}$ *be such that* $ij \geq N$ *and* $\hat{\mathcal{P}}_\varepsilon$ *be the matrix whose entries are given, for* $k, l \in \{1, \dots, 4\}$*, by*

$$\hat{\mathcal{P}}^{k,l}_{\varepsilon}(\mathbf{r}) = \left\{\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big)_k, \Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big)_l\right\}_{\bar{\mathbf{r}}}\Big(\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\mathbf{r})\Big). \tag{4.52}$$

Then, for any $\mathfrak{b}^{\#}(\mathbf{r}^{\star}_0, R^{\star}_{\mathbf{r}_0})$ *such that* $\overline{\mathfrak{b}^{\#}(\mathbf{r}^{\star}_0, R^{\star}_{\mathbf{r}_0})} \subset \mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})$*, there exists a real number* $\bar{\eta}_3 > 0$ *such that for any* $\varepsilon \in [-\bar{\eta}_3, \bar{\eta}_3]$

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^{\#}(\mathbf{r}^{\star}_0, R^{\star}_{\mathbf{r}_0})\Big) \subset \mathfrak{b}^{\#}(\mathbf{r}_0, R_0), \tag{4.53}$$

and for any $\mathbf{r} \in \mathfrak{b}^{\#}(\mathbf{r}_0, R_0)$ *the following equality holds true:*

$$\forall k, l \in \{1, 2, 3, 4\}, \ \hat{\mathcal{T}}^{k,l}_{\varepsilon}(\mathbf{r}) = \bar{\mathcal{T}}^{k,l}_{\varepsilon}(\mathbf{r}) + \varepsilon^{N+1}\rho^{N,k,l}_{S}(\varepsilon, \mathbf{r}), \tag{4.54}$$

where $\rho^{N,k,l}_{S}$ *is in* $\mathcal{C}^{\infty}_{\#}\Big([-\bar{\eta}_3, \bar{\eta}_3] \times \mathfrak{b}^{\#}(\mathbf{r}_0, R_0)\Big)$ *and where* $\hat{\mathcal{T}}_\varepsilon$ *stands for the matrix which satisfies*

$$\hat{\mathcal{P}}_\varepsilon = \frac{1}{\varepsilon}\hat{\mathcal{T}}_\varepsilon. \tag{4.55}$$

Remark 4.16. *Formula* (4.54) *is written in terms of* $\hat{\mathcal{T}}_\varepsilon$ *because* $\hat{\mathcal{P}}_\varepsilon$ *has a singularity in* $\varepsilon = 0$*. Considering* $\hat{\mathcal{T}}_\varepsilon$ *instead of* $\hat{\mathcal{P}}_\varepsilon$ *allows us to avoid having to distinguish between the cases* $\varepsilon = 0$ *and* $\varepsilon \neq 0$*. In formula* (4.52)*, we have rather written the formula using that* $\hat{\mathcal{P}}_\varepsilon$ *is the Poisson Matrix in the coordinate system* $\hat{\mathbf{r}}$*. This allows to understand why* $\hat{\mathcal{P}}_\varepsilon$ *is defined on a subset larger than* $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^{\#}(\bar{\mathbf{r}}^{*}_0, R^{*}_0)\Big)$*.*

The proof of these theorems are given in subsections 4.4, 4.5 and 4.6. They are essentially based on the basic properties of the Partial Lie Sums we will expose in the next subsection.

4.3 Basic Properties of the Partial Lie Sums

We will start this subsection with some topological properties.

Lemma 4.17. *Let $\mathfrak{W} \subset \mathbb{R}^4$ be an open set, $f = f(\mathfrak{r}) \in \mathcal{C}_b^\infty(\mathfrak{W})$ and $\mathbf{X}_{\varepsilon f}^\varepsilon$ be the Hamiltonian vector field associated with εf. Then for any $p \in \{1, \ldots, 4\}$ and any $k \geq 1$, $\left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot \mathbf{r}_p \in \mathcal{C}_b^\infty(\mathfrak{W})$, where $\mathbf{r}_p$ is the p-th coordinate function and $\left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot$ is the iterated application of $\mathbf{X}_{\varepsilon f}^\varepsilon$ as a differential operator acting on functions.*

Proof. By definition, $\mathbf{X}_{\varepsilon f}^\varepsilon = \bar{\mathcal{P}}_\varepsilon \nabla(\varepsilon f)$ and $\left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^1 \cdot \mathbf{r}_p = \left(\bar{\mathcal{T}}_\varepsilon \nabla f\right)_p$. Hence, as all entries of $\bar{\mathcal{T}}_\varepsilon$ are in $\mathcal{C}_b^\infty\left(\mathbb{R}^4\right)$ and as f is in $\mathcal{C}_b^\infty(\mathfrak{W})$, so is $\left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^1 \cdot \mathbf{r}_p$. An easy induction gives then the result. This ends the proof of Lemma 4.17. □

Lemma 4.18. *Let $\mathfrak{b}^\#(\mathfrak{r}_0, R_0)$ be an open periodic ball defined by (4.36), f be a function in $\mathcal{C}_\#^\infty\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0)\Big)$ and i, j be two positive integers. Then, for every $R_0' > R_0$, there exists a real number $\eta_4 > 0$ such that*

$$\forall \varepsilon \in [-\eta_4, \eta_4]\,,\ \boldsymbol{\vartheta}_{\varepsilon,f}^{i,j}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0)\Big) \subset \mathfrak{b}^\#(\mathfrak{r}_0, R_0'). \tag{4.56}$$

Proof. From definitions (4.45) and (4.46), for any $\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_0)$:

$$\begin{aligned} \boldsymbol{\vartheta}_{\varepsilon,f}^{i,j}(\mathfrak{r}) &= \left(\left(\sum_{k=0}^{i} \frac{\varepsilon^{jk}}{k!} \left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot \mathbf{r}_1\right)(\mathfrak{r})\,, \ldots, \left(\sum_{k=0}^{i} \frac{\varepsilon^{jk}}{k!} \left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot \mathbf{r}_4\right)(\mathfrak{r})\right) \\ &= \mathfrak{r} + \varepsilon \boldsymbol{\nu}_{\varepsilon,f}^{i,j}(\mathfrak{r})\,, \end{aligned} \tag{4.57}$$

where

$$\boldsymbol{\nu}_{\varepsilon,f}^{i,j}(\mathfrak{r}) = \left(\left(\sum_{k=1}^{i} \frac{\varepsilon^{jk-1}}{k!} \left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot \mathbf{r}_1\right)(\mathfrak{r})\,, \ldots, \left(\sum_{k=1}^{i} \frac{\varepsilon^{jk-1}}{k!} \left(\mathbf{X}_{\varepsilon f}^\varepsilon\right)^k \cdot \mathbf{r}_4\right)(\mathfrak{r})\right). \tag{4.58}$$

According to Lemma 4.17, as f is in $\mathcal{C}_b^\infty\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0)\Big)$, $\boldsymbol{\nu}_{\varepsilon,f}^{i,j}$ is in $\mathcal{C}_b^\infty\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0)\Big)$, we can define

$$\left\|\boldsymbol{\nu}_{\varepsilon,f}^{i,j}\right\|_\infty = \sup_{\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_0)} \left|\boldsymbol{\nu}_{\varepsilon,f}^{i,j}(\mathfrak{r})\right|. \tag{4.59}$$

Since $\varepsilon \mapsto \left\|\boldsymbol{\nu}_{\varepsilon,f}^{i,j}\right\|_\infty$ is smooth, $\varepsilon \left\|\boldsymbol{\nu}_{\varepsilon,f}^{i,j}\right\|_\infty \to 0$ when $\varepsilon \to 0$ and for all $c > 0$ there exists a real number η_4 such that for any $\varepsilon \in [-\eta_4, \eta_4]$, $\left|\varepsilon \left\|\boldsymbol{\nu}_{\varepsilon,f}^{i,j}\right\|_\infty\right| < c$. Let $c > 0$ such that $c < R_0' - R_0$, for any $\varepsilon \in [-\eta, \eta]$ and for any $\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_0)$ we have

$$\left|\boldsymbol{\vartheta}_{\varepsilon,f}^{i,j}(\mathfrak{r}) - \mathfrak{r}_0\right|_{1,2,4} \leq |\mathfrak{r} - \mathfrak{r}_0|_{1,2,4} + \left|\varepsilon \boldsymbol{\nu}_{\varepsilon,f}^{i,j}(\mathfrak{r})\right|_{1,2,4} \leq R_0 + c < R_0' \tag{4.60}$$

This proves that $\boldsymbol{\vartheta}_{\varepsilon,f}^{i,j}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0)\Big) \subset \mathfrak{b}^\#(\mathfrak{r}_0, R_0')$ and ends the proof of Lemma 4.18. □

Lemma 4.19. *Let $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ be an open periodic ball defined by (4.36), f be a function in $\mathcal{C}^{\infty}_{\#}\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\Big)$ and i, j be two positive integers. Then, for any real number R_0' such that $0 < R_0' < R_0$, there exists a real number η such that for any $\varepsilon \in [-\eta, \eta]$*

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\Big) \supset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0'). \tag{4.61}$$

Proof. Here again, we will use expression (4.57) of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$. The proof of Lemma 4.19 is based on the Brouwer Theorem (see Brouwer [3] or Istratescu [21]). Let $R_0^{(2)}$, $R_0^{(3)}$, $\alpha_0^{(2)}$ and $\alpha_0^{(3)}$ be real numbers satisfying $R_0' < R_0^{(2)} < R_0^{(3)} < R_0$ and $0 < \alpha_0^{(2)} < \alpha_0^{(3)}$, l be an integer, and let $\boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(2)}, \alpha_0^{(2)}}$ and $\boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(3)}, \alpha_0^{(3)}}$ be the compact and convex subsets of $\mathbb{R}^4$ defined by

$$\boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(2)}, \alpha_0^{(2)}} = \Big\{\mathfrak{r} \in \mathbb{R}^4,\ |\mathfrak{r} - \mathfrak{r}_0|_{1,2,4} \leq R_0^{(2)} \text{ and } \mathfrak{r}_3 \in \Big[(l-1)\,\pi - \alpha_0^{(2)}, (l+1)\,\pi + \alpha_0^{(2)}\Big]\Big\} \tag{4.62}$$

and

$$\boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(3)}, \alpha_0^{(3)}} = \Big\{\mathfrak{r} \in \mathbb{R}^4,\ |\mathfrak{r} - \mathfrak{r}_0|_{1,2,4} \leq R_0^{(3)} \text{ and } \mathfrak{r}_3 \in \Big[(l-1)\,\pi - \alpha_0^{(3)}, (l+1)\,\pi + \alpha_0^{(3)}\Big]\Big\}. \tag{4.63}$$

Since $\varepsilon \left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,f}\right\|_{\infty} \to 0$ when $\varepsilon \to 0$, we can define $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$ and for any $\mathfrak{r}' \in \boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(2)}, \alpha_0^{(2)}}$,

$$\left|\mathfrak{r}' - \mathfrak{r}_0\right|_{1,2,4} + \varepsilon \left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,f}\right\|_{\infty} \leq R_0^{(3)}, \tag{4.64}$$

and

$$\left|\mathfrak{r}_3' - l\pi\right| + \varepsilon \left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,f}\right\|_{\infty} \leq \alpha_0^{(3)}. \tag{4.65}$$

Now, for all $\mathfrak{r}' \in \boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(2)}, \alpha_0^{(2)}}$, we define the function $F^{\varepsilon}_{\mathfrak{r}'}$ by

$$F^{\varepsilon}_{\mathfrak{r}'} :\ \boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(3)}, \alpha_0^{(3)}} \to \mathbb{R}^4;\ \mathfrak{r} \mapsto \mathfrak{r}' - \varepsilon \boldsymbol{\nu}^{i,j}_{\varepsilon,f}(\mathfrak{r}). \tag{4.66}$$

By construction and because of the properties of $\boldsymbol{\nu}^{i,j}_{\varepsilon,f}$, $F^{\varepsilon}_{\mathfrak{r}'}$ is continuous on $\boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(3)}, \alpha_0^{(3)}}$ and for any $\varepsilon \in [-\eta, \eta]$ and any $\mathfrak{r} \in \boldsymbol{\mathfrak{K}}^l_{\mathfrak{r}_0, R_0^{(3)}, \alpha_0^{(3)}}$,

$$|F^{\varepsilon}_{\mathfrak{r}'}(\mathfrak{r}) - \mathfrak{r}_0|_{1,2,4} \leq \left|\mathfrak{r}' - \mathfrak{r}_0\right|_{1,2,4} + |\varepsilon| \left|\boldsymbol{\nu}^{i,j}_{\varepsilon,f}(\mathfrak{r})\right|_{1,2,3} < R_0^{(3)}, \tag{4.67}$$

and

$$|(F^{\varepsilon}_{\mathfrak{r}'}(\mathfrak{r}))_3 - l\pi| \leq \left|\mathfrak{r}_3' - l\pi\right| + |\varepsilon| \left|\Big(\boldsymbol{\nu}^{i,j}_{\varepsilon,f}(\mathfrak{r})\Big)_3\right| \leq \alpha_0^{(3)}, \tag{4.68}$$

meaning $F^{\varepsilon}_{\mathfrak{r}'}\left(\mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}\right) \subset \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}$. Hence, invoking the Brouwer Theorem and more precisely its convex compact version, function $F^{\varepsilon}_{\mathfrak{r}'}$ has a fixed point in $\mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}$. So we have proven that

$$\exists \eta > 0,\ \forall \varepsilon, |\varepsilon| < \eta,\ \forall \mathfrak{r}' \in \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(2)},\alpha_0^{(2)}},\ \exists \mathfrak{r} \in \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}},\ \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}(\mathfrak{r}) = \mathfrak{r}', \tag{4.69}$$

proving $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}\right) \supset \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(2)},\alpha_0^{(2)}}$ and consequently that

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\bigcup_{l\in\mathbb{Z}} \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}\right) \supset \bigcup_{l\in\mathbb{Z}} \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(2)},\alpha_0^{(2)}}. \tag{4.70}$$

Since $\overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(3)}\right)} = \bigcup_{l\in\mathbb{Z}} \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(3)},\alpha_0^{(3)}}$ and $\overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(2)}\right)} = \bigcup_{l\in\mathbb{Z}} \mathfrak{K}^{l}_{\mathfrak{r}_0,R_0^{(2)},\alpha_0^{(2)}}$, (4.70) can be rewritten as $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(3)}\right)}\right) \supset \overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(2)}\right)}$. Finally, since $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0') \subset \overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(2)}\right)}$ and $\overline{\mathfrak{b}^{\#}\left(\mathfrak{r}_0, R_0^{(3)}\right)} \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ we obtain (4.61). This ends the proof of Lemma 4.19. □

Lemma 4.19 ensures that, if $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ is invertible with $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ as inverse function, for sufficiently small ε we have:

$$\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0')\right) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0). \tag{4.71}$$

Now, we will focus on more algebraic properties. In definition 4.7 we used Hamiltonian vector fields. The reason why chosing this class of vector fields is related to the following lemma and its consequences.

Lemma 4.20. *Let $f = f(\mathfrak{r})$, $g = g(\mathfrak{r})$ and $h = h(\mathfrak{r})$ be three smooth functions defined on an open subset $\mathfrak{M} \subset \mathbb{R}^4$. Then, for all $n \geq 1$, the following equality holds true on $\mathfrak{M}$:*

$$\left(\mathbf{X}^{\varepsilon}_{\varepsilon f}\right)^n \cdot \{g,h\} = \sum_{k=0}^{n} C_n^k \left\{ \left(\mathbf{X}^{\varepsilon}_{\varepsilon f}\right)^k \cdot g, \left(\mathbf{X}^{\varepsilon}_{\varepsilon f}\right)^{n-k} \cdot h \right\}, \tag{4.72}$$

where $\mathbf{X}^{\varepsilon}_{\varepsilon f}$ is the Hamiltonian vector field associated with εf, $\left(\mathbf{X}^{\varepsilon}_{\varepsilon f}\right)^n \cdot$ its iterated application as a differential operator and $\{g,h\}$ the Poisson Bracket between functions g and h.

Proof. This proof is easily obtained by induction. The key point of the proof is the following equality

$$\mathbf{X}^{\varepsilon}_{\varepsilon f} \cdot \{g,h\} = \left\{\mathbf{X}^{\varepsilon}_{\varepsilon f} \cdot g, h\right\} + \left\{g, \mathbf{X}^{\varepsilon}_{\varepsilon f} \cdot h\right\}, \tag{4.73}$$

which is a direct consequence of the Jacoby identity and which is specific to Hamiltonian vector fields. □

Another result which does not require the Hamiltonian nature of the vector field, is the following lemma.

Lemma 4.21. *Let $g = g(\mathfrak{r})$ and $h = h(\mathfrak{r})$ be two smooth functions defined on an open subset $\mathfrak{M} \subset \mathbb{R}^4$ and $\boldsymbol{\mathfrak{X}}$ be a smooth vector field defined on $\mathfrak{M}$. Then, for all $p \geq 0$, the following equality holds true on $\mathfrak{M}$:*

$$(\boldsymbol{\mathfrak{X}})^p \cdot (gh) = \sum_{k=0}^{p} C_p^k \left((\boldsymbol{\mathfrak{X}})^k \cdot g \right) \left((\boldsymbol{\mathfrak{X}})^{p-k} \cdot h \right). \tag{4.74}$$

As consequence of the previous lemmas, we will now see that operator $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot$ almost commutes with the Poisson Bracket and the product between two functions. More precisely, we have

Property 4.22. *Let $\mathfrak{M} \subset \mathbb{R}^4$ be an open subset, $f = f(\mathfrak{r})$, $g = g(\mathfrak{r})$ and $h = h(\mathfrak{r})$ be three functions in $\mathcal{C}^\infty(\mathfrak{M})$ and i, j be two positive integers. If $ij \geq N$, then the following equality holds true for every $\mathfrak{r}$ in $\mathfrak{M}$:*

$$\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot \{g, h\} \right)(\mathfrak{r}) = \left\{ \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot g, \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot h \right\}(\mathfrak{r}) + \varepsilon^N \rho^{N,i,j}_{PC}(\varepsilon, \mathfrak{r}), \tag{4.75}$$

where operator $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ is defined by (4.45) and where $\rho^{N,i,j}_{PC}$ is in $\mathcal{C}^\infty(\mathbb{R} \times \mathfrak{M})$.

Proof. Firstly, we will define on $\mathfrak{M}$ the function $\{g, h\}^{\bar{\mathcal{T}}_\varepsilon} = \{g, h\}^{\bar{\mathcal{T}}_\varepsilon}(\mathfrak{r})$ by

$$\{g, h\}^{\bar{\mathcal{T}}_\varepsilon}(\mathfrak{r}) = \left(\bar{\mathcal{T}}_\varepsilon(\mathfrak{r}) \, \nabla h(\mathfrak{r}) \right) \cdot \left(\nabla g(\mathfrak{r}) \right), \tag{4.76}$$

and we notice that $\varepsilon \mapsto \{g, h\}^{\bar{\mathcal{T}}_\varepsilon}(\mathfrak{r})$ is in $\mathcal{C}^\infty(\mathbb{R})$ for any $\mathfrak{r} \in \mathfrak{M}$.

Now, expanding $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot \{g, h\}$ using Lemma 4.20, expanding $\left\{ \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot g, \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot h \right\}$, and making the difference between these two expansions yields (4.75) with

$$\rho^{N,i,j}_{PC}(\varepsilon, \mathfrak{r}) = -\sum_{k=i+1}^{2i} \varepsilon^{jk-(N+1)} \sum_{(m,p) \in \{1,\dots,i\}^2 \ s.t. \ m+p=k} \frac{1}{m! p!} \left\{ \left(\mathbf{X}^\varepsilon_{\varepsilon f} \right)^m \cdot g, \left(\mathbf{X}^\varepsilon_{\varepsilon f} \right)^p \cdot h \right\}^{\bar{\mathcal{T}}_\varepsilon}(\mathfrak{r}). \tag{4.77}$$

As $ij \geq N$, all $k \geq i+1$ satisfy $jk \geq N+1$. Hence $\varepsilon \mapsto \rho^{N,i,j}_{PC}(\varepsilon, \mathfrak{r})$ is in $\mathcal{C}^\infty(\mathbb{R})$ for any $\mathfrak{r} \in \mathfrak{M}$. In addition, $\mathfrak{r} \mapsto \rho^{N,i,j}_{PC}(\varepsilon, \mathfrak{r})$ is clearly in $\mathcal{C}^\infty(\mathfrak{M})$ for any $\varepsilon \in \mathbb{R}$. This ends the proof of Property 4.22. □

Property 4.23. *Let $\mathfrak{M} \subset \mathbb{R}^4$ be an open subset, $f = f(\mathfrak{r})$, $g = g(\mathfrak{r})$ and $h = h(\mathfrak{r})$ be three functions in $\mathcal{C}^\infty(\mathfrak{M})$ and i, j be two positive integers. Then, if $ij \geq N$, the following equality holds true on $\mathfrak{M}$:*

$$\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot (gh) \right)(\mathfrak{r}) = \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot g \right)(\mathfrak{r}) \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot h \right)(\mathfrak{r}) + \varepsilon^{N+1} \rho^{N,i,j}_{FP}(\varepsilon, \mathfrak{r}), \tag{4.78}$$

where $\rho^{N,i,j}_{FP}$ is in $\mathcal{C}^\infty(\mathbb{R} \times \mathfrak{M})$.

We will now end this subsection by giving the following important property claiming that $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot$ and $\circ\, \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ almost commute.

Theorem 4.24. *Let $\mathfrak{N} \in \mathbb{R}^3$ be an open subset such that $\overline{\mathfrak{N}}$ is a compact subset of $\mathbb{R}^2 \times (0,+\infty)$; $\mathfrak{M}^\#$ be the open subset of $\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$ defined by*

$$\mathfrak{M}^\# = \big\{ \mathfrak{r} \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\,,\ (\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4) \in \mathfrak{N} \text{ and } \mathfrak{r}_3 \in \mathbb{R} \big\}\,; \tag{4.79}$$

$\mathcal{O} \subset \mathbb{R}^2 \times (0,+\infty)$ be an open subset such that $\overline{\mathfrak{N}} \subset \mathcal{O}$ and $\mathcal{O}^\#$ be the open subset of $\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$ defined by

$$\mathcal{O}^\# = \big\{ \mathfrak{r} \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\,,\ (\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4) \in \mathcal{O} \text{ and } \mathfrak{r}_3 \in \mathbb{R} \big\}\,; \tag{4.80}$$

$f = f(\mathfrak{r}) \in \mathcal{C}^\infty_\#\big(\mathcal{O}^\#\big)$ (see (3.31)); $g_\varepsilon = g_\varepsilon(\mathfrak{r}) \in \mathcal{A}\big(\mathbb{R}^3 \times (0,+\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$ (see Definition 4.2 and (3.118)) for every ε in some interval I containing 0 and $\varepsilon \mapsto g_\varepsilon(\mathfrak{r})$ be in $\mathcal{C}^\infty(I)$ for any $\mathfrak{r} \in \mathfrak{M}^\#$; and i,j be two positive integers such that $ij \geq N$. Then, there exists a real number $\bar\eta_5 > 0$ such that for any $\varepsilon \in [-\bar\eta_5, \bar\eta_5] \cap I$ the following equality holds true for any $\mathfrak{r}$ in $\mathfrak{M}^\#$:

$$\Big(g_\varepsilon \circ \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\Big)(\mathfrak{r}) = \Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f} \cdot g_\varepsilon\Big)(\mathfrak{r}) + \varepsilon^{N+1} \rho^{N,i,j}_{FC}(\varepsilon,\mathfrak{r})\,, \tag{4.81}$$

where $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ stands for the function defined by (4.46) in the left hand side of the equality and for operator defined by (4.45) in the right hand side and where $\rho^{N,i,j}_{FC}$ is in $\mathcal{C}^\infty_\#\big(I \times \mathfrak{M}^\#\big)$.

Proof. Since $g_\varepsilon \in \mathcal{Q}^\infty_{T,b}$, there exists a finite set $\mathbb{I}_{g_\varepsilon} \subset \mathbb{Z}$ and $(c^\varepsilon_n)_{n \in \mathbb{I}_{g_\varepsilon}} \in \Big(\mathcal{O}^\infty_{T,b}\Big)^{\mathbb{I}_{g_\varepsilon}}$ such that $g_\varepsilon(\mathfrak{r}) = \sum_{n \in \mathbb{I}_{g_\varepsilon}} c^\varepsilon_n(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3)\sqrt{\mathfrak{r}_4}^{\,n}$. Moreover, for each $n \in \mathbb{I}_{g_\varepsilon}$, c^ε_n corresponds to a finite sum of terms of the form $\cos^{n_l}(\mathfrak{r}_3)\sin^{n_m}(\mathfrak{r}_3)\, d^\varepsilon_{n_p}(\mathfrak{r}_1,\mathfrak{r}_2)$, where $d^\varepsilon_{n_p} \in \mathcal{C}^\infty_b\big(\mathbb{R}^2\big)$. Now, as $g_\varepsilon \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\big)$, the $d^\varepsilon_{n_p}$ belong to $\mathcal{A}\big(\mathbb{R}^2\big)$. Consequently, by linearity, the proof of the theorem reduces to prove formula (4.81) with function g_ε of the form $g_\varepsilon(\mathfrak{r}) = \cos^l(\mathfrak{r}_3)\sin^m(\mathfrak{r}_3)\, d^\varepsilon(\mathfrak{r}_1,\mathfrak{r}_2)\sqrt{\mathfrak{r}_4}^{\,n}$, where $d^\varepsilon = d^\varepsilon(\mathfrak{r}_1,\mathfrak{r}_2) \in \mathcal{A}\big(\mathbb{R}^2\big) \cap \mathcal{C}^\infty_b\big(\mathbb{R}^2\big)$. This is what we will do.

