Optimal control of a continuous-in-time financial model *

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Abstract. The achievement of a project requires tools to monitor and adjust its evolution over time. Rather than to check at mid-term whether the objectives will be achieved or not, and adjust them, it is interesting to develop a control tool in order to effectively conduct the project's objectives. In this paper, we improve the continuous-in-time financial model developed in Frénod & Chakkour [3], that describes working of loan and repayment, in order to prepare its capability to be used in control theory approach. The aim is to determine the optimal loan schedule taking into account the objective of the project, the income and the spending. For that, we set out an optimal control method for the strategy elaboration phase to better adjust the project implementation.

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1　Introduction

A project is a temporary activity group designed to produce a product, service or result. A project is temporary in that sense it has a defined beginning and end in time and so a lifetime. When the project involves a loan, it is necessary to find the best strategy in order to achieve the goals of the project, minimizing the cost of the loan. This strategy should be able to achieve the project goals on time under constraints that may be imposed by law, taxes, banking institutions and more generally by the project environment. It gives rise to the loan scheme which is the way that the amounts concerned by the loan will be borrowed.

Once this strategy set out, and all along the project implementation, it is needed to control the way the loan scheme is implemented. The objective is to counterbalance the gap between the reality of the environment in which the project takes place and the forecast made during the strategy elaboration.

There are several references in the literature dealing with continuous-in-time financial model. Among them we find: R. Merton [6] which provides an overview and synthesis of finance theory from the perspective of continuous-in-time analysis. In [8], S. Sundaresan surveys and assesses the development of continuous-in-time methods in finance during the period between 1970 and 2000. In addition, many studies have used control engineering methods and techniques in finance. For example Keel [5] explored and extended optimal portfolio construction techniques currently found in the literature. Grigorieva & Khailov [4] built a controlled system of differential equations modeling a firm that takes a loan in order to expand its production activities.

In this paper we base on a continuous-in-time financial model developed in Frénod & Chakkour [3] that describes working of loan and repayment. This model is not designed for the financial market but for the public institutions. The main objective of this paper is to use this model in the framework of the control theory. For that, we set out an optimal control method for the strategy elaboration phase and an adaptive control method to better adjust the project implementation.

2　Continuous-in-time financial model

In this section we improve the continuous-in-time financial model of Frénod & Chakkour [3] in order to account for the project notion and to prepare its capability to be used in control theory approach. The time domain is the interval $[0, \Theta]$, where $\Theta > 0$ is the lifetime of the project. We consider that beyond Θ the spending associated with the project are done, the loan associated with the project is completely paid off and the project is finished.

2.1　Variables of the model

To characterize the budget of a project, we introduce the loan density κ_E and the density of repayment ρ_K which is connected, as explained in Frénod & Chakkour [3], to the loan density by a convolution operator:

$$\rho_K(t) = (\kappa_E * \gamma)(t), \tag{1}$$

where γ is the repayment pattern. Since the whole amount associated with the loan has to be repaid, γ has to satisfies:

$$\int_{-\infty}^{+\infty} \gamma(t)dt = 1. \tag{2}$$

We denote by $\mathcal{K}_{RD}$ the current debt, given as the solution to the following differential equation:

$$\frac{d\mathcal{K}_{RD}(t)}{dt} = \kappa_E(t) - \rho_K(t) - \rho_K^{\mathcal{I}}(t), \tag{3}$$

where $\rho_K^{\mathcal{I}}(t)$ is the density of repayment of the current debt $\mathcal{K}_{RD}(0)$ at the beginning of the period. It is called initial debt repayment scheme. Initial condition for equation (3) is given by:

$$\mathcal{K}_{RD}(0) = \int_0^{+\infty} \rho_K^{\mathcal{I}}(t)dt. \tag{4}$$

We denote by $\rho_I(t)$ the density of interest defined by:

$$\rho_I(t) = \alpha(t)\mathcal{K}_{RD}(t), \tag{5}$$

where $\alpha(t)$ is the floating rate interest. The algebraic spending density is denoted $\sigma(t)$, it takes into account the spending and the income and it is given by:

$$\sigma(t) = \beta(t) + \sigma_g(t), \tag{6}$$

where $\beta(t) \geq 0$ is the "financing needs density", it is the density of spending that are intended for the project only. $\sigma_g(t)$ is the current spending density. We assume that $\beta(t) \geq 0$ because only spending are concerned.

The fact that the initial time of the project is 0 and the lifetime is Θ translates as:

$$\text{supp } (\kappa_E) \subset [0, \Theta], \quad \text{supp } (\kappa_E) + \text{supp } (\gamma) \subset [0, \Theta], \quad \text{supp } (\rho_K^{\mathcal{I}}) \subset [0, \Theta], \tag{7}$$

where supp (f) is the support of f.

2.2 Objectives of the project

Integrating (3) over $[0, t]$, we obtain using (1) the following relation:

$$\mathcal{K}_{RD}(t) = \mathcal{K}_{RD}(0) + \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds - \int_0^t \rho_K^{\mathcal{I}}(s)ds, \tag{8}$$

and using (4), we obtain:

$$\mathcal{K}_{RD}(t) = \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds + \int_t^\infty \rho_K^{\mathcal{I}}(s)ds. \tag{9}$$

We want that the spending density balances the income density. In our model we have the following densities: σ which, depending on its sign, stands alternately for income or spending, κ_E which is an income density and $\rho_K, \rho_I, \tilde{\rho}_K^{\mathcal{I}}$ which are spending densities. Hence the balance relation reads:

$$\sigma(t) = \kappa_E(t) - \rho_K(t) - \rho_I(t) - \rho_K^{\mathcal{I}}(t). \tag{10}$$

Using (9) and (5), we deduce the following relation:

$$\sigma(t) = \kappa_E(t) - (\kappa_E * \gamma)(t) - \alpha(t)\mathcal{K}_{RD}(t) - \rho_K^{\mathcal{I}}(t)$$
$$= (\mathcal{L}[\kappa_E])(t) - \alpha(t)\int_t^\infty \rho_K^{\mathcal{I}}(s)ds - \rho_K^{\mathcal{I}}(t), \tag{11}$$

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where the operator $\mathcal{L}$ is defined by:

$$(\mathcal{L}[\kappa_E])(t) \doteq \kappa_E(t) - (\kappa_E * \gamma)(t) - \alpha(t) \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds, \tag{12}$$

is the algebraic income density associated to the loan. In other words, it is the difference between the income density induced by the loan density on the one hand and the spending density associated with the repayment density and the interest payment density on the other hand. Using (6) we have then:

$$\beta(t) = (\mathcal{L}[\kappa_E])(t) - \alpha(t) \int_t^\infty \rho_K^{\mathcal{I}}(s)ds - \rho_K^{\mathcal{I}}(t) - \sigma_g(t). \tag{13}$$

The isolated spending density $\beta(t)$ is the difference between the algebraic income density associated with the loan, and the spending densities related to the following: current spending, initial debt repayment and payment of the interests of this latter.

