Hamiltonian Formalism for the Vlasov-Maxwell System

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December 6, 2010

Abstract -

Keywords -

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Part I
Towards Hamiltonian description of a charged particle

1 Departure point (in terms of scientific knowledge)

The following is taken for granted.

In a flat space filled with vacuum, the most usual way to write the Maxwell equations is:

$$-\frac{1}{c^2}\frac{\partial \mathbb{E}}{\partial t} + \nabla \times \mathbb{B} = \mu_0 \mathbb{J}, \tag{1.1}$$
$$\frac{\partial \mathbb{B}}{\partial t} + \nabla \times \mathbb{E} = 0, \tag{1.2}$$
$$\nabla \cdot \mathbb{E} = \frac{1}{\varepsilon_0}\boldsymbol{\rho}, \tag{1.3}$$
$$\nabla \cdot \mathbb{B} = 0. \tag{1.4}$$

Equation (1.1) is called "Ampere's Theorem", equation (1.2) "Faraday's Law" and (1.3) "Poisson's Equation".
ε_0 is the "Vacuum Electric Permittivity" and μ_0 the "Vacuum Magnetic Permeability". They are linked together by: $\mu_0\varepsilon_0 = 1/c^2$, where c is the "Light Velocity in the Vacuum".
System ((1.1) - (1.4)) models the time-space evolution of the "Electric Field" $\mathbb{E} = \mathbb{E}(t, \mathbf{x})$, and the "Magnetic Field" $\mathbb{B} = \mathbb{B}(t, \mathbf{x})$, seen by an observer in an inertial frame of reference. $\boldsymbol{\rho} = \boldsymbol{\rho}(t, \mathbf{x})$ stands for the "Charge Density" filling the space and $\mathbb{J} = \mathbb{J}(t, \mathbf{x})$ for the "Current Density" (seen in the inertial frame of reference).

If a particle of mass m, charge q and velocity (in the inertial frame of reference) $\mathbf{v}_0$ is situated in position $\mathbf{x}_0$ at time t_0, it feels the following "Lorentz's Force" .

$$\mathbb{F}(t_0, \mathbf{x}_0, \mathbf{v}_0) = q(\mathbb{E}(t_0, \mathbf{x}_0) + \mathbf{v}_0 \times \mathbb{B}(t_0, \mathbf{x}_0)). \tag{1.5}$$

Hence applying "Newton's Law", the considered particle follows a trajectory in the "Position-Velocity Space" $(\mathbf{X}, \mathbf{V}) = (\mathbf{X}(t), \mathbf{V}(t)) = (\mathbf{X}(t; \mathbf{x}_0, \mathbf{v}_0, t_0), \mathbf{V}(t; \mathbf{x}_0, \mathbf{v}_0, t_0))$ which is solution to:

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}, \qquad \mathbf{X}(t_0; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbf{x}_0, \tag{1.6}$$
$$m\frac{\partial \mathbf{V}}{\partial t} = \mathbb{F}(t, \mathbf{X}, \mathbf{V}), \qquad \mathbf{V}(t_0; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbf{v}_0, \tag{1.7}$$

or, in its expanded shapes:

$$\frac{\partial \mathbf{X}}{\partial t}(t) = \mathbf{V}(t), \qquad \mathbf{X}(t_0) = \mathbf{x}_0, \tag{1.8}$$
$$m\frac{\partial \mathbf{V}}{\partial t}(t) = \mathbb{F}(t, \mathbf{X}(t), \mathbf{V}(t)), \qquad \mathbf{V}(t_0) = \mathbf{v}_0, \tag{1.9}$$

or

$$\frac{\partial \mathbf{X}}{\partial t}(t; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbf{V}(t; \mathbf{x}_0, \mathbf{v}_0, t_0), \qquad \mathbf{X}(t_0; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbf{x}_0, \tag{1.10}$$
$$m\frac{\partial \mathbf{V}}{\partial t}(t; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbb{F}(t, \mathbf{X}(t; \mathbf{x}_0, \mathbf{v}_0, t_0), \mathbf{V}(t; \mathbf{x}_0, \mathbf{v}_0, t_0)), \qquad \mathbf{V}(t_0; \mathbf{x}_0, \mathbf{v}_0, t_0) = \mathbf{v}_0. \tag{1.11}$$

2 Status of mathematical objects in game I

Above, all the mathematical objects in game are the same. They are scalar or vector fields defined on the position space $\mathcal{X}$, which is a subset of $\mathbb{R}^N$ or $\mathbb{R}^N$ itself, with $N = 3$ (or may be 2 or 1 in simplified models that will no be considered in the near sequel).
In view of generalizing the point of view, it is clever to question on their intrinsic status. For this, first, a weakened version of Newton's Law saying: "mass times acceleration of a particle equals the force acting on it" has to be considered.

2.1 D'Alembert's version of Newton's Law

This version which is sometimes called "D'Alembert's Principle" consists in considering in any point $\mathbf{x}$ of $\mathcal{X}$ all the "Virtual Displacements" or "Admissible Displacements" which are the elements of the tangent space $\mathbf{T_x}\mathcal{X}$ of $\mathcal{X}$ in $\mathbf{x}$. Here, in any point $\mathbf{x}$, $\mathbf{T_x}\mathcal{X}$ and $\mathcal{X}$ may be considered as the same space.
It says: in any point $\mathbf{x}$ of $\mathcal{X}$ such that, for a given time s, $\mathbf{X}(s;\mathbf{x}_0,\mathbf{v}_0,t_0) = \mathbf{x}$, for any Admissible Displacement ν of $\mathbf{T_x}\mathcal{X}$ the "Virtual Work" of the force $\mathbb{F}$ which writes $\mathbb{F}(t,\mathbf{X}(s),\mathbf{V}(s))\cdot\nu$ equals the "Virtual Work" of the inertial force $m\dfrac{\partial \mathbf{V}}{\partial t}$ which writes $m\dfrac{\partial \mathbf{V}}{\partial t}(s)\cdot\nu$
Defining the "Momentum" $\mathbb{M} = \mathbb{M}(t) = \mathbb{M}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$ associated with the trajectory $(\mathbf{X}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$, $\mathbf{V}(t;\mathbf{x}_0,\mathbf{v}_0,t_0))$, by

$$\mathbb{M}(t;\mathbf{x}_0,\mathbf{v}_0,t_0) = m\mathbf{V}(t;\mathbf{x}_0,\mathbf{v}_0,t_0) \tag{2.1}$$

"D'Alembert's Principle" writes :

Principle 2.1 *In any point* $\mathbf{x}$ *of* $\mathcal{X}$ *such that, for a given time* s, $\mathbf{X}(s;\mathbf{x}_0,\mathbf{v}_0,t_0) = \mathbf{x}$, *the following equality holds for any* ν *of* $\mathbf{T_x}\mathcal{X}$,

$$\Big(\mathbb{F}(s,\mathbf{X}(s),\mathbf{V}(s)) - \frac{\partial \mathbb{M}}{\partial t}(s)\Big)\cdot\nu = 0. \tag{2.2}$$

Moreover, this equality characterizes trajectory $\mathbf{X}(.)$. ∎

Remark 2.2 Defining $\mathbf{m}_0$ as $m\mathbf{v}_0$, it is relevant to use the following notation $\mathbb{M}(t;\mathbf{x}_0,\mathbf{m}_0,t_0)$ in place of $\mathbb{M}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$. From now, this notation will be used. ∎

2.2 Status of the force $\mathbb{F}$ and of the momentum $\mathbb{M}$ in a given point x of $\mathbf{T_x}\mathcal{X}$

Once (2.2) is set, the force $\mathbb{F}(s,\mathbf{X}(s),\mathbf{V}(s))$ and the inertial force $\dfrac{\partial \mathbb{M}}{\partial t}(s)$ may be seen as objects acting on elements in $\mathbf{T_x}\mathcal{X}$ via an inner product (here denoted by "$\cdot$"). Hence, intrinsically they they can be represented by elements denoted $\mathbf{F}(s,\mathbf{X}(s),\mathbf{V}(s))$ and $\dfrac{\partial \tilde{\mathbf{M}}}{\partial t}(s)$ of the cotangent space $\mathbf{T^*_x}\mathcal{X}$ of $\mathcal{X}$ in $\mathbf{x}$. This allows to avoid the inner product. Since $\dfrac{\partial \tilde{\mathbf{M}}}{\partial t}(s)$ is in $\mathbf{T^*_x}\mathcal{X}$, $\tilde{\mathbf{M}}$ has to be in a manifold whose tangent space is $\mathbf{T^*_x}\mathcal{X}$, then it is not irrelevant to consider that $\tilde{\mathbf{M}}$ is also in $\mathbf{T^*_x}\mathcal{X}$.

As a conclusion : In a point $\mathbf{x}$ of $\mathcal{X}$ such that, for a given time s, $\mathbf{X}(s;\mathbf{x}_0,\mathbf{v}_0,t_0) = \mathbf{x}$ the force $\mathbf{F}(s,\mathbf{X}(s;\mathbf{x}_0,\mathbf{v}_0,t_0),\mathbf{V}(s;\mathbf{x}_0,\mathbf{v}_0,t_0))$ and of the momentum $\tilde{\mathbf{M}}(s;\mathbf{x}_0,\mathbf{v}_0,t_0)$ may be considered as elements of the cotangent space $\mathbf{T^*_x}\mathcal{X}$.

Once set this conclusion, the question of the definition of the momentum is asked. It has to be defined intrinsically and (2.1) has to be an expression of this intrinsic definition within a coordinates system inside which the inner product has its usual expression. Hence, a mapping $\mathcal{M}_{\mathbf{x}}$ from $\mathbf{T_x}\mathcal{X}$ to $\mathbf{T^*_x}\mathcal{X}$ has to be built.
A way to build this mapping consists, for any $\mathbf{v}$ in $\mathbf{T_x}\mathcal{X}$, in defining $\mathcal{M}_{\mathbf{x}}(\mathbf{v})$ as the unique linear form or 1-form such as $\langle \mathcal{M}_{\mathbf{x}}(\mathbf{v}), \mathbf{v}/|\mathbf{v}|\rangle = \sup\{\langle \mathcal{M}_{\mathbf{x}}(\mathbf{v}), \nu\rangle, \nu \in \mathbf{T_x}\mathcal{X}\}$ and $\langle \mathcal{M}_{\mathbf{x}}(\mathbf{v}), \mathbf{v}\rangle = m|\mathbf{v}|^2$.
Another way, which is equivalent, consists in considering the differentiable function $\bar{L}_{\mathbf{x}} : \mathbf{T_x}\mathcal{X} \to \mathbb{R}$ defined by

$$\bar{L}_{\mathbf{x}}(\mathbf{v}) = \frac{1}{2} m|\mathbf{v}|^2, \tag{2.3}$$

and in saying that $\mathcal{M}_{\mathbf{x}}(\mathbf{v})$ is the differential of $\bar{L}_{\mathbf{x}}$ in $\mathbf{v}$, i.e.

$$\mathcal{M}_{\mathbf{x}}(\mathbf{v}) = d_{\mathbf{v}} \bar{L}_{\mathbf{x}}. \tag{2.4}$$

(Since $\bar{L}_{\mathbf{x}} : \mathbf{T_x}\mathcal{X} \to \mathbb{R}$, $d_{\mathbf{v}}\bar{L}_{\mathbf{x}} : \mathbf{T_v}(\mathbf{T_x}\mathcal{X}) \to \mathbb{R}$ and since $\mathbf{T_x}\mathcal{X}$ is a vector space, $\mathbf{T_v}(\mathbf{T_x}\mathcal{X})$ may be identified with $\mathbf{T_x}\mathcal{X}$ for any $\mathbf{v}$ in $\mathbf{T_x}\mathcal{X}$.)

Remark 2.3 Notice that defining such a one-to-one linear application $\mathcal{M}_{\mathbf{x}}$ from $\mathbf{T_x}\mathcal{X}$ onto $\mathbf{T^*_x}\mathcal{X}$ in any $\mathbf{x} \in \mathcal{X}$, in a way regular enough with respect to $\mathbf{x}$ and such that $\langle \mathcal{M}_{\mathbf{x}}(\nu), \nu'\rangle = \langle \mathcal{M}_{\mathbf{x}}(\nu'), \nu\rangle$ for any $\nu \in \mathbf{T_x}\mathcal{X}$ and any $\nu' \in \mathbf{T_x}\mathcal{X}$ for any $\mathbf{x} \in \mathcal{X}$, defines a metric on $\mathcal{X}$ by considering $\langle \mathcal{M}_{\mathbf{x}}(\nu), \nu'\rangle$ as the inner product of ν by ν'. ■

With this mapping at hand, the momentum $\tilde{\mathbf{M}}(t; \mathbf{x}_0, \mathbf{m}_0, t_0)$ associated with trajectory $(\mathbf{X}(t; \mathbf{x}_0, \mathbf{v}_0, t_0),$ $\mathbf{V}(t; \mathbf{x}_0, \mathbf{v}_0, t_0))$ is defined as

$$\tilde{\mathbf{M}}(t; \mathbf{x}_0, \mathbf{m}_0, t_0) = \mathcal{M}_{\mathbf{X}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)}(\mathbf{V}(t; \mathbf{x}_0, \mathbf{v}_0, t_0)). \tag{2.5}$$

Remark 2.4 If in place of classical mechanics, the framework is special theory of relativity, then (2.1) is replaced by:

$$\mathbb{M} = \left(1 - \frac{|\mathbf{V}|^2}{c^2}\right)^{-1/2} m\mathbf{V}. \tag{2.6}$$

Using as function $\bar{L}_{\mathbf{x}} : \mathbf{T_x}\mathcal{X} \mapsto \mathbb{R}$: $\bar{L}_{\mathbf{x}}(\mathbf{v}) = mc^2\left(1 - \sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}\right)$, setting again $\mathcal{M}_{\mathbf{x}}(\mathbf{v}) = d_{\mathbf{v}}\bar{L}_{\mathbf{x}}$, the momentum $\mathbb{M}$ defined by (2.6) is associated with the following element of $\mathbf{T^*_x}\mathcal{X}$: $\tilde{\mathbf{M}} = \mathcal{M}_{\mathbf{X}}(\mathbf{V})$. ■

Remark 2.5 Above, if, in place of choosing $\bar{L}_{\mathbf{x}}(\mathbf{v}) = \frac{1}{2}m|\mathbf{v}|^2$, for any $\phi(\mathbf{x})$ depending only on the position variable $\bar{L}_{\mathbf{x}}(\mathbf{v}) = \frac{1}{2}m|\mathbf{v}|^2 - \phi(\mathbf{x})$ is chosen, $d_{\mathbf{v}}\bar{L}_{\mathbf{x}}$ is left unchanged and then setting again $\mathcal{M}_{\mathbf{x}}(\mathbf{v}) = d_{\mathbf{v}}\bar{L}_{\mathbf{x}}$ the definition of the momentum $\tilde{\mathbf{M}} = \mathcal{M}_{\mathbf{X}}(\mathbf{V})$ is valid again. ■

2.3 Status of the force F and of the momentum $\tilde{\mathbf{M}}$

Firstly, reinterpreting the Lorentz's Force expression (1.5), it may be deduced the following: From the fields $\mathbb{E}$ and $\mathbb{B}$ defined on $\mathcal{X}$ at any time t, for any vector field $\mathbf{W}$ defined on $\mathcal{X}$, at any time t, a Lorentz's Force Field is defined on $\mathcal{X}$ by

$$\mathbb{F}(t, \mathbf{x}, \mathbf{W}) = q(\mathbb{E}(t, \mathbf{x}) + \mathbf{W} \times \mathbb{B}(t, \mathbf{x})), \tag{2.7}$$

for any $\mathbf{x}$ in $\mathcal{X}$.