Let $\mathfrak{r}_0 \in \mathcal{O}^\#$. As $d^\varepsilon \in \mathcal{A}\big(\mathbb{R}^2\big)$, and as $(\mathfrak{r}_4 \mapsto \sqrt{\mathfrak{r}_4}^{\,n}) \in \mathcal{A}((0,+\infty))$, there exists a real number $R_{\mathfrak{r}_0} > 0$ and a formal power series $T_{\mathfrak{r}_0}$ of three variables which set of convergence contains the closure $\overline{\mathfrak{b}^3(0,R_{\mathfrak{r}_0})}$ of the Euclidian ball of dimension 3, which is such that $\mathfrak{b}^3(((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_4),R_{\mathfrak{r}_0}) \subset \mathcal{O}$ and such that $\forall\,(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4) \in \mathfrak{b}^3(((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_4),R_{\mathfrak{r}_0})$,

$$\begin{aligned} d(\mathfrak{r}_1,\mathfrak{r}_2)\sqrt{\mathfrak{r}_4}^{\,n} =& T_{\mathfrak{r}_0}((\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4) - ((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_4)) & (4.82)\\ =& \sum_{\mathfrak{l} \in \mathbb{N}^3} a^{l,\mathfrak{r}_0}_\varepsilon ((\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4) - ((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_4))^{\mathfrak{l}}\,. & (4.83) \end{aligned}$$

In addition, since $\big(\mathfrak{r}_3 \mapsto \cos^l(\mathfrak{r}_3)\sin^m(\mathfrak{r}_3)\big)$ is a power series of radius $+\infty$ with respect to $\mathfrak{r}_3$, there exists a formal power series $S_{\mathfrak{r}_0}$ such that $\overline{\mathfrak{b}^\#(0,R_{\mathfrak{r}_0})} \subset \boldsymbol{\Sigma}_{S_{\mathfrak{r}_0}}$ (see (4.41)), $\mathfrak{b}^\#(\mathfrak{r}_0,R_{\mathfrak{r}_0}) \subset \mathcal{O}^\#$ and such that

$$\forall \mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0,R_{\mathfrak{r}_0})\,,\ g_\varepsilon(\mathfrak{r}) = S_{\mathfrak{r}_0}(\mathfrak{r} - \mathfrak{r}_0) = \sum_{\mathfrak{l} \in \mathbb{N}^4} g^{l,\mathfrak{r}_0}_\varepsilon (\mathfrak{r} - \mathfrak{r}_0)^{\mathfrak{l}}\,. \tag{4.84}$$

Let $R'_{\mathfrak{r}_0} \in (0, R_{\mathfrak{r}_0})$. According to Lemma 4.18, there exists a real number $\eta_{R_{\mathfrak{r}_0},R'_{\mathfrak{r}_0}} > 0$ such that for any $\varepsilon \in \left[-\eta_{R_{\mathfrak{r}_0},R'_{\mathfrak{r}_0}}, \eta_{R_{\mathfrak{r}_0},R'_{\mathfrak{r}_0}}\right]$, $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. Hence, for any $\mathfrak{r} \in \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$, we have

$$g_\varepsilon\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\,(\mathfrak{r})\Big) = \sum_{\mathbf{l}\in\mathbb{N}^4} g^{l,\mathfrak{r}_0}_{\varepsilon}\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\,(\mathfrak{r}) - \mathfrak{r}_0\Big)^{\mathbf{l}}. \tag{4.85}$$

On another hand, let $\Theta_\varepsilon = \Theta_\varepsilon(\mathfrak{r}) = (\Theta_{\varepsilon,\mathbf{m}}(\mathfrak{r}))_{\mathbf{m}\in\mathbb{N}^4\ s.t.\ |\mathbf{m}|\leq i}$ be the smooth function satisfying for all smooth functions h_ε,

$$\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot h_\varepsilon\Big)(\mathfrak{r}) = \sum_{|\mathbf{m}|\leq i}\Theta_{\varepsilon,\mathbf{m}}(\mathfrak{r})\,\frac{\partial h_\varepsilon}{\partial \mathfrak{r}^{\mathbf{m}}}(\mathfrak{r})\,. \tag{4.86}$$

We have, for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$,

$$\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot g_\varepsilon\Big)(\mathfrak{r}) = \sum_{|\mathbf{m}|\leq i}\Theta_{\varepsilon,\mathbf{m}}\,(\mathfrak{r})\frac{\partial g_\varepsilon}{\partial \mathfrak{r}^{\mathbf{m}}}(\mathfrak{r}) = \left(\sum_{|\mathbf{m}|\leq i}\Theta_{\varepsilon,\mathbf{m}}(\mathfrak{r})\,\frac{\partial}{\partial \mathfrak{r}^{\mathbf{m}}}\right)\left(\sum_{\mathbf{l}\in\mathbb{N}^4} g^{l,\mathfrak{r}_0}_{\varepsilon}\,\mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}\right)(\mathfrak{r})\,. \tag{4.87}$$

where $\mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}$ stand for the function $\mathfrak{r} \mapsto (\mathfrak{r}_1 - (\mathfrak{r}_0)_1)^{l_1}(\mathfrak{r}_2 - (\mathfrak{r}_0)_2)^{l_2}(\mathfrak{r}_3 - (\mathfrak{r}_0)_3)^{l_3}(\mathfrak{r}_4 - (\mathfrak{r}_0)_4)^{l_4}$.

Since $\mathfrak{b}^{\#}(0, R_{\mathfrak{r}_0}) \subset \boldsymbol{\Sigma}_{S_{\mathfrak{r}_0}}$, we can permute sum and derivatives and we obtain:

$$\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot g_\varepsilon\Big)(\mathfrak{r}) = \sum_{|\mathbf{m}|\leq i}\Theta_{\varepsilon,\mathbf{m}}(\mathfrak{r})\sum_{\mathbf{l}\in\mathbb{N}^4} g^{l,\mathfrak{r}_0}_{\varepsilon}\frac{\partial \mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}}{\partial \mathfrak{r}^{m}}\,(\mathfrak{r}) = \sum_{\mathbf{l}\in\mathbb{N}^4} g^{\mathbf{l},\mathfrak{r}_0}_{\varepsilon}\,\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot\mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}\Big)(\mathfrak{r})\,. \tag{4.88}$$

Besides, using Property 4.23 and the link (4.46) between function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ and operator $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$, we obtain that, for any $\mathbf{l} \in \mathbb{N}^4$,

$$\begin{aligned}\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot\mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}\Big)\,(\mathfrak{r}) &= \Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot(\mathbf{r}_1 - (\mathfrak{r}_0)_1), \ldots, \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot(\mathbf{r}_4 - (\mathfrak{r}_0)_4)\Big)^{\mathbf{l}}(\mathfrak{r}) + \varepsilon^{N+1}\rho^{N,i,j}_{\mathbf{l},\mathfrak{r}_0}(\varepsilon,\mathfrak{r})\\ &= \Big(\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot\mathbf{r}_1, \ldots, \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot\mathbf{r}_4\Big)(\mathfrak{r}) - \mathfrak{r}_0\Big)^{\mathbf{l}} + \varepsilon^{N+1}\rho^{N,i,j}_{\mathbf{l},\mathfrak{r}_0}(\varepsilon,\mathfrak{r})\\ &= \Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\,(\mathfrak{r}) - \mathfrak{r}_0\Big)^{\mathbf{l}} + \varepsilon^{N+1}\rho^{N,i,j}_{\mathbf{l},\mathfrak{r}_0}(\varepsilon,\mathfrak{r})\,.\end{aligned} \tag{4.89}$$

As both $\sum_{\mathbf{l}\in\mathbb{N}^4} g^{l,\mathfrak{r}_0}_{\varepsilon}\,\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\,(\mathfrak{r}) - \mathfrak{r}_0\Big)^{\mathbf{l}}$ and $\sum_{\mathbf{l}\in\mathbb{N}^4} g^{\mathbf{l},\mathfrak{r}_0}_{\varepsilon}\,\Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot\mathbf{r}^{\mathbf{l}}_{\mathfrak{r}_0}\Big)(\mathfrak{r})$ converge normally on subset $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big) \cap \mathbb{R}^2 \times [a,b] \times \mathbb{R}$ for any compact set $[a,b] \subset \mathbb{R}$, their difference

$$\varepsilon^{N+1}\left(-\sum_{\mathbf{l}\in\mathbb{N}^4} g^{\mathbf{l},\mathfrak{r}_0}_{\varepsilon}\rho^{N,i,j}_{\mathbf{l},\mathfrak{r}_0}\,(\varepsilon,\mathfrak{r})\right) \tag{4.90}$$

also converges normally on this subset and we can deduce that, for any $\mathfrak{r} \in \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$,

$$\Big(g_\varepsilon \circ \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\Big)\,(\mathfrak{r}) = \Big(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\cdot g_\varepsilon\Big)\,(\mathfrak{r}) + \varepsilon^{N+1}\left(-\sum_{\mathbf{l}\in\mathbb{N}^4} g^{\mathbf{l},\mathfrak{r}_0}_{\varepsilon}\rho^{N,i,j}_{\mathbf{l},\mathfrak{r}_0}\,(\varepsilon,\mathfrak{r})\right). \tag{4.91}$$

Finally, as

$$\overline{\mathfrak{N}} \subset \underset{\left((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_4\right)\in\mathfrak{O}}{\cup} \mathfrak{b}^3\left(\left((\mathfrak{r}_0)_1, (\mathfrak{r}_0)_2, (\mathfrak{r}_0)_4\right), R'_{\mathfrak{r}_0}\right) \tag{4.92}$$

and as $\overline{\mathfrak{N}}$ is compact, there exists $\mathfrak{r}_0^1, \ldots, \mathfrak{r}_0^p$ such that

$$\overline{\mathfrak{N}} \subset \overset{p}{\underset{i=1}{\cup}} \mathfrak{b}^3\left(\left((\mathfrak{r}_0^i)_1, (\mathfrak{r}_0^i)_2, (\mathfrak{r}_0^i)_4\right), R'_{\mathfrak{r}_0^i}\right). \tag{4.93}$$

Setting $\bar{\eta}_5 = \min_{i=1,\ldots,p} \eta_{R_{\mathfrak{r}_0}, R'_{\mathfrak{r}_0}}$, we obtain equality (4.81) for all $\mathfrak{r} \in \mathfrak{M}^{\#}$ and for all $\varepsilon \in [-\bar{\eta}_5, \bar{\eta}_5] \cap I$. This ends the proof of Theorem 4.24. □

4.4 Proof of Theorems 4.9 and 4.13

The proof of Theorem 4.9 consists in checking that there exists a real number $\bar{\eta}_1$ such that for any $\varepsilon \in [-\bar{\eta}_1, \bar{\eta}_1]$, the map $\bar{\mathbf{r}} \mapsto \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\bar{\mathbf{r}})$, defined by (4.46), satisfies the assumptions of the global inversion Theorem (see Theorem 4.5) on $\mathfrak{C}^{\#}$.

In the first place, function $\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$, defined by (4.58), is differentiable and, according to Lemma 4.17, its differential is bounded. Moreover, according to formula (4.58), $\varepsilon \mapsto \boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}(\bar{\mathbf{r}})$ is clearly in $\mathcal{C}^\infty(\mathbb{R})$ for any $\bar{\mathbf{r}} \in \mathfrak{C}^{\#}$. Hence, we can define

$$\left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right\|_{1,\infty} = \sup_{\bar{\mathbf{r}}\in\mathfrak{C}^{\#}} \left|\left(d\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right)_{\bar{\mathbf{r}}}\right|_\infty, \tag{4.94}$$

where, here, $|.|_\infty$ stands for norm infinity in $\mathbb{R}^{4\times4}$, and function $\varepsilon \mapsto \left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right\|_{1,\infty}$ is clearly in $\mathcal{C}^\infty([-\bar{\eta}_1, \bar{\eta}_1])$. Now, since $\varepsilon\left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right\|_{1,\infty} \to 0$ when $\varepsilon \to 0$, there exists a real number $\bar{\eta}_1 > 0$ such that

$$\forall \varepsilon \in [-\bar{\eta}_1, \bar{\eta}_1], \quad \left|\varepsilon\left\|\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right\|_{1,\infty}\right| < 1. \tag{4.95}$$

Hence, since $\mathfrak{C}^{\#}$ is convex, we deduce that $\varepsilon\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$ is Lipschitz continuous on $\mathfrak{C}^{\#}$.

The second step consists in checking that for any $\bar{\mathbf{r}} \in \mathfrak{C}^{\#}$, $\left(d\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\right)_{\bar{\mathbf{r}}}$ is an isomorphism. As

$$\left(d\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\right)_{\bar{\mathbf{r}}} = id + \varepsilon\left(d\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right)_{\bar{\mathbf{r}}}, \tag{4.96}$$

the Jacobian Matrix of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ in $\bar{\mathbf{r}} \in \mathfrak{C}^{\#}$ can be rewritten as

$$\mathrm{Jac}(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}})(\bar{\mathbf{r}}) = 1 + \varepsilon\chi(\varepsilon, \bar{\mathbf{r}}), \tag{4.97}$$

where χ is bounded with respect to $\bar{\mathbf{r}} \in \mathfrak{C}^{\#}$ and $\varepsilon \mapsto \chi(\varepsilon, \bar{\mathbf{r}})$ is in $\mathcal{C}^\infty(\mathbb{R})$ for any $\bar{\mathbf{r}} \in \mathfrak{C}^{\#}$. Hence, there exists a real number $\eta_2 > 0$ such that for any $\varepsilon \in [-\eta_2, \eta_2]$ $|\varepsilon\,\|\chi(\varepsilon,.)\|_\infty| < 1$.

Hence, for $|\varepsilon| < \min(\bar{\eta}_1, \eta_2)$ the assumptions of the global inversion Theorem are satisfied leading to the conclusion that $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ is a diffeomorphism on $\mathfrak{C}^{\#}$. This ends the proof of Theorem 4.9. □

Theorem 4.13 is a direct consequence of Theorem 4.9 and Lemma 4.18. □

4.5 Proof of Theorem 4.14

Once $\bar{\mathfrak{r}}_0^*$, R_0^*, $\mathfrak{r}_0$ and R_0 are set, let $R_0^\bullet$, R_0' and R_0'' be three real numbers satisfying $0 < R_0^* < R_0^\bullet < R_0 < R_0' < R_0''$ and such that $\overline{\mathfrak{b}^4(\bar{\mathfrak{r}}_0^*, R_0^*)} \subset \mathfrak{b}^4(\mathfrak{r}_0, R_0^\bullet)$ and $\overline{\mathfrak{b}^4(\mathfrak{r}_0, R_0'')} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$.

In this proof, we will apply Theorem 4.24 with $f = -\bar{f}$ and $g_\varepsilon = \boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$. Using this Theorem requires that $\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}} \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$ and that $\varepsilon \mapsto \boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$ is smooth on some interval I containing 0. Obviously $\varepsilon \mapsto \boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$ is in $\mathcal{C}^\infty(\mathbb{R})$. On another hand, as $\bar{f}$ and all entries of the Poisson Matrix $\bar{\mathcal{P}}_\varepsilon$ are real analytic functions and as product and partial derivatives of real analytic functions are real analytic functions, function $\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$ is real analytic on $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. In addition, since $\mathcal{Q}^\infty_{T,b}$ is an algebra, it is stable by addition, multiplication by a scalar and by product. Moreover, $\mathcal{Q}^\infty_{T,b}$ is clearly stable by derivation. Consequently, $\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}} \in \mathcal{Q}^\infty_{T,b}$.

In a first place, we will show that there exists a real number $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$:

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^\#(\bar{\mathfrak{r}}_0^*, R_0^*)\Big) \subset \mathfrak{b}^\#(\mathfrak{r}_0, R_0), \tag{4.98}$$

$$\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}} \text{ is well defined and analytic on } \mathfrak{b}^\#(\mathfrak{r}_0, R_0). \tag{4.99}$$

According to Lemma 4.18, there exists a real number η_6 such that, for any $\varepsilon \in [-\eta_6, \eta_6]$, $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0^\bullet)\Big) \subset \mathfrak{b}^\#(\mathfrak{r}_0, R_0)$ and hence such that, for any $\varepsilon \in [-\eta_6, \eta_6]$, (4.98) holds true.

According to Lemma 4.19, there exists a real number η_7 such that, for any $\varepsilon \in [-\eta_7, \eta_7]$,

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0'')\Big) \supset \mathfrak{b}^\#(\mathfrak{r}_0, R_0'). \tag{4.100}$$

Applying Theorem 4.5 as in the proof of Theorem 4.9, and applying Theorem 4.4, yields that there exists a real number η_8 such that for any $\varepsilon \in [-\eta_8, \eta_8]$, $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ is well defined and analytic on $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_0'')\Big)$. Hence, for any $\varepsilon \in [-\min(\eta_7, \eta_8), \min(\eta_7, \eta_8)]$, $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ is well defined and analytic on $\mathfrak{b}^\#(\mathfrak{r}_0, R_0)$. Setting $\eta = \min(\eta_6, \eta_7, \eta_8)$ yields (4.99) and the first part of the theorem.

Secondly, we will show that there exists a real number $\eta' > 0$ such that $\forall \varepsilon \in [-\eta', \eta']$, and for any $\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_0)$ equality (4.51) holds true. Applying Theorem 4.24 with $f = -\bar{f}$, $\mathfrak{N} = \mathfrak{b}^3\left((\mathfrak{r}_0)_1, (\mathfrak{r}_0)_2, (\mathfrak{r}_0)_3, R_0\right) \subset \mathbb{R}^3$ (the closure of $\mathfrak{N}$ is clearly compact) and $g_\varepsilon = \varepsilon\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}$, we deduce that there exists a real number $\bar{\eta}_8$ such that for any $\varepsilon \in [-\bar{\eta}_8, \bar{\eta}_8]$ and any $\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_0)$:

$$\varepsilon\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathfrak{r})\right) = \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}} \cdot \varepsilon\boldsymbol{\nu}^{i,j}_{\varepsilon,\bar{f}}\right)(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^{N,i,j}_{FC}(\varepsilon, \mathfrak{r})\,. \tag{4.101}$$

Moreover, by definition of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}$,

$$id\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathfrak{r})\right) = \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}} \cdot id\right)(\mathfrak{r})\,, \tag{4.102}$$

and consequently

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathfrak{r})\right) = \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}} \cdot \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\right)(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^{N,i,j}_{FC}(\varepsilon,\mathfrak{r}). \tag{4.103}$$

An easy computation leads to

$$\begin{aligned}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}} \cdot \boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\right)(\mathfrak{r}) &= \left(\sum_{l=0}^{i}\frac{\left(-\varepsilon^j\right)^l}{l!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^l\cdot\right)\left(\sum_{k=0}^{i}\frac{\left(\varepsilon^j\right)^k}{k!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^k\cdot\mathfrak{r}\right)(\mathfrak{r})\\ &= \mathfrak{r} + \varepsilon^{N+1}\boldsymbol{\rho}^{N,i,j}_{c}(\varepsilon,\mathfrak{r}),\end{aligned} \tag{4.104}$$

where $\boldsymbol{\rho}^{N,i,j}_{c}$ is in $\mathcal{C}^{\infty}_{\#}\Big(\mathbb{R}\times\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)\Big)$.
Hence, we have shown that for any $\varepsilon\in[-\bar{\eta}_8,\bar{\eta}_8]$ and for any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$ we have

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathfrak{r})\right) = \mathfrak{r} + \varepsilon^{N+1}\left(\boldsymbol{\rho}^{N,i,j}_{c}(\varepsilon,\mathfrak{r}) + \boldsymbol{\rho}^{N,i,j}_{FC}(\varepsilon,\mathfrak{r})\right). \tag{4.105}$$

Now, there exists a real number $\eta_9>0$ such that for any $\varepsilon\in[-\eta_9,\eta_9]$,

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)\right)\right)\subset\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0'). \tag{4.106}$$

Let $\eta'=\min(\bar{\eta}_8,\eta_9,\eta)$. Then, for any $\varepsilon\in[-\eta',\eta']$, $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0')$ and applying $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ to both sides of formula (4.105), we obtain for any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$:

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}(\mathfrak{r}) = \boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{r}+\varepsilon^{N+1}\left(\boldsymbol{\rho}^{N,i,j}_{c}(\varepsilon,\mathfrak{r}) + \boldsymbol{\rho}^{N,i,j}_{FC}(\varepsilon,\mathfrak{r})\right)\right). \tag{4.107}$$

As $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ is well defined on both $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-\bar{f}}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)\right)\right)$ and $\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$ and as for small ε, they are both included in $\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0')$, which is convex, we can make a Taylor expansion of the right hand side of formula (4.107). Hence we obtain formula (4.51) for sufficiently small ε and for any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$.

Thirdly, we will show that the components 1, 2 and 4 of $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ are in $\mathcal{C}^{\infty}_{\#}\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)\Big)$ and that the penultimate coordinate satisfies (4.50). Let $\bar{\mathfrak{r}}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$. As $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0'')\right)\supset\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$, there exists $\bar{\mathfrak{r}}'\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0'')$ satisfying $\bar{\mathfrak{r}}=\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}(\bar{\mathfrak{r}}')$. We also define $\bar{\mathfrak{r}}^{\#}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)$ and $\bar{\mathfrak{r}}'^{\#}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0'')$ by $\bar{\mathfrak{r}}^{\#}=(\bar{\mathfrak{r}}_1,\bar{\mathfrak{r}}_2,\bar{\mathfrak{r}}_3+2\pi,\bar{\mathfrak{r}}_4)$ and $\bar{\mathfrak{r}}'^{\#}=(\bar{\mathfrak{r}}_1',\bar{\mathfrak{r}}_2',\bar{\mathfrak{r}}_3'+2\pi,\bar{\mathfrak{r}}_4')$. Since the components 1, 2 and 4 of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$ are in $\mathcal{C}^{\infty}_{\#}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0'')\right)$ and since the penultimate component satisfy

$$\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\right)_3\left(\bar{\mathfrak{r}}'^{\#}\right) = \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\right)_3(\bar{\mathfrak{r}}') + 2\pi, \tag{4.108}$$

we obtain:

$$\bar{\mathfrak{r}}^{\#} = \boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\bar{\mathfrak{r}}'^{\#}\right), \tag{4.109}$$

and consequently

$$\mathbf{\Xi}^{i,j}_{\varepsilon,f}\left(\bar{\mathfrak{r}}^{\#}\right)=\mathbf{\Xi}^{i,j}_{\varepsilon,f}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}\left(\bar{\mathfrak{r}}'^{\#}\right)\right)=\bar{\mathfrak{r}}'^{\#}. \tag{4.110}$$

Eventually, to end this proof we need to show that $\boldsymbol{\rho}^N_{\mathbf{\Xi}^{i,j}}$ is in $\mathcal{C}^\infty_{\#}\Big([-\bar{\eta}_2;\bar{\eta}_2]\times\mathfrak{b}^{\#}(\mathfrak{r}_0,R_0)\Big)$. The components 1, 2 and 4 of $\mathbf{\Xi}^{i,j}_{\varepsilon,f}$ and $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-f}$ are 2π-periodic with respect to $\mathfrak{r}_3$. Consequently, the components 1, 2 and 4 of $\boldsymbol{\rho}^N_{\mathbf{\Xi}^{i,j}}$ are 2π-periodic with respect to $\mathfrak{r}_3$. Moreover, the penultimate component of $\mathbf{\Xi}^{i,j}_{\varepsilon,f}$ and $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-f}$ satisfy

$$\begin{aligned}\left(\mathbf{\Xi}^{i,j}_{\varepsilon,f}\right)_3(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3+2\pi,\mathfrak{r}_4)&=\left(\mathbf{\Xi}^{i,j}_{\varepsilon,f}\right)_3(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3,\mathfrak{r}_4)+2\pi,\\ \left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-f}\right)_3(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3+2\pi,\mathfrak{r}_4)&=\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,-f}\right)_3(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3,\mathfrak{r}_4)+2\pi.\end{aligned}$$

Consequently, the penultimate component of $\boldsymbol{\rho}^N_{\mathbf{\Xi}^{i,j}}$ is 2π-periodic. This ends the proof of Theorem 4.14. □

4.6 Proof of Theorem 4.15

Since (4.53) is a consequence of Theorem 4.14, for any $k,l\in\{1,\dots,4\}$, ones $\hat{\mathcal{P}}^{k,l}_{\varepsilon}$ is set by (4.52) the only thing to prove is (4.54). For this, let $R'_{\mathfrak{r}_0}$ and $R''_{\mathfrak{r}_0}$ be two real numbers such that $0<R_0<R'_{\mathfrak{r}_0}<R''_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0,R''_{\mathfrak{r}_0})}\subset\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$.