We define an objective as a couple collection (c_i, Θ_i), $i \in \{1, \ldots, N\}$, where c_i is the amount which has to be spent for the project at the moment $\Theta_i \in]0, \Theta[$. We suppose that $0 < \Theta_1 < \Theta_2 < \ldots < \Theta_N < \Theta$, and to be consistant we need that $0 \le c_1 \le c_2 \le \ldots \le c_N$. We say that the objective is reached if:

$$\int_0^{\Theta_i} \beta(t)dt \ge c_i, \ \forall i = 1, \ldots, N. \tag{14}$$

The above equation indicating that at any Θ_i the amount allocated to the project is at least the amount needed for the project.

Using this model we will establish the strategy, i.e. find the loan which allows the objectives (c_i, Θ_i) to be reached. Furthermore, this loan is not chosen at random but have to satisfy some conditions. Typically, it must minimize the cost of the loan. This strategy can be written as an optimal control problem which is developed in the next section.

3 Strategy elaboration: optimize loan coast

Among all the variables introduced in the previous section, the only one that can be modified is the loan density κ_E. We consider that the repayment pattern γ and the interest rate α are essentially imposed by banking institutions. The density of algebraic spending σ is imposed by the project environment. Hence γ, α and σ are, at the level of the strategy elaboration, foresights of what they will be when the project will be implemented.

We introduce the loan cost which is induced by interest payments $C_I[\kappa_E]$ and which is defined as:

$$C_I[\kappa_E] = \int_0^\infty (\boldsymbol{\rho_I}[\kappa_E])(t)dt, \tag{15a}$$

where $\boldsymbol{\rho_I}$ is the functional that links loan density κ_E to density of interest ρ_I:

$$(\boldsymbol{\rho_I}[\kappa_E])(t) = \alpha(t)\mathcal{K}_{RD}(t) = \alpha(t)\left(\int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds + \int_t^\infty \rho_K^{\mathcal{I}}(s)ds\right), \tag{15b}$$

and the total amount of the loan $C_E[\kappa_E]$ is defined by:

$$C_E[\kappa_E] = \int_0^\infty \kappa_E(t)dt. \tag{16}$$

Minimizing loan coast consists in finding loan density κ_E that minimizes the loan coast $C_I[\kappa_E]$.

We introduce now the following two optimization problems in a intuitive and not mathematically completely rigorous way. Those problems called (O1) and (O2) will then be inserted within rigorous framework in the next section.

O1. Optimization of the spending. Given (c_i, Θ_i), $i \in \{1, \ldots, N\}$, find the loan density $\check{\kappa}_E$ solution to:

$$C_I[\check{\kappa}_E] = \min_{\kappa_E} C_I[\kappa_E], \tag{17a}$$

subject to:

$$\int_0^{\Theta_i} \beta(t)dt \geq c_i, \ \forall i = 1, \ldots, N. \tag{17b}$$

In this problem we seek the optimal loan density $\kappa_E(t)$ that minimizes the cost of the loan, considering that the amount allocated to the project until $\Theta_i \in]0, \Theta[$ is at least the amount c_i needed for the project at Θ_i. There is no uniqueness of the solution to problem (O1). Among its solution of problem (O1) some are subjected to perturbations which are numerical artifices that are irrelevant from financial slant. We will therefore build a method based on filtering theory to eliminate those perturbations. Several filters will be investigated in the Section 5.2 to choose a suitable one to be incorporated in our method. We introduce here the general framework of the filtering based problem which is set in terms of an optimization problem:

O2. Filtering perturbed solutions to (O1) by an optimization problem. From all solutions $\check{\kappa}_E(t)$ resulting from (O1), we denote by $\hat{\kappa}_E(t)$ the desired unperturbed solution obtained by eliminating the perturbations in $\check{\kappa}_E(t)$, which is defined by:

$$\hat{\kappa}_E(t) = \check{\kappa}_E(t) - \hat{G}(t), \tag{18}$$

where $\hat{G}(t)$ represents the perturbations density. $\hat{G}(t)$ is obtained as the result of the following optimization problem (called (O2)) which can be written generically as: given $\check{\kappa}_E(t)$, find $\hat{G}(t)$ solution of:

$$\min_G F(\check{\kappa}_E, G), \tag{19a}$$

where the definition of the functional F depends on the filter which is used. The problem (19a) is subject to the following constraints:

$$\int_0^{\Theta_i} \beta(t)dt \geq c_i, \ \forall i = 1, \ldots, N. \tag{19b}$$

Moreover, since G is the perturbation to be removed, it satisfies:

$$\int_0^{\Theta} G(t)dt = 0. \tag{19c}$$

This is then a supplementary constraint for optimization problem (O2).

4 Optimization problem (O1)

4.1 Mathematical framework of optimization problem (O1)

The objective of this section is to build the mathematical framework of the optimal control problem. We suppose that $\kappa_E(t), \gamma(t)$ and $\rho_K^{\mathcal{T}}(t)$ introduced in section 2.1 are in $L^2([0, \Theta])$ and $\alpha(t), \mathcal{K}_{RD}(t)$ are in $L^\infty([0, \Theta])$. Let U be the set of admissible loan densities $\kappa_E(t) \in L^2[0, \Theta]$, such that $\kappa_E^{\max}(t) \geq \kappa_E(t) \geq 0$, where $\kappa_E^{\max}(t) \in L^\infty([0, \Theta])$ is the maximal loan densities that can be borrowed. The admissible loan density should allow to have a positive financing needs:

$$\beta(t) \geq 0. \tag{20}$$

By using (13), equation (20) is equivalent to write:

$$(\mathcal{L}[\kappa_E])(t) \geq \sigma_g(t) + \alpha(t) \int_t^\infty \rho_K^{\mathcal{I}}(s)ds + \rho_K^{\mathcal{I}}(t), \tag{21}$$

and equation (19b) can be written in the next form:

$$\int_0^{\Theta_i} (\mathcal{L}[\kappa_E])(t)dt \geq c_i + \int_0^{\Theta_i} \left[\alpha(t) \int_t^\infty \rho_K^{\mathcal{I}}(s)ds + \rho_K^{\mathcal{I}}(t) + \sigma_g(t) \right] dt, \ \forall i = 1, \ldots, N. \tag{22}$$