Remark 2.6 As $\mathbf{W}$ is here a vector field, it would be more correct to use an expanded notation $\mathbf{W}(\mathbf{x})$ in expression (2.7) to give:

$$\mathbb{F}(t,\mathbf{x},\mathbf{W}(\mathbf{x})) = q(\mathbb{E}(t,\mathbf{x}) + \mathbf{W}(\mathbf{x}) \times \mathbb{B}(t,\mathbf{x})). \tag{2.8}$$

■

As a conclusion : The force felt by a particle of charge q being in $\mathbf{x}_0$ with velocity $\mathbf{v}_0$ is then the value $\mathbb{F}(t,\mathbf{x}_0,\mathbf{W}(\mathbf{x}_0))$ of $\mathbb{F}$ in $\mathbf{x}_0$, where the force field $\mathbb{F}(t,\mathbf{x},\mathbf{W})$ is computed with a vector field $\mathbf{W}$ which is such that $\mathbf{W}(\mathbf{x}_0) = \mathbf{v}_0$.
Secondly, in view of the conclusion of subsection 2.2, in any point $\mathbf{x}$ of $\mathcal{X}$, $\mathbb{F}$ and $\mathbb{M}$ may be represented by linear forms or 1-forms on $\mathbf{T_x}\mathcal{X}$.

As a conclusion : Hence, for any time, the two vector fields $\mathbb{F}(t,.,.)$ and $\mathbb{M}(t)$ defined on $\mathcal{X}$ may be, more relevantly, represented by two differential 1-forms on $\mathcal{X}$: $\mathbf{F}(t)$ and $\tilde{\mathbf{M}}(t)$.

In the sequel, for any $n \leq N$, for a time-dependent differential n-form $\mathbf{K}(t)$ defined on $\mathcal{X}$, in any $\mathbf{x}$ of $\mathcal{X}$, $\{\mathbf{K}(t,\mathbf{x})\}$ will denote the n-form $(\mathbf{T_x}\mathcal{X})^n \mapsto \mathbb{R}$ which is the value of the differential n-form at $\mathbf{x}$.

2.4 Status of the electric field

In view of Lorentz's Force expression (2.7), it seems to be reasonable to consider the force and the Electric Field to be represented by objects of the same nature.

As a conclusion : Hence the Electric Field $\mathbb{E}(t,.)$ will be represented by a differential 1-form $\mathbf{E}(t)$ on $\mathcal{X}$.
(As a consequence, the charge q have the status of a number or of a linear operator from $\mathbf{T_x}\mathcal{X}$ to $\mathbf{T_x}\mathcal{X}$ in every $\mathbf{x}$ of $\mathcal{X}$.)

2.5 Status of the magnetic field

If, in any $\mathbf{x}$ of $\mathcal{X}$ the tangent space $\mathbf{T_x}\mathcal{X}$ is equipped with an inner product "$\cdot$" and a cross product "$\times$", any differential 2-form (possibly time-dependant) $\mathbf{B}(t)$ defined on $\mathcal{X}$ can be represented by the vector field $\mathbb{B}(t,.)$ on $\mathcal{X}$ whose value in any $\mathbf{x}$ of $\mathcal{X}$ is such that

$$\{\mathbf{B}(t,\mathbf{x})\}(\nu,\nu') = \mathbb{B}(t,\mathbf{x}) \cdot (\nu \times \nu') = (\mathbb{B}(t,\mathbf{x}) \times \nu) \cdot \nu' = (\nu' \times \mathbb{B}(t,\mathbf{x})) \cdot \nu, \quad \text{for any } \nu \text{ and } \nu' \text{ in } \mathbf{T_x}\mathcal{X}. \tag{2.9}$$

Moreover, for any vector field $\mathbf{W}$ defined on $\mathcal{X}$, the interior product $i_{\mathbf{W}}\mathbf{B}(t)$ of $\mathbf{B}(t)$ by $\mathbf{W}$ and which results as the differential 1-form whose value in any $\mathbf{x}$ of $\mathcal{X}$ is:

$$i_{\mathbf{W}}\mathbf{B}(t,\mathbf{x}) : \nu \mapsto \{\mathbf{B}(t,\mathbf{x})\}(\mathbf{W}(\mathbf{x}),\nu), \tag{2.10}$$

for any ν in $\mathbf{T_x}\mathcal{X}$, is represented by the vector field $\mathbb{B}(t,.) \times \mathbf{W} = -\mathbf{W} \times \mathbb{B}(t,.)$ (whose value in a point $\mathbf{x}$ of $\mathcal{X}$ is $-\mathbf{W}(\mathbf{x}) \times \mathbb{B}(t,\mathbf{x})$).

As a conclusion : Using these remarks, the Magnetic Field $\mathbb{B}(t,.)$ is represented by the differential 2-form $\mathbf{B}(t)$. $\mathbf{B}(t)$ and $\mathbb{B}(t,.)$ are linked by (2.9).

Reinterpreting, once again, the expression of the Lorentz's Force within the framework of differential forms, it may be deduced the following: From the differential 1-form $\mathbf{E}(t)$ and the differential 2-form

$\mathbf{B}(t)$ defined on $\mathcal{X}$ at any time t, for any vector field $\mathbf{W}$ defined on $\mathcal{X}$, at any time t, a Lorentz's Force differential 1-Form is defined on $\mathcal{X}$ by

$$\mathbf{F}(t) = q(\mathbf{E}(t) - \dot{\iota}_{\mathbf{W}}\mathbf{B}(t)). \tag{2.11}$$

As a conclusion: The object permitting to compute, in any $\mathbf{x}_0$ of $\mathcal{X}$, the Virtual Work of the Lorentz's Force for any Admissible Displacement of a particle of mass m being in $\mathbf{x}_0$ with velocity $\mathbf{v}_0$ is then the value $\{\mathbf{F}(t,\mathbf{x}_0)\}$ where $\mathbf{F}$ is defined by (2.11) with a vector field $\mathbf{W}$ which is such that $\mathbf{W}(\mathbf{x}_0) = \mathbf{v}_0$.

2.6 On Lorentz's Force writing

The first way to write Lorentz's Force is the one presented via formula (2.11), lines above it and interior product definition (2.10). There is another way to define the interior product, more complicated, but useful in the sequel and involving the tangent bundle $\mathbf{T}\mathcal{X}$ which is defined, at any time t, by $\mathbf{T}\mathcal{X} = \cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x}, \mathbf{T}_{\mathbf{x}}\mathcal{X})$.

First, having a time-dependent differential 1-form $\mathbf{A}(t)$ defined on $\mathcal{X}$, a function or differential 0-form $\mathring{\iota}\mathbf{A}(t)$ may be defined on $\mathbf{T}\mathcal{X}$ by setting

$$\mathring{\iota}\mathbf{A}(t,\mathbf{x},\mathbf{v}) = \mathring{\iota}_{\mathbf{v}}\mathbf{A}(t,\mathbf{x}) = \{\mathbf{A}(t,\mathbf{x})\}(\mathbf{v}). \tag{2.12}$$

More generally, if $\mathbf{K}(t)$ is a time-dependent differential n-form defined on $\mathcal{X}$, time-dependant differential $(n-1)$-form $\mathring{\iota}\mathbf{K}(t)$ may be defined on $\mathbf{T}\mathcal{X}$. To do this, it has to be noticed that, in any point $(\mathbf{x},\mathbf{v})$ of $\mathbf{T}\mathcal{X}$, tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be identified with $\mathbf{T}_{\mathbf{x}}\mathcal{X} \times \mathbf{T}_{\mathbf{x}}\mathcal{X}$, and making such an identification leads to the fact that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be written as (ν,υ) with $\nu \in \mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\upsilon \in \mathbf{T}_{\mathbf{x}}\mathcal{X}$. Then, $\mathring{\iota}\mathbf{K}(t)$ is defined by setting, for any $((\nu_1,\upsilon_1),\dots,(\nu_{n-1},\upsilon_{n-1}))$ in $\big(\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)\big)^{n-1}$,

$$\begin{aligned}\{\mathring{\iota}\mathbf{K}(t,\mathbf{x},\mathbf{v})\}((\nu_1,\upsilon_1),\dots,(\nu_{n-1},\upsilon_{n-1})) &= \{\mathring{\iota}_{\mathbf{v}}\mathbf{K}(t,\mathbf{x})\}((\nu_1,\upsilon_1),\dots,(\nu_{n-1},\upsilon_{n-1}))\\ &= \{\mathbf{K}(t,\mathbf{x})\}(\mathbf{v},\nu_1,\dots,\nu_{n-1}).\end{aligned} \tag{2.13}$$

Beside this, if $\mathcal{K}(t)$ is a differential n-form on $\mathbf{T}\mathcal{X}$, for any $(\mathbf{x},\mathbf{v})$ of $\mathbf{T}\mathcal{X}$ it defines a n-form on $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ $(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)\mathcal{K}$

$$\{(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)\mathcal{K}(t)\}(\nu_1,\dots,\nu_n) = \{\mathcal{K}(t,\mathbf{x},\mathbf{v})\}((\nu_1,0),\dots,(\nu_n,0)), \tag{2.14}$$

for any $(\nu_1,\dots,\nu_n)$ in $(\mathbf{T}_{\mathbf{x}}\mathcal{X})^n$.

Then, the interior product $\dot{\iota}_{\mathbf{W}}\mathbf{B}(t)$ may be defined using

$$\dot{\iota}_{\mathbf{W}}\mathbf{B}(t,\mathbf{x}) = \big(\mathcal{D}_{(\mathbf{x},\mathbf{W}(\mathbf{x}))}\Pi\big)\big(\mathring{\iota}\mathbf{B}\big). \tag{2.15}$$

and Lorentz's Force may be written as

$$\mathbf{F}(t,\mathbf{x}) = q\Big(\mathbf{E}(t) - \big(\mathcal{D}_{(\mathbf{x},\mathbf{W}(\mathbf{x}))}\Pi\big)\big(\mathring{\iota}\mathbf{B}\big)\Big). \tag{2.16}$$

The interest of this shape is that it enables to write the force $\mathbf{F}(t,\mathbf{X}(t))$ in a given point of the trajectory $\mathbf{X}(t)$, as an element of $\mathbf{T}^*_{\mathbf{X}(t)}\mathcal{X}$ (considering the Position-Velocity trajectory $(\mathbf{X}(t),\mathbf{V}(t))$) by

$$\mathbf{F}(t,\mathbf{X}(t)) = q\Big(\mathbf{E}(\mathbf{X}(t)) - \big(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi\big)\big(\mathring{\iota}\mathbf{B}\big)\Big). \tag{2.17}$$

without introducing any vector field $\mathbf{W}$.

2.7 Remark on notation, Pushforward and Pullback

The goal of this subsection is to justify the notation $(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)\mathcal{K}(t)$ used in formula (2.14) and after. Having the tangent bundle $\mathbf{T}\mathcal{X}$, the following projection may be defined:

$$\begin{array}{rccl} \Pi: & \mathbf{T}\mathcal{X} & \to & \mathcal{X} \\ & (\mathbf{x},\mathbf{v}) & \mapsto & \mathbf{x}\ . \end{array} \tag{2.18}$$

Once this is done, in any $(\mathbf{x},\mathbf{v})$ of $\mathbf{T}\mathcal{X}$, the differential $d_{(\mathbf{x},\mathbf{v})}\Pi$ of Π is defined by

$$\begin{array}{rccl} d_{(\mathbf{x},\mathbf{v})}\Pi: & \mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big) & \to & \mathbf{T}_{\mathbf{x}}\mathcal{X} \\ & (\nu,\upsilon) & \mapsto & \nu\ , \end{array} \tag{2.19}$$

since, as already noticed, $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be identified with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}$, meaning that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be written as (ν,υ) with $\nu\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\upsilon\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$. The following application

$$\begin{array}{rccl} \Pi_{*(\mathbf{x},\mathbf{v})}: & \mathbf{T}^*_{\mathbf{x}}\mathcal{X} & \to & \mathbf{T}^*_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big) \\ & \pi & \mapsto & \Pi_{*(\mathbf{x},\mathbf{v})}(\pi)\ , \end{array} \tag{2.20}$$

may also be defined by setting, for any (ν,υ) in $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$,

$$\langle\Pi_{*(\mathbf{x},\mathbf{v})}(\pi),(\nu,\upsilon)\rangle=\langle\pi,d_{(\mathbf{x},\mathbf{v})}\Pi((\nu,\upsilon))\rangle=\langle\pi,\nu\rangle. \tag{2.21}$$

It is clear that this application is one-to-one onto its image $\Pi_{*(\mathbf{x},\mathbf{v})}(\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$. It may be noticed that

$$\ker\big(d_{(\mathbf{x},\mathbf{v})}\Pi\big)=\big\{(\nu,\upsilon)\in\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big),\nu=0\big\}=\big\{(0,\upsilon),\upsilon\in\mathbf{T}_{\mathbf{x}}\mathcal{X}\big\}, \tag{2.22}$$

$$\Pi_{*(\mathbf{x},\mathbf{v})}(\mathbf{T}^*_{\mathbf{x}}\mathcal{X})=\big\{\pi\in\mathbf{T}^*_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big),\forall(\nu,\upsilon)\in\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big),\langle\pi,(\nu,\upsilon)\rangle=\langle\pi,(\nu,0)\rangle\big\}, \tag{2.23}$$

and consequently, that

$$\pi\in\Pi_{*(\mathbf{x},\mathbf{v})}(\mathbf{T}^*_{\mathbf{x}}\mathcal{X})\text{ if and only if }\langle\pi,(\nu,\upsilon)\rangle=0,\forall(\nu,\upsilon)\in\ker\big(d_{(\mathbf{x},\mathbf{v})}\Pi\big), \tag{2.24}$$

$$(\nu,\upsilon)\in\ker\big(d_{(\mathbf{x},\mathbf{v})}\Pi\big)\text{ if and only if }\langle\pi,(\nu,\upsilon)\rangle=0,\forall\pi\in\Pi_{*(\mathbf{x},\mathbf{v})}(\mathbf{T}^*_{\mathbf{x}}\mathcal{X}). \tag{2.25}$$

Then, as there exits a natural projection

$$\begin{array}{rccl} \overline{\mathcal{D}}^{\perp}_{(\mathbf{x},\mathbf{v})}\Pi: & \mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big) & \to & \ker\big(d_{(\mathbf{x},\mathbf{v})}\Pi\big) \\ & (\nu,\upsilon) & \mapsto & (0,\upsilon), \end{array} \tag{2.26}$$

a second one may be defined:

$$\begin{array}{rccl} \overline{\mathcal{D}}_{(\mathbf{x},\mathbf{v})}\Pi: & \mathbf{T}^*_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big) & \to & \Pi_{*(\mathbf{x},\mathbf{v})}(\mathbf{T}^*_{\mathbf{x}}\mathcal{X}) \\ & \pi & \mapsto & \pi-\pi\circ\overline{\mathcal{D}}^{\perp}_{(\mathbf{x},\mathbf{v})}\ . \end{array} \tag{2.27}$$

In other words, $(\overline{\mathcal{D}}_{(\mathbf{x},\mathbf{v})}\Pi)(\pi)$ is defined by $\langle(\overline{\mathcal{D}}_{(\mathbf{x},\mathbf{v})}\Pi)(\pi),(\nu,\upsilon)\rangle=\langle\pi,(\nu,0)\rangle$ for any (ν,υ) in $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$.
Having $(\overline{\mathcal{D}}_{(\mathbf{x},\mathbf{v})}\Pi)$ on hand, it is easy to see that $(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)$ defined by (2.14), when it applies on 1-forms, is

$$\begin{array}{rccl} \mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi: & \mathbf{T}^*_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big) & \to & \mathbf{T}^*_{\mathbf{x}}\mathcal{X} \\ & \pi & \mapsto & \big((\Pi_{*(\mathbf{x},\mathbf{v})})^{-1}\circ(\overline{\mathcal{D}}_{(\mathbf{x},\mathbf{v})}\Pi)\big)\pi. \end{array} \tag{2.28}$$

Indication : $(d_{(\mathbf{x},\mathbf{v})}\Pi)(\nu,\upsilon)$ is called the "Pushforward" of (ν,υ) by Π in $(\mathbf{x},\mathbf{v})$ and $\Pi_{*(\mathbf{x},\mathbf{v})}(\pi)$ the "Pullback" of π by Π in $(\mathbf{x},\mathbf{v})$.