Applying Property 4.22 with $\mathfrak{M}=\mathfrak{b}^{\#}(\mathfrak{r}_0,R''_{\mathfrak{r}_0})$, $f=\bar{f}$, $g=\bar{\mathbf{r}}_k$ and $h=\bar{\mathbf{r}}_l$ we obtain for any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R''_{\mathfrak{r}_0})$

$$\left[\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\{\bar{\mathbf{r}}_k,\bar{\mathbf{r}}_l\}_{\bar{\mathbf{r}}}\right](\mathfrak{r})=\left\{\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\bar{\mathbf{r}}_k,\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\bar{\mathbf{r}}_l\right\}_{\bar{\mathbf{r}}}(\mathfrak{r})+\varepsilon^N\rho^{N,k,l}_{PC}(\varepsilon,\mathfrak{r}), \tag{4.111}$$

where $\rho^{n,k,l}_{PC}$ is $\mathcal{C}^\infty(\mathbb{R}\times\mathfrak{b}^{\#}(\mathfrak{r}_0,R''_{\mathfrak{r}_0}))$. Moreover, from expression (4.77) of the remainder $\rho^{n,k,l}_{PC}$ we obviously see that it is 2π-periodic with respect to $\mathfrak{r}_3=\theta$ and in $\mathcal{Q}^\infty_{T,b}$.

Applying Theorem 4.24 with $f=\bar{f}$, $\mathfrak{N}=\mathfrak{b}^3\big((\mathfrak{r}_0)_1,(\mathfrak{r}_0)_2,(\mathfrak{r}_0)_3,R'_{\mathfrak{r}_0}\big)\subset\mathbb{R}^3$ (the closure of $\mathfrak{N}$ is clearly compact) and $g_\varepsilon=\bar{\mathcal{T}}^{k,l}_{\varepsilon}$, there exists a real number $\bar{\eta}_8>0$ such that for any $\varepsilon\in[-\bar{\eta}_8,\bar{\eta}_8]$ and any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R'_{\mathfrak{r}_0})$:

$$\left(\bar{\mathcal{T}}^{k,l}_{\varepsilon}\circ\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\right)(\mathfrak{r})=\left[\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\bar{\mathcal{T}}^{k,l}_{\varepsilon}\right](\mathfrak{r})+\varepsilon^{N+1}\rho^{N,k,l}_{FC}(\varepsilon,\mathfrak{r}), \tag{4.112}$$

where $\rho^{N,k,l}_{FC}$ is in $\mathcal{C}^\infty_{\#}(\mathbb{R}\times\mathfrak{b}^{\#}(\mathfrak{r}_0,R'_{\mathfrak{r}_0}))$.

Combining equations (4.111) and (4.112) yields for any $\varepsilon\in[-\bar{\eta}_8,\bar{\eta}_8]$ and any $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R'_{\mathfrak{r}_0})$:

$$\varepsilon\left\{\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\bar{\mathbf{r}}_k,\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\cdot\bar{\mathbf{r}}_l\right\}_{\bar{\mathbf{r}}}(\mathfrak{r})=\bar{\mathcal{T}}^{k,l}_{\varepsilon}\left(\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\mathfrak{r})\right)-\varepsilon^{N+1}\left[\rho^{N,k,l}_{PC}(\varepsilon,\mathfrak{r})+\rho^{N,k,l}_{FC}(\varepsilon,\mathfrak{r})\right]. \tag{4.113}$$

Now, according to Theorem 4.9, there exists a real number $\bar{\eta}_9 > 0$ such that for any $\varepsilon \in [-\bar{\eta}_9, \bar{\eta}_9]$, $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ is invertible on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$, and a real number $\bar{\eta}_{10} > 0$ such that for any $\varepsilon \in [-\bar{\eta}_{10}, \bar{\eta}_{10}]$,

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\right) \supset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\,. \tag{4.114}$$

Formula (4.114) also means that for any $\varepsilon \in [-\min(\eta_9, \eta_{10}), \min(\eta_9, \eta_{10})]$, $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\right) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$ and hence that $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ is well defined on $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\right)$.

Let $\bar{\eta}_{11} = \min\left(\bar{\eta}_8, \bar{\eta}_9, \bar{\eta}_{10}\right)$, then for any $\varepsilon \in [-\bar{\eta}_{11}, \bar{\eta}_{11}]$ and any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$, we have

$$\hat{\mathcal{T}}^{k,l}_{\varepsilon}(\mathfrak{r}) = \bar{\mathcal{T}}^{k,l}_{\varepsilon}(\mathfrak{r}) - \varepsilon^{N+1}\left[\rho^{N,k,l}_{PC}\Big(\varepsilon, \boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\mathfrak{r})\Big) + \rho^{N,k,l}_{FC}\Big(\varepsilon, \boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\mathfrak{r})\Big)\right]. \tag{4.115}$$

Now, as $\rho^{N,k,l}_{PC}$ and $\rho^{N,k,l}_{FC}$ are smooth with respect to $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ and as $\boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}$ is smooth with respect to $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ and with respect to ε, formula (4.115) can be rewritten for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)$ as

$$\hat{\mathcal{T}}^{k,l}_{\varepsilon}(\mathfrak{r}) = \bar{\mathcal{T}}^{k,l}_{\varepsilon}(\mathfrak{r}) + \varepsilon^{N+1}\rho^{N,k,l}_{S}(\varepsilon, \mathfrak{r})\,, \tag{4.116}$$

where $\rho^{N,k,l}_{S}(\varepsilon, \mathfrak{r}) = -\left(\rho^{N,k,l}_{PC} + \rho^{N,k,l}_{FC}\right)\left(\varepsilon, \boldsymbol{\Xi}^{i,j}_{\varepsilon,\bar{f}}(\mathfrak{r})\right)$ is in $\mathcal{C}^{\infty}(\mathbb{R} \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0))$. Moreover, since the 1, 2 and 4 components of $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ are 2π-periodic with respect to $\mathfrak{r}_3$, the penultimate component of $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ satisfies formula (4.50) and $\rho^{N,k,l}_{PC}$ and $\rho^{N,k,l}_{FC}$ are 2π-periodic with respect to $\mathfrak{r}_3$, the remainder $\rho^{N,k,l}_{S}$ is 2π-periodic with respect to $\mathfrak{r}_3$. This ends the proof of Theorem 4.15. □

4.7 Extension of Lemmas 4.18 and 4.19, Properties 4.22 and 4.23, and Theorem 4.24

The purpose of this subsection is to extend Lemmas 4.18 and 4.19, Properties 4.22 and 4.23, and Theorem 4.24 to $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$. Notice also that viewed as a function $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$ is defined by:

$$\left(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\right)(\mathfrak{r}) = \left(\left(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\right) \cdot \mathfrak{r}_1, \ldots, \left(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\right) \cdot \mathfrak{r}_4\right), \tag{4.117}$$

Lemma 4.25. *Let $k \in \mathbb{N}^{\star}$, $f_1,\ f_2, \ldots, f_k \in \mathcal{C}^{\infty}_{\#}\left(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\right)$, $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ be positive integers, $\mathfrak{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, and $R_0^{\bullet}$, R_0 and R'_0 be three real numbers satisfying $R_0^{\bullet} < R_0 < R'_0$, and such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R'_0)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. Then, there exists a real number $\eta > 0$ such that for any $\varepsilon \in (-\eta, \eta)$:*

$$\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0^{\bullet}) \subset \boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\left(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\right) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_0\big)\,. \tag{4.118}$$

Proof. The proof of Lemma 4.25 is very similar to the proofs of Lemmas 4.18 and 4.19. In fact, we just have to replace in these proofs $\boldsymbol{\nu}^{i,j}_{\varepsilon,f}$ by $\boldsymbol{\nu}^{k}_{\varepsilon}$, where $\boldsymbol{\nu}^{k}_{\varepsilon}$ is such that

$$\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}(\mathfrak{r}) = \mathfrak{r} + \varepsilon\boldsymbol{\nu}^{k}_{\varepsilon}(\mathfrak{r})\,. \tag{4.119}$$

This ends the proof of Lemma 4.25. □

Lemma 4.26. *Let* $k \in \mathbb{N}^\star$, $f_1,\ f_2, \dots, f_k \in \mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, $i_1,\dots,i_k$ *and* $j_1,\dots,j_k$ *be positive integers such that for any* $k \in \{1,\dots,N\}$, $i_k j_l \geq N$, *and* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)$ *such that* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)\subset\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. *Then, for any functions* h *and* g *in* $\mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, *the following equality holds true on* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)$

$$\begin{aligned}&\Big(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot\{g,h\}\Big)(\mathfrak{r})\\&=\Big\{\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot g,\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot h\Big\}(\mathfrak{r})+\varepsilon^N\rho^{N,k}_{PC}(\varepsilon,\mathfrak{r}),\end{aligned}\tag{4.120}$$

where $\rho^{N,k}_{PC}$ *is* $\mathcal{C}^\infty_\#\Big(\mathbb{R}\times\mathfrak{b}^\#(\mathfrak{r}_0,R_0)\Big)$.

Proof. For $k=1$, formula (4.120) is given by Property 4.22. Moreover, since $f_1,\ g$ and h are in $\mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, and according to formula (4.77), the remainder is clearly 2π-periodic with respect to the penultimate variable. The rest of the proof is an easy induction. This ends the proof of Lemma 4.26. □

Lemma 4.27. *Let* $k \in \mathbb{N}^\star$, $f_1,\ f_2, \dots, f_k \in \mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, $i_1,\dots,i_k$ *and* $j_1,\dots,j_k$ *be positive integers such that for any* $k \in \{1,\dots,N\}$, $i_k j_l \geq N$, *and* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)$ *such that* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)\subset\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$. *Then, for any functions* h *and* g *in* $\mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, *the following equality holds true on* $\mathfrak{b}^\#(\mathfrak{r}_0,R_0)$

$$\begin{aligned}&\Big(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot(gh)\Big)(\mathfrak{r})=\\&\qquad\Big(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot g\Big)(\mathfrak{r})\Big(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1}\cdot\ldots\cdot\boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\cdot h\Big)(\mathfrak{r})+\varepsilon^{N+1}\rho^{N,k}_{FP}(\varepsilon,\mathfrak{r}),\end{aligned}\tag{4.121}$$

where $\rho^{N,k}_{FP}$ *is in* $\mathcal{C}^\infty_\#\Big(\mathbb{R}\times\mathfrak{b}^\#(\mathfrak{r}_0,R_0)\Big)$.

Proof. For $k=1$, formula (4.121) is given by Property 4.23. Moreover, since $f_1,\ g$ and h are in $\mathcal{C}^\infty_\#\left(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\right)$, the remainder is clearly 2π-periodic with respect to the penultimate variable. The rest of the proof is an easy induction. This ends the proof of Lemma 4.27. □

Lemma 4.28. *Let* $\mathfrak{N}\subset\mathbb{R}^3$ *be an open set such that* $\overline{\mathfrak{N}}$ *is a compact subset of* $\mathbb{R}^2\times(0,+\infty)$; $\mathfrak{M}^\#$ *the open subset of* $\mathbb{R}^4$ *defined by*

$$\mathfrak{M}^\# = \left\{\mathfrak{r}\in\mathbb{R}^4,\ (\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4)\in\mathfrak{N} \text{ and } \mathfrak{r}_3\in\mathbb{R}\right\};\tag{4.122}$$

$\mathcal{O}\subset\mathbb{R}^2\times(0,+\infty)$ *be an open subset such that* $\overline{\mathfrak{N}}\subset\mathcal{O}$; $\mathcal{O}^\#$ *the be open subset of* $\mathbb{R}^4$ *defined by*

$$\mathcal{O}^\# = \left\{\mathfrak{r}\in\mathbb{R}^4,\ (\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_4)\in\mathcal{O} \text{ and } \mathfrak{r}_3\in\mathbb{R}\right\};\tag{4.123}$$

$k \in \mathbb{N}^\star$, $f_1,\ f_2,\dots,f_k\in\mathcal{C}^\infty_\#\left(\mathcal{O}^\#\right)$, $i_1,\dots,i_k$ *and* $j_1,\dots,j_k$ *be positive integers such that for any* $k\in\{1,\dots,N\}$, $i_k j_l\geq N$; *and* $g_\varepsilon = g_\varepsilon(\mathfrak{r})\in\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$ *for every* ε *in* *some interval I containing 0 and $\varepsilon \mapsto g_\varepsilon(\mathfrak{r})$ be in $\mathcal{C}^\infty(I)$ for any $\mathfrak{r} \in \mathfrak{M}^\#$. Then, there exists a real number $\bar{\eta}_5 > 0$ such that for any $\varepsilon \in [-\bar{\eta}_5, \bar{\eta}_5] \cap I$ the following equality holds true for any $\mathfrak{r}$ in $\mathfrak{M}^\#$:*

$$\left(g_\varepsilon \circ \left(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}\right)\right)(\mathfrak{r}) = \left(\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k} \cdot g_\varepsilon\right)(\mathfrak{r}) + \varepsilon^{N+1}\rho^{N,k}_{FC}(\varepsilon, \mathfrak{r}), \qquad (4.124)$$

where $\rho^{N,k}_{FC}$ is in $\mathcal{C}^\infty_\#\big(I \times \mathfrak{M}^\#\big)$.

Proof. In order to prove Lemma 4.28, we only have to check that all the steps of the proof of Theorem 4.24 are valid with $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$ instead of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$.

Firstly, and with the same arguments as in the proof of Theorem 4.24, we have only to show formula (4.124) with functions g_ε of the form $g_\varepsilon(\mathfrak{r}) = \cos^l(\mathfrak{r}_3)\sin^m(\mathfrak{r}_3)\, d^\varepsilon(\mathfrak{r}_1, \mathfrak{r}_2)\, \sqrt{\mathfrak{r}_4}^{\,n}$, where $d^\varepsilon = d^\varepsilon(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathcal{A}\big(\mathbb{R}^2\big) \cap \mathcal{C}^\infty_b\left(\mathbb{R}^2\right)$.

Since Lemma 4.18 is extended by Lemma 4.25, formula (4.85) is also valid with $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$ instead of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$.

Eventually, since Property 4.23 is extended by Lemma 4.27, and since $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$ is defined by (4.117), formula (4.89) is also valid with $\boldsymbol{\vartheta}^{i_1,j_1}_{\varepsilon,f_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{i_k,j_k}_{\varepsilon,f_k}$ instead of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$.

The rest of the proof is stricto-sensu the same. This ends the proof of Lemma 4.28 □

5 The Partial Lie Transform Method

In the previous section we introduced the partial Lie Sum functions and the Partial Lie Sums of order (i,j) and we have set out their properties. In this section, from these Sums, we will build for $N \in \mathbb{N}^\star$ fixed by (4.33), whatever its worth, a change of coordinates such that in the yielding coordinate system, the Hamiltonian function does not depend on θ up to order N in ε. Moreover, thanks to Theorem 4.15, we will construct this change of coordinates such that the Poisson Matrix (regarded as a function) remains unchanged, up to order $N-1$ in ε, under this change of coordinates. More precisely, we will prove that such a coordinate system exists and give a constructive algorithm (Algorithm 5.11) in order to build it. In the first subsection, we will introduce the general Partial Lie Transform change of coordinates of order N (see formula (5.2)), and set out its properties. In the second subsection, we will give an algorithm to find the $\bar{g}_i$, for $i \in \{0, \ldots, N\}$ such that the Hamiltonian becomes under its partial normal form (see subsection 5.2 for a definition of the partial normal forms).

5.1 The Partial Lie Transform Change of Coordinates of order N

Let $N \in \mathbb{N}^*$ be set by (4.33). For $i \in \{1, \ldots, N\}$, we define the positive integer α_i by

$$\alpha_i = \min\{k \in \mathbb{N} \text{ s.t. } ki \geq N\} \, (= \mathbb{E}\left(\frac{N}{i}\right)), \qquad (5.1)$$

where $\mathbb{E}$ stands for the integer part, and let us state the following lemma which will allow us to define the change of coordinates.

Lemma 5.1. *Let $\mathfrak{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$, $R_{\mathbf{r}_0}$ be a positive real number; $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0})$ be defined by* (4.36) *and satisfying $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0}) \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$; $R'_{\mathbf{r}_0}$ be a real number such that $0 < R'_{\mathbf{r}_0} < R_{\mathbf{r}_0}$; and $\bar{g}_1, \dots, \bar{g}_N \in \mathcal{C}^\infty_{\#}\left(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\right)$. Then, there exists a real number η such that for any $\varepsilon \in [-\eta, \eta]$, the function $\boldsymbol{\chi}^N_\varepsilon$ defined by:*

$$\boldsymbol{\chi}^N_\varepsilon = \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,-\bar{g}_1} \circ \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,-\bar{g}_2} \circ \dots \circ \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,-\bar{g}_N}, \tag{5.2}$$

with the $\boldsymbol{\vartheta}^{\alpha_i,i}_{\varepsilon,-\bar{g}_i}$ defined by (4.46)*, is well defined on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0})$, and satisfies*

$$\boldsymbol{\chi}^N_\varepsilon\Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathbf{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0}), \tag{5.3}$$

and

$$\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathbf{r}_0}\big) \subset \boldsymbol{\chi}^N_\varepsilon\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0})\Big). \tag{5.4}$$

Proof. We will prove this lemma by induction on N. More precisely, we will show by induction on N, that for any $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0}) \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$, for any $f_1, f_2, \dots, f_N \in \mathcal{C}^\infty_{\#}\left(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\right)$, for any positive integers $\beta_1, \dots, \beta_N$ and $\gamma_1, \dots, \gamma_N$, for any $R'_{\mathbf{r}_0}$ and $R^\bullet_{\mathbf{r}_0}$ such that $0 < R'_{\mathbf{r}_0} < R^\bullet_{\mathbf{r}_0} < R_{\mathbf{r}_0}$, there exists a real number $\bar{\eta}_N > 0$ such that for all $\varepsilon \in [-\bar{\eta}_N, \bar{\eta}_N]$ the function

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \dots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N}, \tag{5.5}$$

is well defined on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\bullet_{\mathfrak{r}_0}\big)$ and is such that

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \dots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\bullet_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0}), \tag{5.6}$$

and

$$\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big) \subset \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \dots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\bullet_{\mathfrak{r}_0}\big)\Big). \tag{5.7}$$

For $N = 1$ it is simply Lemmas 4.18 and 4.19.

Now, we assume the result for some $N \geq 1$. Let $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathbf{r}_0}) \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$, $f_1, f_2, \dots, f_{N+1} \in \mathcal{C}^\infty_{\#}\left(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\right)$, $\beta_1, \dots, \beta_{N+1}$ and $\gamma_1, \dots, \gamma_{N+1}$ be positive integers and $R'_{\mathbf{r}_0}$, $R^\bullet_{\mathbf{r}_0}$ be such that $0 < R'_{\mathbf{r}_0} < R^\bullet_{\mathbf{r}_0} < R_{\mathbf{r}_0}$.

Let $R^\star_{\mathfrak{r}_0} \in (R^\bullet_{\mathbf{r}_0}, R_{\mathbf{r}_0})$ and $R''_{\mathfrak{r}_0} \in (R'_{\mathbf{r}_0}, R^\bullet_{\mathbf{r}_0})$. Then according to Lemmas 4.18 and 4.19, there exists a real number $\eta > 0$ such that for all $\varepsilon \in [-\eta, \eta]$

$$\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\bullet_{\mathfrak{r}_0}\big) \subset \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}} \Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\star_{\mathfrak{r}_0}\big)\Big), \tag{5.8}$$

$$\boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}} \Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\bullet_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R^\star_{\mathfrak{r}_0}\big). \tag{5.9}$$

By induction hypothesis (applied with the triplet of periodic balls of radius ($R^{\bullet}_{\mathbf{r}_0}$, $R^{\star}_{\mathbf{r}_0}$, $R_{\mathbf{r}_0}$) and ($R''_{\mathbf{r}_0}$, $R^{\bullet}_{\mathbf{r}_0}$, $R^{\star}_{\mathbf{r}_0}$)), there exists a real number $\eta' > 0$ such that for all $\varepsilon \in [-\eta', \eta']$

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \tag{5.10}$$

is well defined on $\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\star}_{\mathbf{r}_0}\big)$ and such that

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N}\Big(\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\star}_{\mathbf{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})\,, \tag{5.11}$$

and

$$\mathfrak{b}^{\#}\big(\mathbf{r}_0, R''_{\mathbf{r}_0}\big) \subset \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N}\Big(\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\bullet}_{\mathbf{r}_0}\big)\Big) \tag{5.12}$$

Let $\bar{\eta}_{N+1} = \min(\eta, \eta')$. For any $\varepsilon \in [-\bar{\eta}_{N+1}, \bar{\eta}_{N+1}]$,

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}, \tag{5.13}$$

is well defined on $\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\bullet}_{\mathbf{r}_0}\big)$,

$$\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\Big(\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\bullet}_{\mathbf{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})\,, \tag{5.14}$$

and

$$\mathfrak{b}^{\#}\big(\mathbf{r}_0, R''_{\mathbf{r}_0}\big) \subset \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\Big(\mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\bullet}_{\mathbf{r}_0}\big)\Big)\,. \tag{5.15}$$

Since $\mathfrak{b}^{\#}\big(\mathbf{r}_0, R'_{\mathbf{r}_0}\big) \subset \mathfrak{b}^{\#}\big(\mathbf{r}_0, R^{\bullet}_{\mathbf{r}_0}\big)$, (5.14) implies (5.3) and since $\mathfrak{b}^{\#}\big(\mathbf{r}_0, R''_{\mathbf{r}_0}\big) \subset \mathfrak{b}^{\#}\big(\mathbf{r}_0, R'_{\mathbf{r}_0}\big)$, (5.15) implies (5.7). This ends the proof of Lemma 5.1. □

In particular, Lemma 5.1 ensures that, if $\boldsymbol{\chi}^N_\varepsilon$ is invertible with $\boldsymbol{\lambda}^N_\varepsilon$ as inverse function, for sufficiently small ε we have: $\boldsymbol{\lambda}^N_\varepsilon\Big(\mathfrak{b}^{\#}\big(\mathbf{r}_0, R'_{\mathbf{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})$.

Definition 5.2. *Function $\boldsymbol{\chi}^N_\varepsilon$ is called the Partial Lie Transform map of order N.*

The reason to choose $1, 2, \ldots, N$ as the indexes in $\boldsymbol{\vartheta}^{\alpha_i,i}_{\varepsilon,-\bar{g}_i}$ will be clarified in the next subsection. The first reason for the choice of α_i is that they satisfy $i\alpha_i \geq N$ for $i \in \{1, \ldots, N\}$. Hence, the theorems of the previous subsection may apply to functions $\boldsymbol{\vartheta}^{\alpha_i,i}_{\varepsilon,-\bar{g}_i}$. The second reason for this choice is that with this definition the number of terms in the sum that defines $\boldsymbol{\vartheta}^{\alpha_i,i}_{\varepsilon,-\bar{g}_i}$ is minimal.

We now give a theorem concerning the inverse of the Partial Lie Transform map.

Theorem 5.3. *Let $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N \in \mathcal{C}^\infty_{\#}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big)$, $\mathbf{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and $R_{\mathbf{r}_0} > 0$ be a real number such that $\overline{\mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. Then, for any $R'_{\mathbf{r}_0} > 0$ such that $0 < R_{\mathbf{r}_0} < R'_{\mathbf{r}_0}$ and such that $\overline{\mathfrak{b}^{\#}\big(\mathbf{r}_0, R'_{\mathbf{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, there exists a real number $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$, the restriction $\boldsymbol{\chi}^N_{\varepsilon\,|\mathfrak{b}^{\#}(\mathbf{r}_0, R_{\mathbf{r}_0})}$ of $\boldsymbol{\chi}^N_\varepsilon$ defined by*

(5.2) to $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ is a diffeomorphism, $\boldsymbol{\chi}_\varepsilon^N\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$, and the inverse function $\boldsymbol{\lambda}_\varepsilon^N = \big(\boldsymbol{\chi}_\varepsilon^N\big)^{-1}$ of $\boldsymbol{\chi}_\varepsilon^N$ is well defined on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ and expresses as

$$\boldsymbol{\lambda}_\varepsilon^N = \boldsymbol{\Xi}_{\varepsilon,-\bar{g}_N}^{\alpha_N,N} \circ \ldots \circ \boldsymbol{\Xi}_{\varepsilon,-\bar{g}_1}^{\alpha_1,1}. \tag{5.16}$$

where for $i = 1, \ldots, N$, $\boldsymbol{\Xi}_{\varepsilon,-\bar{g}_i}^{\alpha_i,i}$ is the inverse function of $\boldsymbol{\vartheta}_{\varepsilon,-\bar{g}_i}^{\alpha_i,i}$, given by Theorem 4.14 Moreover, the components 1, 2 and 4 of $\boldsymbol{\lambda}_\varepsilon^N$ are in $\mathcal{C}_\#^\infty\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$ and the penultimate component satisfies for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ and for any $\varepsilon \in [-\eta, \eta]$:

$$\big(\boldsymbol{\lambda}_\varepsilon^N\big)_3(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3 + 2\pi, \mathfrak{r}_4) = \big(\boldsymbol{\lambda}_\varepsilon^N\big)_3(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4) + 2\pi. \tag{5.17}$$

Proof. Firstly, we will prove by induction that for any $N \in \mathbb{N}^\star$, any $f_1, \ldots, f_N \in \mathcal{C}_\#^\infty\Big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\Big)$, any $\beta_1, \ldots, \beta_N \in \mathbb{N}^\star$, any $\gamma_1, \ldots, \gamma_N \in \mathbb{N}^\star$, any $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and for any $R'_{\mathfrak{r}_0} > 0$ such that $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$ there exists a real number $\eta_N > 0$ such that, for any $\varepsilon \in [-\eta_N, \eta_N]$, $\Big(\boldsymbol{\vartheta}_{\varepsilon,f_N}^{\beta_N,\gamma_N} \circ \boldsymbol{\vartheta}_{\varepsilon,f_{N-1}}^{\beta_{N-1},\gamma_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big)_{|\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})}$ is a diffeomorphism, such that

$$\Big(\boldsymbol{\vartheta}_{\varepsilon,f_N}^{\beta_N,\gamma_N} \circ \boldsymbol{\vartheta}_{\varepsilon,f_{N-1}}^{\beta_{N-1},\gamma_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big)\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big) \tag{5.18}$$

and such that $\Big(\boldsymbol{\vartheta}_{\varepsilon,f_N}^{\beta_N,\gamma_N} \circ \boldsymbol{\vartheta}_{\varepsilon,f_{N-1}}^{\beta_{N-1},\gamma_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big)^{-1}$ expresses

$$\Big(\boldsymbol{\vartheta}_{\varepsilon,f_N}^{\beta_N,\gamma_N} \circ \boldsymbol{\vartheta}_{\varepsilon,f_{N-1}}^{\beta_{N-1},\gamma_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big)^{-1} = \boldsymbol{\Xi}_{\varepsilon,f_1}^{\beta_1,\gamma_1} \circ \boldsymbol{\Xi}_{\varepsilon,f_2}^{\beta_2,\gamma_2} \circ \ldots \circ \boldsymbol{\Xi}_{\varepsilon,f_N}^{\beta_N,\gamma_N} \tag{5.19}$$

and is well defined on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$.