We can rewrite the optimization problem in a rigorous form. Denoting U the set of function $\kappa_E(t) \in L^2[0, \Theta]$ which satisfy the conditions:

$$\kappa_E^{\max} \geq \kappa_E(t) \geq 0, \tag{23a}$$
$$(\mathcal{L}[\kappa_E])(t) \geq \mu(t), \tag{23b}$$
$$\int_0^{\Theta_i} (\mathcal{L}[\kappa_E])(t)dt \geq c_i + \int_0^{\Theta_i} \mu(t)dt, \ \forall i = 1, \ldots, N, \tag{23c}$$

where

$$\mu(t) = \sigma_g(t) + \alpha(t) \int_t^\infty \rho_K^{\mathcal{I}}(s)ds + \rho_K^{\mathcal{I}}(t), \tag{24}$$

we search a solution $\kappa_E(t)$ of the following optimization problem :

$$C_I[\check{\kappa}_E] = \min_{\kappa_E \in U} C_I[\kappa_E] \tag{25}$$

It is recalled that functions $\sigma_g(t)$, $\rho_K^{\mathcal{I}}(t)$, $\alpha(t)$ and $\gamma(t)$ are considered as known and are model inputs.

From now on, we work with a fixed interest rate α, in this case we can show that solving the problem (25) is equivalent to solve the following problem:

$$C_E[\check{\kappa}_E] = \min_{\kappa_E \in U} C_E[\kappa_E], \tag{26}$$

which means: minimize the cost of the loan is equivalent to minimize the total amount of the loan. The latter is the problem that we will be interested in. We have the following lemma:

Lemma 1. *For a fixed $\alpha \geq 0$, each function $\kappa_E \in L^2([0, \Theta])$ solution of (25) is also solution of (26) and vice versa.*

Proof. According to (15b), we have

$$C_I[\kappa_E] = \int_0^\infty (\rho_I[\kappa_E])(t)dt = \alpha \int_0^\infty \int_0^t (\kappa_E - \kappa_E * \gamma)(s)dsdt + \alpha \int_0^\infty \int_t^\infty \rho_K^{\mathcal{I}}(s)dsdt. \tag{27}$$

Using (7), the previous equation can be written in the form:

$$C_I[\kappa_E] = \alpha \int_0^\infty \int_0^\Theta (\kappa_E - \kappa_E * \gamma)(s)\mathbb{1}_{[0,t]}(s)dsdt + K_1, \tag{28}$$

where

$$K_1 = \alpha \int_0^\Theta \int_t^\Theta \rho_K^{\mathcal{I}}(s)dsdt \geq 0.$$

Using the fact that

$$\int_0^\Theta \mathbb{1}_{[0,t]}(s)dt = \int_0^\Theta \mathbb{1}_{\{t \geq s\}}(t)dt = \int_s^\Theta dt = \Theta - s, \qquad (29)$$

equation (28) can be written in the next form:

$$C_I[\kappa_E] = \alpha \int_0^\infty (\kappa_E - \kappa_E * \gamma)(s) \left(\int_0^\Theta \mathbb{1}_{[0,t]}(s)dt \right) ds + K_1$$

$$= \alpha \int_0^\infty (\kappa_E - \kappa_E * \gamma)(s)(\Theta - s)ds + K_1. \qquad (30)$$

Thus

$$C_I[\kappa_E] = \alpha\Theta \int_0^\infty (\kappa_E - \kappa_E * \gamma)(s)ds - \alpha \int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds + K_1$$

$$= -\alpha \int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds + K_2, \qquad (31)$$

where

$$K_2 = K_1 + \alpha\Theta \int_0^\Theta (\kappa_E - \kappa_E * \gamma)(s)ds,$$

Besides this, we have:

$$\int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds = \int_0^\infty s\kappa_E(s)ds - \int_0^\infty s \int_0^\infty \kappa_E(s-x)\gamma(x)dxds$$

$$= \int_0^\infty s\kappa_E(s)ds - \int_0^\infty s \int_0^\infty \kappa_E(s-x)\gamma(x)\mathbb{1}_{[0,\Theta]}(x)dxds. \qquad (32)$$

With the change of variable: $y = s - x$ we obtain,

$$\int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds = \int_0^\infty s\kappa_E(s)ds - \int_0^\infty s \int_0^\infty \kappa_E(y)\gamma(s-y)\mathbb{1}_{[s-\Theta,s]}(y)dyds$$

$$= \int_0^\infty y\kappa_E(y)dy - \int_0^\infty \kappa_E(y) \int_0^\infty s\gamma(s-y)\mathbb{1}_{[s-\Theta,s]}(y)dsdy$$

$$= \int_0^\infty \kappa_E(y) \left[y - \int_0^\infty s\gamma(s-y)\mathbb{1}_{[s-\Theta,s]}(y)ds \right] dy, \qquad (33)$$

and with the change of variable: $z = s - y$, we obtain:

$$\int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds = \int_0^\infty \kappa_E(y) \left[y - \int_0^\infty (z+y)\gamma(z)\mathbb{1}_{[0,\Theta]}(z)dz \right] dy,$$

using the fact that:

$$\int_0^\infty \gamma(z)\mathbb{1}_{[0,\Theta]}(z)dz = 1, \qquad (34)$$

we get:

$$\int_0^\infty s(\kappa_E - \kappa_E * \gamma)(s)ds = -c_\gamma \int_0^\infty \kappa_E(y)dy = -c_\gamma C_E[\kappa_E], \qquad (35)$$

where

$$c_\gamma = \int_0^\infty z\gamma(z)\mathbb{1}_{[0,\Theta]}(z)dz \geq 0. \qquad (36)$$

We finally obtain from (31) and (35):

$$C_I[\kappa_E] = \alpha c_\gamma C_E[\kappa_E] + K_2. \qquad (37)$$

This ends the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4.2 Existence of solutions for optimization problem (O1)

In this section we show the existence of solutions for optimization problem (26). By introducing the following notation:

$$\mathcal{F}_g \doteq \left\{ f \in L^2[0, \Theta]; f \geq g \right\}, \quad \text{for } g \in L^2[0, \Theta], \tag{38}$$

and the continuous functionals Ψ_i, $i = 1, \ldots, N$, defined by:

$$
\begin{aligned}
\Psi_i : L^2[0, \Theta] &\to \mathbb{R} \\
f &\longmapsto \Psi_i(f) = \int_0^{\Theta_i} \mathcal{L}(f(t)) dt,
\end{aligned}
\tag{39}
$$

we can then rewrite U in the next form:

$$U = \mathcal{F}_0 \cap -\mathcal{F}_{-\kappa_E^{\max}} \cap \mathcal{L}^{-1}(\mathcal{F}_{\mu(t)}) \cap \left[\bigcap_{i=1}^N \Psi_i^{-1} \left(\left[c_i + \int_0^{\Theta_i} \mu(s) ds, +\infty \right[\right) \right]. \tag{40}$$

We will invoke the following theorem which proof can be found in Cohen [1].