2.8 Faraday's Law from differential form point of view

As, for any $\mathbf{x}$ of $\mathcal{X}$, $\{\mathbf{B}(t,\mathbf{x})\}$ is a 2-form on $\mathbf{T_x}\mathcal{X}$, $\left\{\frac{\partial \mathbf{B}}{\partial t}(t,\mathbf{x})\right\}$ is also a 2-form on $\mathbf{T_x}\mathcal{X}$. Hence, if the differential 2-form $\mathbf{B}$ is regular enough, $\frac{\partial \mathbf{B}}{\partial t}(t)$ is also a differential 2-form on $\mathcal{X}$.
Beside this, if $\mathbf{E}$ is the differential 1-form represented by the vector field $\mathbb{E}$ (or, in other words, if in any $\mathbf{x}$ of $\mathcal{X}$, the 1-form $\{\mathbf{E}(t,\mathbf{x})\}$ on $\mathbf{T_x}\mathcal{X}$, which is equipped with an inner product "$\cdot$" and a cross product "$\times$", writes

$$\{\mathbf{E}(t,\mathbf{x})\} = (\mathbb{E}(t,\mathbf{x}))\cdot\,, \tag{2.29}$$

then the exterior derivative $d\mathbf{E}$ of $\mathbf{E}$ (which is a differential 2-form if $\mathbf{E}$ is regular enough) is represented by the vector field $\nabla \times \mathbb{E}$. (This means that, in any $\mathbf{x}$ of $\mathcal{X}$, the following formula holds: $\{d_{\mathbf{x}}\mathbf{E}(t)\}(\nu,\nu') = (\nabla\times\mathbb{E})(t,\mathbf{x})\cdot(\nu\times\nu') = ((\nabla\times\mathbb{E})(t,\mathbf{x})\times\nu)\cdot\nu'$ for any ν and ν' in $\mathbf{T_x}\mathcal{X}$.)
As a consequence of what is just said, Faraday's Law (1.2) may be seen as the following differential 2-form equality:

$$\frac{\partial \mathbf{B}}{\partial t} + d\mathbf{E} = 0. \tag{2.30}$$

2.9 On Ampere's Theorem compatibility with differential form viewpoint

As, for any $\mathbf{x}$ of $\mathcal{X}$, $\{\mathbf{E}(t,\mathbf{x})\}$ is a 1-form on $\mathbf{T_x}\mathcal{X}$, or equivalently is in $\mathbf{T^*_x}\mathcal{X}$, $\left\{\frac{\partial \mathbf{E}}{\partial t}(t,\mathbf{x})\right\}$ is also in $\mathbf{T^*_x}\mathcal{X}$. Another way to see this consists in noticing that naturally, $(\mathbf{x},\{\mathbf{E}(t,\mathbf{x})\})$ is in the cotangent bundle $\mathbf{T}^*\mathcal{X}$. (By the way, $\mathbf{T}^*\mathcal{X} = \cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x},\mathbf{T^*_x}\mathcal{X})$ has a manifold structure.) Then $\frac{\partial(\mathbf{x},\{\mathbf{E}(t,\mathbf{x})\})}{\partial t} = \left(\frac{\partial \mathbf{x}}{\partial t},\frac{\partial\{\mathbf{E}(t,\mathbf{x})\}}{\partial t}\right) = \left(0,\left\{\frac{\partial \mathbf{E}(t,\mathbf{x})}{\partial t}\right\}\right)$ is in $\mathbf{T}_{(\mathbf{x},\{\mathbf{E}(t,\mathbf{x})\})}\big(\mathbf{T}^*\mathcal{X}\big)$; and this tangent space may be identified with $\mathbf{T_x}\mathcal{X}\times\mathbf{T^*_x}\mathcal{X}$.
Beside this, if the vector field $\mathbb{B}$ on $\mathcal{X}$ represents the differential 2-form on $\mathcal{X}$, $\nabla\times\mathbb{B}$ cannot be clearly interpreted in terms of differential forms.
Then interpreting Ampere's Theorem (1.1) in the framework of the differential forms is not as easy as for the Faraday's Law.

Nonetheless, a vector field may represent a differential 1-form (see(2.29)) or a differential 2-form (see(2.9)). Hence rewriting Ampere's Theorem as

$$-\frac{\partial(\varepsilon_0\mathbb{E})}{\partial t} + \nabla\times({\mu_0}^{-1}\mathbb{B}) = \mathbb{J}, \tag{2.31}$$

and considering the new vector fields $\mathbb{D}$ and $\mathbb{H}$ defined from $\mathbb{E}$ and $\mathbb{B}$ by:

$$\mathbb{D} = \varepsilon_0\mathbb{E}, \quad \mathbb{B} = \mu_0\mathbb{H}, \tag{2.32}$$

the following is reached

$$-\frac{\partial\mathbb{D}}{\partial t} + \nabla\times\mathbb{H} = \mathbb{J}, \tag{2.33}$$

In (2.33) $\mathbb{D}$ and $\mathbb{H}$ are to be considered as vector fields representing respectively a differential 2-form $\mathbf{D}$ and a differential 1-form $\mathbf{H}$. Consistently, in (2.32) the multiplications by ε_0 and μ_0 have to be considered as operators transforming vector fields representing differential 1-forms into vector fields representing differential 2-forms.

2.10 Status of the Current Density

Watching equation (2.33), if, as announced, $\mathbb{D}$ represents a differential 2-form and $\mathbb{H}$ a differential 1-form then $\nabla \times \mathbb{H}$ represents a differential 2-form and then $\mathbb{J}$ has to represent a differential 2-form. The question is now: does this last assertion make sense? or may-be more: why does this last assertion make sense? To help the providing of an answer to this question, it has to be noticed that the essential role to be played by a current density is to be integrated on surfaces to give current fluxes, and that precisely, differential 2-forms may be integrated on surfaces.
Then, a current density may be represented by a differential 2-form $\mathbf{J}$ on $\mathcal{X}$ by setting that the current flux (associated with the considered current density) through any surface is the integral of $\mathbf{J}$ over this surface.

2.11 Ampere's Theorem from differential form point of view

Defining two "Hodge Operators" $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ that transform differential 1-forms into differential $(N-1)$-forms (differential 2-forms here since $N=3$) and setting

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}\mathbf{H} \tag{2.34}$$

Ampere's Theorem may be interpreted in terms of differential forms exactly as Faraday's Law may and then reads:

$$-\frac{\partial \mathbf{D}}{\partial t} + d\mathbf{H} = \mathbf{J}, \tag{2.35}$$

(or

$$-\frac{\partial (\boldsymbol{\varepsilon}\mathbf{E})}{\partial t} + d(\boldsymbol{\mu}^{-1}\mathbf{B}) = \mathbf{J}\ ). \tag{2.36}$$

2.12 Magnetic Field divergence free equation from differential form point of view

if $\mathbf{B}$ is the differential 2-form represented by the vector field $\mathbb{B}$ (or, in other words, if in any $\mathbf{x}$ of $\mathcal{X}$, the 2-form $\{\mathbf{B}(t,\mathbf{x})\}$ on $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ is defined by (2.9)) then the exterior derivative $d\mathbf{B}$ of $\mathbf{B}$ (which is a differential 3-form if $\mathbf{B}$ is regular enough) is represented by the function $\nabla \cdot \mathbb{B}$. This means that in any $\mathbf{x}$ of $\mathcal{X}$, the following formula holds:

$$\{d_{\mathbf{x}}\mathbf{B}(t)\}(\nu,\nu',\nu'') = (\nabla \cdot \mathbb{B}(t,\mathbf{x}))((\nu \times \nu')\cdot \nu'') \text{ for any } \nu, \nu' \text{ and } \nu'' \text{ in } \mathbf{T}_{\mathbf{x}}\mathcal{X} \tag{2.37}$$

With this remark, equation (1.4) reads

$$d\mathbf{B} = 0. \tag{2.38}$$

2.13 Poisson's Equation from differential form point of view

Rewriting equation (1.3) as

$$\nabla \cdot (\varepsilon_0 \mathbb{E}) = \boldsymbol{\rho}, \tag{2.39}$$

introducing the charge density differential 3-form ρ, it can be read:

$$d(\boldsymbol{\varepsilon}\mathbf{E}) = \rho, \tag{2.40}$$

or

$$d\mathbf{D} = \rho. \tag{2.41}$$

2.14 Status of the Charge Density

The role of the Charge Density is to be integrated over any tridimensional set in order to give charge contained within this set. Precisely, differential 3-forms may be integrated over tridimensional sets. Then, it is consistent to represent a charge density by the differential 3-form ρ.

2.15 Maxwell System in the differential forms framework

Summarizing all the above versions of the equations of the Maxwell System, it gives

$$-\frac{\partial \mathbf{D}}{\partial t} + d\mathbf{H} = \mathbf{J}, \tag{2.42}$$

$$\frac{\partial \mathbf{B}}{\partial t} + d\mathbf{E} = 0, \tag{2.43}$$

$$d\mathbf{D} = \rho, \tag{2.44}$$

$$d\mathbf{B} = 0. \tag{2.45}$$

with

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}\mathbf{H}. \tag{2.46}$$

In ((2.42)-(2.46)) $\mathbf{E}$ and $\mathbf{H}$ are 1-forms on the position space $\mathcal{X}$; $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{J}$ are 2-forms on $\mathcal{X}$; and; ρ is a 3-form on $\mathcal{X}$. Operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are Hodge Operators mapping one-to-one the 1-forms on $\mathcal{X}$ onto the 2-forms on $\mathcal{X}$.

3 From Newton to Lagrange

3.1 Maxwell System with Electric and Magnetic Potentials

As on $\mathcal{X}$ the forms with zero exterior derivatives are the exterior derivatives, from (2.45) it can be deduced that there exists a differential 1-form $\mathbf{A}$ on $\mathcal{X}$ such that

$$\mathbf{B} = d\mathbf{A} \tag{3.1}$$

Then, inserting this in (2.43) yields $d\Big(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}\Big) = 0$. From this last equality, it can be deduced that there exists a 0-form Φ such that $\Big(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}\Big) = -d\Phi$, or

$$\mathbf{E} = -d\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{3.2}$$

Equations (2.42) and (2.44) yield:

$$-\frac{\partial\Big(-\boldsymbol{\varepsilon}d\Phi - \boldsymbol{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\Big)}{\partial t} + d(\boldsymbol{\mu}^{-1}d\mathbf{A}) = \mathbf{J} \tag{3.3}$$

$$d\Big(-\boldsymbol{\varepsilon}d\Phi - \boldsymbol{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\Big) = \rho, \tag{3.4}$$

or

$$\frac{\partial\Big(\boldsymbol{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\Big)}{\partial t}+\frac{\partial(\boldsymbol{\varepsilon} d\Phi)}{\partial t}+d(\boldsymbol{\mu}^{-1}d\mathbf{A})=\mathbf{J} \tag{3.5}$$

$$-d\Big(\boldsymbol{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\Big)-d(\boldsymbol{\varepsilon} d\Phi)=\rho, \tag{3.6}$$

(If Hodge operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ depend neither on time nor on position, these last equations read

$$\boldsymbol{\varepsilon}\frac{\partial^2 \mathbf{A}}{\partial t^2}+\boldsymbol{\varepsilon}\frac{\partial(d\Phi)}{\partial t}+d(\boldsymbol{\mu}^{-1}d\mathbf{A})=\mathbf{J}, \tag{3.7}$$

$$-\frac{\partial(d(\boldsymbol{\varepsilon}\mathbf{A}))}{\partial t}-d(\boldsymbol{\varepsilon} d\Phi)=\rho.\ ) \tag{3.8}$$

3.2 On momentum choice

3.2.1 Momentum possibly associated with a differential 1-form

Reinterpreting equality (2.2) with the differential form point of view, consists, once $\tilde{\mathbf{M}}(t;\mathbf{x}_0,\mathbf{m}_0,t_0)$ is defined from the Position-Velocity trajectory $(\mathbf{X}(t),\mathbf{V}(t))=(\mathbf{X}(t;\mathbf{x}_0,\mathbf{v}_0,t_0),\mathbf{V}(t;\mathbf{x}_0,\mathbf{v}_0,t_0))$ by (2.5), (2.4) and (2.3), in considering that the Position-Velocity trajectory $(\mathbf{X}(t),\mathbf{V}(t))$ is solution to

$$\Big\{(\mathbf{F}-\frac{\partial\tilde{\mathbf{M}}}{\partial t})(t,\mathbf{X}(t))\Big\}\nu=0, \tag{3.9}$$

for all ν in $\mathbf{T}_{\mathbf{X}(t)}\mathcal{X}$, where $\mathbf{F}(t)$ is defined, at any time t, from the differential 1-form $\mathbf{E}(t)$ and the differential 2-form $\mathbf{B}(t)$ by (2.11) with a vector field $\mathbf{v}$ which is such that $\mathbf{v}((\mathbf{X}(t))=\mathbf{V}(t)$.
But, if a regular differential 1-form $\mathbf{A}$ is defined on $\mathcal{X}$, it is not forbidden to associate with Position-Velocity trajectory $(\mathbf{X}(t),\mathbf{V}(t))$ the following momentum

$$\mathbf{M}(t)=\mathcal{M}^{[\mathbf{A}]}_{\mathbf{X}(t)}(\mathbf{V}(t)), \tag{3.10}$$

with

$$\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v})=\mathcal{M}_{\mathbf{x}}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}=d_{\mathbf{v}}\bar{L}_{\mathbf{x}}+\{q\mathbf{A}(t,\mathbf{x})\}, \tag{3.11}$$

where $\mathcal{M}_{\mathbf{x}}$ is defined by (2.4) and $\bar{L}_{\mathbf{x}}$ by (2.3).