For $N = 1$: Let $R''_{\mathfrak{r}_0} > 0$ be such that $0 < R_{\mathfrak{r}_0} < R''_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$. Applying Theorem 4.13 and Lemma 4.19 yield that there exists a real number $\eta > 0$ such that $\forall \varepsilon \in [-\eta, \eta]$, $\boldsymbol{\vartheta}_{\varepsilon,f_1\ |\mathfrak{b}^{\#}(\mathfrak{r}_0, R''_{\mathfrak{r}_0})}^{\beta_1,\gamma_1}$ is a diffeomorphism and such that

$$\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0}) \subset \boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R''_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big).$$

Consequently, $\boldsymbol{\vartheta}_{\varepsilon,f_1}^{\beta_1,\gamma_1}\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$, and $\boldsymbol{\Xi}_{\varepsilon,f_1}^{\beta_1,\gamma_1}$ is well defined on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$.

Now, we assume that the result is true for some $N \geq 1$. Let $f_1, \ldots, f_{N+1} \in \mathcal{C}_\#^\infty\Big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\Big)$, $\beta_1, \ldots, \beta_{N+1} \in \mathbb{N}^\star$, $\gamma_1, \ldots, \gamma_{N+1} \in \mathbb{N}^\star$, $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ satisfying $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, and $R'_{\mathfrak{r}_0} > 0$ a real number such that $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$.
Let $R''_{\mathfrak{r}_0}$, $R^{(3)}_{\mathfrak{r}_0}$ and $R^{(4)}_{\mathfrak{r}_0}$ be such that $0 < R_{\mathfrak{r}_0} < R''_{\mathfrak{r}_0} < R^{(3)}_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0} < R^{(4)}_{\mathfrak{r}_0}$, and $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R^{(4)}_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. By induction hypothesis (applied successively with the couples of ball of

radius $(R'_{\mathfrak{r}_0}, R^{(4)}_{\mathfrak{r}_0})$, $(R_{\mathfrak{r}_0}, R''_{\mathfrak{r}_0})$, and again with $(R'_{\mathfrak{r}_0}, R^{(4)}_{\mathfrak{r}_0}))$, there exists a real number $\eta_N > 0$ such that for any $\varepsilon \in [-\eta_N, \eta_N]$, function $\left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \boldsymbol{\vartheta}^{\beta_{N-1},\gamma_{N-1}}_{\varepsilon,f_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)_{|\mathfrak{b}^\#\left(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\right)}$ is a diffeomorphism,

$$\left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \boldsymbol{\vartheta}^{\beta_{N-1},\gamma_{N-1}}_{\varepsilon,f_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)\left(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\right) \subset \mathfrak{b}^\#\left(\mathfrak{r}_0, R''_{\mathfrak{r}_0}\right) \tag{5.20}$$

and

$$\left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \boldsymbol{\vartheta}^{\beta_{N-1},\gamma_{N-1}}_{\varepsilon,f_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)^{-1} = \boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \tag{5.21}$$

is well defined on $\mathfrak{b}^\#\left(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\right)$.

Applying Lemma 4.18, there exists a real number $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$,

$$\boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\left(\mathfrak{b}^\#\left(\mathfrak{r}_0, R''_{\mathfrak{r}_0}\right)\right) \subset \mathfrak{b}^\#\left(\mathfrak{r}_0, R^{(3)}_{\mathfrak{r}_0}\right).$$

Let $\eta' = \min(\eta_N, \eta)$, then for any $\varepsilon \in [-\eta', \eta']$,

$$\boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\left(\left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \boldsymbol{\vartheta}^{\beta_{N-1},\gamma_{N-1}}_{\varepsilon,f_{N-1}} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)\left(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\right)\right) \subset \mathfrak{b}^\#\left(\mathfrak{r}_0, R^{(3)}_{\mathfrak{r}_0}\right). \tag{5.22}$$

Applying Lemma 4.19 and Theorem 4.9, there exists a real number $\bar{\eta}_{12} > 0$ such that $\forall \varepsilon \in [-\bar{\eta}_{12}, \bar{\eta}_{12}]$,

$$\mathfrak{b}^\#\left(\mathfrak{r}_0, R^{(3)}_{\mathfrak{r}_0}\right) \subset \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\left(\mathfrak{b}^\#\left(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\right)\right) \tag{5.23}$$

and such that $\boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}$ is invertible on $\mathfrak{b}^\#\left(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\right)$. Consequently,

$$\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \boldsymbol{\Xi}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}} \tag{5.24}$$

is well defined on $\mathfrak{b}^\#\left(\mathfrak{r}_0, R^{(3)}_{\mathfrak{r}_0}\right)$.

Let $\eta_{N+1} = \min\left(\eta_N, \eta, \eta_{12}\right)$, then for any $\varepsilon \in [-\bar{\eta}_{N+1}, \bar{\eta}_{N+1}]$, the following equalities hold true on $\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})$:

$$\begin{aligned}
&\left[\left(\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_N,\gamma_N}_{\varepsilon,f_N}\right) \circ \boldsymbol{\Xi}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\right] \circ \left(\boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}} \circ \boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right) \\
&= \left(\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_N,\gamma_N}_{\varepsilon,f_N}\right) \circ \left(\boldsymbol{\Xi}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}} \circ \boldsymbol{\vartheta}^{\beta_{N+1},\gamma_{N+1}}_{\varepsilon,f_{N+1}}\right) \circ \left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right) \\
&= \left(\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_N,\gamma_N}_{\varepsilon,f_N}\right) \circ \left(\boldsymbol{\vartheta}^{\beta_N,\gamma_N}_{\varepsilon,f_N} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right) \\
&= id_{|\mathfrak{b}^\#\left(\mathfrak{r}_0, R_{\mathfrak{r}_0}\right)}.
\end{aligned} \tag{5.25}$$

Starting with the 2π-periodicity of the components 1, 2 and 4 of $\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}$ and formula (4.50), the proofs of the 2π-periodicity of the components 1, 2 and 4 of $\boldsymbol{\lambda}^N_\varepsilon$ and formula (5.17) are easily obtained by induction. This ends the proof of Theorem 5.3. □

Theorem 5.4. *Let $\bar{g}_1, \dots, \bar{g}_N \in \mathcal{Q}^{\infty}_{T,b} \cap \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\big)$ (see Definition 4.2 and (3.118)), $\mathfrak{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$ and $R_{\mathfrak{r}_0} > 0$ be a real number such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$. Then, for any $\mathfrak{b}^{\#}\big(\mathfrak{r}_0^{\star}, R^{\star}_{\mathfrak{r}_0}\big)$ such that $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0^{\star}, R^{\star}_{\mathfrak{r}_0}\big)} \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ there exists a real number $\bar{\eta}_{13} > 0$ such that for any $\varepsilon \in [-\bar{\eta}_{13}; \bar{\eta}_{13}]$*

$$\boldsymbol{\chi}^N_{\varepsilon}\Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0^{\star}, R^{\star}_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0}) \tag{5.26}$$

(where $\boldsymbol{\chi}^N_{\varepsilon}$ is defined by (5.2)), and such that the inverse function $\boldsymbol{\lambda}^N_{\varepsilon} = \big(\boldsymbol{\chi}^N_{\varepsilon}\big)^{-1}$ of $\boldsymbol{\chi}^N_{\varepsilon}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ and expresses as

$$\boldsymbol{\lambda}^N_{\varepsilon}(\mathfrak{r}) = \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r}) + \varepsilon^{N+1} \boldsymbol{\rho}^N_{\boldsymbol{\lambda}}(\varepsilon, \mathfrak{r}), \tag{5.27}$$

where $\boldsymbol{\rho}^N_{\boldsymbol{\lambda}}$ is in $\mathcal{C}^{\infty}_{\#}\Big([-\eta_{12}, \eta_{12}] \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$.

Proof. We will start the proof of Theorem 5.4 by proving formula (5.26). Since $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0^{\star}, R^{\star}_{\mathfrak{r}_0}\big)} \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$, there exists a real number $R'_{\mathfrak{r}_0} > 0$ such that $0 < R'_{\mathfrak{r}_0} < R_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0^{\star}, R^{\star}_{\mathfrak{r}_0}\big)} \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$. Applying Theorem 5.3, there exists a real number $\bar{\eta} > 0$ such that for any $\varepsilon \in [-\bar{\eta}; \bar{\eta}]$, $\boldsymbol{\chi}^N_{\varepsilon}\Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\Big) \subset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$, and consequently such that formula (5.26) is satisfied.

The fact that there exists a real number $\bar{\eta}' > 0$ such that for any $\varepsilon \in [-\bar{\eta}'; \bar{\eta}']$, $\boldsymbol{\lambda}^N_{\varepsilon}$ is well defined on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ is also obtained by applying Theorem 5.3.

Consequently, to end the proof of Theorem 5.4 we need to check that there exists a real number $\eta_N > 0$ such that for any $\varepsilon \in [-\eta_N; \eta_N]$, $\boldsymbol{\lambda}^N_{\varepsilon}$ is analytic and satisfies formula (5.27) on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. Consequently, we will prove by induction on $p \in \mathbb{N}^{\star}$ that for any $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$; for any $\beta_1, \dots, \beta_p \in \mathbb{N}^{\star}$, for any $\gamma_1, \dots, \gamma_p \in \mathbb{N}^{\star}$, such that for any $i \in \{1, \dots, p\}$, $\beta_i \gamma_i \geq N$; for any $f_1, f_2, \dots, f_p \in \mathcal{A}\left(\mathbb{R}^3 \times (0,+\infty)\right) \cap \mathcal{Q}^{\infty}_{T,b}$; there exists a real number $\eta_p > 0$ such that for any $\varepsilon \in [-\eta_p, \eta_p]$, $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_p,\gamma_p}_{\varepsilon,f_p}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ and for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$

$$\Big(\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\Big)(\mathfrak{r}) = \Big(\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,-f_p} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,-f_1}\Big)(\mathfrak{r}) + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p}(\varepsilon, \mathfrak{r}), \tag{5.28}$$

where $\boldsymbol{\rho}^{N,p}$ is in $\mathcal{C}^{\infty}_{\#}\Big([-\eta_p, \eta_p] \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$.

The initialization of the induction is given by Theorem 4.14.

Now, we assume that the result is true for some fixed $p \geq 1$. Let $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ be such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$; $\beta_1, \dots, \beta_{p+1} \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_{p+1} \in \mathbb{N}$ such that for any $i \in \{1, \dots, p+1\}$, $\beta_i \gamma_i \geq N$; and $f_1, f_2, \dots, f_{p+1} \in \mathcal{A}\left(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\right) \cap \mathcal{Q}^{\infty}_{T,b}$.

Let $R'_{\mathfrak{r}_0}$ be a positive real numbers such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$. In a first place we will show that there exists a real number $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$, $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. According to Theorem 4.14, there exists a real number $\eta_1 > 0$ such that for any $\varepsilon \in [-\eta_1, \eta_1]$, $\boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. Moreover, according to Lemma 4.19, there exists a real number $\eta_2 > 0$ such that for any $\varepsilon \in [-\eta_2, \eta_2]$, $\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \Big(\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\Big) \supset \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. Let $\eta_3 = \min(\eta_1, \eta_2)$, then for any $\varepsilon \in [-\eta_3, \eta_3]$, $\boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ and

$$\boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\,. \tag{5.29}$$

By the induction hypothesis, there exists a real number $\eta_4 > 0$ such that for any $\varepsilon \in [-\eta_4, \eta_4]$, $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_p,\gamma_p}_{\varepsilon,f_p}$ is well defined and analytic on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$. Now, let $\eta = \min(\eta_3, \eta_4)$. According to Theorem 4.3, for any $\varepsilon \in [-\eta, \eta]$, $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$.

On another hand, according to Lemma 5.1, there exists a real number η' such that for any $\varepsilon \in [-\eta', \eta']$

$$\boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\,, \tag{5.30}$$

and such that $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1}$ is well defined on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$. Moreover, according to Lemma 4.25, there exists a real number $\eta'' > 0$ such that for any $\varepsilon \in [-\eta'', \eta'']$

$$\Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}\Big)\Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\,. \tag{5.31}$$

Let $\eta^{(3)} = \min(\eta, \eta', \eta'')$. Then for any $\varepsilon \in \big[-\eta^{(3)}, \eta^{(3)}\big]$ and for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$,

$$\begin{aligned}
&\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}(\mathfrak{r}) \\
&= \boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big(\boldsymbol{\Xi}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\Xi}^{\beta_p,\gamma_p}_{\varepsilon,f_p} \circ \boldsymbol{\Xi}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}(\mathfrak{r})\Big) \\
&= \boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big(\Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}\Big)(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{\beta,\gamma}(\varepsilon, \mathfrak{r})\Big) \\
&= \boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big(\Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}\Big)(\mathfrak{r})\Big) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{int}(\varepsilon, \mathfrak{r}) \\
&= \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,-f_1}\Big(\Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}\Big)(\mathfrak{r})\Big) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{int2}(\varepsilon, \mathfrak{r}) \\
&= \Big(\Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}\Big) \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,-f_1}\Big)(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{\beta,\gamma}(\varepsilon, \mathfrak{r})\,.
\end{aligned} \tag{5.32}$$

In this formula, the second equality is obtained from the induction hypothesis, and the third equality is obtained using a Taylor expansion. Notice that this expansion is valid because of formulas (5.30) and (5.31) and since $\boldsymbol{\Xi}^{\beta_1,\gamma_1}_{\varepsilon,f_1}$ is well defined on the convex subset $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$. The fourth equality is obtained by applying the case $p = 1$, and the last equality by applying the generalization of Theorem 4.24, given by Lemma 4.28, with $g_\varepsilon = \boldsymbol{\nu}^{\beta_1,\gamma_1}_{\varepsilon,-f_1}$ and $\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,-f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,-f_2}$.

The proof that $\boldsymbol{\rho}^N_{\beta,\gamma}$ is in $\mathcal{C}^\infty_\#\Big(\big[-\eta^{(3)}, \eta^{(3)}\big] \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$ is very similar to the proof that $\boldsymbol{\rho}^N_{\boldsymbol{\Xi}^{i,j}_{\varepsilon,f}}$ is in $\mathcal{C}^\infty_\#\Big([-\bar\eta_2, \bar\eta_2] \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$ in Theorem 4.14. This ends the proof of Theorem 5.4. $\square$

Now, we can consider the change of coordinate $\hat{\mathbf{r}} = \boldsymbol{\chi}_\varepsilon^N(\bar{\mathbf{r}})$ from $\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$ onto its range and formula (5.27) allows us to compute easily an expansion of the inverse change of coordinates. In order to obtain the expression of the Hamiltonian dynamical system in the Partial Lie Transform Coordinate System of order N, we need both the expressions of the Poisson Matrix and of the Hamiltonian function in this coordinate system.

Theorem 5.5. *Let $\mathbf{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and $R_{\mathbf{r}_0} > 0$ be such that $\overline{\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$; $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_N \in \mathcal{A}\left(\mathbb{R}^3 \times (0, +\infty)\right) \cap \mathcal{Q}_{T,b}^\infty$; and $\hat{\mathcal{P}}_\varepsilon$ be the Matrix which entries are given by:*

$$\hat{\mathcal{P}}_\varepsilon^{k,l}(\mathbf{r}) = \left\{ \left(\boldsymbol{\chi}_\varepsilon^N\right)_k, \left(\boldsymbol{\chi}_\varepsilon^N\right)_l \right\}_{\bar{\mathbf{r}}} \left(\boldsymbol{\lambda}_\varepsilon^N(\mathbf{r})\right). \tag{5.33}$$

Then, for any $\mathfrak{b}^\#(\mathbf{r}_0^\star, R_{\mathbf{r}_0}^\star)$ such that $\overline{\mathfrak{b}^\#(\mathbf{r}_0^\star, R_{\mathbf{r}_0}^\star)} \subset \mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$, there exists a real number $\bar{\eta}_{14} > 0$, such that for any $\varepsilon \in [-\bar{\eta}_{14}; \bar{\eta}_{14}]$,

$$\boldsymbol{\chi}_\varepsilon^N\left(\mathfrak{b}^\#(\mathbf{r}_0^\star, R_{\mathbf{r}_0}^\star)\right) \subset \mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0}) \tag{5.34}$$

and for any $\mathbf{r} \in \mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$ the following equality holds true:

$$\forall i, j \in \{1, 2, 3, 4\}, \ \hat{\mathcal{T}}_\varepsilon^{i,j}(\mathbf{r}) = \bar{\mathcal{T}}_\varepsilon^{i,j}(\mathbf{r}) + \varepsilon^{N+1} \rho_{\hat{\mathcal{T}}^{i,j}}^N(\varepsilon, \mathbf{r}), \tag{5.35}$$

where $\rho_{\hat{\mathcal{T}}^{i,j}}^N$ is in $\mathcal{C}_\#^\infty\left([-\bar{\eta}_{14}; \bar{\eta}_{14}] \times \mathfrak{b}^\#(\mathbf{r}_0, R_0)\right)$, and $\hat{\mathcal{T}}_\varepsilon$ stands for the matrix satisfying:

$$\hat{\mathcal{P}}_\varepsilon = \frac{1}{\varepsilon} \hat{\mathcal{T}}_\varepsilon. \tag{5.36}$$

Proof. In a first place, we will show by induction on $p \in \mathbb{N}^\star$ that for any $\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$ such that $\overline{\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$; for any $\beta_1, \dots, \beta_p \in \mathbb{N}$, for any $\gamma_1, \dots, \gamma_p \in \mathbb{N}$, such that $\forall i \in \{1, \dots, p\}, \beta_i \gamma_i \geq N$; for any $f_1, f_2, \dots, f_p \in \mathcal{A}\left(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\right) \cap \mathcal{Q}_{T,b}^\infty$; there exists a real number $\eta_p > 0$ such that for any $\varepsilon \in [-\eta_p, \eta_p]$, $\boldsymbol{\vartheta}_{\varepsilon, f_1}^{\beta_1, \gamma_1} \circ \dots \circ \boldsymbol{\vartheta}_{\varepsilon, f_p}^{\beta_p, \gamma_p}$ is well defined and analytic on $\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$ and the $k-$th component of $\boldsymbol{\vartheta}_{\varepsilon, f_1}^{\beta_1, \gamma_1} \circ \boldsymbol{\vartheta}_{\varepsilon, f_2}^{\beta_2, \gamma_2} \circ \dots \circ \boldsymbol{\vartheta}_{\varepsilon, f_p}^{\beta_p, \gamma_p}$ is given by

$$\begin{aligned} \left(\boldsymbol{\vartheta}_{\varepsilon, f_1}^{\beta_1, \gamma_1} \circ \boldsymbol{\vartheta}_{\varepsilon, f_2}^{\beta_2, \gamma_2} \circ \dots \circ \boldsymbol{\vartheta}_{\varepsilon, f_p}^{\beta_p, \gamma_p}\right)_k(\mathbf{r}) = \Big(\boldsymbol{\vartheta}_{\varepsilon, f_p}^{\beta_p, \gamma_p} \cdot \boldsymbol{\vartheta}_{\varepsilon, f_{p-1}}^{\beta_{p-1}, \gamma_{p-1}} \cdot \dots \cdot \boldsymbol{\vartheta}_{\varepsilon, f_1}^{\beta_1, \gamma_1} \cdot \bar{\mathbf{r}}_k\Big)(\mathbf{r}) \\ + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p}(\varepsilon, \mathbf{r}), \end{aligned} \tag{5.37}$$

where $\boldsymbol{\rho}^{N,p}$ is in $\mathcal{C}_\#^\infty\left([-\eta_p; \eta_p] \times \mathfrak{b}^\#(\mathbf{r}_0, R_0)\right)$.

For, $p = 1$, equality (5.37) is direct consequence of definition 4.7 and the analyticity is obvious (Notice that for $p = 1$ the remainder is zero).

Now, we assume the result for some fixed $p \geq 1$. Let $\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})$ be such that $\overline{\mathfrak{b}^\#(\mathbf{r}_0, R_{\mathbf{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, $\beta_1, \dots, \beta_{p+1} \in \mathbb{N}$, $\gamma_1, \dots, \gamma_{p+1} \in \mathbb{N}$, such that for any $i \in \{1, \dots, p+1\}$, $\beta_i \gamma_i \geq N$, and $f_1, f_2, \dots, f_{p+1} \in \mathcal{A}\left(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\right) \cap \mathcal{Q}_{T,b}^\infty$.

Let $R_{\mathfrak{r}_0}$, and $R'_{\mathfrak{r}_0}$ be two real numbers such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$. We will show that there exists a real number $\eta > 0$ such that for any $\varepsilon \in [-\eta, \eta]$, $\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. According to Lemma 4.18, there exists $\eta_1 > 0$ such that for any $\varepsilon \in [-\eta_1, \eta_1]$,

$$\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big).$$

Obviously, $\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$. According to the induction assumption, there exists $\eta_2 > 0$ such that for any $\varepsilon \in [-\eta_2, \eta_2]$, $\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}$ is well defined and analytic on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$. Setting $\eta = \min(\eta_1, \eta_2)$, and according to Theorem 4.3, for any $\varepsilon \in [-\eta, \eta]$, $\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}$ is well defined and analytic on $\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$.

To end this induction, we will show that there exists a real number $\eta' > 0$ such that for any $\varepsilon \in [-\eta', \eta']$ and for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$:

$$\Big(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\Big)_k(\mathfrak{r}) = \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \boldsymbol{\vartheta}^{\beta_{p-1},\gamma_{p-1}}_{\varepsilon,f_{p-1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \cdot \bar{\mathbf{r}}_k\Big)(\mathfrak{r}) + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p+1}(\varepsilon, \mathfrak{r}), \tag{5.38}$$

where $\boldsymbol{\rho}^{N,p+1}$ is is in $\mathcal{C}^{\infty}_{\#}\Big([-\eta'; \eta'] \times \mathfrak{b}^{\#}(\mathfrak{r}_0, R_0)\Big)$.

According to the induction step, there exists a real number η_2 such that for any $\varepsilon \in [-\eta_2, \eta_2]$, $\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}$ is well defined and analytic on $\mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$ and the $k-$th component of $\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}$ is given by (5.37). According to Lemma 4.18, there exists a real number η_3 such that for any $\varepsilon \in [-\eta_3, \eta_3]$,

$$\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \Big(\mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^{\#}\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big). \tag{5.39}$$

Setting $\eta' = \min(\eta_2, \eta_3)$ we obtain for any $\varepsilon \in [-\eta', \eta']$ and for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$:

$$\begin{aligned}
&\Big(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\Big)_k(\mathfrak{r}) \\
&= \Big(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\Big)_k \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}(\mathfrak{r})\Big) \\
&= \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big)_k \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}(\mathfrak{r})\Big) + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p}\Big(\varepsilon, \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}(\mathfrak{r})\Big) \\
&= \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big)_k(\mathfrak{r}) + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p+1}(\varepsilon, \mathfrak{r}) \\
&= \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big)_k(\mathfrak{r}) + \varepsilon^{N+1} \boldsymbol{\rho}^{N,p+1}(\varepsilon, \mathfrak{r}).
\end{aligned} \tag{5.40}$$

In the third equality we have used Theorem 4.24 with $g_\varepsilon = \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big)_k - \mathfrak{r}_k$.

Since for $k = 1,\ 2,$ and 4,

$$\Big(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1} \circ \boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2} \circ \ldots \circ \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\Big)_k \text{ and } \Big(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\Big)_k \tag{5.41}$$

are 2π-periodic, so is the remainder. Since for $k=3$

$$\left(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\circ\boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2}\circ\ldots\circ\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\right)_3\left(\mathfrak{r}^{\#}\right)=\left(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\circ\boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2}\circ\ldots\circ\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\right)_3(\mathfrak{r})+2\pi, \tag{5.42}$$

$$\left(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\cdot\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)_3\left(\mathfrak{r}^{\#}\right)=\left(\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\cdot\boldsymbol{\vartheta}^{\beta_{p+1},\gamma_{p+1}}_{\varepsilon,f_{p+1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\right)_3(\mathfrak{r})+2\pi, \tag{5.43}$$

the reminder is also 2π-periodic. This ends this first proof by induction.