Theorem 2. *Let V be a subspace of a Hilbert space. If $J : V \mapsto \bar{\mathbb{R}}$ is a lower semi-continuous function, convexe and coercive on V, if V is convexe and closed, if*

$$Dom(J) \cap V \neq \emptyset,$$

where

$$Dom(J) = \{x \in V; J(x) < +\infty\}.$$

then there exists at least a solution of the optimization problem:

$$\min_{x \in V} J(x).$$

Using Theorem 2 we obtain the next theorem:

Theorem 3. *If set U given by (40) is such that $U \neq \emptyset$, then optimization problem (26) has at least one solution.*

Proof. The set U given by (40) is convex and closed because it is the intersection of closed sets. In fact $\mathcal{F}_g$ is a closed set, $\mathcal{L}$ is a continuous linear operator and Ψ is a continuous operator. The functional:

$$
\begin{aligned}
C_E : L^2[0, \Theta] &\to \mathbb{R} \\
f &\longmapsto C_E[f] = \int_0^{\Theta} f(t) dt,
\end{aligned}
\tag{41}
$$

is a continuous function, so it is a lower semi-continuous function. In addition, C_E is a convex function because it is linear on U and coercive because U is a bounded set (constraint (23a)). Since $Dom(C_E) = L^2([0, \Theta])$ and according to Theorem 2, if $U \neq \emptyset$ then optimization problem (26) has at least one solution. $\qquad \square$

4.3   Discretization of optimization problem (O1)

From now, we suppose that the interest rate α is constant and $\kappa_E^{\max}$ is infinite. The way to discretize problem (26) consists in introducing on $[0, \Theta]$ n points $(t_i)_{i=1\dots,n}$ such that $t_i = (i-1)h$, where $h = \frac{\Theta}{n-1}$. We assume that for any $j = 1, \ldots, N$, there exists a t_i such that $\Theta_j = t_i$. At the operational level, if this condition is not verified, we substitute Θ_j by the closest t_i. Then we define n intervals $(I_i)_{i=1,\ldots,n}$, such that $\bigcup_{i=1}^{n} I_i = [0, \Theta]$, defined by:

$$I_i = \left[(i - \frac{3}{2})h, (i - \frac{1}{2})h\right], \text{ for } i = 2, \ldots, n-1, \quad I_1 = \left[0, \frac{h}{2}\right], \quad I_n = \left[(n - \frac{3}{2})h, (n-1)h\right].$$

We consider then the space S made of functions that are defined on $[0, \Theta]$ and constant on each I_i, for $i = 1, \ldots, n$. We introduce the approximation operator Π defined by:

$$
\begin{aligned}
\Pi : L^2[0, \Theta] &\to S \\
f(t) &\longmapsto \bar{f}(t) = \sum_{i=1}^{n} f_i \mathbb{1}_{I_i}(t),
\end{aligned}
\tag{42}
$$

where $f_i = \dfrac{1}{|I_i|} \displaystyle\int_{I_i} f(t)dt$ is the mean value of the function f on interval I_i. We denote $\tilde{f} = (f_1, \ldots, f_n)^\mathsf{T}$ and we will use these notations afterward for the quantities of the model. We have:

$$\int_0^{t_i} \bar{f}(t)dt = \frac{h}{2}f_1 + h\sum_{j=2}^{i-1} f_j + \frac{h}{2}f_i, \quad i = 2, \ldots, n, \tag{43a}$$

$$\int_{t_i}^{\Theta} \bar{f}(t)dt = \frac{h}{2}f_i + h\sum_{j=i+1}^{n-1} f_j + \frac{h}{2}f_n, \quad i = 1, \ldots, n-1. \tag{43b}$$

With this discret space on hands, we consider the discret optimization problem that consists in finding $\bar{\kappa}_E(t)$ solution of (26) under constraints (23), with γ replaced by $\bar{\gamma}$, $\rho_K^{\mathcal{I}}$ by $\bar{\rho}_K^{\mathcal{I}}$ and σ_g by $\bar{\sigma}_g$.

It is straightforward to show that solving the optimization problem (26) is equivalent to solve the next one.

$$C_E[\check{\kappa}_E] = \min_{\bar{\kappa}_E \in \bar{U}} C_E[\bar{\kappa}_E], \tag{44}$$

where $\bar{U}$ is the set of function $\bar{\kappa}_E \in S$ satisfying:

$$\bar{\kappa}_E(t) \geq 0, \tag{45a}$$

$$(\mathcal{L}[\bar{\kappa}_E])(t) \geq \bar{\sigma}_g(t) + \alpha \int_t^{\Theta} \bar{\rho}_K^{\mathcal{I}}(s)ds + \bar{\rho}_K^{\mathcal{I}}(t), \tag{45b}$$

$$\int_0^{t_i} (\mathcal{L}[\bar{\kappa}_E])(t)dt \geq u(t_i) + \int_0^{t_i} \left[\alpha \int_t^{\Theta} \bar{\rho}_K^{\mathcal{I}}(s)ds + \bar{\rho}_K^{\mathcal{I}}(t) + \bar{\sigma}_g(t)\right]dt, \ \forall i = 1, \ldots, n, \tag{45c}$$

where u is a function that is equal to c_i in each interval $[\Theta_i, \Theta_{i+1}]$, i.e:

$$u(t) = \sum_{i=1}^{N-1} c_i \mathbb{1}_{[\Theta_i, \Theta_{i+1}]}(t) + c_N \mathbb{1}_{[\Theta_N, \Theta]}(t). \tag{46}$$