Remark 3.1 Following remark 2.3, replacing $\mathcal{M}_{\mathbf{x}}$ by $\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}$ may be seen as replacing the metric on $\mathcal{X}$ by another operator on $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}$ which is no more an inner product. ■

3.2.2 Time derivative of the momentum

Considering $\mathbf{T}\mathcal{X}=\cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x},\mathbf{T}_{\mathbf{x}}\mathcal{X})$, at any time t,

$$\mathfrak{B}\mathcal{M}^{[\mathbf{A}]}:(\mathbf{x},\mathbf{v})\mapsto\big(\mathbf{x},\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v})\big)=\big(\mathbf{x},\mathcal{M}^{[\mathbf{A}]}(\mathbf{x},\mathbf{v})\big), \tag{3.12}$$

may be seen as a one-to-one mapping from $\mathbf{T}\mathcal{X}$ onto $\mathbf{T}^*\mathcal{X}$. ($\mathfrak{B}\mathcal{M}^{[\mathbf{A}]}$ could be called the bundlization of $\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}$ and $\mathcal{M}^{[\mathbf{A}]}(\mathbf{x},\mathbf{v})$ is the tangent space's componant of this bundlization in $(\mathbf{x},\mathbf{v})$.) The value $d_{(\mathbf{x},\mathbf{v})}\mathcal{M}^{[\mathbf{A}]}$ of its differential $d\mathcal{M}^{[\mathbf{A}]}$ in $(\mathbf{x},\mathbf{v})$ maps $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ onto $\mathbf{T}_{(\mathbf{x},\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v}))}\big(\mathbf{T}^*\mathcal{X}\big)$. Tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be identified with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\mathbf{T}_{(\mathbf{x},\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v}))}\big(\mathbf{T}^*\mathcal{X}\big)$ with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$. Looking at $(\mathbf{X}(.),\mathbf{V}(.))$ as a trajectory on $\mathbf{T}\mathcal{X}$, it is mapped to a trajectory $(\mathbf{X}(.),\mathbf{M}(.))$ on $\mathbf{T}^*\mathcal{X}$, and its tangent vector $\Big(\dfrac{\partial \mathbf{X}}{\partial t}(t), \dfrac{\partial \mathbf{V}}{\partial t}(t)\Big)$ in $(\mathbf{X}(t), \mathbf{V}(t))$ is mapped to the tangent vector $\Big(\dfrac{\partial \mathbf{X}}{\partial t}(t), \dfrac{\partial \mathbf{M}}{\partial t}(t)\Big)$ of trajectory $(\mathbf{X}(.), \mathbf{M}(.))$ in $(\mathbf{X}(t), \mathbf{M}(t))$ by the differential $d_{(\mathbf{X}(t),\mathbf{V}(t))}\mathcal{M}^{[\mathbf{A}]}$ to which the time partial derivative of $\{q\mathbf{A}(t,\mathbf{x})\}$ has to be added. In other words,

$$\Big(\frac{\partial \mathbf{X}}{\partial t}(t), \frac{\partial \mathbf{M}}{\partial t}(t)\Big) = d_{(\mathbf{X}(t),\mathbf{V}(t))}(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]})\Big(\frac{\partial \mathbf{X}}{\partial t}(t), \frac{\partial \mathbf{V}}{\partial t}(t)\Big) + \Big(0, \Big\{q\frac{\partial \mathbf{A}}{\partial t}(t, \mathbf{X}(t))\Big\}\Big). \tag{3.13}$$

To compute $d_{(\mathbf{X}(t),\mathbf{V}(t))}(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]})$ the following chart of a neighborhood $\mathcal{B}(\mathbf{X}(t))$ of $\mathbf{X}(t)$ on $\mathcal{X}$ is used

$$\begin{array}{rccl} \mathcal{C}: & \mathcal{B}(\mathbf{X}(t)) & \to & \mathbb{R}^N \\ & \mathbf{x} & \mapsto & \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3). \end{array} \tag{3.14}$$

On $\mathbb{R}^N$ stands the canonical frame $(\mathbf{e}_{\mathbf{q}_1}, \mathbf{e}_{\mathbf{q}_2}, \mathbf{e}_{\mathbf{q}_3})$. Then (3.15) may be rewritten as $\mathcal{C}(\mathbf{x}) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3}$. For every $\mathbf{x} \in \mathcal{X}$, the following chart is used on $\mathbf{T}_\mathbf{x}\mathcal{X}$:

$$\begin{array}{rccl} d_\mathbf{x}\mathcal{C}: & \mathbf{T}_\mathbf{x}\mathcal{X} & \to & \mathbb{R}^N \ (= \mathbf{T}_\mathbf{q}\mathbb{R}^N) \\ & \mathbf{v} & \mapsto & \dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3). \end{array} \tag{3.15}$$

Taking the same frame as previously but naming it $(\partial_{\mathbf{q}_1}, \partial_{\mathbf{q}_2}, \partial_{\mathbf{q}_3})$, (3.15) reads also $d_\mathbf{x}\mathcal{C}(\mathbf{v}) = \dot{\mathbf{q}}_1\partial_{\mathbf{q}_1} + \dot{\mathbf{q}}_2\partial_{\mathbf{q}_2} + \dot{\mathbf{q}}_3\partial_{\mathbf{q}_3}$

Another way to formulate this consists in saying that the following chart in the following neighborhood $\cup_{\mathbf{x}\in\mathcal{B}(\mathbf{X}(t))}(\mathbf{x}, \mathbf{T}_\mathbf{x}\mathcal{X})$ of $(\mathbf{X}(t), \mathbf{V}(t))$ on $\mathbf{T}\mathcal{X}$ is used:

$$\begin{array}{rccl} (\mathcal{C}, d_\mathbf{x}\mathcal{C}): & \cup_{\mathbf{x}\in\mathcal{B}((\mathbf{X}(t))}(\mathbf{x}, \mathbf{T}_\mathbf{x}\mathcal{X}) & \to & \mathbb{R}^N \times \mathbb{R}^N \\ & (\mathbf{x},\mathbf{v}) & \mapsto & (\mathbf{q}, \dot{\mathbf{q}}) = ((\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), (\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3)), \end{array} \tag{3.16}$$

which also reads : $(\mathcal{C}, d_\mathbf{x}\mathcal{C})(\mathbf{x}, \mathbf{v}) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \dot{\mathbf{q}}_1\partial_{\mathbf{q}_1} + \dot{\mathbf{q}}_2\partial_{\mathbf{q}_2} + \dot{\mathbf{q}}_3\partial_{\mathbf{q}_3}$.

Beside this, since $\mathcal{C}: \mathcal{B}(\mathbf{X}(t)) \to \mathbb{R}^N$ it may be seen as N or 3 regular functions or differential 0-forms $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$, the exterior derivatives or differentials $d_\mathbf{x}\mathcal{C}_1 = d_\mathbf{x}\mathcal{C}_1$, $d_\mathbf{x}\mathcal{C}_2 = d_\mathbf{x}\mathcal{C}_2$, $d_\mathbf{x}\mathcal{C}_3 = d_\mathbf{x}\mathcal{C}_3$ may be considered in every $\mathbf{x} \in \mathcal{X}$. They define a frame on $\mathbf{T}^*_\mathbf{x}\mathcal{X}$ which is denoted $(d\mathbf{q}_1, d\mathbf{q}_2, d\mathbf{q}_3)$. ($d\mathbf{q}_1$ is nothing but $\mathbf{v} \mapsto \dot{\mathbf{q}}_1$.) In this frame any 1-form $\mathbf{m}$ has the following coordinates $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3$. Then, the following chart is built on $\mathbf{T}^*_\mathbf{x}\mathcal{X}$:

$$\begin{array}{rccl} \mathcal{D}_\mathbf{x}\mathcal{C}: & \mathbf{T}^*_\mathbf{x}\mathcal{X} & \to & \mathbb{R}^N \ (= \mathbf{T}^*_\mathbf{q}\mathbb{R}^N) \\ & \mathbf{m} & \mapsto & \mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3). \end{array} \tag{3.17}$$

Another way to formulate this consists in following, in a simpler manner, the way followed in subsection 2.7 while defining $(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)$ from $\Pi_{*(\mathbf{x},\mathbf{v})}$. Indeed, $\mathcal{C}_{*\mathbf{x}}$ may be defined as

$$\mathcal{C}_{*\mathbf{x}}: \quad \mathbb{R}^N \ (= \mathbf{T}^*_\mathbf{q}\mathbb{R}^N) \quad \to \quad \mathbf{T}^*_\mathbf{x}\mathcal{X}, \tag{3.18}$$

in defining the Pullback $\mathcal{C}_{*\mathbf{x}}(\mathbf{p})$ of $\mathbf{p}$ in $\mathbf{x}$, by setting for any ν in $\mathbf{T}_\mathbf{x}\mathcal{X}$,

$$\begin{aligned} \langle \mathcal{C}_{*\mathbf{x}}(\mathbf{p}), \nu\rangle &= \langle \mathcal{C}_{*\mathbf{x}}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3), \nu\rangle = \langle \mathcal{C}_{*\mathbf{x}}(\mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3), \nu\rangle \\ &= \langle \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3, d_\mathbf{x}\mathcal{C}(\nu)\rangle = \langle \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3, (\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3)\rangle \\ &= \langle \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3, \dot{\mathbf{q}}_1\partial_{\mathbf{q}_1} + \dot{\mathbf{q}}_2\partial_{\mathbf{q}_2} + \dot{\mathbf{q}}_3\partial_{\mathbf{q}_3}\rangle = \mathbf{p}_1\dot{\mathbf{q}}_1 + \mathbf{p}_2\dot{\mathbf{q}}_2 + \mathbf{p}_3\dot{\mathbf{q}}_3. \end{aligned} \tag{3.19}$$

Easyly, $\mathcal{C}_{*\mathbf{x}}$ is one-to-one onto $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and $\mathcal{D}_{\mathbf{x}}\mathcal{C} = (\mathcal{C}_{*\mathbf{x}})^{-1}$. This may of course be taken as the defininition of $\mathcal{D}_{\mathbf{x}}\mathcal{C}$.
Once $\mathcal{D}_{\mathbf{x}}\mathcal{C}$ is properly introduced, it is possible to consider that the following chart, in neighborhood $\cup_{\mathbf{x}\in\mathcal{B}(\mathbf{X}(t))}(\mathbf{x}, \mathbf{T}^*_{\mathbf{x}}\mathcal{X})$ of $(\mathbf{X}(t), \mathbf{M}(t))$ on $\mathbf{T}^*\mathcal{X}$, is used:

$$\begin{array}{rccl} (\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C}): & \cup_{\mathbf{x}\in\mathcal{B}((\mathbf{X}(t))}(\mathbf{x}, \mathbf{T}^*_{\mathbf{x}}\mathcal{X}) & \to & \mathbb{R}^N \times \mathbb{R}^N \\ & (\mathbf{x},\mathbf{m}) & \mapsto & (\mathbf{q}, \mathbf{p}) = ((\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)), \end{array} \tag{3.20}$$

or $(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \mathbf{m}) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3$.

Using those charts, $\breve{\mathbf{A}}_1(t, \mathbf{q})$, $\breve{\mathbf{A}}_2(t, \mathbf{q})$ and $\breve{\mathbf{A}}_3(t, \mathbf{q})$ are functions on $\mathbb{R}^N$ which are the coordinates of $\mathbf{A}$, they are defined to be such that

$$(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \mathbf{A}(t, \mathbf{x})) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \breve{\mathbf{A}}_1(t, \mathbf{q})d\mathbf{q}_1 + \breve{\mathbf{A}}_2(t, \mathbf{q})d\mathbf{q}_2 + \breve{\mathbf{A}}_3(t, \mathbf{q})d\mathbf{q}_3, \tag{3.21}$$

and $\breve{\mathbf{A}}$ is the associated vector, i.e. $\breve{\mathbf{A}} = (\breve{\mathbf{A}}_1, \breve{\mathbf{A}}_2, \breve{\mathbf{A}}_3)$. Function $\breve{L}_{\mathbf{q}}$ is the expression of $\bar{L}_{\mathbf{x}}$ within the coordinate system. It is such that

$$\bar{L}_{\mathbf{x}}(\mathbf{v}) = \breve{L}_{\mathbf{q}}(\dot{\mathbf{q}}), \quad \text{with} \quad (\mathbf{q}, \dot{\mathbf{q}}) = (\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \mathbf{v}), \tag{3.22}$$

Mapping $\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}$ which maps one-to-one $\mathbb{R}^N \times \mathbb{R}^N$ onto $\mathbb{R}^N \times \mathbb{R}^N$ is the expression of $\mathfrak{B}\mathcal{M}^{[\mathbf{A}]}$ within the coordinate systems. It is such that

$$\begin{aligned} &(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]}(\mathbf{x}, \mathbf{v})) \\ &\qquad = \mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}, \breve{\mathcal{M}}_{\mathbf{q}}^{[\mathbf{A}]}(\dot{\mathbf{q}})) = (\mathbf{q}, \breve{\mathcal{M}}^{[\mathbf{A}]}(\mathbf{q}, \dot{\mathbf{q}})), \quad \text{with} \quad (\mathbf{q}, \dot{\mathbf{q}}) = (\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \mathbf{v}). \end{aligned} \tag{3.23}$$

Mapping $\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}$ may be expressed as

$$\begin{aligned} \mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}(\mathbf{q}, \dot{\mathbf{q}}) &= \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} \\ &+ \left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}(\dot{\mathbf{q}}) + q\breve{\mathbf{A}}_1(\mathbf{q})\right) d\mathbf{q}_1 + \left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}(\dot{\mathbf{q}}) + q\breve{\mathbf{A}}_2(\mathbf{q})\right) d\mathbf{q}_2 + \left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}(\dot{\mathbf{q}}) + q\breve{\mathbf{A}}_3(\mathbf{q})\right) d\mathbf{q}_3. \end{aligned} \tag{3.24}$$