Beside this, Lemma 4.26, yields:

$$\begin{aligned}&\left\{\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\cdot\boldsymbol{\vartheta}^{\beta_{p-1},\gamma_{p-1}}_{\varepsilon,f_{p-1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\cdot\bar{\mathbf{r}}_k,\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\cdot\boldsymbol{\vartheta}^{\beta_{p-1},\gamma_{p-1}}_{\varepsilon,f_{p-1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\cdot\bar{\mathbf{r}}_l\right\}_{\bar{\mathbf{r}}}(\mathfrak{r})\\&=\left(\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\cdot\boldsymbol{\vartheta}^{\beta_{p-1},\gamma_{p-1}}_{\varepsilon,f_{p-1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\cdot\{\bar{\mathbf{r}}_k,\bar{\mathbf{r}}_l\}_{\bar{\mathbf{r}}}\right)(\mathfrak{r})+\varepsilon^N\boldsymbol{\rho}_2^{N,p}(\varepsilon,\mathfrak{r}),\end{aligned} \tag{5.44}$$

with $\boldsymbol{\rho}_2^{N,p}$ 2π-periodic with respect to the penultimate variable.

Afterwards, the bilinearity of the Poisson Bracket, formulas (5.37) and (5.44) yield for any $\varepsilon\in[-\eta',\eta']$ and for all $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_{\mathfrak{r}_0})$:

$$\begin{aligned}&\left\{\left(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\circ\boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2}\circ\ldots\circ\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\right)_k,\left(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\circ\boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2}\circ\ldots\circ\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\right)_l\right\}_{\bar{\mathbf{r}}}(\mathfrak{r})\\&=\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\cdot\boldsymbol{\vartheta}^{\beta_{p-1},\gamma_{p-1}}_{\varepsilon,f_{p-1}}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\cdot\{\bar{\mathbf{r}}_k,\bar{\mathbf{r}}_l\}_{\bar{\mathbf{r}}}(\mathfrak{r})+\varepsilon^N\boldsymbol{\rho}^N(\varepsilon,\mathfrak{r}),\end{aligned} \tag{5.45}$$

with $\boldsymbol{\rho}^N$ 2π-periodic with respect to the penultimate variable.

Lastly formula (5.37), a Taylor expansion, and Lemma 4.28, yield that for any $p\in\mathbb{N}^\star$, there exists a real number $\eta_4>0$ such that for any $\varepsilon\in[-\eta_4,\eta_4]$, and for all $\mathfrak{r}\in\mathfrak{b}^{\#}(\mathfrak{r}_0,R_{\mathfrak{r}_0})$:

$$\begin{aligned}&\bar{\mathcal{T}}^{k,l}_\varepsilon\left(\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\circ\boldsymbol{\vartheta}^{\beta_2,\gamma_2}_{\varepsilon,f_2}\circ\ldots\circ\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}(\mathfrak{r})\right)\\&=\left(\boldsymbol{\vartheta}^{\beta_p,\gamma_p}_{\varepsilon,f_p}\cdot\ldots\cdot\boldsymbol{\vartheta}^{\beta_1,\gamma_1}_{\varepsilon,f_1}\cdot\bar{\mathcal{T}}^{k,l}_\varepsilon\right)(\mathfrak{r})+\varepsilon^{N+1}\boldsymbol{\rho}_p^N(\varepsilon,\mathfrak{r})\end{aligned} \tag{5.46}$$

where $\boldsymbol{\rho}_p^N$ is 2π-periodic with respect to the penultimate variable. Combining (5.45) and (5.46) yields the result. This ends the proof of Theorem 5.5. □

The way that map $\boldsymbol{\lambda}_\varepsilon^N$, which expression is given by (5.16), transforms functions is now tackled.

Theorem 5.6. *Let $\mathfrak{r}_0\in\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$, and $R_{\mathfrak{r}_0}>0$ be such that $\overline{\mathfrak{b}^{\#}(\mathfrak{r}_0,R_{\mathfrak{r}_0})}\subset\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)$; $\bar{g}_1,\bar{g}_2,\ldots,\bar{g}_N\in\mathcal{Q}^\infty_{T,b}\cap\mathcal{A}(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty))$; $\bar{G}=\bar{G}(\varepsilon,\mathfrak{r})\in\mathcal{C}^\infty_{\#}(I\times\mathbb{R}^2\times\mathbb{R}\times(0,+\infty))$ where I is an interval containing 0; and $\hat{G}$ be defined by $\hat{G}(\varepsilon,\mathfrak{r})=\bar{G}\left(\varepsilon,\boldsymbol{\lambda}_\varepsilon^N(\mathfrak{r})\right)$. Assume that $\bar{G}$ admits the following decomposition*

$$\bar{G}=\bar{G}^N+\varepsilon^{N+1}\iota_{N+1}, \tag{5.47}$$

where $\iota_{N+1} \in \mathcal{C}^\infty_\#\big(I \times \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\big)$, $\bar{G}^N \in \mathcal{Q}^\infty_{T,b} \cap \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)\big)$ *for any* $\varepsilon \in \mathbb{R}$ *and* $\varepsilon \mapsto \bar{G}^N(\varepsilon,\mathfrak{r})$ *is in* $\mathcal{C}^\infty(\mathbb{R})$ *for any* $\mathfrak{r} \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$. *Then, there exists a real number* $\bar{\eta}_{20} > 0$ *such that for any* $\varepsilon \in [-\bar{\eta}_{20};\bar{\eta}_{20}] \cap I$, *and for any* $\mathfrak{r} \in \mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})$*:*

$$\hat{G}(\varepsilon,\mathfrak{r}) = \Big(\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N} \cdot \bar{G}^N\Big)(\varepsilon,\mathfrak{r}) + \varepsilon^{N+1}\rho^N_{\bar{G}}(\varepsilon,\mathfrak{r}), \tag{5.48}$$

where $\rho^N_{\bar{G}} \in \mathcal{C}^\infty_\#\Big(([-\bar{\eta}_{20};\bar{\eta}_{20}] \cap I) \times \mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$.

Proof. Let $\mathfrak{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$, $R_{\mathfrak{r}_0} > 0$ and $R'_{\mathfrak{r}_0}$ be a real number such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^\#(\mathfrak{r}_0, R'_{\mathfrak{r}_0})} \subset \mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)$. According to Lemma 5.1, there exists a real number $\eta' > 0$ such that $\forall \varepsilon \in [-\eta', \eta']$

$$\boldsymbol{\lambda}^N_\varepsilon\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big). \tag{5.49}$$

According to Lemma 4.25 there exists a real number $\eta'' > 0$ such that $\forall \varepsilon \in [-\eta', \eta']$

$$\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big) \subset \mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big). \tag{5.50}$$

Consequently, both $\boldsymbol{\lambda}^N_\varepsilon\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$ and $\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$ are included in $\mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$ which is convex, and consequently, applying a Taylor Theorem we obtain:

$$\begin{aligned} &\bar{G}^N\Big(\varepsilon, \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{\boldsymbol{\lambda}}(\varepsilon,\mathfrak{r})\Big) \\ &\qquad = \bar{G}^N\Big(\varepsilon, \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r})\Big) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{int1}(\varepsilon,\mathfrak{r}). \end{aligned} \tag{5.51}$$

Eventually, applying formula (5.27) of Theorem 5.4, and the extension, given in Lemma 4.28, of Theorem 4.24 with $g_\varepsilon = \bar{G}^N(\varepsilon,\cdot)$ and $\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}$ instead of $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,f}$, we obtain:

$$\begin{aligned} \hat{G}(\varepsilon,\mathfrak{r}) &= \bar{G}\left(\varepsilon, \boldsymbol{\lambda}^N_\varepsilon(\mathfrak{r})\right) \\ &= \bar{G}\Big(\varepsilon, \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{\boldsymbol{\lambda}}(\varepsilon,\mathfrak{r})\Big) \\ &= \bar{G}^N\Big(\varepsilon, \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r})\Big) + \varepsilon^{N+1}\boldsymbol{\rho}^N_{int1}(\varepsilon,\mathfrak{r}) \\ &= \Big(\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}\Big) \cdot \bar{G}^N(\varepsilon,\mathfrak{r}) + \varepsilon^{N+1}\rho^N_{\bar{G}}(\varepsilon,\mathfrak{r}). \end{aligned} \tag{5.52}$$

Since $\bar{G} \in \mathcal{C}^\infty_\#\Big(([-\bar{\eta}_{20};\bar{\eta}_{20}] \cap I) \times \mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)$, and since

$$\Big(\boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \ldots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}\Big) \cdot \bar{G}^N \in \mathcal{C}^\infty_\#\Big([-\bar{\eta}_{20};\bar{\eta}_{20}] \times \mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big),$$

the remainder $\rho^N_{\bar{G}}$ is also 2π-periodic with respect to the penultimate variable. This ends the proof of Theorem 5.6. □

To end this subsection, we will give an useful expression of $\hat{G}(\varepsilon, \mathfrak{r})$ that we will use in the next subsection. For $p \in \mathbb{N}$, we define the subset $\mathcal{U}_p \subset \mathbb{N}^p$ by

$$\mathcal{U}_p = \left\{ (m_1, \dots, m_p) \in \mathbb{N}^p \text{ s.t. } \sum_{k=1}^{p} k m_k = p \right\}. \tag{5.53}$$

Proposition 5.7. *With the same notations and under the same assumptions as in Theorem 5.6, if $\bar{G}^N$ can be written as*

$$\bar{G}^N(\varepsilon, \mathfrak{r}) = \sum_{k=0}^{N} \bar{G}_k(\mathfrak{r})\, \varepsilon^k, \tag{5.54}$$

then, there exists a real number $\bar{\eta}_{21} > 0$ such that for any $\varepsilon \in [-\bar{\eta}_{21}; \bar{\eta}_{21}] \cap I$ and for any $\mathfrak{r} \in \mathfrak{b}^{\#}(\mathfrak{r}_0, R_{\mathfrak{r}_0})$:

$$\hat{G}(\varepsilon, \mathfrak{r}) = \sum_{n=0}^{N} \left(\sum_{k=0}^{n} \mathbf{V}^{\varepsilon}_{n-k} \cdot \bar{G}_k \right)(\mathfrak{r})\, \varepsilon^n + \varepsilon^{N+1} \iota^{N,\bullet}_{\bar{G}}(\varepsilon, \mathfrak{r}), \tag{5.55}$$

where, for any $l \in \{0, \dots, N\}$,

$$\mathbf{V}^{\varepsilon}_l = \sum_{(m_1,\dots,m_l)\in\mathcal{U}_l} \frac{\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^{m_1} \cdot \ldots \cdot \left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_l}\right)^{m_l} \cdot}{m_1! \dots m_l!} \tag{5.56}$$

Proof. The proof is obvious. We just have to expand formula (5.48). □

5.2 The Partial Lie Transform Method

The results we set out in the former subsection will be used to set out what we call the Partial Lie Transform Method of order N. This method applies to Hamiltonian Function $\bar{H}_\varepsilon$, since its first term does not depend on the penultimate coordinate; i.e. $\bar{H}_0 \in \bar{\mathcal{K}}$, where

$$\overline{\mathcal{K}} = \left\{ \bar{f} = \bar{f}(\bar{\mathbf{r}}) \in \mathcal{C}^\infty\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \text{ s.t. } \frac{\partial \bar{f}}{\partial \bar{r}_3} = 0 \right\}. \tag{5.57}$$

The goal of the Partial Lie Transform Method is to find $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_N \in \mathcal{Q}^\infty_{T,b} \cap \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big)$ such that under the change of coordinates (5.2) the Hamiltonian Function does not depend, up to order N, on the penultimate coordinate; i.e. such that the first terms $\hat{H}_0, \hat{H}_1, \dots, \hat{H}_N$ of the expansion of $\hat{H}_\varepsilon$, are in

$$\hat{\mathcal{K}} = \left\{ \hat{f} \in \mathcal{C}^\infty\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \text{ s.t. } \frac{\partial \hat{f}}{\partial \hat{r}_3} = 0 \right\}. \tag{5.58}$$

In this case, we will say that the Hamiltonian function is under its partial normal form of order N.

We will start the construction of the algorithm. The matrix $\bar{\mathcal{T}}_\varepsilon$, defined by $\bar{\mathcal{P}}_\varepsilon = \frac{1}{\varepsilon}\bar{\mathcal{T}}_\varepsilon$, is given by:

$$\bar{\mathcal{T}}_\varepsilon(\mathfrak{r}_1, \mathfrak{r}_2) = \bar{\mathcal{T}}_0 + \varepsilon^2 \bar{\mathcal{T}}_2(\mathfrak{r}_1, \mathfrak{r}_2), \tag{5.59}$$

where

$$\bar{\mathcal{T}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } \bar{\mathcal{T}}_2(\mathfrak{r}_1, \mathfrak{r}_2) = \begin{bmatrix} 0 & \frac{-1}{B(\mathfrak{r}_1,\mathfrak{r}_2)} & 0 & 0 \\ \frac{1}{B(\mathfrak{r}_1,\mathfrak{r}_2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.60}$$

Hence Hamiltonian vector fields $\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_i}$ defined by (4.3) are given by:

$$\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_i} = \bar{\mathcal{T}}_0 \nabla \bar{g}_i + \varepsilon^2 \bar{\mathcal{T}}_2 \nabla \bar{g}_i, \tag{5.61}$$

or equivalently by

$$\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_i} = \overline{\mathbf{M}}_i + \varepsilon^2 \overline{\mathbf{N}}_{i+2}, \tag{5.62}$$

where

$$\overline{\mathbf{M}}_i = \frac{\partial \bar{g}_i}{\partial \mathfrak{r}_4}\frac{\partial}{\partial \mathfrak{r}_3} - \frac{\partial \bar{g}_i}{\partial \mathfrak{r}_3}\frac{\partial}{\partial \mathfrak{r}_4} \text{ and } \overline{\mathbf{N}}_{i+2} = -\frac{1}{B(\mathfrak{r}_1, \mathfrak{r}_2)}\left(\frac{\partial \bar{g}_i}{\partial \mathfrak{r}_2}\frac{\partial}{\partial \mathfrak{r}_1} - \frac{\partial \bar{g}_i}{\partial \mathfrak{r}_1}\frac{\partial}{\partial \mathfrak{r}_2}\right). \tag{5.63}$$

The first result on which the method is based is the following theorem.

Theorem 5.8. *Let $\mathcal{M}_0\mathcal{C}^\infty_{per}$ be the space defined by*

$$\mathcal{M}_0\mathcal{C}^\infty_{per} = \left\{ u \in \mathcal{C}^\infty_\#\left(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\right) \text{ such that } \int_0^{2\pi} u(\mathfrak{r})\, d\mathfrak{r}_3 = 0 \right\}, \tag{5.64}$$

and $\bar{H}_0$ be the function defined by $\bar{H}_0(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_4) = B(\mathfrak{r}_1, \mathfrak{r}_2)\,\mathfrak{r}_4$ (see formula (3.119) *of corollary 3.23). Then, for any $\bar{u} \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b} \cap \mathcal{M}_0\mathcal{C}^\infty_{per}$, the function*

$$\bar{g}(\mathfrak{r}) = -\frac{1}{B(\mathfrak{r}_1, \mathfrak{r}_2)} \int_0^{\mathfrak{r}_3} \bar{u}(\mathfrak{r}')\, d\mathfrak{r}'_3 \tag{5.65}$$

is solution of the PDE

$$\left(\bar{\mathcal{T}}_0 \nabla \bar{g}\right) \cdot \nabla \bar{H}_0 = \bar{u}, \tag{5.66}$$

and $\bar{g} \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$ (see Definition 4.2 and (3.118)*).*

Proof. On the one hand, PDE (5.66) can be rewritten

$$B(\mathfrak{r}_1, \mathfrak{r}_2)\frac{\partial \bar{g}}{\partial \mathfrak{r}_3}(\mathfrak{r}) = \bar{u}(\mathfrak{r}). \tag{5.67}$$

Consequently, the function $\bar{g}$ defined by formula (5.65) is solution of (5.66). Now, we need to check that this function is in $\mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$. Firstly, since $\mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big)$ is stable by integration, function $\bar{g}$ is clearly in $\mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big)$. On the other hand, since $\bar{u} \in \mathcal{Q}^\infty_{T,b} \cap \mathcal{M}_0\mathcal{C}^\infty_{per}$ function $\bar{g}$ is in $\mathcal{Q}^\infty_{T,b}$. □

Theorem 5.9. *Let $\bar{H}_\varepsilon$ be the Hamiltonian function expressed in the Darboux coordinate system and whose expansion of order N is given by Corollary 3.22. Then, there exists $\hat{H}_1, \dots, \hat{H}_N \in \hat{\mathcal{K}} \cap \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$ and $\bar{g}_1, \dots, \bar{g}_N \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$ such that for any $\mathfrak{r}_0 \in \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, for any $R_{\mathfrak{r}_0}$ and $R'_{\mathfrak{r}_0}$ such that $0 < R_{\mathfrak{r}_0} < R'_{\mathfrak{r}_0}$ and $\overline{\mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ there exists a real number $\bar{\eta}_{14} > 0$, such that for any $\varepsilon \in [-\bar{\eta}_{14}; \bar{\eta}_{14}]$, function $\boldsymbol{\chi}^N_\varepsilon$ defined by (5.2) is well defined on $\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})$,*

$$\overline{\boldsymbol{\chi}^N_\varepsilon\Big(\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})\Big)} \subset \mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\,, \tag{5.68}$$

function $\boldsymbol{\lambda}^N_\varepsilon$ defined by (5.16) is well defined on $\mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$,

$$\overline{\boldsymbol{\lambda}^N_\varepsilon\Big(\mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\Big)} \subset \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\,, \tag{5.69}$$

and for any $\varepsilon \in [-\bar{\eta}_{14}; \bar{\eta}_{14}] \cap \mathbb{R}_+$ function $\hat{H}^N_\varepsilon$ defined by $\hat{H}^N_\varepsilon(\mathfrak{r}) = \bar{H}_\varepsilon\big(\boldsymbol{\lambda}^N_\varepsilon(\mathfrak{r})\big)$ writes:

$$\hat{H}^N_\varepsilon(\mathfrak{r}) = \sum_{k=0}^{n} \hat{H}_k(\mathfrak{r})\, \varepsilon^k + \varepsilon^{N+1} \iota^N_{\bar{H}}(\varepsilon, \mathfrak{r})\,, \tag{5.70}$$

where $\iota^N_{\bar{H}} \in \mathcal{C}^\infty_\#\big([0; \bar{\eta}_{14}] \times \mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big)$ (see (3.30)).

Remark 5.10. *In order to be precise, if we work on the Darboux Coordinate chart or on the Partial Lie Transform Coordinate chart we will write $\bar{\mathfrak{r}}$, instead of $\mathfrak{r}$, for $\mathfrak{r}$ in the Darboux coordinate chart; and $\hat{\mathfrak{r}}$, instead of $\mathfrak{r}$, for $\mathfrak{r}$ in the Partial Lie Transform coordinate chart. In Theorem 5.9, the periodic ball $\mathfrak{b}^\#(\mathfrak{r}_0, R_{\mathfrak{r}_0})$ is viewed as a subset of the open subset on which the Darboux Coordinate System is defined (precisely $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$), and $\mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$ is viewed as a subset of the open subset on which the Partial Lie Transform Coordinate System is defined.*

Proof. Let $\bar{g}_1, \dots, \bar{g}_N \in \mathcal{A}\big(\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)\big) \cap \mathcal{Q}^\infty_{T,b}$. According to Lemma 5.1 and Theorem 5.3, there exists a real number $\bar{\eta}_{20} > 0$ such that for any $\varepsilon \in [-\bar{\eta}_{20}; \bar{\eta}_{20}]$, formulas (5.68) and (5.69) hold true. Moreover, according to Proposition 5.7, there exists a real number $\bar{\eta}_{21} > 0$ such that for any $\varepsilon \in [-\bar{\eta}_{21}; \bar{\eta}_{21}] \cap \mathbb{R}_+$ and for any $\hat{\mathfrak{r}} \in \mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)$,

$$\hat{H}^N_\varepsilon(\hat{\mathfrak{r}}) = \sum_{n=0}^{N} \left(\sum_{k=0}^{n} \mathbf{V}^\varepsilon_{n-k} \cdot \bar{H}_k \right)(\hat{\mathfrak{r}})\, \varepsilon^n + \varepsilon^{N+1} \iota^{N,\bullet}_{\bar{H}}(\varepsilon, \hat{\mathfrak{r}})\,, \tag{5.71}$$

where $\iota^N_{\hat{H}} \in \mathcal{C}^\infty_\#\Big([-\bar{\eta}_{21}; \bar{\eta}_{21}] \cap \mathbb{R}_+ \times \mathfrak{b}^\#\big(\mathfrak{r}_0, R'_{\mathfrak{r}_0}\big)\Big)$, and for any $l \in \{0, \dots, N\}$, $\mathbf{V}^\varepsilon_l$ is defined by formula (5.56).
The only possible values that m_l can have in formula (5.56) are 0 and 1. If $m_l = 1$, then $m_1 = m_2 = \ldots = m_{l-1} = 0$. Hence, the only term in the sum of the right hand side of (5.56) that involves function $\bar{g}_l$ is $\bar{\mathbf{X}}^\varepsilon_{\varepsilon \bar{g}_l} \cdot$. Consequently the only term of

$$\sum_{k=0}^{n} \mathbf{V}^\varepsilon_{n-k} \cdot \bar{H}_k \tag{5.72}$$

that involves function $\bar{g}_n$ is $\mathbf{\bar{X}}^{\varepsilon}_{\varepsilon\bar{g}_n} \cdot \bar{H}_0$.

Injecting formula (5.62) in the right hand side of (5.71), gathering terms of the same power of ε, and comparing the result with the desired form of $\hat{H}^N_\varepsilon(\hat{\mathbf{r}})$:

$$\hat{H}^N_\varepsilon(\hat{\mathbf{r}}) = \bar{H}_0(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \varepsilon\hat{H}_1(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \ldots + \varepsilon^N\hat{H}_N(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \varepsilon^{N+1}\iota_{\hat{H}}(\varepsilon,\hat{\mathbf{r}}), \quad (5.73)$$

we obtain that $\bar{g}_1$ must be such that

$$\hat{H}_1 = \bar{H}_1 + \big(\bar{\mathcal{T}}_0\nabla\bar{g}_1\big)\cdot\nabla\bar{H}_0, \quad (5.74)$$

and, for any $i\in\{2,\ldots,N\}$, that $\bar{g}_i$ must satisfy

$$\hat{H}_i = \big(\bar{\mathcal{T}}_0\nabla\bar{g}_i\big)\cdot\nabla\bar{H}_0 - \mathcal{V}\left(\bar{g}_1,\ldots,\bar{g}_{i-1}\right), \quad (5.75)$$

with $\mathcal{V}\left(\bar{g}_1,\ldots,\bar{g}_{i-1}\right)$ depending only on $\bar{g}_1,\ldots,\bar{g}_{i-1}$ and their derivatives (and of course of the entries of the Poisson Matrix) and, consequently, is in $\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$.

Now, to end the proof, we need to check that using (5.74) and (5.75) we can build recursively $\hat{H}_1,\ldots,\hat{H}_N$ and $\bar{g}_1,\ldots,\bar{g}_N \in \mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$ such that for any $i\in\{1,\ldots,N\}$, $\hat{H}_i$ is in $\hat{\mathcal{K}}\cap\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$. We will prove it by induction on $i\in\{1,\ldots,N\}$.

For $i=1$, setting

$$\hat{H}_1 = \frac{1}{2\pi}\int_0^{2\pi}\bar{H}_1\,d\hat{r}_3 \quad\text{and}\quad u_1 = -\bar{H}_1 + \frac{1}{2\pi}\int_0^{2\pi}\bar{H}_1\,d\hat{r}_3, \quad (5.76)$$

yields that $\hat{H}_1\in\hat{\mathcal{K}}\cap\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}$ and $u_1\in\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}\cap\mathcal{M}_0\mathcal{C}^\infty_{per}$. Hence, equation (5.74) yields

$$\big(\bar{\mathcal{T}}_0\nabla\bar{g}_1\big)\cdot\nabla\bar{H}_0 = u_1, \quad (5.77)$$

and Theorem 5.8 gives $\bar{g}_1$ as a solution to this PDE.