Because of the piecewise constante nature of $\bar{\kappa}_E$, we have:

$$\bar{\kappa}_E(t) - (\bar{\kappa}_E * \bar{\gamma})(t) = (I_n - C)\tilde{\kappa}_E, \tag{47}$$

where matrix $C \in \mathbb{R}^{n \times n}$, resulting from the convolution, is defined by:

$$C_{i+1,j+1} = h\gamma((i - j \equiv n) + 1), \quad i, j \in \{0, \ldots, n - 1\},$$

and $\tilde{\kappa}_E = (\bar{\kappa}_E(t_1), \ldots, \bar{\kappa}_E(t_n))^\top \in \mathbb{R}^n$. Using (47) and (43a), operator $\mathcal{L}$ defined by (12) when applied to $\bar{\kappa}_E$ yields operator L acting on $\tilde{\kappa}_E$, defined by:

$$\mathcal{L}(\bar{\kappa}_E)(t) = L\tilde{\kappa}_E, \tag{48a}$$

its expression is:

$$L = (I_n - C) - \alpha M(I_n - C) = (I_n - \alpha M)(I_n - C) \in \mathbb{R}^{n \times n}, \tag{48b}$$

with, by using (43a):

$$M = \frac{h}{2} \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 2 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ 1 & 2 & \ldots & 2 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

is the matrix resulting from the approximation of the integral in the formula (12). The constraints given by (45) may then be written as:

$$\tilde{\kappa}_E \geq 0, \tag{49a}$$
$$L\tilde{\kappa}_E \geq \tilde{\sigma}_g + (\alpha P + I_n)\tilde{\rho}_K^{\mathcal{I}}, \tag{49b}$$
$$ML\tilde{\kappa}_E \geq \nu + M\big[(\alpha P + I_n)\tilde{\rho}_K^{\mathcal{I}} + \tilde{\sigma}_g\big], \tag{49c}$$

where by using (43b):

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & \ldots & 2 & 1 \\ 0 & 1 & \ldots & 2 & 1 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

and ν is the discretization of (46) given by:

$$\nu = \begin{pmatrix} u(t_1) \\ \vdots \\ u(t_n) \end{pmatrix} \in \mathbb{R}^n. \tag{50}$$

The discretization of the optimization problem (26) is finally written as the following one. For $J = (1, \ldots, 1)^\top \in \mathbb{R}^n$:

$$C_E(\check{\kappa}_E) = \min_{\tilde{\kappa}_E} J^\top \tilde{\kappa}_E, \tag{51a}$$

subject to:

$$\tilde{\kappa}_E \geq 0, \tag{51b}$$
$$L\tilde{\kappa}_E \geq \tilde{\sigma}_g + (\alpha P + I_n)\tilde{\rho}_K^{\mathcal{I}}, \tag{51c}$$
$$ML\tilde{\kappa}_E \geq \nu + M\big[(\alpha P + I_n)\tilde{\rho}_K^{\mathcal{I}} + \tilde{\sigma}_g\big], \tag{51d}$$

Problem (51) is a linear optimization problem.

5 Optimization problem (O2)

5.1 Mathematical framework of optimization problem (O2)

In this section we build a mathematical framework for the optimization problem (O2) introduced in Section 3. We suppose the optimal loan density $\check{\kappa}_E(t)$ and the perturbation density $\hat{G}(t)$ are in $L^2([0,\Theta])$. The functional F introduced in Section 3 is a map from $L^2[0,\Theta] \times L^2[0,\Theta]$ to $\mathbb{R}$. Having $\check{\kappa}_E(t)$ resulting from (O1), we search $\hat{G}(t)$ solution of the next optimization problem. Given $\check{\kappa}_E(t)$, find $\hat{G}(t)$ solution of:

$$\min_G F(\check{\kappa}_E, G) \tag{52}$$

subjected to the same constraints of (O1), where $\kappa_E(t)$ is replaced by $\check{\kappa}_E(t) - G(t)$ in formulas (23), and to constraint (19c). We can rewrite this optimization problem called (O2) in a rigorous form. Denoting U_2 the set of function $G \in L^2([0,\Theta])$ which satisfy the conditions:

$$\kappa_E^{\max} \geq \check{\kappa}_E(t) - G(t) \geq 0, \tag{53a}$$

$$(\mathcal{L}[\check{\kappa}_E - G])(t) \geq \mu(t), \tag{53b}$$

$$\int_0^{\Theta_i} (\mathcal{L}[\check{\kappa}_E - G])(t)dt \geq c_i + \int_0^{\Theta_i} \mu(t)dt, \ \forall i = 1, \ldots, N, \tag{53c}$$

where $\mu(t)$ is defined by (24). For given $\check{\kappa}_E(t)$ obtained from the optimization problem (O1), we search $\hat{G}(t)$ solution of the next optimization problem:

$$\min_{G \in U_2} F(\check{\kappa}_E, G), \tag{54}$$

and we thus obtain $\hat{\kappa}_E(t) = \check{\kappa}_E(t) - \hat{G}(t)$. The type of filter that we will use depends on the choice of F. We will investigated two types of filters: the first one is based on the thresholding method and the second on the total variation denoising (TVD).

Thresholding methode. It consists in eliminating the low and very high frequency of a signal, by writing the functional F, for all Y and G in $L^2[0,\Theta]$, as the following form:

$$F(Y, G) = \lambda_1 \int_{-\infty}^{+\infty} |G(t)|^2 \, dt + \lambda_2 \int_{-\infty}^{+\infty} \left| \mathcal{F}^{-1} \left(\mathcal{F} \left(\frac{\partial (Y - G)}{\partial t} \right) \rho_\eta \right) \right| dt, \tag{55}$$

where $\mathcal{F}$ is the Fourier transform operator and for fixed $\eta > 0$, $\rho_\eta(t) = 0$ if $|t| > \eta$. The second member of (55) collects the mid-frequency of the derivative of $(Y - G)(t)$.

Total Variation Denoising (TVD). It is a tool for noise reduction developed so as to preserve sharp edges in the underlying signal [7]. In such cases we can define the functional F as:

$$F(Y, G) = \lambda_1 \int_{-\infty}^{+\infty} |G(t)|^2 \, dt + \lambda_2 \int_{-\infty}^{+\infty} \left| \frac{\partial (Y - G)}{\partial t} \right| dt. \tag{56}$$

In (55) and (56), $\lambda_1 > 0$ and $\lambda_2 > 0$ are two regularization parameters to control the degree of smoothing. Increasing λ_2 gives more weight to the second term which measures the total variation of $(Y - G)(t)$. In the next section we will test the efficiency of each of these filters.