Matrix $\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})$ is an expression of $d_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})$ which is an expression of $d_{(\mathbf{x},\mathbf{v})}(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]})$ within the coordinates. This means, differentiating (3.23), that

$$\left(d_{(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]}(\mathbf{x},\mathbf{v}))}(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})\right)\left(d_{(\mathbf{x},\mathbf{v})}(\mathfrak{B}\mathcal{M}^{[\mathbf{A}]})\right) = \left(d_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})\right)\left(d_{(\mathbf{x},\mathbf{v})}(\mathcal{C}, d_{\mathbf{x}}\mathcal{C})\right), \tag{3.25}$$

and matrix $\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})$ has the following expression:

$$\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}) =$$

$$\begin{pmatrix} I_{\mathbb{R}^N} & 0 \\ \begin{pmatrix} \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}+q\breve{\mathbf{A}}_1\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}+q\breve{\mathbf{A}}_1\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}+q\breve{\mathbf{A}}_1\right)}{\partial \mathbf{q}_3} \\ \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}+q\breve{\mathbf{A}}_2\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}+q\breve{\mathbf{A}}_2\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}+q\breve{\mathbf{A}}_2\right)}{\partial \mathbf{q}_3} \\ \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}+q\breve{\mathbf{A}}_3\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}+q\breve{\mathbf{A}}_3\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}+q\breve{\mathbf{A}}_3\right)}{\partial \mathbf{q}_3} \end{pmatrix} & \begin{pmatrix} \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_1} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_1} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_1} \\ \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_2} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_2} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_2} \\ \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_3} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_3} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_3} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} I_{\mathbb{R}^N} & 0 \\ \nabla_{\mathbf{q}}\breve{\mathcal{M}}^{[\mathbf{A}]} & \nabla_{\dot{\mathbf{q}}}\breve{\mathcal{M}}^{[\mathbf{A}]} \end{pmatrix}. \tag{3.26}$$

Defining trajectories $\mathbf{Q}(t) = \mathcal{C}(\mathbf{X}(t))$, $\dot{\mathbf{Q}}(t) = (d_{\mathbf{X}(t)}\mathcal{C})(\mathbf{V}(t))$ and $\mathbf{P}(t) = (\mathcal{D}_{\mathbf{X}(t)}\mathcal{C})(\mathbf{M}(t))$ within the coordinate systems, equation (3.13) may be translated into

$$\begin{aligned} \Big(\frac{\partial \mathbf{Q}}{\partial t}(t), \frac{\partial \mathbf{P}}{\partial t}(t)\Big) &= d_{(\mathbf{Q}(t),\dot{\mathbf{Q}}(t))}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})\Big(\frac{\partial \mathbf{Q}}{\partial t}(t), \frac{\partial \dot{\mathbf{Q}}}{\partial t}(t)\Big) + \Big(0, \Big\{q\frac{\partial \breve{\mathbf{A}}}{\partial t}(t,\mathbf{Q}(t))\Big\}\Big), \\ &= \big(\nabla_{(\mathbf{Q}(t),\dot{\mathbf{Q}}(t))}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]})\big)\Big(\frac{\partial \mathbf{Q}}{\partial t}(t), \frac{\partial \dot{\mathbf{Q}}}{\partial t}(t)\Big) + \Big(0, \Big\{q\frac{\partial \breve{\mathbf{A}}}{\partial t}(t,\mathbf{Q}(t))\Big\}\Big). \end{aligned} \tag{3.27}$$

Introducing $\mathfrak{B}\mathcal{M}^{[\,]} : (\mathbf{x},\mathbf{v}) \mapsto \big(\mathbf{x}, \mathcal{M}_{\mathbf{x}}(\mathbf{v})\big) = \big(\mathbf{x}, \mathcal{M}^{[\,]}(\mathbf{x},\mathbf{v})\big)$ mapping $\mathbf{T}\mathcal{X}$ to $\mathbf{T}^*\mathcal{X}$, whose differential $d_{(\mathbf{x},\mathbf{v})}(\mathfrak{B}\mathcal{M}^{[\,]})$ is represented, within the coordinate systems, by

$$\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\,]}) = \begin{pmatrix} I_{\mathbb{R}^N} & 0 \\ \begin{pmatrix} \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_1}\right)}{\partial \mathbf{q}_3} \\ \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_2}\right)}{\partial \mathbf{q}_3} \\ \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}\right)}{\partial \mathbf{q}_1} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}\right)}{\partial \mathbf{q}_2} & \frac{\partial\left(\frac{\partial \breve{L}_{\mathbf{q}}}{\partial \dot{\mathbf{q}}_3}\right)}{\partial \mathbf{q}_3} \end{pmatrix} & \begin{pmatrix} \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_1} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_1} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_1} \\ \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_2} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_2} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_2} \\ \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_1 \partial \dot{\mathbf{q}}_3} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_2 \partial \dot{\mathbf{q}}_3} & \frac{\partial^2 \breve{L}_{\mathbf{q}}}{\partial \mathbf{q}_3 \partial \dot{\mathbf{q}}_3} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} I_{\mathbb{R}^N} & 0 \\ \nabla_{\mathbf{q}}\breve{\mathcal{M}}^{[\,]} & \nabla_{\dot{\mathbf{q}}}\breve{\mathcal{M}}^{[\,]} \end{pmatrix}, \tag{3.28}$$

(which means this matrix is a representation of $d_{(\mathbf{x},\mathbf{v})}(\mathfrak{B}\mathcal{M}^{[\,]})$ defined as

$$\big(d_{(\mathfrak{B}\mathcal{M}^{[\,]}(\mathbf{x},\mathbf{v}))}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})\big)\Big(d_{(\mathbf{x},\mathbf{v})}(\mathfrak{B}\mathcal{M}^{[\,]})\Big) = \Big(d_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\,]})\Big)\big(d_{(\mathbf{x},\mathbf{v})}(\mathcal{C},d_{\mathbf{x}}\mathcal{C})\big) \quad) \tag{3.29}$$

equality (3.26) reads

$$\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\mathbf{A}]}) = \nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\breve{\mathcal{M}}^{[\,]}) + \begin{pmatrix} 0 & 0 \\ q\begin{pmatrix} \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_1} & \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} & \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3} \\ \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} & \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_2} & \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3} \\ \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} & \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} & \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_3} \end{pmatrix} & 0 \end{pmatrix}. \tag{3.30}$$

Another shape will now be given to (3.27). Since $\breve{\mathbf{A}}(t,\mathbf{q}) = \breve{\mathbf{A}}_1(t,\mathbf{q})d\mathbf{q}_1 + \breve{\mathbf{A}}_2(t,\mathbf{q})d\mathbf{q}_2 + \breve{\mathbf{A}}_3(t,\mathbf{q})d\mathbf{q}_3$,

$$d_{\mathbf{q}}\breve{\mathbf{A}} = \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3}\right) d\mathbf{q}_2 \wedge d\mathbf{q}_3 + \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3}\right) d\mathbf{q}_1 \wedge d\mathbf{q}_3 + \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2}\right) d\mathbf{q}_1 \wedge d\mathbf{q}_2, \tag{3.31}$$

which is the expression of $d_{\mathbf{x}}\mathbf{A}$ in the coordinate system associated with frame $(d\mathbf{q}_2 \wedge d\mathbf{q}_3, d\mathbf{q}_1 \wedge d\mathbf{q}_3, d\mathbf{q}_1 \wedge d\mathbf{q}_2)$, then the expression of differential 1-form $i_{\mathbf{W}}(d\mathbf{A})$, which is the interior product of $d\mathbf{A}$ by any vector field $\mathbf{W}$, has the following expression within the coordinate system associated with frame $(d\mathbf{q}_1, d\mathbf{q}_2, d\mathbf{q}_3)$:

$$\begin{aligned} i_{\breve{\mathbf{W}}}(d\breve{\mathbf{A}}) = &\left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3}\right)(\breve{\mathbf{W}}_2\, d\mathbf{q}_3 - \breve{\mathbf{W}}_3\, d\mathbf{q}_2) + \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3}\right)(\breve{\mathbf{W}}_1\, d\mathbf{q}_3 - \breve{\mathbf{W}}_3\, d\mathbf{q}_1) \\ &+ \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2}\right)(\breve{\mathbf{W}}_1\, d\mathbf{q}_2 - \breve{\mathbf{W}}_2\, d\mathbf{q}_1) = \left(\left[\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1}\right]\breve{\mathbf{W}}_2 + \left[\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3} - \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1}\right]\breve{\mathbf{W}}_3\right) d\mathbf{q}_1 \\ &+ \left(\left[\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3} - \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2}\right]\breve{\mathbf{W}}_3 + \left[\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2}\right]\breve{\mathbf{W}}_1\right) d\mathbf{q}_2 \\ &+ \left(\left[\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3}\right]\breve{\mathbf{W}}_1 + \left[\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3}\right]\breve{\mathbf{W}}_2\right) d\mathbf{q}_3. \end{aligned} \tag{3.32}$$

On the other hand, $i_{\mathbf{W}}\mathbf{A}$ has the following expression:

$$i_{\breve{\mathbf{W}}}\breve{\mathbf{A}} = \breve{\mathbf{A}}_1(t,\mathbf{q})\breve{\mathbf{W}}_1(\mathbf{q}) + \breve{\mathbf{A}}_2(t,\mathbf{q})\breve{\mathbf{W}}_2(\mathbf{q}) + \breve{\mathbf{A}}_3(t,\mathbf{q})\breve{\mathbf{W}}_3(\mathbf{q}), \tag{3.33}$$

and, as a consequence, if $\breve{\mathbf{W}}$ is independent of $\mathbf{q}$, the exterior derivative $d_{\mathbf{x}}(i_{\mathbf{W}}\mathbf{A})$ of $i_{\mathbf{W}}\mathbf{A}$ in $\mathbf{x}$ writes, within the coordinate system,

$$\begin{aligned} d_{\mathbf{q}}(i_{\breve{\mathbf{W}}}\breve{\mathbf{A}}) = &\left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_1}\breve{\mathbf{W}}_1 + \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1}\breve{\mathbf{W}}_2 + \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1}\breve{\mathbf{W}}_3\right) d\mathbf{q}_1 \\ &+ \left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2}\breve{\mathbf{W}}_1 + \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_2}\breve{\mathbf{W}}_2 + \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2}\breve{\mathbf{W}}_3\right) d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3}\breve{\mathbf{W}}_1 + \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3}\breve{\mathbf{W}}_2 + \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_3}\breve{\mathbf{W}}_3\right) d\mathbf{q}_3. \end{aligned} \tag{3.34}$$

Coupling (3.27) (with (3.24) or (3.30)) on the one hand, and, on the other hand, (3.32) and (3.33), the following equality holds:

$$\begin{aligned} \left(\frac{\partial \mathbf{Q}}{\partial t}(t), \frac{\partial \mathbf{P}}{\partial t}(t)\right) = &\left(0, \left\{q\frac{\partial \breve{\mathbf{A}}}{\partial t}(t,\mathbf{Q}(t))\right\}\right) + \left(\nabla_{(\mathbf{q},\dot{\mathbf{q}})}(\mathfrak{B}\mathcal{M}^{[\,]}_{\mathbf{q}})\right)\left(\frac{\partial \mathbf{Q}}{\partial t}(t), \frac{\partial \dot{\mathbf{Q}}}{\partial t}(t)\right) \\ &+ \left(0, \left\{q\, i_{\breve{\breve{\mathbf{V}}}}(d\breve{\mathbf{A}})(t,\mathbf{Q}(t))\right\}\right) + \left(0, \left\{q\, d(i_{\breve{\breve{\mathbf{V}}}}\breve{\mathbf{A}})(t,\mathbf{Q}(t))\right\}\right), \end{aligned} \tag{3.35}$$

where $\breve{\breve{\mathbf{V}}}$ is a the vector field which is such that $\breve{\breve{\mathbf{V}}}(\mathbf{X}(t)) = \mathbf{V}(t)$ and whose coordinates $d_{\mathbf{x}}\mathcal{C}(\breve{\breve{\mathbf{V}}}(\mathbf{x}))$ are independant of $\mathbf{x}$ in a neighborhood of $\mathbf{X}(t)$.

Remark 3.2 Notice that the above computation suppose that vector field $\breve{\mathbf{W}}$ is independent of $\mathbf{q}$. But this computation is independent of the coordinate system within which the computation is done. Then, vector field $\breve{\breve{\mathbf{V}}}$ that needs to be used in formula (3.35) may be any vector field which does not depend on the space variable in a given coordinate system. ■

As a consequence,

$$\begin{aligned}\frac{\partial \mathbf{M}}{\partial t}(t) &= \frac{\partial \tilde{\mathbf{M}}}{\partial t}(t) + \Big\{q\frac{\partial \mathbf{A}}{\partial t}(t,\mathbf{X}(t))\Big\} + \{q\, i_{\widetilde{\mathbf{V}}}(d\mathbf{A})(t,\mathbf{X}(t))\} + \{q\, d(i_{\widetilde{\mathbf{V}}}\mathbf{A})(t,\mathbf{X}(t))\}, \\ &= \frac{\partial \tilde{\mathbf{M}}}{\partial t}(t) + \Big\{q\frac{\partial \mathbf{A}}{\partial t}(t,\mathbf{X}(t))\Big\} + \{q\, l_{\widetilde{\mathbf{V}}}\mathbf{A}(t,\mathbf{X}(t))\}\end{aligned} \tag{3.36}$$

where the "Lie Derivative" $l_{\widetilde{\mathbf{V}}}\mathbf{A}$ of differential 1-form $\mathbf{A}$ is along the vector field $\widetilde{\mathbf{V}}$ which is such that $\widetilde{\mathbf{V}}(\mathbf{x}) = \mathbf{V}(t)$ for any $\mathbf{x}$ in $\mathcal{X}$.