Let $i\in\{1,\ldots,N-1\}$. Assume the result for all $k\in\{1,\ldots,i\}$. $\hat{H}_{i+1}$ is given by $\hat{H}_{i+1} = \big(\bar{\mathcal{T}}_0\nabla\bar{g}_{i+1}\big)\cdot\nabla\bar{H}_0 - \mathcal{V}_{i+1}\left(\bar{g}_1,\ldots,\bar{g}_i\right)$. Setting

$$\hat{H}_{i+1} = -\frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_{i+1}\left(\bar{g}_1,\ldots,\bar{g}_i\right)d\hat{r}_3 \in \hat{\mathcal{K}}\cap\mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}, \quad (5.78)$$

and

$$u_{i+1} = \mathcal{V}_{i+1}\left(\bar{g}_1,\ldots,\bar{g}_i\right) - \frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}\left(\bar{g}_1,\ldots,\bar{g}_i\right)d\hat{r}_3 \in \mathcal{A}\big(\mathbb{R}^2\times\mathbb{R}\times(0,+\infty)\big)\cap\mathcal{Q}^\infty_{T,b}\cap\mathcal{M}_0\mathcal{C}^\infty_{per}, \quad (5.79)$$

transform this equation into

$$\big(\bar{\mathcal{T}}_0\nabla\bar{g}_{i+1}\big)\cdot\nabla\bar{H}_0 = u_{i+1}. \quad (5.80)$$

Here again Theorem 5.8 gives $\bar{g}_{i+1}$ as a solution to this PDE. This ends the proof of Theorem 5.9. □

Theorem 5.9 and its proof makes up the Partial Lie Transform Algorithm. This algorithm can be summarized as follow:

Algorithm 5.11.

- *step 1: Inject formula* (5.62) *in the right hand side of*

$$\hat{H}_{\varepsilon}^{N}(\hat{\mathbf{r}}) = \sum_{n=0}^{N}\left(\sum_{k=0}^{n}\mathbf{V}_{n-k}^{\varepsilon}\cdot\bar{H}_k\right)(\hat{\mathbf{r}})\,\varepsilon^n + \varepsilon^{N+1}\iota_{\bar{H}}^{N,\bullet}(\varepsilon,\hat{\mathbf{r}})\,, \tag{5.81}$$

where for any $l \in \{0,\dots,N\}$, $\mathbf{V}_l^{\varepsilon}$ *is defined by formula* (5.56).

- *step 2: Gather terms according to their power of* ε *and compare the result with the following desired expression:*

$$\hat{H}_{\varepsilon}^{N}(\hat{\mathbf{r}}) = \hat{H}_{\varepsilon,T}^{N}(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \varepsilon^{N+1}\iota_{\hat{H}^N}^{N}(\varepsilon,\hat{\mathbf{r}})\,, \tag{5.82}$$

with

$$\hat{H}_{\varepsilon,T}^{N}(\hat{r}_1,\hat{r}_2,\hat{r}_4) = \bar{H}_0(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \varepsilon\hat{H}_1(\hat{r}_1,\hat{r}_2,\hat{r}_4) + \ldots + \varepsilon^N\hat{H}_N(\hat{r}_1,\hat{r}_2,\hat{r}_4)\,, \tag{5.83}$$

to obtain

$$\hat{H}_1 = \bar{H}_1 + \left(\bar{\mathcal{T}}_0\nabla\bar{g}_1\right)\cdot\nabla\bar{H}_0 \tag{5.84}$$

and, for any $i \in \{1,\dots,N\}$*:*

$$\hat{H}_i = \left(\bar{\mathcal{T}}_0\nabla\bar{g}_i\right)\cdot\nabla\bar{H}_0 - \mathcal{V}_i(\bar{g}_1,\dots,\bar{g}_{i-1})\,. \tag{5.85}$$

- *step 3: Set*

$$\hat{H}_1 = -\frac{1}{2\pi}\int_0^{2\pi}\bar{H}_1\,d\hat{r}_3, \tag{5.86}$$

and get $\bar{g}_1$ *by solving*

$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_1\right)\cdot\nabla\bar{H}_0 = \bar{H}_1 - \frac{1}{2\pi}\int_0^{2\pi}\bar{H}_1\,d\hat{r}_3. \tag{5.87}$$

Then for $i \in \{1,\dots,N\}$*, set*

$$\begin{aligned}\hat{H}_i &= -\frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_i\left(\bar{g}_1,\dots,\bar{g}_{i-1}\right)d\hat{r}_3,\\ u_i &= \mathcal{V}_i\left(\bar{g}_1,\dots,\bar{g}_{i-1}\right) - \frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_i\left(\bar{g}_1,\dots,\bar{g}_{i-1}\right)d\hat{r}_3,\end{aligned} \tag{5.88}$$

and get $\bar{g}_i$ *and by solving:*

$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_i\right)\cdot\nabla\bar{H}_0 = u_i, \tag{5.89}$$

with the help of formula (5.65).

This Algorithm is given in detail up to $N=5$ in Appendix B.

Applying this algorithm, we obtain functions $\bar{g}_i$ and hence the change of coordinates

$$\hat{\mathbf{r}} = \boldsymbol{\chi}_\varepsilon^N(\varepsilon, \bar{\mathbf{r}}), \tag{5.90}$$

where $\boldsymbol{\chi}_\varepsilon^N$ is defined by (5.2) and (4.46). Moreover, in the yielding coordinate system the Hamiltonian function writes

$$\hat{H}_\varepsilon^N(\hat{\mathbf{r}}) = \bar{H}_0(\hat{r}_1, \hat{r}_2, \hat{r}_4) + \sum_{n=0}^{N} \varepsilon^n \hat{H}_n(\hat{r}_1, \hat{r}_2, \hat{r}_4) + \varepsilon^{N+1} \iota_{\hat{H}}^N(\hat{\mathbf{r}}), \tag{5.91}$$

and the Poisson Matrix is given by formulas (5.35) and (5.36).

6 The Gyro-Kinetic Coordinate System - Proof of Theorem 1.3

In the present section, we begin by giving the following Theorem 6.1 from which the part of Theorem 1.3 that concerns any value of N is then deducted. In the next subsection, we prove Theorem 6.1 and finally, we consider the case when $N=2$ to end the proof of Theorem 1.3.

Theorem 6.1. *Assume that the magnetic field B satisfies assumptions* (1.42) *and* (1.43) *and that all its derivatives are bounded. Let $\mathfrak{Pol}$ be the polar in velocity change of coordinate defined by* (2.21)*; $\boldsymbol{\Upsilon}$ be the diffeomorphism of $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$, whose components are solutions to the system of PDEs* (3.11)*, whose third component is given by $\boldsymbol{\Upsilon}_3(\mathbf{x}, \theta, v) = \theta$ and which expansions of its components* 1, 2 *and* 4 *are given by Theorems 3.4, 3.10 and 3.11. Let N be a positive integer; and $\bar{g}_1, \ldots, \bar{g}_N$ and $H_{\varepsilon,T}$ be the maps defined on $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ and obtained by Algorithm 5.11. Then, for any open subset $\mathcal{O}(\mathbf{x}_0, R_{\mathbf{x}_0}; a, b)$ (see formula* (1.69)*), for any $R'_{\mathbf{x}_0}$ such that $R'_{\mathbf{x}_0} > R_{\mathbf{x}_0}$, and for any (c,d) such that $\left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right] \subset (c,d)$, there exists a real number $\eta \in (0, +\infty)$, such that for any $\varepsilon \in (0, \eta)$, there exists a real number $t_e^\varepsilon > \frac{\alpha_0}{\varepsilon}$, where α_0 depends only on B and the difference between $R_{\mathbf{x}_0}$ and $R'_{\mathbf{x}_0}$, such that for any $(\mathbf{y}_0, \theta_0, k_0) \in \boldsymbol{\Upsilon}(\mathfrak{b}^2(\mathbf{x}_0, R_{\mathbf{x}_0}) \times \mathbb{R} \times (a,b))$, and for any $t \in (-t_e^\varepsilon, t_e^\varepsilon)$, the characteristic $(\mathbf{Y}^\varepsilon, \boldsymbol{\mathcal{K}}_{\mathfrak{Dar}}^\varepsilon, \Theta_{\mathfrak{Dar}}^\varepsilon)(t; \mathbf{y}_0, \theta_0, k_0)$ expressed in the Darboux Coordinate System $(\mathbf{y}, \theta, k) = \boldsymbol{\Upsilon}(\mathbf{x}, \theta, v)$ satisfies*

$$(\mathbf{Y}^\varepsilon, \boldsymbol{\mathcal{K}}_{\mathfrak{Dar}}^\varepsilon, \Theta_{\mathfrak{Dar}}^\varepsilon)(t; \mathbf{y}_0, \theta_0, k_0) \in \mathfrak{b}^2(\mathbf{x}_0, R'_{\mathbf{x}_0}) \times \mathbb{R} \times \left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right]; \tag{6.1}$$

and such that the map $\boldsymbol{\chi}_\varepsilon^N$, defined by

$$\boldsymbol{\chi}_\varepsilon^N = \boldsymbol{\vartheta}_{\varepsilon,-\bar{g}_1}^{\alpha_1,1} \circ \boldsymbol{\vartheta}_{\varepsilon,-\bar{g}_2}^{\alpha_2,2} \circ \ldots \circ \boldsymbol{\vartheta}_{\varepsilon,-\bar{g}_N}^{\alpha_N,N}, \tag{6.2}$$

where for any $i \in \{1, \ldots, N\}$, the function $\boldsymbol{\vartheta}_{\varepsilon,-\bar{g}_i}^{\alpha_i,i}$ is defined on $\mathbb{R}^2 \times \mathbb{R} \times (0, +\infty)$ by formula (4.45) *and where the α_i are defined by* (5.1)*, is a diffeomorphism from*

$$\mathcal{O}_{\mathfrak{Dar}} = \mathfrak{b}^2(\mathbf{x}_0, R'_{\mathbf{x}_0}) \times \mathbb{R} \times (c,d), \tag{6.3}$$

onto its range.
Moreover, for any $\varepsilon \in (0,\eta)$*, for any* $t \in (-t_e^\varepsilon, t_e^\varepsilon)$*, for any* $(\mathbf{z}_0,\gamma_0,j_0) \in \boldsymbol{\chi}_\varepsilon^N \circ \boldsymbol{\Upsilon}(\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})\times \mathbb{R}\times(a,b))$*, the solution* $\big(\mathbf{Z}^{T,\#},\Gamma^{T,\#},\mathcal{J}^{T,\#}\big) = \big(\mathbf{Z}^{T,\#},\Gamma^{T,\#},\mathcal{J}^{T,\#}\big)(t,\mathbf{z}_0,\gamma_0,j_0)$ *of the following dynamical system, written within system of coordinates* $(\mathbf{z},\gamma,j) = \boldsymbol{\chi}_\varepsilon^N(\mathbf{y},\theta,k)$ *on* $\boldsymbol{\chi}_\varepsilon^N(\mathcal{O}_{\mathfrak{Dar}})$*,*

$$\frac{\partial \mathbf{Z}^{T,\#}}{\partial t} = -\frac{\varepsilon}{B\left(\mathbf{Z}^{T,\#}\right)}\begin{pmatrix} \dfrac{\partial \hat{H}^N_{\varepsilon,T}}{\partial z_2}\Big(\mathbf{Z}^{T,\#},j_0\Big) \\ -\dfrac{\partial \hat{H}^N_{\varepsilon,T}}{\partial z_1}\Big(\mathbf{Z}^{T,\#},j_0\Big)\end{pmatrix}, \qquad \mathbf{Z}^{T,\#}(0;\mathbf{z}_0,j_0) = \mathbf{z}_0, \tag{6.4}$$

$$\frac{\partial \Gamma^{T,\#}}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial \hat{H}^N_{\varepsilon,T}}{\partial j}\Big(\mathbf{Z}^{T,\#},j_0\Big), \qquad \boldsymbol{\Gamma}^{T,\#}(0;\mathbf{z}_0,j_0,\gamma_0) = \gamma_0, \tag{6.5}$$

$$\frac{\partial \mathcal{J}^{T,\#}}{\partial t} = 0, \qquad \mathcal{J}^{T,\#}(0;\mathbf{z}_0,j_0) = j_0, \tag{6.6}$$

where $\hat{H}^N_{\varepsilon,T}$ *is defined by formula*(5.83)*, satisfies*

$$\Big\|(\mathbf{Z}^{\#},\mathcal{J}^{\#}) - (\mathbf{Z}^{T,\#},\mathcal{J}^{T,\#})\Big\|_{\infty,\mathbf{init}^{\#}} \le C\varepsilon^{N-1}, \tag{6.7}$$

where C *is a constant depending only on* B *and the difference between* $R_{\mathbf{x}_0}$ *and* $R'_{\mathbf{x}_0}$ *(and not on* ε*), where*

$$\|g\|_{\infty,\mathbf{init}^{\#}} = \sup_{(\mathbf{z}_0,j_0,\gamma_0)\in\boldsymbol{\chi}_\varepsilon^N\circ\boldsymbol{\Upsilon}(\mathfrak{b}^2(\mathbf{x}_0,R_{\mathbf{x}_0})\times\mathbb{R}\times(a,b))} |g(\mathbf{z}_0,j_0,\gamma_0)|, \tag{6.8}$$

and where $\big(\mathbf{Z}^{\#},\Gamma^{\#},\mathcal{J}^{\#}\big) = \boldsymbol{\chi}_\varepsilon^N(\mathbf{Y}^\varepsilon,\boldsymbol{\mathcal{K}}^\varepsilon_{\mathfrak{Dar}},\Theta^\varepsilon_{\mathfrak{Dar}})$ *corresponds to the expression of* $(\mathbf{Y}^\varepsilon,\boldsymbol{\mathcal{K}}^\varepsilon_{\mathfrak{Dar}},$ $\Theta^\varepsilon_{\mathfrak{Dar}})$ *in the coordinate system* $(\mathbf{z},\gamma,j)$*.*

6.1 Proof of Theorem 1.3 for any fixed N

Once N is fixed, in view of definitions of $\mathcal{O}(\mathbf{x}_0,R_{\mathbf{x}_0};a,b)$ and $\mathcal{O}\big(\mathbf{x}_0,R'_{\mathbf{x}_0};c,d\big)$ and having in mind that $\boldsymbol{\Upsilon}$ and $\boldsymbol{\chi}_\varepsilon^N$ are the Darboux change of coordinate map and the Lie change of coordinate map, the proof of the first part of Theorem 1.3 is a direct consequence of Theorem 6.1, as soon as there exists a diffeomorphism

$$\boldsymbol{\chi}_\varepsilon^{N,\circ} \;:\; \mathfrak{b}^2\big(\mathbf{x}_0,R'_{\mathbf{x}_0}\big)\times(\mathbb{R}/2\pi\mathbb{Z})\times(c,d) \to \boldsymbol{\chi}_\varepsilon^{N,\circ}\big(\mathfrak{b}^2\big(\mathbf{x}_0,R'_{\mathbf{x}_0}\big)\times(\mathbb{R}/2\pi\mathbb{Z})\times(c,d)\big) \tag{6.9}$$

such that $\mathfrak{p}\circ\boldsymbol{\chi}_\varepsilon^N(t,\cdot) = \boldsymbol{\chi}_\varepsilon^{N,\circ}(t,\cdot)\circ\mathfrak{p}$.

Since the components $1,\ 2$ and 4 of $\boldsymbol{\chi}_\varepsilon^N$ are 2π-periodic with respect to θ, and since the penultimate components satisfies

$$\big(\boldsymbol{\chi}_\varepsilon^N\big)_3(\mathbf{y},\theta+2\pi,k) = \big(\boldsymbol{\chi}_\varepsilon^N\big)_3(\mathbf{y},\theta,k) + 2\pi, \tag{6.10}$$

it is obviously the case.

Notice also that map $\mathcal{C}G_\varepsilon$ is given by

$$\mathcal{C}G_\varepsilon = \boldsymbol{\chi}_\varepsilon^{N,\circ}\circ\boldsymbol{\Upsilon}\circ\mathfrak{Pol}. \tag{6.11}$$

6.2 Proof of Theorem 6.1

The proof of Theorem 6.1 consists essentially in showing that we are under the conditions of application of Theorem 4.1, and to apply it.

Once N is fixed, let $\hat{H}_1, \dots, \hat{H}_N \in \hat{\mathcal{K}} \cap \mathcal{A}(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)) \cap \mathcal{Q}^{\infty}_{T,b}$ and $\bar{g}_1, \dots, \bar{g}_N \in \mathcal{A}(\mathbb{R}^2 \times \mathbb{R} \times (0,+\infty)) \cap \mathcal{Q}^{\infty}_{T,b}$ (see Definition 4.2 and (3.118)) be the functions obtained by applying Algorithm 5.11; $\mathcal{O}(\mathbf{x}_0, R_{\mathbf{x}_0};\, a, b)$ be the open subset defined by (1.69); $\mathcal{O}_{\mathfrak{Pol}} = \mathfrak{b}^2(\mathbf{x}_0, R_{\mathbf{x}_0}) \times \mathbb{R} \times (a,b)$; and $\mathcal{O}_{\mathfrak{Dar}} = \mathfrak{b}^2(\mathbf{x}_0, R'_{\mathbf{x}_0}) \times \mathbb{R} \times (c,d)$, where $R'_{\mathbf{x}_0} > R_{\mathbf{x}_0}$ and $\left[\frac{a^2}{2\|B\|_\infty}, \frac{b^2}{2}\right] \subset (c,d)$.

Let $\mathcal{O}^{\mathfrak{D},\star}_{\mathfrak{Int}} = \mathfrak{b}^2\big(\mathbf{x}_0, R^\star_{\mathbf{x}_0}\big) \times \mathbb{R} \times (c^\star, d^\star)$, and $\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}} = \mathfrak{b}^2\big(\mathbf{x}_0, R^\bullet_{\mathbf{x}_0}\big) \times \mathbb{R} \times (c^\bullet, d^\bullet)$, where $R^\star_{\mathbf{x}_0}$ and $R^\bullet_{\mathbf{x}_0}$ are positive real numbers satisfying $0 < R'_{\mathbf{x}_0} < R^\star_{\mathbf{x}_0} < R^\bullet_{\mathbf{x}_0}$, and

$$[c,d] \subset (c^\star, d^\star) \subset [c^\star, d^\star] \subset (c^\bullet, d^\bullet) \subset [c^\bullet, d^\bullet] \subset (0,+\infty)\,. \tag{6.12}$$

Firstly, we will show that the Lie change of coordinates of order N is well defined on $\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}$ and that the inverse map $\boldsymbol{\lambda}^N_\varepsilon$ is well defined on $\mathcal{O}^{\mathfrak{D},\star}_{\mathfrak{Int}}$.

Applying Theorem 4.9 with $\mathfrak{C} = \mathfrak{b}^2\big(\mathbf{x}_0, R^\bullet_{\mathbf{x}_0}\big) \times (c^\bullet, d^\bullet)$ and $\mathfrak{C}^\# = \mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}$ yields that for any positive integers i,j and for any function f in $\mathcal{C}^\infty_\#\Big(\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}\Big) \cap \mathcal{C}^\infty_b\Big(\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}\Big)$, there exists a real number η_1 such that for any $\varepsilon \in (-\eta_1, \eta_1)$ function $\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}$ defined by (4.46) is a diffeomorphism from $\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}$ onto its range. Afterwards, a direct induction yields that there exists a real number η_N such that for any $\varepsilon \in (-\eta_N, \eta_N)$ the Partial Lie Transform map of order N is a diffeomorphism from $\mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}$ onto its range.

Now, by compacity, we can cover $\mathcal{O}^{\mathfrak{D},\star}_{\mathfrak{Int}}$ by a finite number $p^\star$ of periodic balls:

$$\mathcal{O}^{\mathfrak{D},\star}_{\mathfrak{Int}} = \bigcup_{k=1}^{p^\star} \mathfrak{b}^\#\Big(\mathfrak{r}^k_0, R^\star_{\mathfrak{r}^k_0}\Big)\,, \tag{6.13}$$

such that for any $k \in \{1,\dots,p^\star\}$, there exists $R^\bullet_{\mathfrak{r}^k_0}$ such that $R^\star_{\mathfrak{r}^k_0} < R^\bullet_{\mathfrak{r}^k_0}$ and $\overline{\mathfrak{b}^\#\Big(\mathfrak{r}^k_0, R^\bullet_{\mathfrak{r}^k_0}\Big)} \subset \mathcal{O}^{\mathfrak{D},\bullet}_{\mathfrak{Int}}$.

According to Lemma 5.1, for each $k \in \{1,\dots,p^\star\}$, there exists a real number $\eta^\star_k$ such that for any $\varepsilon \in (-\eta^\star_k, \eta^\star_k)$

$$\mathfrak{b}^\#\Big(\mathfrak{r}^k_0, R^\star_{\mathfrak{r}^k_0}\Big) \subset \boldsymbol{\chi}^N_\varepsilon\Big(\mathfrak{b}^\#\Big(\mathfrak{r}^k_0, R^\bullet_{\mathfrak{r}^k_0}\Big)\Big)\,. \tag{6.14}$$

Consequently, for sufficiently small ε, $\boldsymbol{\lambda}^N_\varepsilon$ is well defined on $\mathcal{O}^{\mathfrak{D},\star}_{\mathfrak{Int}}$.

According to Theorems 5.4, 5.5, and 5.9, for any $k \in \{1,\dots,p^\star\}$, there exists a real number $\bar{\eta}_k$ such that for any $\varepsilon \in [-\bar{\eta}_k, \bar{\eta}_k]$, the inverse function $\boldsymbol{\lambda}^N_\varepsilon = \big(\boldsymbol{\chi}^N_\varepsilon\big)^{-1}$ of $\boldsymbol{\chi}^N_\varepsilon$ is analytic on $\mathfrak{b}^\#\Big(\mathfrak{r}^k_0, R^\star_{\mathfrak{r}^k_0}\Big)$ and expresses as

$$\boldsymbol{\lambda}^N_\varepsilon(\mathfrak{r}) = \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,\bar{g}_1} \cdot \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,\bar{g}_2} \cdot \dots \cdot \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,\bar{g}_N}(\mathfrak{r}) + \varepsilon^{N+1}\boldsymbol{\rho}^{N,k}_{\boldsymbol{\lambda}}(\varepsilon,\mathfrak{r})\,, \tag{6.15}$$

where $\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{N,k}$ is in $\mathcal{C}_{\#}^{\infty}\Big([-\bar{\eta}_k,\bar{\eta}_k]\times\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)\Big)$; the matrix $\hat{\mathcal{P}}_{\varepsilon}$, defined by (5.33), is well defined on $\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)$ and

$$\forall i,j\in\{1,2,3,4\}\,,\ \hat{\mathcal{T}}_{\varepsilon}^{i,j}(\mathfrak{r})=\bar{\mathcal{T}}_{\varepsilon}^{i,j}(\mathfrak{r})+\varepsilon^{N+1}\rho_{\hat{\mathcal{T}}^{i,j}}^{N,k}(\varepsilon,\mathfrak{r})\,, \tag{6.16}$$

where $\rho_{\hat{\mathcal{T}}^{i,j}}^{N,k}$ is in $\mathcal{C}_{\#}^{\infty}\Big([-\bar{\eta}_k;\bar{\eta}_k]\times\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)\Big)$ and where $\hat{\mathcal{T}}_{\varepsilon}$ stands for the matrix satisfying (5.36); and, for any $\varepsilon\in[0,\bar{\eta}_k]$ the function $\hat{H}_{\varepsilon}^{N}$ defined by $\hat{H}_{\varepsilon}^{N}(\mathfrak{r})=\bar{H}\left(\varepsilon,\boldsymbol{\lambda}_{\varepsilon}^{N}(\mathfrak{r})\right)$, is well defined on $\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)$, and expresses

$$\hat{H}_{\varepsilon}^{N}=\bar{H}_0+\sum_{n=0}^{N}\varepsilon^n\hat{H}_n+\varepsilon^{N+1}\iota_{PLH}^{N,k}, \tag{6.17}$$

where $\iota_{PLH}^{N,k}$ is in $\mathcal{C}_{\#}^{\infty}\Big([0;\bar{\eta}_k]\times\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)\Big)$. Let $\bar{\eta}=\min_{k=1,\dots,p^\star}\bar{\eta}_k$. Since the functions

$$\begin{aligned}&\boldsymbol{\lambda}_{\varepsilon}^{N}-\boldsymbol{\vartheta}_{\varepsilon,\bar{g}_1}^{\alpha_1,1}\cdot\boldsymbol{\vartheta}_{\varepsilon,\bar{g}_2}^{\alpha_2,2}\cdot\ldots\cdot\boldsymbol{\vartheta}_{\varepsilon,\bar{g}_N}^{\alpha_N,N},\\&\hat{\mathcal{T}}_{\varepsilon}^{i,j}-\bar{\mathcal{T}}_{\varepsilon}^{i,j},\ i,j\in\{1,2,3,4\}\,,\end{aligned} \tag{6.18}$$

are in $\mathcal{C}_{\#}^{\infty}\Big([-\bar{\eta};\bar{\eta}]\times\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}\Big)$ and since the function

$$\hat{H}_{\varepsilon}-\bar{H}_0-\sum_{n=0}^{N}\varepsilon^n\hat{H}_n, \tag{6.19}$$

is in $\mathcal{C}_{\#}^{\infty}\Big([0;\bar{\eta}]\times\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}\Big)$, functions $\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{N}$, $\rho_{\hat{\mathcal{T}}^{i,j}}^{N}$ and ι_{PLH}^{N}, defined on $\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}$, and whose restrictions to each $\mathfrak{b}^{\#}\Big(\mathfrak{r}_0^k,R^{\star}_{\mathfrak{r}_0^k}\Big)$ are given by $\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{N,k}$, $\rho_{\hat{\mathcal{T}}^{i,j}}^{N,k}$ and $\iota_{PLH}^{N,k}$, are in $\mathcal{C}_{\#}^{\infty}\Big([-\bar{\eta};\bar{\eta}]\times\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}\Big)$ for the two firsts and $\iota_{PLH}^{N,k}\in\mathcal{C}_{\#}^{\infty}\Big([0;\bar{\eta}]\times\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}\Big)$