5.2 Numerical algorithm for filtering

In order to test the efficiency of the filters introduced in Section 5.1, we build a class of periodic signals perturbed by some oscillations as shown in Figure 1, and we study the ability of filters to eliminate this oscillations. The goal is to estimate an original signal X from a perturbed version of X called Y and modeled as:

$$Y = X + G,$$

where G is the oscillations. Having Y, TVD estimates unperturbed signal X by solving the next optimization problem. For a given perturbed signal Y, we search G solution to:

$$\min_G F(Y, G), \tag{57}$$

and we have then the unperturbed signal $X = Y - G$. We solve the last optimization problem by discretizing (55) or (56) and then using a finite-dimensional optimization algorithm. For that, we use the Majorization-Minimization (MM) algorithms developed by Figueiredo, *and al* [2].
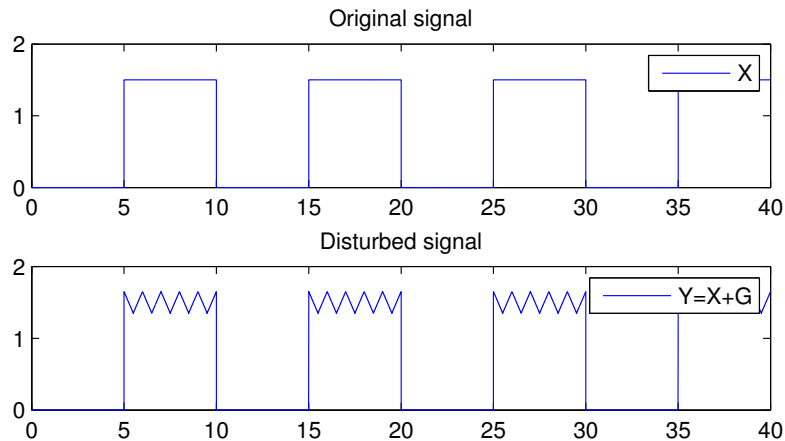
Figure 1: Type of signals used to test the filters.

As we can see from Figures 2 and 3, the filtering method based on the optimization problem (57) allows us to remove the perturbations. Thresholding method seems more effective for the type of signals that we have in Figure 1.
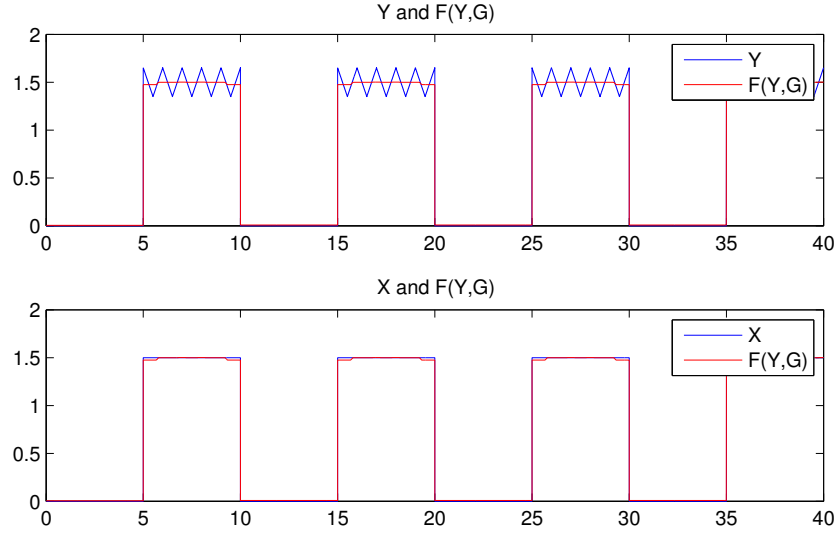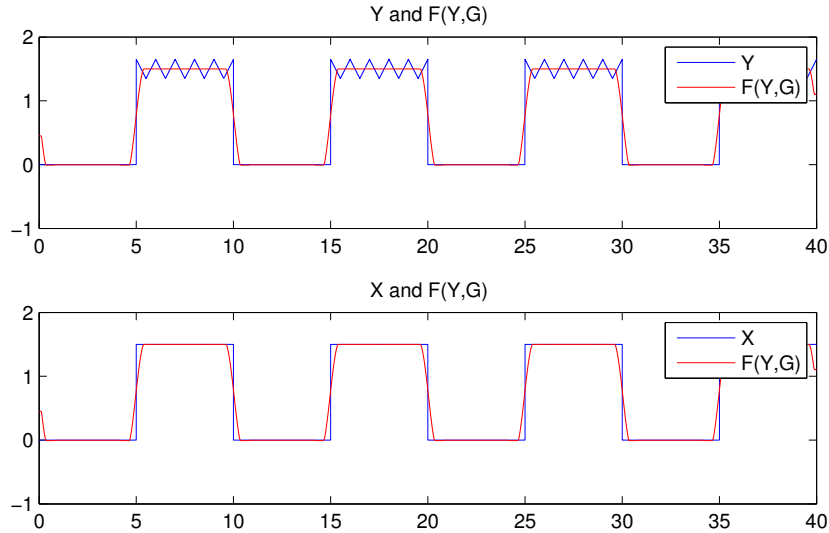
Figure 2: Filtering by using TVD.

Figure 3: Filtering by using thresholding methode.

6 Numerical simulations

We solve the optimization problem (51) by using the simplex method. The numerical simulations were performed by using the mathematical software of scientific computing Matlab. We impose the repayment pattern γ, the current spending σ_g and we seek the optimal loan κ_E which allows the objectives of the project to be realized. We check that the constraints are verified.

6.1 Example 1

In Example 1, the repayment is done in a constant way between the first and 6th year after borrowing. We pay off nothing outside this period. The current spending density σ_g alternates between positive values, corresponding to periods where income is less than spending and neg-

ative values, corresponding to periods where income is larger than spending. The objective to be achieved are fixed at the end of the 4nd and 14th year. The optimal loan density is given in Figure 4. Figure 5 shows that the constraints are satisfied. For this example we have taken:

$$\Theta = 20, \quad n = 1000, \quad \alpha = 0.01, \quad \rho_K^{\mathcal{I}} = 0,$$