3.2.3 Pullback, Lie Derivative and Cartan's Formula

The Lie Derivative of a differential time independent 1-form $\mathbf{A}$ along a regular vector field $\mathbf{v}$, is defined in any point $\mathbf{x}$ of $\mathcal{X}$ by

$$\{l_{\mathbf{W}}\mathbf{A}(\mathbf{x})\} = \frac{\partial(\{g^s_{*\mathbf{x}}\}(\mathbf{A}(g^s(\mathbf{x})))}{\partial s}(0), \tag{3.37}$$

or

$$\{l_{\mathbf{W}}\mathbf{A}(\mathbf{x})\}(\nu) = \frac{\partial(\{\mathbf{A}(g^s(\mathbf{x}))\}(\{d_{\mathbf{x}}g^s\}(\nu)))}{\partial s}(0), \tag{3.38}$$

for every ν in $\mathbf{T}_{\mathbf{x}}\mathcal{X}$, where $g^s_{*\mathbf{x}}$ is the "Pullback in $\mathbf{x}$" by the flow g^s of the field $\mathbf{W}$. To be precise, g^s is such that for any s, g^s maps one-to-one $\mathcal{X}$ onto $\mathcal{X}$ and

$$\frac{\partial(g^s(\mathbf{x}))}{\partial s} = \mathbf{W}(g^s(\mathbf{x})), \quad g^0(\mathbf{x}) = \mathbf{x}. \tag{3.39}$$

For any s, its differential dg^s is such that for any $\mathbf{x}$ in $\mathcal{X}$, $d_{\mathbf{x}}g^s$ maps one-to-one $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ onto $\mathbf{T}_{g^s(\mathbf{x})}\mathcal{X}$ (or, in other words (g^s, dg^s) maps one-to-one $\mathbf{T}\mathcal{X}$ onto $\mathbf{T}\mathcal{X}$). Then, for any π in $\mathbf{T}^*_{g^s(\mathbf{x})}\mathcal{X}$ and any ν in $\mathbf{T}_{\mathbf{x}}\mathcal{X}$, the following quantity may be computed:

$$\langle \pi, d_{\mathbf{x}}g^s(\nu)\rangle. \tag{3.40}$$

This defines the Pullback $g^s_{*\mathbf{x}}$ of any π in $\mathbf{T}^*_{g^s(\mathbf{x})}\mathcal{X}$ towards $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ or

$$\{g^s_{*\mathbf{x}}(\pi)\}(\nu) = \langle g^s_{*\mathbf{x}}(\pi), \nu\rangle = \langle \pi, d_{\mathbf{x}}g^s(\nu)\rangle = \pi\Big(d_{\mathbf{x}}g^s(\nu)\Big). \tag{3.41}$$

Moreover, since for any s, g^s is a diffeomorphism with $(g^s)^{-1} = g^{-s}$, $(g^{-s}, g^s_{*g^{-s}})$ maps one-to-one $\mathbf{T}^*\mathcal{X}$ onto $\mathbf{T}^*\mathcal{X}$ $((g^{-s}, g^s_{*g^{-s}})(\mathbf{x},\pi)) = (g^{-s}(\mathbf{x}), g^s_{*g^{-s}(\mathbf{x})}(\pi))$.
If now $\pi^{(k)}$ is a k-form on $\mathbf{T}_{g^s(\mathbf{x})}\mathcal{X}$ the Pullback $g^s_{*\mathbf{x}}$ of $\pi^{(k)}$ may also be set by

$$\{g^s_{*\mathbf{x}}(\pi^{(k)})\}(\nu_1,\dots,\nu_k) = \pi^{(k)}\Big(d_{\mathbf{x}}g^s(\nu_1),\dots,d_{\mathbf{x}}g^s(\nu_k)\Big), \tag{3.42}$$

for any $\nu_1,\dots,\nu_k$ in $\mathbf{T}_{\mathbf{x}}\mathcal{X}$, The Lie Derivative of a differential time dependent 1-form $\mathbf{A}(t)$ along a regular vector field $\mathbf{W}$, is defined in any point $\mathbf{x}$ of $\mathcal{X}$ by

$$\{l_{\mathbf{W}}\mathbf{A}(t,\mathbf{x})\} = \frac{\partial(\{g^s_{*\mathbf{x}}\}(\mathbf{A}(t,g^s(\mathbf{x})))}{\partial s}(0), \tag{3.43}$$

for every ν in $\mathbf{T_x}\mathcal{X}$,
"Cartan's Formula" gives the following expression of the Lie Derivative:

$$l_{\mathbf{W}}\mathbf{A} = i_{\mathbf{W}}(d\mathbf{A}) + d(i_{\mathbf{W}}\mathbf{A}). \tag{3.44}$$

REMARK 3.3 The following formulas are vector field analogous of "Cartan's Formula". For a vector fields $\mathbb{A}$ and a constant vector field $\mathbf{W}$, $\nabla\mathbb{A}$ standing for the Jacobian Matrix of $\mathbb{A}$ and $\nabla(\mathbb{A}\cdot\mathbf{W})$ for the gradient of $\mathbb{A}\cdot\mathbf{W}$,

$$(\nabla\mathbb{A})\mathbf{W} = (\nabla\times\mathbb{A})\times\mathbf{W} + \nabla(\mathbb{A}\cdot\mathbf{W}). \tag{3.45}$$

In the case when $\mathbf{W}$ is not constant, the following holds:

$$\nabla(\mathbb{A}\cdot\mathbf{W}) = ((\nabla\mathbb{A})^T)\mathbf{W} + ((\nabla\mathbf{W})^T)\mathbb{A}, \tag{3.46}$$

$$(\nabla\mathbb{A})\mathbf{W} = (\nabla\times\mathbb{A})\times\mathbf{W} + (\nabla\mathbb{A})^T\mathbf{W}. \tag{3.47}$$

Then,

$$\begin{aligned}(\nabla\mathbb{A})\mathbf{W} + ((\nabla\mathbf{W})^T)\mathbb{A} &= (\nabla\mathbb{A} - (\nabla\mathbb{A})^T)\mathbf{W} + ((\nabla\mathbb{A})^T)\mathbf{W} + ((\nabla\mathbf{W})^T)\mathbb{A} \\ &= (\nabla\times\mathbb{A})\times\mathbf{W} + \nabla(\mathbb{A}\cdot\mathbf{W}).\end{aligned} \tag{3.48}$$

Clearly, the left hand sides of (3.45) and (3.48) are the vector representations, by (2.9), of differential 1-form $l_{\mathbf{W}}\mathbf{A}$ if $\mathbb{A}$ is the vector representation of differential 1-form $\mathbf{A}$. Beside this, $(\nabla\times\mathbb{A})\times\mathbf{W}$ is the vector representation of differential 2-form $i_{\mathbf{W}}(d\mathbf{A})$ and $\nabla(\mathbb{A}\cdot\mathbf{W})$ is the vector representation of differential 2-form $d(i_{\mathbf{W}}\mathbf{A})$. ■

REMARK 3.4 The Lie Derivative of any differential n-form $\mathbf{K}$ or any time-dependent differential n-form $\mathbf{K}(t)$ on $\mathcal{X}$ may be defined by (3.37) and (3.43) replacing $\mathbf{A}$ by $\mathbf{K}$. Always replacing $\mathbf{A}$ by $\mathbf{K}$, Cartan's Formula (3.44) is valid for any differential n-form $\mathbf{K}$, with the convention that the interior product of any differential 0-form with any vector field is 0. ■

REMARK 3.5 The value of the Lie Derivative of a 0-form along any vector field $\mathbf{W}$ in any $\mathbf{x}$ of $\mathcal{X}$ does not depend on the way $\mathbf{W}$ vary around $\mathbf{x}$. if $n\geq 1$, the value of the Lie Derivative of a n-form along any vector field $\mathbf{W}$ in any $\mathbf{x}$ of $\mathcal{X}$ depends on the way $\mathbf{W}$ vary around $\mathbf{x}$. ■

3.2.4 Another expression of the time-derivative of the momentum

Watching (3.36) reveals a kind of incoherence since it gives an expression of $\frac{\partial\mathbf{M}}{\partial t}(t)$, which is pointwise, involving an operator using a vector field (defined in a neighborhood of the considered point). As a consequence, the involved vector field has to be chosen independent of the position variable within the coordinate system in which the computation is led. Then (3.36) cannot give rise to an intrinsic formulation of $\frac{\partial\mathbf{M}}{\partial t}(t)$, although such a kind of intrinsic formulation exists, since (3.27) may be written in the following shape, which does not depend on coordinate systems:

$$\Big(\frac{\partial\mathbf{X}}{\partial t}(t), \frac{\partial\mathbf{M}}{\partial t}(t)\Big) = d_{(\mathbf{X}(t),\mathbf{V}(t))}\mathcal{M}^{[\mathbf{A}]}\Big(\frac{\partial\mathbf{X}}{\partial t}(t), \frac{\partial\mathbf{V}}{\partial t}(t)\Big) + \Big(0, \Big\{q\frac{\partial\mathbf{A}}{\partial t}(t,\mathbf{X}(t))\Big\}\Big). \tag{3.49}$$

The goal is is now to write a version of (3.36) which may give rise to an intrinsic formulation. For this, in place of computing $\dot{\iota}_{\breve{\mathbf{W}}}\breve{\mathbf{A}}$ (see (3.33)), $\mathring{\iota}_{\breve{\mathbf{W}}}\breve{\mathbf{A}}$, which is an expression of $\mathring{\iota}_{\mathbf{W}}\mathbf{A}$, is computed. $\mathring{\iota}_{\mathbf{W}}\mathbf{A}$ is defined as differential 0-form on $\mathbf{T}\mathcal{X}$ by

$$\mathring{\iota}\mathbf{A}(t,\mathbf{x},\mathbf{v}) = \mathring{\iota}_{\mathbf{v}}\mathbf{A}(t,\mathbf{x}) = \{\mathbf{A}(t,\mathbf{x})\}(\mathbf{v}). \tag{3.50}$$

This, expressed in coordinate systems, gives:

$$\mathring{\iota}_{\dot{\mathbf{q}}}\breve{\mathbf{A}}(t,\mathbf{q}) = \breve{\mathbf{A}}_1(t,\mathbf{q})\dot{\mathbf{q}}_1 + \breve{\mathbf{A}}_2(t,\mathbf{q})\dot{\mathbf{q}}_2 + \breve{\mathbf{A}}_3(t,\mathbf{q})\dot{\mathbf{q}}_3. \tag{3.51}$$

Computing the exterior derivative $d(\mathring{\iota}\breve{\mathbf{A}})$ of the differential 0-form $\mathring{\iota}\breve{\mathbf{A}}$, which is a differential 1-form on $\mathbf{T}\mathcal{X}$, yields, in the coordinates systems:

$$\begin{aligned} d_{(\mathbf{q},\dot{\mathbf{q}})}(\mathring{\iota}\breve{\mathbf{A}}) = & \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_1 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_2 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_3 \\ & + \breve{\mathbf{A}}_1(t,\mathbf{q})d\dot{\mathbf{q}}_1 + \breve{\mathbf{A}}_2(t,\mathbf{q})d\dot{\mathbf{q}}_2 + \breve{\mathbf{A}}_3(t,\mathbf{q})d\dot{\mathbf{q}}_3, \end{aligned} \tag{3.52}$$

and using the operator $\mathcal{D}_{(\mathbf{q},\dot{\mathbf{q}})}\Pi : \mathbf{T}^*_{(\mathbf{q},\dot{\mathbf{q}})}(\mathbb{R}^N \times \mathbb{R}^N)) \to \mathbf{T}^*_{\mathbf{q}}(\mathbb{R}^N)$, defined by (2.28) in a more general setting,

$$\begin{aligned} (\mathcal{D}_{(\mathbf{q},\dot{\mathbf{q}})}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big) = & \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_1 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_2 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_3}(t,\mathbf{q})\,\dot{\mathbf{q}}_3\right) d\mathbf{q}_3, \end{aligned} \tag{3.53}$$

is obtained. $(\mathcal{D}_{(\mathbf{q},\dot{\mathbf{q}})}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big)$ is the expression, in the coordinate systems, of $(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)\Big(d(\mathring{\iota}\mathbf{A})\Big)$. As a consequence, for a given vector field $\mathbf{W}$ defined on $\mathcal{X}$, $(\mathcal{D}_{(\mathbf{q},\breve{\mathbf{W}}(\mathbf{q}))}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big)$ which is an expression of $(\mathcal{D}_{(\mathbf{x},\mathbf{W}(\mathbf{x}))}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big)$ has the following shape:

$$\begin{aligned} (\mathcal{D}_{(\mathbf{q},\breve{\mathbf{W}}(\mathbf{q}))}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big) = & \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_1}(t,\mathbf{q})\breve{\mathbf{W}}_1(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1}(t,\mathbf{q})\breve{\mathbf{W}}_2(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1}(t,\mathbf{q})\breve{\mathbf{W}}_3(\mathbf{q})\right) d\mathbf{q}_1 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2}(t,\mathbf{q})\breve{\mathbf{W}}_1(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_2}(t,\mathbf{q})\breve{\mathbf{W}}_2(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2}(t,\mathbf{q})\breve{\mathbf{W}}_3(\mathbf{q})\right) d\mathbf{q}_2 \\ & + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3}(t,\mathbf{q})\breve{\mathbf{W}}_1(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3}(t,\mathbf{q})\breve{\mathbf{W}}_2(\mathbf{q}) + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_3}(t,\mathbf{q})\breve{\mathbf{W}}_3(\mathbf{q})\right) d\mathbf{q}_3, \end{aligned} \tag{3.54}$$

which is similar to the shape of (3.34), but without assuming anything like an independence of the vector field with respect to the variable.

Computationally, in the same way as (3.36) is gotten, the following holds:

$$\frac{\partial \mathbf{M}}{\partial t}(t) = \frac{\partial \tilde{\mathbf{M}}}{\partial t}(t) + \Big\{ q \frac{\partial \mathbf{A}}{\partial t}(t, \mathbf{X}(t)) \Big\} + \{ q \, (\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(\overset{\circ}{\iota}(d\mathbf{A}(t))\big)\} + \{ q \, (\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(d(\overset{\circ}{\iota}\mathbf{A}(t))\big)\}, \quad (3.55)$$

3.2.5 Pseudo-Force possibly associated with a differential 1-form

As a consequence of (3.36), or preferably (3.55), equation (3.9) is equivalent to

$$\Big\{ (\mathbf{F} + q \frac{\partial \mathbf{A}}{\partial t} + q \, i_{\tilde{\mathbf{V}}} d\mathbf{A} + q \, d(i_{\tilde{\mathbf{V}}}\mathbf{A}) - \frac{\partial \mathbf{M}}{\partial t})(t, \mathbf{X}(t)) \Big\} \nu = 0, \quad (3.56)$$

for a vector field $\check{\tilde{\mathbf{V}}}$ such that $\check{\tilde{\mathbf{V}}}(\mathbf{X}(t)) = \mathbf{V}(t)$ and whose coordinates $d_{\mathbf{x}}\mathcal{C}(\check{\tilde{\mathbf{V}}}(\mathbf{x}))$ are independant of $\mathbf{x}$ in a neighborhood of $\mathbf{X}(t)$, or to

$$\Big\{ (\mathbf{F}(t) + q \frac{\partial \mathbf{A}}{\partial t}(t) + q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(\overset{\circ}{\iota}(d\mathbf{A}(t))\big) + q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(d(\overset{\circ}{\iota}\mathbf{A}(t))\big) - \frac{\partial \mathbf{M}}{\partial t}(t) \Big\} \nu = 0, \quad (3.57)$$

for all ν in $\mathbf{T}_{\mathbf{X}(t)}\mathcal{X}$. Defining the following 1-form $\mathbf{F}^{[\mathbf{A}]}$ by:

$$\mathbf{F}^{[\mathbf{A}]}(t) = \mathbf{F}(t) + q \frac{\partial \mathbf{A}}{\partial t}(t) + q \, d(i_{\tilde{\mathbf{V}}}\mathbf{A}(t)) + q \, d(i_{\tilde{\mathbf{V}}}\mathbf{A}(t)), \quad (3.58)$$

or

$$\mathbf{F}^{[\mathbf{A}]}(t) = \mathbf{F}(t) + q \frac{\partial \mathbf{A}}{\partial t}(t) + q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(\overset{\circ}{\iota}(d\mathbf{A}(t))\big) + q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(d(\overset{\circ}{\iota}\mathbf{A}(t))\big), \quad (3.59)$$

for every time t, (3.56) reads also

$$\Big\{ (\mathbf{F}^{[\mathbf{A}]} - \frac{\partial \mathbf{M}}{\partial t})(t, \mathbf{X}(t)) \Big\} \nu = 0, \quad (3.60)$$

for all ν in $\mathbf{T}_{\mathbf{X}(t)}\mathcal{X}$.