According to Theorems 3.24 and 3.25, there exists a real number $\eta'>0$ and a real number $\alpha_0>0$, depending only to the difference between $R_{\mathbf{x}_0}$ and $R'_{\mathbf{x}_0}$, b and B (and not ε), such that for any $\varepsilon\in(-\eta',\eta')$,

$$\boldsymbol{\Upsilon}\left(\overline{\mathcal{O}_{\mathfrak{Pol}}}\right)\subset\mathfrak{b}^2\big(\mathbf{x}_0,R'_{\mathbf{x}_0}\big)\times\mathbb{R}\times\left[\frac{a^2}{2\left\|B\right\|_{\infty}},\frac{b^2}{2}\right], \tag{6.20}$$

and such that for any $\varepsilon\in(0,\eta)$ there exists a real number $t_e^{\varepsilon}>\frac{\alpha_0}{\varepsilon}$ such that for any $t\in(-t_e^{\varepsilon},t_e^{\varepsilon})$ and for any $(\mathbf{y},\theta,k)\in\boldsymbol{\Upsilon}\left(\overline{\mathcal{O}_{\mathfrak{Pol}}}\right)$,

$$\left(\mathbf{Y}^{\varepsilon}(t;\mathbf{y},\theta,k),\mathcal{K}_{\mathfrak{Dar}}^{\varepsilon}(t;\mathbf{y},\theta,k)\right)\in\mathfrak{b}^2\big(\mathbf{x}_0,R'_{\mathbf{x}_0}\big)\times\left[\frac{a^2}{2\left\|B\right\|_{\infty}},\frac{b^2}{2}\right]. \tag{6.21}$$

Let $\mathcal{O}'=\boldsymbol{\chi}_{\varepsilon}^{N}\circ\boldsymbol{\Upsilon}\left(\mathcal{O}_{\mathfrak{Pol}}\right)$, and $\mathcal{O}=\mathcal{O}_{\mathfrak{Int}}^{\mathfrak{D},\star}$. Then, there exists a real number η'', such that for any $\varepsilon\in(-\eta'',\eta'')$,

$$\boldsymbol{\chi}_{\varepsilon}^{N}\left(\boldsymbol{\Upsilon}\left(\overline{\mathcal{O}_{\mathfrak{Pol}}}\right)\right)\subset\mathcal{O}_{\mathfrak{Dar}}, \tag{6.22}$$

and

$$\boldsymbol{\chi}_\varepsilon^N \left(\overline{\mathfrak{b}^2\left(\mathbf{x}_0, R'_{\mathbf{x}_0}\right)} \times \mathbb{R} \times \left[\frac{a^2}{2\left\|B\right\|_\infty}, \frac{b^2}{2} \right] \right) \subset \mathcal{O}. \tag{6.23}$$

This last equality implies that the range by $\boldsymbol{\chi}_\varepsilon^N$ of the characteristics expressed in the Darboux coordinate system and provided with initial conditions in $\boldsymbol{\Upsilon}\left(\overline{\mathcal{O}_{\mathfrak{Pol}}}\right)$, or equivalently that the characteristics expressed in the Lie Coordinate System and provided with initial conditions in $\mathcal{O}'$, belongs to $\mathcal{O}$ for any $t \in (-t_e^\varepsilon, t_e^\varepsilon)$. Let

$$\eta = \min\left(\eta_N, \eta_1^\star, \ldots, \eta_{p^\star}^\star, \bar{\eta}_1, \ldots, \bar{\eta}_{p^\star}, \eta', \eta''\right). \tag{6.24}$$

Choosing

$$R_{\mathbf{x}_0}^\star > 1 + R'_{\mathbf{x}_0} + \sqrt{2} \sup_{\varepsilon\in[-\eta,\eta]} \frac{\left\|\nabla H_{\varepsilon,T}\right\|_\infty}{\left\|B\right\|_\infty}, \tag{6.25}$$

yields that we are under the conditions of application of Theorem 4.1. Applying this theorem yields the main part of Theorem 1.3.

6.3 Application with $N = 2$

Applying Algorithm 5.11 with $N = 2$, using results of Appendix B, yields

$$\bar{g}_1(\mathbf{y}, \theta, k) = \frac{\hat{c}(\theta)\cdot\nabla_{\mathbf{y}}B(\mathbf{y})}{3B(\mathbf{y})^3}\left(2B(\mathbf{y})\,k\right)^{\frac{3}{2}}, \tag{6.26}$$

$$\bar{g}_2(\mathbf{y}, \theta, k) = \frac{\left(2B(\mathbf{y})\,k\right)^2}{12B(\mathbf{y})^5}\hat{a}(\theta)\,\hat{c}(\theta) : \left[3B(\mathbf{y}) : \nabla\nabla B(\mathbf{y}) - \nabla B\nabla B(\mathbf{y})\right], \tag{6.27}$$

where

$$\nabla_{\mathbf{x}}^2 B(\mathbf{x}) = \frac{\partial^2 B}{\partial x_1^2}(\mathbf{x}) + \frac{\partial^2 B}{\partial x_2^2}(\mathbf{x}), \tag{6.28}$$

$$\left(\nabla_{\mathbf{x}}B(\mathbf{x})\right)^2 = \nabla_{\mathbf{x}}B(\mathbf{x})\cdot\nabla_{\mathbf{x}}B(\mathbf{x}), \tag{6.29}$$

$$\nabla\nabla B = \begin{pmatrix} \frac{\partial^2 B}{\partial x_1^2} & \frac{\partial^2 B}{\partial x_1 \partial x_2} \\ \frac{\partial^2 B}{\partial x_1 \partial x_2} & \frac{\partial^2 B}{\partial x_2^2} \end{pmatrix} \text{ and } \nabla B\nabla B = \begin{pmatrix} \left(\frac{\partial B}{\partial x_1}\right)^2 & \frac{\partial B}{\partial x_1}\frac{\partial B}{\partial x_2} \\ \frac{\partial B}{\partial x_1}\frac{\partial B}{\partial x_2} & \left(\frac{\partial B}{\partial x_2}\right)^2 \end{pmatrix} \tag{6.30}$$

(Notice that, if u, v are two vectors and if A is a matrix, the notation $uv : A$ stands for $u^T Av$), and

$$\hat{H}_1(\mathbf{z}, j) = 0, \tag{6.31}$$

$$\hat{H}_2(\mathbf{z}, j) = \frac{j^2}{4B(\mathbf{z})}\left[B(\mathbf{z})\,\nabla_{\mathbf{x}}^2 B(\mathbf{z}) - 3\left(\nabla_{\mathbf{x}}B(\mathbf{z})\right)^2\right]. \tag{6.32}$$

Consequently, the dynamical system (1.66), (1.68), (1.67) approximates, with accuracy ε, the characteristics $(\mathbf{Z}, \Gamma, \mathcal{J}) = (\mathbf{Z}(t, \mathbf{z}_s, \gamma_s, j_s, s), \Gamma(t, \mathbf{z}_s, \gamma_s, j_s, s), \mathcal{J}(t, \mathbf{z}_s, \gamma_s, j_s, s)) = \mathcal{C}\!G(\mathbf{X}(t, \mathbf{x}_s, \mathbf{v}_s, s), \mathbf{V}(t, \mathbf{x}_s, \mathbf{v}_s, s))$, with $(\mathbf{X}, \mathbf{V})$ the solution of dynamical system (1.40)–(1.41). This ends the proof of the part of Theorem 1.3 concerning the application when $N = 2$. □

A Appendix : Example of non-symplectic Hamiltonian vector field flow

In this Appendix, we exhibit an example of a flow - of parameter ε - of Hamiltonian vector field, which - since the Poisson Matrix does depend on ε - is not symplectic.

Let $\varepsilon \in]-1,+\infty[$. Consider on $\mathbb{R}^4$ endowed with coordinate system $\mathbf{x} = (x_1, x_2, x_3, x_4)$ the symplectic two-form $\omega_\varepsilon \in \Omega^2\left(\mathbb{R}^4\right)$, given by

$$\omega_\varepsilon(\mathbf{x}) = \frac{1}{1+\varepsilon} dx_1 \wedge dx_2 + \frac{1}{2+\varepsilon} dx_3 \wedge dx_4. \tag{A.1}$$

The associated Poisson Matrix is given by

$$J_\varepsilon(\mathbf{x}) = \begin{pmatrix} 0 & 1+\varepsilon & 0 & 0 \\ -(1+\varepsilon) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+\varepsilon \\ 0 & 0 & -(2+\varepsilon) & 0 \end{pmatrix}. \tag{A.2}$$

Now, let $G(\mathbf{x}) = x_1 x_3$. The Hamiltonian vector field generated by G is given by

$$\mathbf{X}_G(\mathbf{x}) = J_\varepsilon(\mathbf{x}) \nabla_{\mathbf{x}} G(\mathbf{x}) = \begin{pmatrix} 0 \\ -(1+\varepsilon) x_3 \\ 0 \\ -(2+\varepsilon) x_1 \end{pmatrix}. \tag{A.3}$$

The flow of this Hamiltonian vector field, denoted by φ_ε, is given by

$$\varphi_\varepsilon(\mathbf{x}) = \left(x_1, x_2 - \left(\varepsilon + \frac{\varepsilon^2}{2}\right) x_3, x_3, x_4 - \left(2\varepsilon + \frac{\varepsilon^2}{2}\right) x_1\right). \tag{A.4}$$

Now, if we make the change of coordinate $\mathbf{x} \mapsto \mathbf{z} = \varphi_\varepsilon(\mathbf{x})$, the Poisson Matrix in the coordinate system $\mathbf{z}$ reads

$$\begin{pmatrix} 0 & 1+\varepsilon & 0 & 0 \\ -(1+\varepsilon) & 0 & 0 & \frac{\varepsilon^2}{2} \\ 0 & 0 & 0 & 2+\varepsilon \\ 0 & -\frac{\varepsilon^2}{2} & -(2+\varepsilon) & 0 \end{pmatrix}. \tag{A.5}$$

Thus, although φ_ε is the flow of an Hamiltonian vector field, φ_ε is not symplectic.

B Algorithm 5.11 detailed up to $N = 5$

In this Appendix, we apply Algorithm 5.11 for $N = 1,\ 2,\ 3,\ 4$ and 5. The computations are led using Sage software. We first need to compute $\mathbf{V}_0^\varepsilon$, $\mathbf{V}_1^\varepsilon$, $\mathbf{V}_2^\varepsilon$, $\mathbf{V}_3^\varepsilon$, $\mathbf{V}_4^\varepsilon$ and $\mathbf{V}_5^\varepsilon$ with formula (5.56). The first step consists in expressing $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{U}_4$ and $\mathcal{U}_5$.

$$
\begin{aligned}
\mathcal{U}_2 &= \left\{(m_1,m_2)\in\mathbb{N}^2 \text{ s.t. } m_1+2m_2=2\right\} = \{(0,1)\,;(2,0)\}\,,\\
\mathcal{U}_3 &= \left\{(m_1,m_2,m_3)\in\mathbb{N}^3 \text{ s.t. } m_1+2m_2+3m_3=3\right\}\\
&= \{(0,0,1)\,;(1,1,0)\,;(3,0,0)\}\,,\\
\mathcal{U}_4 &= \left\{(m_1,m_2,m_3,m_4)\in\mathbb{N}^4 \text{ s.t. } m_1+2m_2+3m_3+4m_4=4\right\}\\
&= \{(0,0,0,1)\,;(1,0,1,0)\,;(0,2,0,0)\,;(2,1,0,0)\,;(4,0,0,0)\}\,,\\
\mathcal{U}_5 &= \left\{(m_1,m_2,m_3,m_4,m_5)\in\mathbb{N}^5 \text{ s.t. } m_1+2m_2+3m_3+4m_4+5m_5=5\right\}\\
&= \Big\{\,(0,0,0,0,1)\,;(1,0,0,1,0)\,;(0,1,1,0,0)\,;(2,0,1,0,0)\,;\\
&\qquad (1,2,0,0,0)\,;(3,1,0,0,0)\,;(5,0,0,0,0)\,\Big\}
\end{aligned}
\tag{B.1}
$$

Then, applying formula (5.56) yields:

$$
\begin{aligned}
\mathbf{V}_0^\varepsilon &= id,\\
\mathbf{V}_1^\varepsilon &= \bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\cdot\\
&= \overline{\mathbf{M}}_1 + \varepsilon^2\overline{\mathbf{N}}_3,\\
\mathbf{V}_2^\varepsilon &= \frac{1}{2}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^2\cdot + \bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\cdot\\
&= \left(\overline{\mathbf{M}}_2 + \frac{1}{2}\overline{\mathbf{M}}_1^2\right) + \varepsilon^2\left(\overline{\mathbf{N}}_4 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\right) + \varepsilon^4\frac{1}{2}\overline{\mathbf{N}}_3\\
\mathbf{V}_3^\varepsilon &= \frac{1}{6}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^3\cdot + \bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\cdot\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\cdot + \bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_3}\cdot\\
&= \left(\overline{\mathbf{M}}_3 + \overline{\mathbf{M}}_1\overline{\mathbf{M}}_2 + \frac{1}{6}\overline{\mathbf{M}}_1^3\right)\\
&\quad + \varepsilon^2\left(\overline{\mathbf{N}}_5 + \overline{\mathbf{M}}_1\overline{\mathbf{N}}_4 + \overline{\mathbf{N}}_3\overline{\mathbf{M}}_2 + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 + \frac{1}{6}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\right)\\
&\quad + \varepsilon^4\left(\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 + \frac{1}{6}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\right)\\
&\quad + \frac{1}{6}\varepsilon^6\overline{\mathbf{N}}_3^3
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\mathbf{V}_4^{\varepsilon} &= \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_4}\cdot + \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\cdot \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_3}\cdot + \frac{1}{2}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2}\right)^2\cdot + \frac{1}{2}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^2\cdot \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2}\cdot + \frac{1}{24}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^4\cdot \\
&= \left(\overline{\mathbf{M}}_4 + \overline{\mathbf{M}}_1\overline{\mathbf{M}}_3 + \frac{1}{2}\overline{\mathbf{M}}_1^2\overline{\mathbf{M}}_2 + \frac{1}{2}\overline{\mathbf{M}}_2^2 + \frac{1}{24}\overline{\mathbf{M}}_1^4\right) \\
&\quad + \varepsilon^2\Big(\overline{\mathbf{N}}_6 + \overline{\mathbf{M}}_1\overline{\mathbf{N}}_5 + \frac{1}{2}\overline{\mathbf{M}}_2\overline{\mathbf{N}}_4 + \overline{\mathbf{N}}_3\overline{\mathbf{M}}_3 + \frac{1}{2}\overline{\mathbf{N}}_4\overline{\mathbf{M}}_2 + \frac{1}{2}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_4 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_2 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2 \\
&\qquad + \frac{1}{24}\overline{\mathbf{M}}_1^3\overline{\mathbf{N}}_3 + \frac{1}{24}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 + \frac{1}{24}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2 + \frac{1}{24}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^3\Big) \\
&\quad + \varepsilon^4\Big(\overline{\mathbf{N}}_3\overline{\mathbf{N}}_5 + \frac{1}{2}\overline{\mathbf{N}}_4^2 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_4 + \frac{1}{2}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_2 \\
&\qquad + \frac{1}{24}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3^2 + \frac{1}{24}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 N_3 + \frac{1}{24}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1 + \frac{1}{24}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3 + \frac{1}{24}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 + \frac{1}{24}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1^2\Big) \\
&\quad + \varepsilon^6\Big(\frac{1}{2}\overline{\mathbf{N}}_3^2\overline{\mathbf{N}}_4 + \frac{1}{24}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^3 + \frac{1}{24}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2 + \frac{1}{24}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 + \frac{1}{24}\overline{\mathbf{N}}_3^3\overline{\mathbf{M}}_1\Big) \\
&\quad + \varepsilon^8\frac{1}{24}\overline{\mathbf{N}}_3^4
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\mathbf{V}_5^{\varepsilon} &= \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_5}\cdot + \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\cdot\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_4}\cdot + \bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2}\cdot\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_3}\cdot + \left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^2\cdot\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_3}\cdot \\
&\quad + \frac{1}{2}\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\cdot\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2}\right)^2\cdot + \frac{1}{6}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^3\cdot\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2} + \frac{1}{120}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^5\cdot \\
&= \Big(\overline{\mathbf{M}}_5 + \overline{\mathbf{M}}_1\overline{\mathbf{M}}_4 + \overline{\mathbf{M}}_2\overline{\mathbf{M}}_3 + \overline{\mathbf{M}}_1^2\overline{\mathbf{M}}_3 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2^2 + \frac{1}{6}\overline{\mathbf{M}}_1^3\overline{\mathbf{M}}_2 + \frac{1}{120}\overline{\mathbf{M}}_1^5\Big) \\
&\quad + \varepsilon^2\Big(\frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^4 + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^3 + \frac{1}{120}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2 + \overline{\mathbf{N}}_7 + \overline{\mathbf{M}}_1\overline{\mathbf{N}}_6 + \overline{\mathbf{M}}_2\overline{\mathbf{N}}_5 \\
&\quad + \overline{\mathbf{N}}_3\overline{\mathbf{M}}_4 + \overline{\mathbf{N}}_4\overline{\mathbf{M}}_3 + \overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_5 + \overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_3 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2\overline{\mathbf{N}}_4 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_4\overline{\mathbf{M}}_2 \\
&\quad + \overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{M}}_3 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_2^2 + \frac{1}{6}\overline{\mathbf{M}}_1^3\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_2 \\
&\quad + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\overline{\mathbf{M}}_2 + \frac{1}{120}\overline{\mathbf{M}}_1^4\overline{\mathbf{N}}_3 + \frac{1}{120}\overline{\mathbf{M}}_1^3\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\Big) \\
&\quad + \varepsilon^4\Big(\overline{\mathbf{N}}_3\overline{\mathbf{N}}_6 + \overline{\mathbf{N}}_4\overline{\mathbf{N}}_5 + \overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{N}}_5 + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_4^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_5 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_2\overline{\mathbf{N}}_4 + \overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_3 \\
&\quad + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4\overline{\mathbf{M}}_2 + \frac{1}{6}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_2 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2 N_4 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_2 \\
&\quad + \frac{1}{6}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2 + \frac{1}{120}\overline{\mathbf{M}}_1^3\overline{\mathbf{N}}_3^2 + \frac{1}{120}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 \\
&\quad + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3 + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 + \frac{1}{120}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1 \\
&\quad + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1^2 + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^3\overline{\mathbf{N}}_3 + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 \\
&\quad + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_4 + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2 + \frac{1}{120}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1^3\Big) \\
&\quad + \varepsilon^6\Big(\overline{\mathbf{N}}_3^2\overline{\mathbf{N}}_5 + \frac{1}{2}\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4^2 + \frac{1}{6}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{N}}_4 + \frac{1}{6}\overline{\mathbf{N}}_3^3\overline{\mathbf{M}}_2 \\
&\quad + \frac{1}{120}\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3^3 + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2 + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 \\
&\quad + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^3\overline{\mathbf{M}}_1 + + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3^2 + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 \\
&\quad + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1 + \frac{1}{120}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1^2\overline{\mathbf{N}}_3 + \frac{1}{120}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1 + \frac{1}{120}\overline{\mathbf{N}}_3^3\overline{\mathbf{M}}_1^2\Big) \\
&\quad + \varepsilon^8\Big(\frac{1}{6}\overline{\mathbf{N}}_3^3\overline{\mathbf{N}}_4 + \frac{1}{120}\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^4 + \frac{1}{120}\overline{\mathbf{N}}_3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^3 \\
&\quad + \frac{1}{120}\overline{\mathbf{N}}_3^2\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3^2 + \frac{1}{120}\overline{\mathbf{N}}_3^3\overline{\mathbf{M}}_1\overline{\mathbf{N}}_3 + \frac{1}{120}\overline{\mathbf{N}}_3^4\overline{\mathbf{M}}_1\Big) \\
&\quad + \varepsilon^{10}\frac{1}{120}\overline{\mathbf{N}}_3^5
\end{aligned}
\tag{B.4}
$$

Hence, the expansions in power of ε of $\mathbf{V}_1^{\varepsilon}$, $\mathbf{V}_2^{\varepsilon}$, $\mathbf{V}_3^{\varepsilon}$, $\mathbf{V}_4^{\varepsilon}$ and $\mathbf{V}_5^{\varepsilon}$ and the dependency with respect to the differential operators $\overline{\mathbf{M}}_i$ and $\overline{\mathbf{N}}_j$ of each term of these expansions can be summarized by:

$$
\begin{aligned}
&\mathbf{V}_1^\varepsilon = \mathbf{V}_1^0\left(\overline{\mathbf{M}}_1\right) + \varepsilon^2 \mathbf{V}_1^2\left(\overline{\mathbf{N}}_3\right),\\
&\mathbf{V}_2^\varepsilon = \mathbf{V}_2^0\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2\right) + \varepsilon^2 \mathbf{V}_2^2\left(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\right) + \varepsilon^4 \mathbf{V}_2^4\left(\overline{\mathbf{N}}_3\right),\\
&\mathbf{V}_3^\varepsilon = \mathbf{V}_3^0\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3\right) + \varepsilon^2 \mathbf{V}_3^2\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5\right) + \varepsilon^4 \mathbf{V}_3^4\left(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\right)\\
&\quad + \varepsilon^6 \mathbf{V}_3^6\left(\overline{\mathbf{N}}_3\right),\\
&\mathbf{V}_4^\varepsilon = \mathbf{V}_4^0\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{M}}_4\right) + \varepsilon^2 \mathbf{V}_4^2\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5, \overline{\mathbf{N}}_6\right)\\
&\quad + \varepsilon^4 \mathbf{V}_4^4\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5\right) + \varepsilon^6 \mathbf{V}_4^6\left(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\right) + \varepsilon^8 \mathbf{V}_4^8\left(\overline{\mathbf{N}}_3\right),\\
&\mathbf{V}_5^\varepsilon = \mathbf{V}_5^0\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{M}}_4, \overline{\mathbf{M}}_5\right) + \varepsilon^2 \mathbf{V}_5^2\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{M}}_4, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5, \overline{\mathbf{N}}_6, \overline{\mathbf{N}}_7\right)\\
&\quad + \varepsilon^4 \mathbf{V}_5^4\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5, \overline{\mathbf{N}}_6\right) + \varepsilon^6 \mathbf{V}_5^6\left(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5\right)\\
&\quad + \varepsilon^8 \mathbf{V}_5^8\left(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\right) + \varepsilon^{10} \mathbf{V}_5^{10}\left(\overline{\mathbf{N}}_3\right).
\end{aligned}
\tag{B.5}
$$

B.1 Formulas for $N = 1$

In the present subsection we will apply algorithm 5.11 with $N = 1$. The first step of the algorithm consists in injecting the expression of $\mathbf{V}_1^\varepsilon$ in

$$
\hat{H}_\varepsilon^1(\hat{\mathbf{r}}) = \sum_{n=0}^{1}\left(\sum_{k=0}^{n} \mathbf{V}_{n-k}^\varepsilon \cdot \bar{H}_k\right)(\hat{\mathbf{r}})\,\varepsilon^n + \varepsilon^2 \iota_{\bar{H}}^1(\varepsilon, \hat{\mathbf{r}}),. \tag{B.6}
$$

Applying the second step of the algorithm, we obtain:

$$
\hat{H}_\varepsilon^1(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon\left(\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1)\cdot \bar{H}_0(\hat{\mathbf{r}})\right) + \varepsilon^3 \mathbf{V}_1^2\left(\overline{\mathbf{M}}_1\right)\cdot \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon^2 \iota_{\bar{H}}^1(\varepsilon, \hat{\mathbf{r}}), \tag{B.7}
$$

that have to be compared with the desired expression:

$$
\hat{H}_\varepsilon^1(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon \hat{H}_1(\hat{\mathbf{r}}) + \varepsilon^2 \iota_{\hat{H}}^1(\varepsilon, \hat{\mathbf{r}}), \tag{B.8}
$$

to get

$$
\hat{H}_1(\hat{\mathbf{r}}) = \left(\bar{\mathcal{T}}_0 \nabla \bar{g}_1\right)\cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) + \bar{H}_1(\hat{\mathbf{r}}). \tag{B.9}
$$

Eventually, applying the last step of the algorithm, we set

$$
\hat{H}_1(\hat{\mathbf{r}}) = \frac{1}{2\pi}\int_0^{2\pi} \bar{H}_1(\hat{\mathbf{r}})\,d\hat{r}_3, \tag{B.10}
$$

$$
u_1(\hat{\mathbf{r}}) = -\bar{H}_1(\hat{\mathbf{r}}) + \frac{1}{2\pi}\int_0^{2\pi} \bar{H}_1(\hat{\mathbf{r}})\,d\hat{r}_3, \tag{B.11}
$$

and then we solve equation $\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_1\right)\cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) = u_1$ by setting

$$
\bar{g}_1(\bar{\mathbf{r}}) = -\frac{1}{B(\hat{r}_1, \hat{r}_2)}\int_0^{\bar{r}_3} u_1(\bar{r}_1, \bar{r}_2, s, \bar{r}_4)\,ds. \tag{B.12}
$$