and
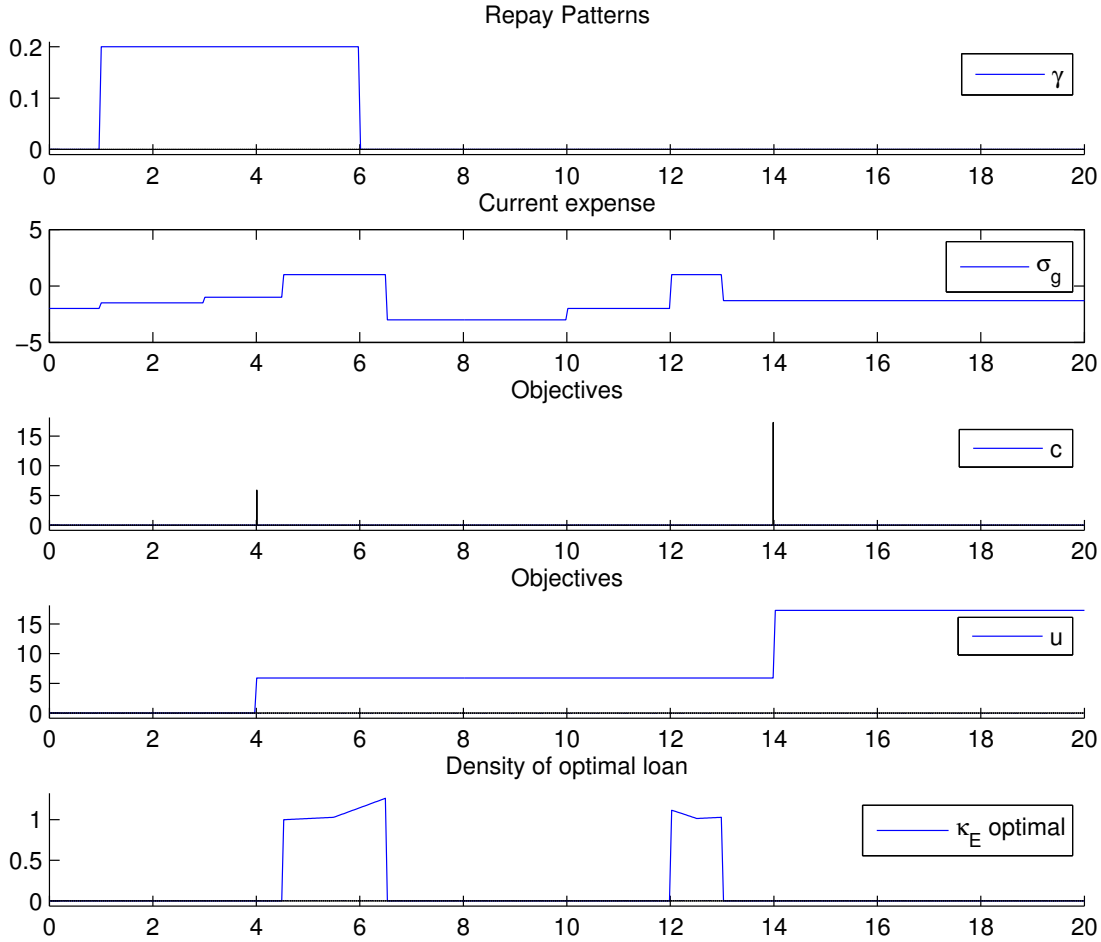
$$\Theta_1 = 4, \quad \Theta_2 = 14, \quad c_1 = 6, \quad c_2 = 17.$$

Figure 4: Example 1. Optimal loan κ_E obtained for given: repayment pattern γ, objectives c and current spending σ_g.

Figure 5 shows the isolated spending density $\beta(t)$ and the isolated spending $\int_0^{\Theta} \beta(t)dt$ obtained from Example 1. As we can see, we get $\beta(t) \geq 0$ and $\int_0^{\Theta_i} \beta(t)dt \geq c_i$, $i = 1, 2$. This means that the constraints of the optimization problem (O1) are satisfied and the project objectives are reached.
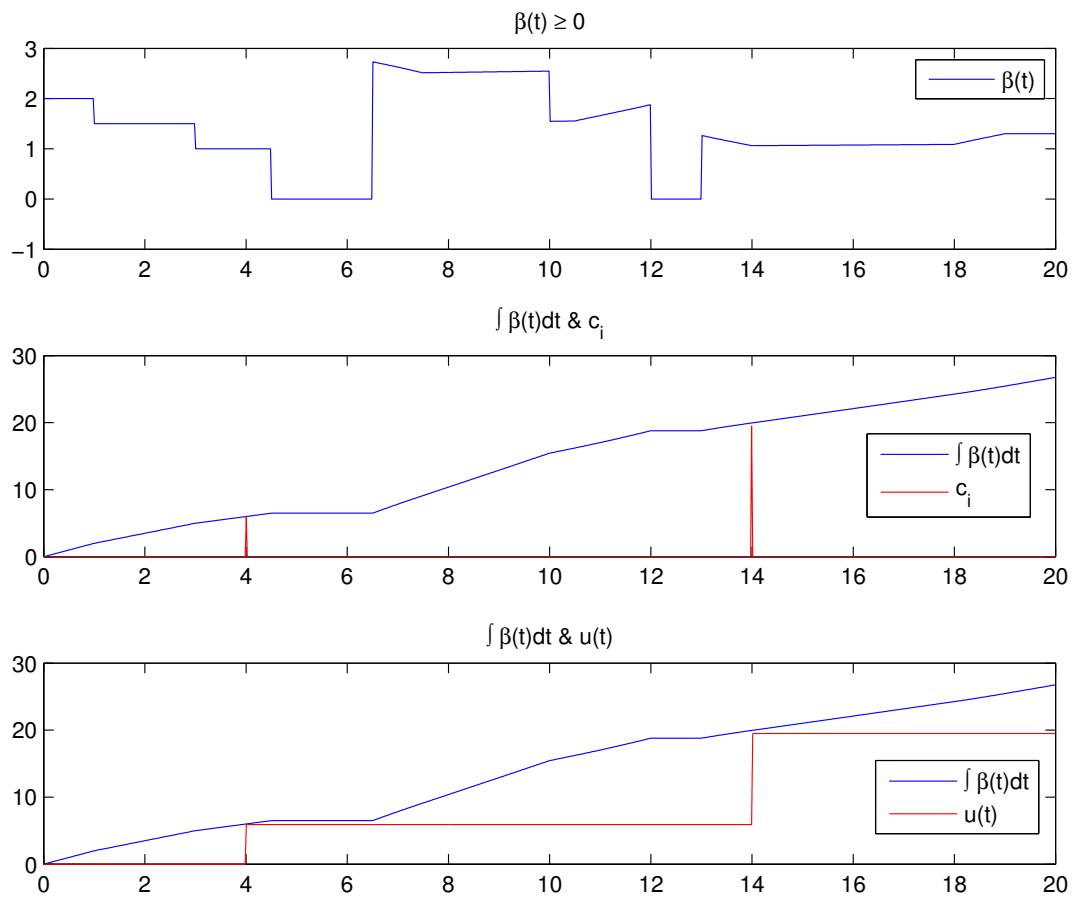
Figure 5: Checking if the constraints are verified for Example 1.