3.3 A pertinent momentum

Correlating formulas (3.58), (3.1), (3.2) and (2.11), leads to consider that linking momentum $\mathbf{M}(t)$ with Position-Velocity trajectory $(\mathbf{X}(t), \mathbf{V}(t))$ by (3.10) with the differential 1-form $\mathbf{A}$ which is the Magnetic Potential, i.e. which is such that (3.1) holds, is pertinent. Indeed, under this assumption, the Lorentz's Force differential 1-form is given by

$$\mathbf{F}(t) = q(-d\Phi(t) - \frac{\partial \mathbf{A}}{\partial t}(t) - i_{\tilde{\mathbf{V}}}(d\mathbf{A})(t)), \quad (3.61)$$

then

$$\mathbf{F}^{[\mathbf{A}]}(t) = q(-d\Phi(t) + q \, d(i_{\tilde{\mathbf{V}}}\mathbf{A}(t)), \quad (3.62)$$

and then (3.60) gives

$$\Big\{(-q\,d\Phi+q\,d(i_{\widetilde{\mathbf{V}}}\mathbf{A})-\frac{\partial\mathbf{M}}{\partial t})(t,\mathbf{X}(t))\Big\}\nu=\Big\{q\,d\big(-\Phi+(i_{\widetilde{\mathbf{V}}}\mathbf{A})\big)-\frac{\partial\mathbf{M}}{\partial t})(t,\mathbf{X}(t))\Big\}\nu=0, \quad (3.63)$$

where vector field $\widetilde{\mathbf{V}}$ is such that $\widetilde{\mathbf{V}}(\mathbf{x})=\mathbf{V}(t)$ for any $\mathbf{x}$ in $\mathcal{X}$.
$\{i_{\widetilde{\mathbf{V}}}\mathbf{A}(t,\mathbf{x})\}(\nu)$ is nothing but $\{\mathbf{A}(t,\mathbf{x})\}(\widetilde{\mathbf{V}}(\mathbf{x}))$. Hence $i_{\widetilde{\mathbf{V}}}\mathbf{A}$ is nothing but the function or differential 0-form whose value in any $\mathbf{x}$ is $\{\mathbf{A}(t,\mathbf{x})\}(\widetilde{\mathbf{V}}(\mathbf{x}))$.

Making the same, but using the viewpoint used for establishing formula (3.59) or (2.17), leads to write

$$\mathbf{F}(t)=q\Big(-d\Phi(t)-\frac{\partial\mathbf{A}}{\partial t}(t)-q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(\overset{\circ}{\iota}(d\mathbf{A}(t))\big)\Big), \quad (3.64)$$

And to define

$$\mathbf{F}^{[\mathbf{A}]}(t)=q(-d\Phi(t)+q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(d(\overset{\circ}{\iota}\mathbf{A}(t))\big). \quad (3.65)$$

Then (3.60) gives

$$\Big\{-q\,d\Phi(t,\mathbf{X}(t))+q(\mathcal{D}_{(\mathbf{X}(t),\mathbf{V}(t))}\Pi)\big(d(\overset{\circ}{\iota}\mathbf{A}(t))\big)-\frac{\partial\mathbf{M}}{\partial t}(t)\Big\}\nu=0, \quad (3.66)$$

3.4 The Lagrange Function

Remembering that the Position-Velocity Space is nothing but $\mathbf{T}\mathcal{X}=\cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x},\mathbf{T}_{\mathbf{x}}\mathcal{X})$, setting on $\mathbf{T}\mathcal{X}$ the following function:

$$\begin{aligned}\mathcal{L}^{[\mathbf{A}]}:\ \mathbf{T}\mathcal{X}\ &\to\mathbb{R}\\ (\mathbf{x},\mathbf{v})\mapsto\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v})&=\bar{L}_{\mathbf{x}}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})=\frac{1}{2}m|\mathbf{v}|^2+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})\\ &=\frac{1}{2}m|\mathbf{v}|^2+q\,\overset{\circ}{\iota}_{\mathbf{v}}\mathbf{A}(t,\mathbf{x})-q\Phi(t,\mathbf{x}),\end{aligned} \quad (3.67)$$

called "Lagrange Function", the two following functions may be considered. They consist in fixing $\mathbf{v}$ or $\mathbf{x}$:

$$\begin{aligned}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{v}}:\mathcal{X}&\to\mathbb{R}\\ \mathbf{x}\ \mapsto\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v})&=\bar{L}_{\mathbf{x}}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})=\frac{1}{2}m|\mathbf{v}|^2+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})\\ &=\frac{1}{2}m|\mathbf{v}|^2+q\,\overset{\circ}{\iota}_{\mathbf{v}}\mathbf{A}(t,\mathbf{x})-q\Phi(t,\mathbf{x}),\end{aligned} \quad (3.68)$$

$$\begin{aligned}\bar{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{x}}:\mathbf{T}_{\mathbf{x}}\mathcal{X}&\to\mathbb{R}\\ \mathbf{v}\ \mapsto\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v})&=\bar{L}_{\mathbf{x}}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})=\frac{1}{2}m|\mathbf{v}|^2+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})-q\Phi(t,\mathbf{x})\\ &=\frac{1}{2}m|\mathbf{v}|^2+q\,\overset{\circ}{\iota}_{\mathbf{v}}\mathbf{A}(t,\mathbf{x})-q\Phi(t,\mathbf{x}).\end{aligned} \quad (3.69)$$

An easy computation gives

$$d_{\mathbf{x}}\widehat{\mathcal{L}}_{\mathbf{v}}^{[\mathbf{A}]} = q(\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi)\big(d(\overset{\circ}{\imath}\mathbf{A}(t))\big) - q d_{\mathbf{x}}(\Phi(t)), \tag{3.70}$$

$$d_{\mathbf{v}}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]} = d_{\mathbf{v}}\bar{L}_{\mathbf{x}} + \{q\mathbf{A}(t,\mathbf{x})\} = \mathcal{M}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{v}), \tag{3.71}$$

and, as a consequence, (3.63) or (3.66) reads

$$\Big\{\Big(\frac{\partial(d_{\mathbf{V}(t)}\bar{\mathcal{L}}_{\mathbf{X}(t)}^{[\mathbf{A}]})}{\partial t} - d_{\mathbf{X}(t)}\widehat{\mathcal{L}}_{\mathbf{V}(t)}^{[\mathbf{A}]}\ \Big)\Big\}(\nu) = 0, \tag{3.72}$$

for all ν in $\mathbf{T}_{\mathbf{X}(t)}\mathcal{X}$.

Concerning the differential $d\mathcal{L}^{[\mathbf{A}]}$ of $\mathcal{L}^{[\mathbf{A}]}$, noticing (as already done) that tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be identified with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}$, and that making such an identification leads to the fact that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be written as (ν,υ) with $\nu\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\upsilon\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$, it may set out the following formula:

$$\big\{d_{(\mathbf{x},\mathbf{v})}\mathcal{L}^{[\mathbf{A}]}\big\}(\nu,\upsilon) = \big\{d_{\mathbf{v}}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}\big\}(\upsilon) + \big\{d_{\mathbf{x}}\widehat{\mathcal{L}}_{\mathbf{v}}^{[\mathbf{A}]}\big\}(\nu). \tag{3.73}$$

3.5 The Lagrange's formulation : Hamilton's Least Action Principle

At this level, D'Alembert's Principle 2.1 may be expressed in the following way: The trajectory $\mathbf{X}(t)=\mathbf{X}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$ starting in $\mathbf{x}_0$ with velocity $\mathbf{v}_0$ at time t_0 of a particle of mass m and charge q traveling in $\mathcal{X}$ endowed with differential N-form ρ and differential 2-form $\mathbf{J}$ representing charge and current densities, is solution to (3.72) coupled with

$$\mathbf{V}(t) = \frac{\partial\mathbf{X}}{\partial t}(t), \quad \mathbf{V}(t_0) = \mathbf{v}_0, \tag{3.74}$$

where $\mathcal{L}^{[\mathbf{A}]}$, $\widehat{\mathcal{L}}_{\mathbf{v}}^{[\mathbf{A}]}$ and $\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}$ are defined by (3.67), (3.68) and (3.69) with $\mathbf{A}$ and Φ solutions to (3.5) and (3.6).

Fixing now a given time $t_1>t_0$, the position $\mathbf{X}(t_1)=\mathbf{X}(t_1;\mathbf{x}_0,\mathbf{v}_0,t_0)$ and the velocity $\mathbf{V}(t_1)=\mathbf{V}(t_1;\mathbf{x}_0,\mathbf{v}_0,t_0)$ of the particle at time t_1 may be considered. Any regular curve $\mathbf{Y}(t)=\mathbf{Y}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$ with associated velocity $\mathbf{U}(t)=\mathbf{U}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$, i.e.

$$\mathbf{U}(t) = \frac{\partial\mathbf{Y}}{\partial t}(t), \tag{3.75}$$

such that $\mathbf{Y}(t_0)=\mathbf{X}(t_0)$ and $\mathbf{Y}(t_1)=\mathbf{X}(t_1)$ may also be considered. It is clear that for any time t

$$\mathbf{U}(t) = \frac{\partial\mathbf{Y}}{\partial t}(t) \text{ is in } \mathbf{T}_{\mathbf{Y}(t)}\mathcal{X}. \tag{3.76}$$

For such a curve $\mathbf{Y}(t)$, with velocity $\mathbf{U}(t)$, in any $\mathbf{Y}(t)$, an element $\mathcal{Y}(t)$ of $\mathbf{T}_{\mathbf{Y}(t)}\mathcal{X}$ is considered making the mapping $t\mapsto(\mathbf{Y}(t),\mathcal{Y}(t))$ to be regular from $[t_0,t_1]$ to $\mathbf{T}\mathcal{X}$ and satisfying $\mathcal{Y}(t_0)=0$ in $\mathbf{T}_{\mathbf{Y}(t_0)}\mathcal{X}=\mathbf{T}_{\mathbf{X}(t_0)}\mathcal{X}$ and $\mathcal{Y}(t_1)=0$ in $\mathbf{T}_{\mathbf{Y}(t_1)}\mathcal{X}=\mathbf{T}_{\mathbf{X}(t_1)}\mathcal{X}$. The vector $\mathcal{U}(t)$, defined by

$$\mathcal{U}(t) = \frac{\partial\mathcal{Y}}{\partial t}(t), \tag{3.77}$$

is also considered.

Computing the following integral, for $\mathbf{Y}(t)$, $\mathbf{U}(t)$ and $\mathcal{Y}(t)$ previously introduced, yields

$$\int_0^1 \Big\{\frac{\partial(d_{\mathbf{U}(t)}\bar{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{Y}(t)})}{\partial t} - d_{\mathbf{Y}(t)}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{U}(t)}\Big)\Big\}(\mathcal{Y}(t))\,dt =$$

$$-\int_0^1 \Big\{d_{\mathbf{U}(t)}\bar{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{Y}(t)}\Big\}(\mathcal{U}(t)) + \Big\{d_{\mathbf{Y}(t)}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{U}(t)}\Big\}(\mathcal{Y}(t))\,dt = -\int_0^1 \Big\{d_{(\mathbf{Y}(t),\mathbf{U}(t))}\mathcal{L}^{[\mathbf{A}]}\Big\}(\mathcal{Y}(t),\mathcal{U}(t))\,dt, \tag{3.78}$$

which quantifies the variation of the following "Action Functional"

$$\mathcal{A}((\mathbf{Y}(.)) = \int_0^1 \mathcal{L}^{[\mathbf{A}]}(\mathbf{Y}(t),\mathbf{U}(t))\,dt, \tag{3.79}$$

under a trajectory variation quantified by $\mathcal{Y}(t)$. Reinterpreting now equation (3.72), it may be deduced that

$$\int_0^1 \Big\{d_{(\mathbf{X}(t),\mathbf{V}(t))}\mathcal{L}^{[\mathbf{A}]}\Big\}(\mathcal{Y}(t),\mathcal{U}(t))\,dt = 0 \tag{3.80}$$

for any trajectory variation quantified by $\mathcal{Y}(t)$. This means that trajectory $(\mathbf{X}(t),\mathbf{V}(t))$ is a critical point (and in fact a minimum) of the Action Functional.

This leads to the following first version of "Hamilton's Least Action Principle".

PRINCIPLE 3.6 *The trajectory $\mathbf{X}(t;\mathbf{x}_0,\mathbf{v}_0,t_0)$ starting in $\mathbf{x}_0$ with velocity $\mathbf{v}_0$ at time t_0 of a particle of mass m and charge q traveling in $\mathcal{X}$ endowed with differential N-form ρ and differential 2-form $\mathbf{J}$ representing charge and current densities, is the minimizer of Action Functional $\mathcal{A}$ defined by (3.79) (coupled with (3.74)) with $\mathcal{L}^{[\mathbf{A}]}$ defined by (3.67), where $\mathbf{A}$ and Φ are solutions to (3.5) and (3.6).*

4 Towards Hamiltonian Formulation

4.1 Legendre's Transform

With every function $\mathcal{L}^{[\mathbf{A}]}$ mapping $\mathbf{T}\mathcal{X}$ to $\mathbb{R}$, such that $\mathbf{v}\mapsto\mathcal{L}^{[\mathbf{A}]}(\mathbf{x},\mathbf{v})$ is convex for any $\mathbf{x}$ in $\mathcal{X}$, the following function $\mathcal{LH}$ defined on $\cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x},\mathbf{T}_\mathbf{x}\mathcal{X}\times\mathbf{T}^*_\mathbf{x}\mathcal{X})$ by

$$\mathcal{LH}(t,\mathbf{x},\mathbf{v},\mathbf{m}) = \langle\mathbf{m},\mathbf{v}\rangle - \mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v}), \tag{4.1}$$

may be associated. Then fixing $\mathbf{x}$ and $\mathbf{m}$, the value $\mathcal{V}^{[\mathbf{A}]}_\mathbf{x}(\mathbf{m})$ of $\mathbf{v}$ minimizing $\mathcal{LH}(\mathbf{x},\mathbf{v},\mathbf{m})$ may be considered. It is characterized as the value of $\mathbf{v}$ minimizing $\mathcal{LH}(\mathbf{x},\mathbf{v},\mathbf{m})$, or, since this function is regular, by differentiating, by

$$d_{(\mathcal{V}^{[\mathbf{A}]}_\mathbf{x}(\mathbf{m}))}\bar{\mathcal{L}}^{[\mathbf{A}]}_\mathbf{x} = \mathbf{m}. \tag{4.2}$$

Now the "Legendre's Transform" $\mathcal{H}^{[\mathbf{A}]}$ of $\mathcal{L}^{[\mathbf{A}]}$ is defined as mapping $\mathbf{T}^*\mathcal{X}$ to $\mathbb{R}$ by setting

$$\mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m}) = \mathcal{LH}(t,\mathbf{x},\mathcal{V}^{[\mathbf{A}]}_\mathbf{x}(\mathbf{m}),\mathbf{m}) = \langle\mathbf{m},\mathcal{V}^{[\mathbf{A}]}_\mathbf{x}(\mathbf{m})\rangle - \mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}^{[\mathbf{A}]}_\mathbf{x}(\mathbf{m})). \tag{4.3}$$

4.2 Hamiltonian Function definition

The manifold $\mathbf{T}^*_\mathbf{x}\mathcal{X}$ in which $(\mathbf{x},\mathbf{m})$ lays is called "Phase Space" and the Legendre's Transform of the Lagrange Function is defined as the "Hamiltonian Function". In other words, if in (4.1) and (4.3), $\mathcal{L}^{[\mathbf{A}]}$ is the Lagrange Function defined by (3.67), $\mathcal{H}^{[\mathbf{A}]}$ is the "Hamiltonian Function" associated with $\mathcal{L}^{[\mathbf{A}]}$ or with the question of determining the behavior of a particle of mass m and charge q traveling in $\mathcal{X}$ endowed with differential N-form ρ and differential 2-form $\mathbf{J}$ representing charge and current densities. "Hamiltonian Function" $\mathcal{H}^{[\mathbf{A}]}$ is defined on manifold $\mathbf{T}^*_\mathbf{x}\mathcal{X}$.