B.2 Formulas for $N = 2$

In this subsection we will apply algorithm 5.11 with $N = 2$. The first step of the algorithm consists in injecting expressions of $\mathbf{V}_1^\varepsilon$ and $\mathbf{V}_2^\varepsilon$ in

$$\hat{H}_\varepsilon^2(\hat{\mathbf{r}}) = \sum_{n=0}^{2} \left(\sum_{k=0}^{n} \mathbf{V}_{n-k}^\varepsilon \cdot \bar{H}_k \right)(\hat{\mathbf{r}})\, \varepsilon^n + \varepsilon^3 \iota_{\bar{H}}^2(\varepsilon, \hat{\mathbf{r}})\,. \tag{B.13}$$

Ordering terms according to their power of ε (second step of the algorithm) we obtain:

$$\begin{aligned}
\hat{H}_\varepsilon^2(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) &+ \varepsilon \left(\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_0(\hat{\mathbf{r}}) \right) \\
&+ \varepsilon^2 \Big(\mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \bar{H}_2(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^3 \Big(\mathbf{V}_1^2(\overline{\mathbf{M}}_1) \cdot \bar{H}_0(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^4 \left(\mathbf{V}_2^2(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4) \cdot \bar{H}_0 + \mathbf{V}_1^2(\overline{\mathbf{N}}_3) \cdot \bar{H}_1 \right) \\
&+ \varepsilon^6 \Big(\mathbf{V}_2^4(\overline{\mathbf{N}}_3) \cdot \bar{H}_0 \Big) \\
&+ \varepsilon^3 \iota_{\bar{H}}^2(\varepsilon, \hat{\mathbf{r}})\,,
\end{aligned} \tag{B.14}$$

that we compare to

$$\hat{H}_\varepsilon^2(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon \hat{H}_1(\hat{\mathbf{r}}) + \varepsilon^2 \hat{H}_2(\hat{\mathbf{r}}) + \varepsilon^3 \iota_{\hat{H}}^2(\varepsilon, \hat{\mathbf{r}})\,, \tag{B.15}$$

to deduce

$$\begin{aligned}
\hat{H}_1(\hat{\mathbf{r}}) &= \left(\bar{\mathcal{T}}_0 \nabla \bar{g}_1 \right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) + \bar{H}_1(\hat{\mathbf{r}})\,, \\
\hat{H}_2(\hat{\mathbf{r}}) &= \left(\bar{\mathcal{T}}_0 \nabla \bar{g}_2 \right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) + \mathcal{V}_2(g_1)(\hat{\mathbf{r}})\,,
\end{aligned} \tag{B.16}$$

with

$$\mathcal{V}_2(g_1) = \frac{1}{2} \overline{\mathbf{M}}_1^2 \cdot \bar{H}_0 + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_1 + \bar{H}_2. \tag{B.17}$$

The last step of the algorithm consists in solving iteratively equations (B.16) with $\bar{g}_1$, $\bar{g}_2$, $\hat{H}_1$ and $\hat{H}_2$ as unknowns. The first equation was tackled in subsection B.1 in which we obtain $\bar{g}_1$ and $\hat{H}_1$. So the job is reduced to solve the second equation of (B.16). Setting

$$\hat{H}_2(\hat{\mathbf{r}}) = -\frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_2(g_1)(\hat{\mathbf{r}})\, d\hat{r}_3, \tag{B.18}$$

we just have to solve

$$\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_2 \right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) = u_2, \tag{B.19}$$

where u_2 is given by

$$u_2(\hat{\mathbf{r}}) = \mathcal{V}_2(g_1)(\hat{\mathbf{r}}) - \frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_2(g_1)(\hat{\mathbf{r}})\, d\hat{r}_3. \tag{B.20}$$

Hence we obtain

$$\bar{g}_2(\bar{\mathbf{r}}) = -\frac{1}{B(\hat{r}_1, \hat{r}_2)} \int_0^{\bar{r}_3} u_2(\bar{r}_1, \bar{r}_2, s, \bar{r}_4)\, ds. \tag{B.21}$$

B.3 Formulas for $N = 3$

In this subsection we will apply algorithm 5.11 with $N = 3$. The first step of the algorithm consists in injecting expressions of $\mathbf{V}_1^\varepsilon$, $\mathbf{V}_2^\varepsilon$ and $\mathbf{V}_3^\varepsilon$ in

$$\hat{H}_\varepsilon^3(\hat{\mathbf{r}}) = \sum_{n=0}^{3} \left(\sum_{k=0}^{n} \mathbf{V}_{n-k}^\varepsilon \cdot \bar{H}_k \right)(\hat{\mathbf{r}})\, \varepsilon^n + \varepsilon^4 \iota_{\bar{H}}^3(\varepsilon, \hat{\mathbf{r}})\,. \tag{B.22}$$

Ordering terms according to their power of ε (second step of the algorithm) we obtain:

$$\begin{aligned}
\hat{H}_\varepsilon^3(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) &+ \varepsilon \left(\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_0(\hat{\mathbf{r}}) \right) \\
&+ \varepsilon^2 \Big(\mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \bar{H}_2(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^3 \Big(\mathbf{V}_1^2(\overline{\mathbf{M}}_1) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_3^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3) \cdot \bar{H}_0(\hat{\mathbf{r}}) \\
&\qquad + \mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_2(\hat{\mathbf{r}}) + \bar{H}_3(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^4 \left(\mathbf{V}_2^2(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^2(\overline{\mathbf{N}}_3) \cdot \bar{H}_1(\hat{\mathbf{r}}) \right) \\
&+ \varepsilon^6 \Big(\mathbf{V}_2^4(\overline{\mathbf{N}}_3) \cdot \bar{H}_0(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^7 \Big(\mathbf{V}_3^4(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_2^4(\overline{\mathbf{N}}_3) \cdot \bar{H}_1(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^9 \Big(\mathbf{V}_3^6(\overline{\mathbf{N}}_3) \cdot \bar{H}_0(\hat{\mathbf{r}}) \Big) \\
&+ \varepsilon^4 \iota_{\bar{H}}^3(\varepsilon, \hat{\mathbf{r}})\,,
\end{aligned} \tag{B.23}$$

that is compared to

$$\hat{H}_\varepsilon^3(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon \hat{H}_1(\hat{\mathbf{r}}) + \varepsilon^2 \hat{H}_2(\hat{\mathbf{r}}) + \varepsilon^3 \hat{H}_3(\hat{\mathbf{r}}) + \varepsilon^4 \iota_{\hat{H}}^3(\varepsilon, \hat{\mathbf{r}})\,, \tag{B.24}$$

so that $\hat{H}_1$ and $\hat{H}_2$ are given by formulas (B.16) and $\hat{H}_3$ is given by:

$$\hat{H}_3(\hat{\mathbf{r}}) = \left(\bar{\mathcal{T}}_0 \nabla \bar{g}_3 \right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) + \mathcal{V}_3(g_1, g_2)(\hat{\mathbf{r}})\,, \tag{B.25}$$

with

$$\mathcal{V}_3(g_1, g_2) = \overline{\mathbf{M}}_1 \overline{\mathbf{M}}_2 \cdot \bar{H}_0(\hat{\mathbf{r}}) + \frac{1}{6} \overline{\mathbf{M}}_1^3 \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^2(\overline{\mathbf{M}}_1) \cdot \bar{H}_0 \tag{B.26}$$

$$+ \mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2) \cdot \bar{H}_1 + \mathbf{V}_1^0(\overline{\mathbf{M}}_1) \cdot \bar{H}_2 + \bar{H}_3. \tag{B.27}$$

Eventually the last step of the algorithm consists in solving iteratively the equations (B.16) and (B.25). The two firsts were tackled in subsection B.1 and B.2 in which we obtain $\bar{g}_1$, $\bar{g}_2$, $\hat{H}_1$ and $\hat{H}_2$. So the job is reduced to solve equation (B.25). Setting

$$\hat{H}_3(\hat{\mathbf{r}}) = -\frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_3(g_1, g_2)(\hat{\mathbf{r}})\, d\hat{r}_3, \tag{B.28}$$

we just have to solve PDE

$$\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_3 \right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) = u_3, \tag{B.29}$$

where u_3 is given by

$$u_3(\hat{\mathbf{r}}) = \mathcal{V}_3(g_1, g_2)(\hat{\mathbf{r}}) - \frac{1}{2\pi}\int_0^{2\pi} \mathcal{V}_3(g_1, g_2)(\hat{\mathbf{r}})\, d\hat{r}_3. \tag{B.30}$$

Hence we obtain

$$\bar{g}_3(\bar{\mathbf{r}}) = -\frac{1}{B(\hat{r}_1, \hat{r}_2)}\int_0^{\bar{r}_3} u_3(\bar{r}_1, \bar{r}_2, s, \bar{r}_4)\, ds. \tag{B.31}$$

B.4 Formulas for $N = 4$

Now we will apply algorithm 5.11 with $N = 4$. The first step of the algorithm consists to inject the expression of $\mathbf{V}_1^\varepsilon$, $\mathbf{V}_2^\varepsilon$, $\mathbf{V}_3^\varepsilon$ and $\mathbf{V}_4^\varepsilon$ in

$$\hat{H}_\varepsilon^4(\hat{\mathbf{r}}) = \sum_{n=0}^{4}\left(\sum_{k=0}^{n}\mathbf{V}_{n-k}^\varepsilon \cdot \bar{H}_k\right)(\hat{\mathbf{r}})\, \varepsilon^n + \varepsilon^5 \iota_{\bar{H}}^4(\varepsilon, \hat{\mathbf{r}}) \tag{B.32}$$

According to the second step of the algorithm we order the terms according to their power of ε and we obtain:

$$\begin{aligned}
\hat{H}_\varepsilon^4(\hat{\mathbf{r}}) = {} & \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon\left(\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1)\cdot\bar{H}_0(\hat{\mathbf{r}})\right) \\
& + \varepsilon^2\Big(\mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \bar{H}_2(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^3\Big(\mathbf{V}_1^2(\overline{\mathbf{M}}_1)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_3^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3)\cdot\bar{H}_0(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \bar{H}_3(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^4\Big(\mathbf{V}_2^2(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_1^2(\overline{\mathbf{N}}_3)\cdot\bar{H}_1(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_3^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_4^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{M}}_4)\cdot\bar{H}_0(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_2^0(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \mathbf{V}_1^0(\overline{\mathbf{M}}_1)\cdot\bar{H}_3(\hat{\mathbf{r}}) + \bar{H}_4(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^5\Big(\mathbf{V}_3^2(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_2^2(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_1(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_1^2(\overline{\mathbf{N}}_3)\cdot\bar{H}_2(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^6\Big(\mathbf{V}_2^4(\overline{\mathbf{N}}_3)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_4^2(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5, \overline{\mathbf{N}}_6)\cdot\bar{H}_0(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_3^2(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}_2^2(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_2(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_1^2(\overline{\mathbf{N}}_3)\cdot\bar{H}_3(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^7\Big(\mathbf{V}_3^4(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_2^4(\overline{\mathbf{N}}_3)\cdot\bar{H}_1(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^8\Big(\mathbf{V}_4^4(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4, \overline{\mathbf{N}}_5)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_3^4(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_1(\hat{\mathbf{r}}) \\
& \qquad + \mathbf{V}_2^4(\overline{\mathbf{N}}_3)\cdot\bar{H}_2(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^9\Big(\mathbf{V}_3^6(\overline{\mathbf{N}}_3)\cdot\bar{H}_0(\hat{\mathbf{r}})\Big) \\
& + \varepsilon^{10}\left(\mathbf{V}_4^6(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}_3^6(\overline{\mathbf{N}}_3)\cdot\bar{H}_1(\hat{\mathbf{r}})\right) \\
& + \varepsilon^{12}\left(\mathbf{V}_4^8(\overline{\mathbf{N}}_3)\cdot\bar{H}_0(\hat{\mathbf{r}})\right) \\
& + \varepsilon^5\iota_{\bar{H}}^4(\varepsilon, \hat{\mathbf{r}})\,.
\end{aligned} \tag{B.33}$$

We compare this expansion with the following desired form

$$
\begin{aligned}
\hat{H}^4_\varepsilon(\hat{\mathbf{r}}) &= \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon \hat{H}_1(\hat{\mathbf{r}}) + \varepsilon^2 \hat{H}_2(\hat{\mathbf{r}}) + \varepsilon^3 \hat{H}_3(\hat{\mathbf{r}}) + \varepsilon^4 \hat{H}_4(\hat{\mathbf{r}}) + \varepsilon^5 \iota^4_{\hat{H}}(\varepsilon, \hat{\mathbf{r}}) \\
&= \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon \left(\bar{H}_1(\hat{\mathbf{r}}) + \overline{\mathbf{M}}_1 \cdot \bar{H}_0(\hat{\mathbf{r}})\right) \\
&\quad + \varepsilon^2 \Big(\mathbf{V}^0_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2\big) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \bar{H}_2(\hat{\mathbf{r}})\Big) \\
&\quad + \varepsilon^3 \Big(\mathbf{V}^2_1\big(\overline{\mathbf{M}}_1\big) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3\big) \cdot \bar{H}_0(\hat{\mathbf{r}}) \\
&\qquad + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2\big) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big) \cdot \bar{H}_2(\hat{\mathbf{r}}) + \bar{H}_3(\hat{\mathbf{r}})\Big) \\
&\quad + \varepsilon^4 \Big(\mathbf{V}^2_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\big) \cdot \bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big) \cdot \bar{H}_1(\hat{\mathbf{r}}) \\
&\qquad + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3\big) \cdot \bar{H}_1(\hat{\mathbf{r}}) + \overline{\mathbf{M}}_4 \cdot \bar{H}_0(\hat{\mathbf{r}}) + \overline{\mathbf{M}}_1 \overline{\mathbf{M}}_3 \cdot \bar{H}_0(\hat{\mathbf{r}}) \\
&\qquad + \frac{1}{2} \overline{\mathbf{M}}^2_1 \overline{\mathbf{M}}_2 \cdot \bar{H}_0(\hat{\mathbf{r}}) + \frac{1}{2} \overline{\mathbf{M}}^2_2 \cdot \bar{H}_0(\hat{\mathbf{r}}) + \frac{1}{24} \overline{\mathbf{M}}^4_1 \cdot \bar{H}_0(\hat{\mathbf{r}}) \\
&\qquad + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2\big) \cdot \bar{H}_2(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big) \cdot \bar{H}_3(\hat{\mathbf{r}}) + \bar{H}_4(\hat{\mathbf{r}})\Big) \\
&\quad + \varepsilon^5 \iota^4_{\hat{H}}(\varepsilon, \hat{\mathbf{r}}) \,.
\end{aligned}
\tag{B.34}
$$

Hence, we obtain that $\hat{H}_1$ and $\hat{H}_2$ are given by formulas (B.16), $\hat{H}_3$ by formula (B.25) and $\hat{H}_4$ by:

$$
\hat{H}_4(\hat{\mathbf{r}}) = \left(\bar{\mathcal{T}}_0 \nabla \bar{g}_4\right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) + \mathcal{V}_4(g_1, g_2, g_3)(\hat{\mathbf{r}})
\tag{B.35}
$$

with

$$
\begin{aligned}
\mathcal{V}_4(g_1, g_2, g_3) &= \mathbf{V}^2_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{N}}_3, \overline{\mathbf{N}}_4\big) \cdot \bar{H}_0 + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big) \cdot \bar{H}_1 \\
&\quad + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \overline{\mathbf{M}}_3\big) \cdot \bar{H}_1 + \overline{\mathbf{M}}_1 \overline{\mathbf{M}}_3 \cdot \bar{H}_0 \\
&\quad + \frac{1}{2} \overline{\mathbf{M}}^2_1 \overline{\mathbf{M}}_2 \cdot \bar{H}_0 + \frac{1}{2} \overline{\mathbf{M}}^2_2 \cdot \bar{H}_0 + \frac{1}{24} \overline{\mathbf{M}}^4_1 \cdot \bar{H}_0 \\
&\quad + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2\big) \cdot \bar{H}_2 + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big) \cdot \bar{H}_3 + \bar{H}_4\Big)
\end{aligned}
\tag{B.36}
$$

Now, the last step of the algorithm consists in solving equations (B.16), (B.25) and (B.35) wit $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$ and $\hat{H}_4$ as unknowns. The three first equations were processed in subsection B.1, B.2 and B.3 in which we obtain the expressions of $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\hat{H}_1$, $\hat{H}_2$, and $\hat{H}_3$. So we just have to solve equation (B.35). Setting

$$
\hat{H}_4(\hat{\mathbf{r}}) = -\frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_4(g_1, g_2, g_3)(\hat{\mathbf{r}}) \, d\hat{r}_3
\tag{B.37}
$$

the job is reduced to solve PDE

$$
\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_4\right) \cdot \nabla \bar{H}_0(\hat{\mathbf{r}}) = u_4,
\tag{B.38}
$$

with $\bar{g}_4$ as unknown, where u_4 is given by

$$
u_4(\hat{\mathbf{r}}) = \mathcal{V}_4(g_1, g_2, g_3)(\hat{\mathbf{r}}) - \frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_4(g_1, g_2, g_3)(\hat{\mathbf{r}}) \, d\hat{r}_3.
\tag{B.39}
$$

Hence we obtain:

$$
\bar{g}_4(\bar{\mathbf{r}}) = -\frac{1}{B(\hat{r}_1, \hat{r}_2)} \int_0^{\bar{r}_3} u_4(\bar{r}_1, \bar{r}_2, s, \bar{r}_4) \, ds
\tag{B.40}
$$

B.5 Formulas for $N = 5$

Presently we will apply algorithm 5.11 with $N = 5$. The first step of the algorithm consists to inject the expression of $\mathbf{V}_1^\varepsilon$, $\mathbf{V}_2^\varepsilon$, $\mathbf{V}_3^\varepsilon$, $\mathbf{V}_4^\varepsilon$ and $\mathbf{V}_5^\varepsilon$ in

$$\hat{H}_\varepsilon^5(\hat{\mathbf{r}}) = \sum_{n=0}^{5} \left(\sum_{k=0}^{n} \mathbf{V}_{n-k}^\varepsilon \cdot \bar{H}_k \right)(\hat{\mathbf{r}})\, \varepsilon^n + \varepsilon^6 \iota_{\bar{H}}^5(\varepsilon, \hat{\mathbf{r}}) \tag{B.41}$$

According to the second step of the algorithm we order the terms according to their power of ε and we obtain:

$$
\begin{aligned}
\hat{H}^5_\varepsilon(\hat{\mathbf{r}}) &= \sum_{n=0}^{5}\left(\sum_{k=0}^{n}\mathbf{V}^\varepsilon_{n-k}\cdot\bar{H}_k\right)(\hat{\mathbf{r}})\,\varepsilon^n + \varepsilon^6\iota^5_{\bar{H}}(\varepsilon,\hat{\mathbf{r}})\\
&= \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon\left(\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\right)\\
&\quad + \varepsilon^2\Big(\mathbf{V}^0_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \bar{H}_2(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^3\Big(\mathbf{V}^2_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \bar{H}_3(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^4\Big(\mathbf{V}^2_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^0_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{M}}_4\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2\big)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_3(\hat{\mathbf{r}}) + \bar{H}_4(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^5\Big(\mathbf{V}^2_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^2_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_2(\hat{\mathbf{r}})\\
&\qquad \mathbf{V}^0_5\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{M}}_4,\overline{\mathbf{M}}_5\big)\,\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^0_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{M}}_4\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\\
&\qquad \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3\big)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2\big)\cdot\bar{H}_3(\hat{\mathbf{r}})\\
&\qquad \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_4(\hat{\mathbf{r}}) + \bar{H}_5(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^6\Big(\mathbf{V}^4_2\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^2_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5,\overline{\mathbf{N}}_6\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^2_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^2_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_2(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_3(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^7\Big(\mathbf{V}^4_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^4_2\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^2_5\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{M}}_4,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5,\overline{\mathbf{N}}_6,\overline{\mathbf{N}}_7\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^2_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5,\overline{\mathbf{N}}_6\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^2_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_2(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^2_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_3(\hat{\mathbf{r}}) + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_4(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^8\Big(\mathbf{V}^4_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^4_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_1(\hat{\mathbf{r}}) + \mathbf{V}^4_2\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_2(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^9\Big(\mathbf{V}^6_3\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^4_5\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5,\overline{\mathbf{N}}_6\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^4_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^4_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_2(\hat{\mathbf{r}}) + \mathbf{V}^4_2\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_4(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^{10}\big(\mathbf{V}^6_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^6_3\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\big)\\
&\quad + \varepsilon^{11}\Big(\mathbf{V}^6_5\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^6_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\\
&\qquad + \mathbf{V}^6_3\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_3(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^{12}\big(\mathbf{V}^8_4\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\big)\\
&\quad + \varepsilon^{13}\Big(\mathbf{V}^8_5\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_0(\hat{\mathbf{r}}) + \mathbf{V}^8_4\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_1(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^{15}\Big(\mathbf{V}^{10}_5\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_0(\hat{\mathbf{r}})\Big)\\
&\quad + \varepsilon^6\iota^5_{\bar{H}}(\varepsilon,\hat{\mathbf{r}})\,,
\end{aligned} \tag{B.42}
$$

that is compared with

$$\hat{H}^5_\varepsilon(\hat{\mathbf{r}}) = \bar{H}_0(\hat{\mathbf{r}}) + \varepsilon\hat{H}_1(\hat{\mathbf{r}}) + \varepsilon^2\hat{H}_2(\hat{\mathbf{r}}) + \varepsilon^3\hat{H}_3(\hat{\mathbf{r}}) + \varepsilon^4\hat{H}_4(\hat{\mathbf{r}}) + \varepsilon^5\hat{H}_5(\hat{\mathbf{r}}) + \varepsilon^6\iota^5_{\hat{H}}(\varepsilon,\hat{\mathbf{r}})\,, \tag{B.43}$$

so that $\hat{H}_1$ and $\hat{H}_2$ are given by formulas (B.16), $\hat{H}_3$ by formula (B.25), $\hat{H}_4$ by formula (B.35) and $\hat{H}_5$ by:

$$\hat{H}_5(\hat{\mathbf{r}}) = \left(\bar{\mathcal{T}}_0\nabla\bar{g}_5\right)\cdot\nabla\bar{H}_0(\hat{\mathbf{r}}) + \mathcal{V}_5(g_1,g_2,g_3,g_4)(\hat{\mathbf{r}})\,, \tag{B.44}$$

with

$$\begin{aligned}\mathcal{V}_5(g_1,g_2,g_3,g_4) = {}& \mathbf{V}^2_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4,\overline{\mathbf{N}}_5\big)\cdot\bar{H}_0 + \mathbf{V}^2_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{N}}_3,\overline{\mathbf{N}}_4\big)\cdot\bar{H}_1 \\ & + \mathbf{V}^2_1\big(\overline{\mathbf{N}}_3\big)\cdot\bar{H}_2 + \overline{\mathbf{M}}_1\overline{\mathbf{M}}_4\cdot\bar{H}_0 + \overline{\mathbf{M}}_2\overline{\mathbf{M}}_3\cdot\bar{H}_0 + \overline{\mathbf{M}}_1^2\overline{\mathbf{M}}_3\cdot\bar{H}_0 \\ & + \frac{1}{2}\overline{\mathbf{M}}_1\overline{\mathbf{M}}_2^2\cdot\bar{H}_0 + \frac{1}{6}\overline{\mathbf{M}}_1^3\overline{\mathbf{M}}_2\cdot\bar{H}_0 + \frac{1}{120}\overline{\mathbf{M}}_1^5\cdot\bar{H}_0 \\ & + \mathbf{V}^0_4\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3,\overline{\mathbf{M}}_4\big)\cdot\bar{H}_1 \\ & + \mathbf{V}^0_3\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2,\overline{\mathbf{M}}_3\big)\cdot\bar{H}_2 + \mathbf{V}^0_2\big(\overline{\mathbf{M}}_1,\overline{\mathbf{M}}_2\big)\cdot\bar{H}_3 \\ & + \mathbf{V}^0_1\big(\overline{\mathbf{M}}_1\big)\cdot\bar{H}_4 + \bar{H}_5.\end{aligned} \tag{B.45}$$

Finally the last step of the algorithm consists to solve iteratively the equations (B.16), (B.25), (B.35) and (B.44) of unknowns $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\bar{g}_5$, $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$, $\hat{H}_4$, and $\hat{H}_5$. The fourth first equations were tackled in subsection B.1, B.2, B.3 and B.4 in which we obtain $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$ and $\hat{H}_4$. So we just have to solve equation (B.44). Setting

$$\hat{H}_5(\hat{\mathbf{r}}) = -\frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_5(g_1,g_2,g_3,g_4)(\hat{\mathbf{r}})\,d\hat{r}_3, \tag{B.46}$$

the job is reduced to solve the PDE

$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_5\right)\cdot\nabla\bar{H}_0(\hat{\mathbf{r}}) = u_5, \tag{B.47}$$

with $\bar{g}_5$ as unknown, where u_5 is given by

$$u_5(\hat{\mathbf{r}}) = \mathcal{V}_5(g_1,g_2,g_3,g_4)(\hat{\mathbf{r}}) - \frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_5(g_1,g_2,g_3,g_4)(\hat{\mathbf{r}})\,d\hat{r}_3. \tag{B.48}$$

We obtain

$$\bar{g}_5(\bar{\mathbf{r}}) = -\frac{1}{B(\hat{r}_1,\hat{r}_2)}\int_0^{\bar{r}_3}u_5(\bar{r}_1,\bar{r}_2,s,\bar{r}_4)\,ds. \tag{B.49}$$

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