6.2 Example 2

In Example 2, the repayment is only made in an increasing way between the 7th and 11th year after borrowing. We pay off nothing outside this period. The optimal loan density is given in Figure 6. Figure 7 shows that the constraints are satisfied. For this example we have taken:

$$\Theta = 20, \quad n = 1000, \quad \alpha = 0.01, \quad \rho_K^{\mathcal{I}} = 0,$$

and

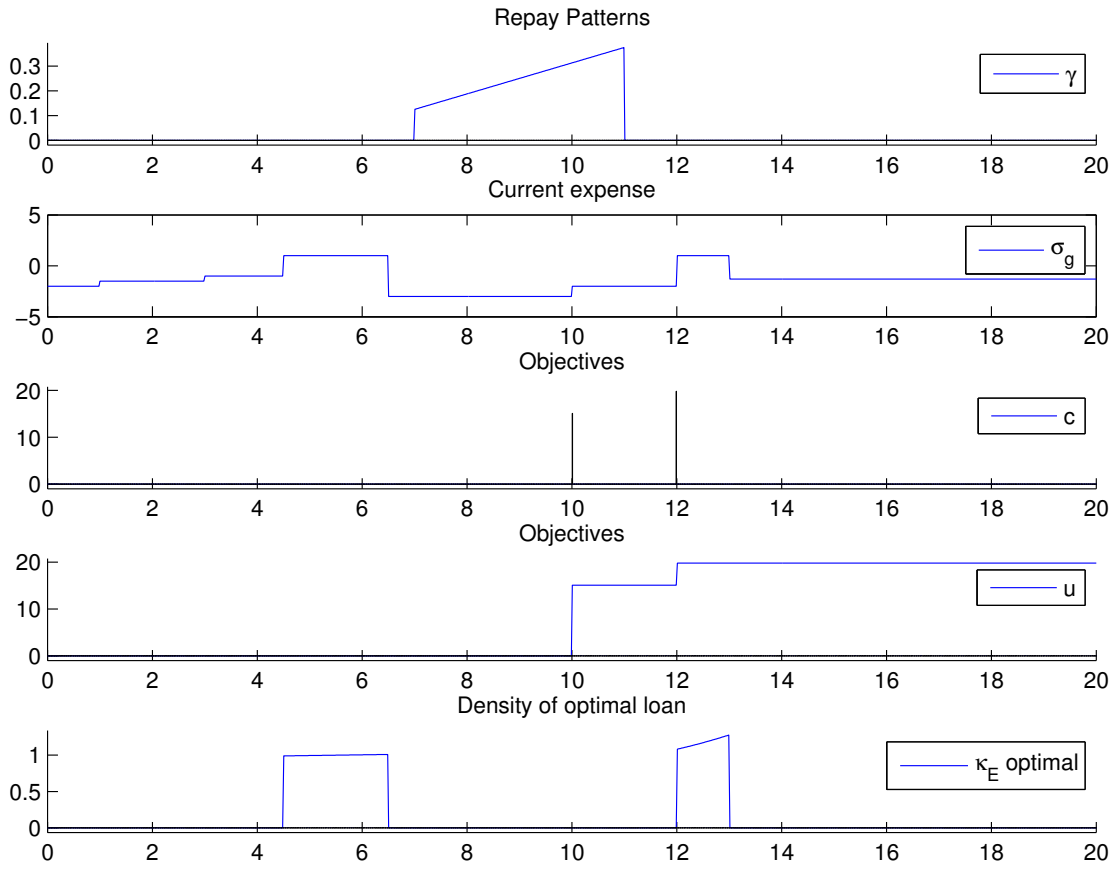$$\Theta_1 = 10, \quad \Theta_2 = 12, \quad c_1 = 15, \quad c_2 = 20.$$

Figure 6: Optimal loan κ_E obtained for given: repayment pattern γ, objectives c and current spending σ_g.

Figure 7 shows the isolated spending density $\beta(t)$ and the isolated spending $\int_0^{\Theta} \beta(t)dt$ obtained from Example 2. As we can see, we get $\beta(t) \geq 0$ and $\int_0^{\Theta_i} \beta(t)dt \geq c_i$, $i = 1, 2$. The constraints of the optimization problem are well checked.
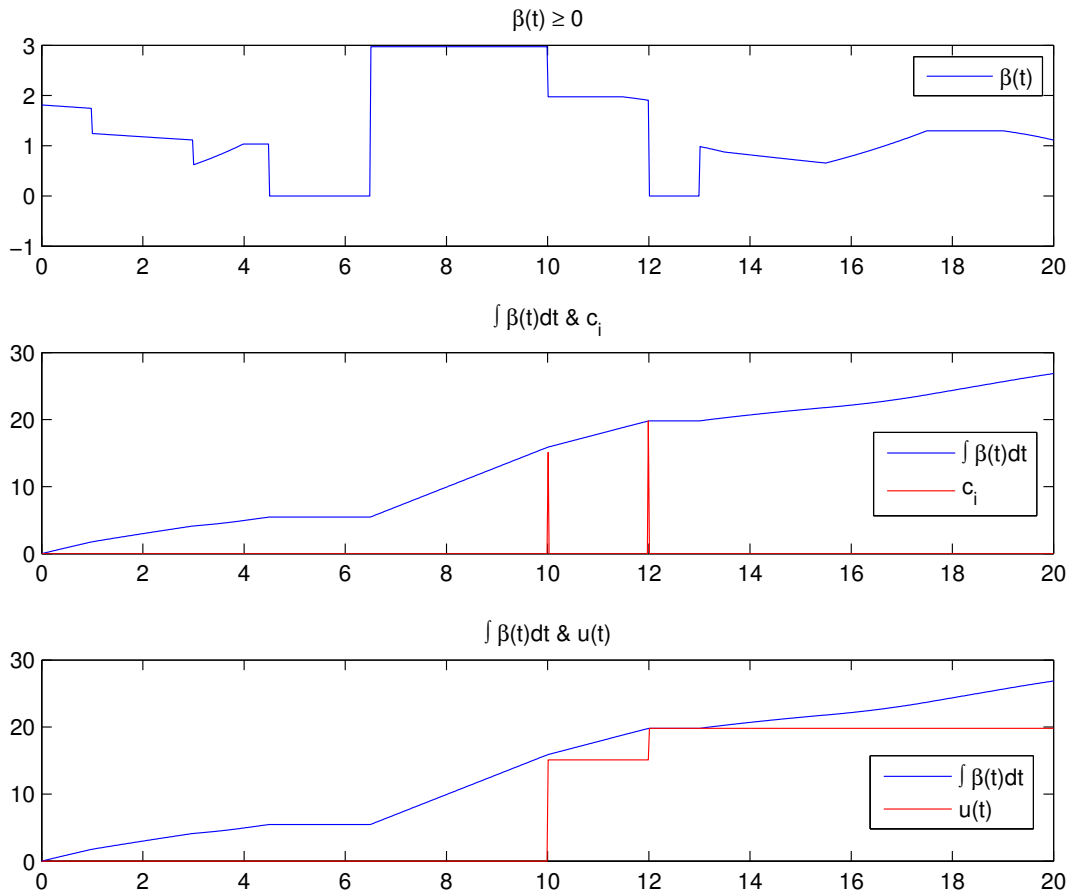
Figure 7: Checking if the constraints are verified for Example 2.

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