4.3 On Hamilton's Least Action Principle in Phase Space

Hamiltonian Function $\mathcal{H}^{[\mathbf{A}]}$ is involved in a version of the Least Action Principle which cannot be stated at this level because it needs Symplectic Geometry or Hamiltonian Mechanics material.

4.4 Hamiltonian shape of Dynamical System in Phase Space

Nevertheless, at this level, a version of Dynamical System (1.6), (1.7) or, which is equivalent, of D'Alembert's Principle (see its successive versions: (2.2), (3.9), (3.56), (3.66)) may be written in a comfortable way which involves the Hamiltonian Function.

Fixing successively $\mathbf{m}$ or $\mathbf{x}$ from $\mathcal{H}^{[\mathbf{A}]}$ the following functions are defined:

$$\begin{aligned} \widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{m}} : \mathcal{X} &\to \mathbb{R} \\ \mathbf{x} &\mapsto \mathcal{H}(\mathbf{x},\mathbf{m}), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{x}} : \mathbf{T}^*_{\mathbf{x}}\mathcal{X} &\to \mathbb{R} \\ \mathbf{m} &\mapsto \mathcal{H}(\mathbf{x},\mathbf{m}). \end{aligned} \tag{4.5}$$

Concerning the differential $d\mathcal{H}^{[\mathbf{A}]}$ of $\mathcal{H}^{[\mathbf{A}]}$, since in any $(\mathbf{x},\mathbf{m})$, tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ may be identified with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$, and since making such an identification leads to the fact that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ writes (ν,π), $\nu\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\pi\in\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$, the following may be written:

$$\big\{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}^{[\mathbf{A}]}\big\}(\nu,\pi) = \big\{d_{\mathbf{x}}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{m}}\big\}(\nu) + \big\{d_{\mathbf{m}}\bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{x}}\big\}(\pi). \tag{4.6}$$

On another hand, the differential of $\mathcal{H}$ may be computed using equality (4.3). In order to achieve this computation, the following mapping $\mathfrak{B}\mathcal{V}^{[\mathbf{A}]}$:

$$\begin{aligned} \mathfrak{B}\mathcal{V}^{[\mathbf{A}]} : \quad \mathbf{T}^*\mathcal{X} \quad &\to \quad \mathbf{T}\mathcal{X} \\ (\mathbf{x},\mathbf{m}) \quad &\mapsto \quad (\mathbf{x},\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})) = (\mathbf{x},\mathcal{V}^{[\mathbf{A}]}(\mathbf{x},\mathbf{m})), \end{aligned} \tag{4.7}$$

is defined for any time t. This definition is in the same spirit as the one of $\mathcal{M}^{[\mathbf{A}]}$ from $\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}$, see paragraph 3.2.2. For fixed $\mathbf{x}$ in $\mathcal{X}$, $\bar{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{x}}$ is $\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}$ seen as a function on $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$:

$$\begin{aligned} \bar{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{x}} : \quad \mathbf{T}^*_{\mathbf{x}}\mathcal{X} \quad &\to \quad \mathbf{T}_{\mathbf{x}}\mathcal{X} \\ \mathbf{m} \quad &\mapsto \quad \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}), \end{aligned} \tag{4.8}$$

and for fixed $\mathbf{m}$, $\mathcal{V}^{[\mathbf{A}]}_{\mathbf{m}}$ is defined as:

$$\begin{aligned} \widehat{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{m}} : \quad \mathcal{X} \quad &\to \quad \mathbf{T}_{\mathbf{x}}\mathcal{X} \\ \mathbf{x} \quad &\mapsto \quad \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}). \end{aligned} \tag{4.9}$$

Then, from (4.3),

$$\begin{aligned} \big\{d_{\mathbf{x}}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{m}}\big\}(\nu) &= \big\langle \mathbf{m}, \{d_{\mathbf{x}}\widehat{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{m}}\}(\nu)\big\rangle - \big\{d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\big\}(\nu) - \big\{d_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\bar{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{x}}\big\}\big(\{d_{\mathbf{x}}\widehat{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{m}}\}(\nu)\big) \\ &= -\big\{d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\big\}(\nu), \end{aligned} \tag{4.10}$$

the last equality being gotten from (4.2); or, in other words,

$$d_{\mathbf{x}}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{m}} = -d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))} . \tag{4.11}$$

In this writing, $d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}$ stands for application $d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{v}}$ computed in $\mathbf{v} = \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})$. (It could be written that $d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))} = (d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]})_{\mathbf{v}=\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})}$ and not $d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))} = d_{\mathbf{x}}(\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})})$.)
In the same spirit,

$$\begin{aligned}\big\{d_{\mathbf{m}}\bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{x}}\big\}(\pi) = \langle \pi, \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})\rangle + \big\langle \mathbf{m}, \{d_{\mathbf{m}}\bar{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{x}}\}(\pi)\big\rangle - \big\{d_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\bar{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{x}}\big\}\big(\{d_{\mathbf{m}}\bar{\mathcal{V}}^{[\mathbf{A}]}_{\mathbf{x}}\}(\pi)\big) \\ = \langle \pi, \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})\rangle, \quad (4.12)\end{aligned}$$

or, in other words,

$$d_{\mathbf{m}}\bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{x}} = \langle\, \cdot\, , \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})\rangle. \tag{4.13}$$

This last equation means that $d_{\mathbf{m}}\mathcal{H}_{\mathbf{x}}$ may be represented by an element of $\mathbf{T}_{\mathbf{x}}\mathcal{X}$: $\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})$, by the help of the duality product between $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and $\mathbf{T}_{\mathbf{x}}\mathcal{X}$.
Using (4.11) and (4.13) in (4.6) give

$$\big\{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}^{[\mathbf{A}]}\big\}(\nu,\pi) = -\big\{d_{\mathbf{x}}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\big\}(\nu) + \langle \pi, \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})\rangle. \tag{4.14}$$

From the equality just gotten, it is possible to deduce an expression of the system satisfied by the Phase Space Trajectory $(\mathbf{X}(t), \mathbf{M}(t))$. This will be done in three step. The first one consists in noticing that from the following equality

$$d_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m}))}\bar{L}_{\mathbf{x}} + q\mathbf{A}(t,\mathbf{x}) = \mathbf{m}, \tag{4.15}$$

which is a direct rewriting of (4.2), and from (3.11) it may be deduced that for any $\mathbf{x}$ laying in $\mathcal{X}$, any $\mathbf{m}$ in $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and any $\mathbf{v}$ in $\mathbf{T}_{\mathbf{x}}\mathcal{X}$,

$$\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{m})) = \mathbf{m} \quad \text{and} \quad \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v})) = \mathbf{v}, \tag{4.16}$$

meaning that $\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}$ and $\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}$ are reverse functions one of each other.
In the second step, from (3.66) and (3.70) it may be set that, for any t and any ν of $\mathbf{T}_{\mathbf{X}(t)}\mathcal{X}$,

$$\Big\{\frac{\partial \mathbf{M}}{\partial t}(t) - d_{\mathbf{X}(t)}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{\mathbf{V}(t)}\Big\}(\nu) = 0, \tag{4.17}$$

or

$$\Big\{\frac{\partial \mathbf{M}}{\partial t}(t) - d_{\mathbf{X}(t)}\widehat{\mathcal{L}}^{[\mathbf{A}]}_{(\mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{M}(t)))}\Big\}(\nu) = 0, \tag{4.18}$$

or, using (4.11),

$$\Big\{\frac{\partial \mathbf{M}}{\partial t}(t) + d_{\mathbf{X}(t)}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{M}(t)}\Big\}(\nu) = 0. \tag{4.19}$$

In the third step, it needs to be noticed that

$$\frac{\partial \mathbf{X}}{\partial t}(t) = \mathbf{V}(t) = \mathcal{V}^{[\mathbf{A}]}_{\mathbf{X}(t)}(\mathbf{M}(t)), \tag{4.20}$$

and then, (4.13) yields

$$\Big\langle \pi, \frac{\partial \mathbf{X}}{\partial t}(t)\Big\rangle = \big\{d_{\mathbf{M}(t)}\bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{X}(t)}\big\}(\pi), \tag{4.21}$$

for any π in $\mathbf{T}^*_{\mathbf{X}(t)}\mathcal{X}$.
(4.19), (4.21) is the standard writing of dynamical system (1.6), (1.7) in Hamiltonian shape.

4.5 Usual Hamiltonian shape in coordinate systems of Dynamical System in Phase Space

Let $\breve{\mathcal{H}}^{[\mathbf{A}]}$ be the expression of $\mathcal{H}^{[\mathbf{A}]}$ in coordinate systems $(\mathbf{q},\mathbf{p})$ induced by charts $\mathcal{C}:\mathcal{B}(\mathbf{x}_0)\to\mathbb{R}^N$, $d_{\mathbf{x}}\mathcal{C}$ and $\mathcal{D}_{\mathbf{x}}\mathcal{C}$ (see (3.14), (3.15) and (3.17)) in a neighborhood of initial position $\mathbf{x}_0$ ($\mathbf{X}(t_0;\mathbf{x}_0,\mathbf{v}_0,t_0)=\mathbf{x}_0$), meaning

$$\mathcal{H}^{[\mathbf{A}]}(\mathbf{x},\mathbf{m})=\breve{\mathcal{H}}^{[\mathbf{A}]}(\mathbf{q},\mathbf{p}),\ \text{with}\ (\mathbf{q},\mathbf{p})=(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m}). \tag{4.22}$$

As soon as $\mathbf{X}(t)$ remains in $\mathcal{B}(\mathbf{x}_0)$, the coordinates $(\mathbf{Q}(t),\mathbf{P}(t))$ of $(\mathbf{X}(t),\mathbf{M}(t))$ (i.e. $(\mathbf{Q}(t),\mathbf{P}(t))=(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{X}(t),\mathbf{M}(t))$) is the solution of the translation of (4.19) and (4.21) in these coordinate systems, i.e.

$$\frac{\partial\mathbf{P}}{\partial t}(t)=-(\nabla_{\mathbf{q}}\breve{\mathcal{H}}^{[\mathbf{A}]})(t,\mathbf{Q}(t),\mathbf{P}(t)), \tag{4.23}$$

$$\frac{\partial\mathbf{Q}}{\partial t}(t)=(\nabla_{\mathbf{p}}\breve{\mathcal{H}}^{[\mathbf{A}]})(t,\mathbf{Q}(t),\mathbf{P}(t)), \tag{4.24}$$

which usually reads:

$$\frac{\partial\mathbf{P}}{\partial t}(t)=-\frac{\partial\breve{\mathcal{H}}^{[\mathbf{A}]}}{\partial\mathbf{q}}(t,\mathbf{Q}(t),\mathbf{P}(t)), \tag{4.25}$$

$$\frac{\partial\mathbf{Q}}{\partial t}(t)=\frac{\partial\breve{\mathcal{H}}^{[\mathbf{A}]}}{\partial\mathbf{p}}(t,\mathbf{Q}(t),\mathbf{P}(t)). \tag{4.26}$$

4.6 Expression of Hamiltonian Function I

Using (2.3) of $\bar{L}_{\mathbf{x}}(\mathbf{v})$ and the fact that $\mathcal{X}$ is a Riemannian Manifold, an expression of Hamiltonian Function $\mathcal{H}$ may be given. Indeed from (2.3) and (3.71),

$$\big\{d_{\mathbf{v}}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}\big\}(\mathbf{v})=\big\{d_{\mathbf{v}}\bar{L}_{\mathbf{x}}\big\}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})=2\bar{L}_{\mathbf{x}}(\mathbf{v})+\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v}). \tag{4.27}$$

Then from (3.68),

$$2\bar{L}_{\mathbf{x}}(\mathbf{v})=2\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v})-2\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})+2q\Phi(t,\mathbf{x}). \tag{4.28}$$

Hence,

$$\big\{d_{\mathbf{v}}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}\big\}(\mathbf{v})=2\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v})-\{q\mathbf{A}(t,\mathbf{x})\}(\mathbf{v})+2q\Phi(t,\mathbf{x}), \tag{4.29}$$

and consequently, because of (4.2),

$$\begin{aligned}\langle\mathbf{m},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m})\rangle&=\big\{(d_{\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}\big\}(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))\\&=2\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))-\{q\mathbf{A}(t,\mathbf{x})\}(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))+2q\Phi(t,\mathbf{x}).\end{aligned} \tag{4.30}$$

Using expression (4.3) of Hamiltonian Function $\mathcal{H}$ and (4.30) yields

$$\begin{aligned}\mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m})&=2\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))-\{q\mathbf{A}(t,\mathbf{x})\}(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))+2q\Phi(t,\mathbf{x})-\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))\\&=\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))-\{q\mathbf{A}(t,\mathbf{x})\}(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))+2q\Phi(t,\mathbf{x})\\&=\bar{L}_{\mathbf{x}}(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))+q\Phi(t,\mathbf{x}).\end{aligned} \tag{4.31}$$

4.7 Expression of Hamiltonian Function II

In order to give an expression of (4.31) within coordinate systems taking into account that $\mathcal{X}$ is flat, $\breve{L}_{\mathbf{q}}$ defined by (3.22), $\breve{\mathbf{A}}$ defined by (3.21) and $\breve{\Phi}$ defined, for any time t, by

$$\Phi(t,\mathbf{x}) = \breve{\Phi}(t,\mathbf{q}), \quad \text{with} \quad \mathbf{q} = \mathcal{C}\mathbf{x}, \tag{4.32}$$

are used. This leads to the following definition of $\breve{\mathcal{L}}^{[\mathbf{A}]}$

$$\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v}) = \breve{\mathcal{L}}^{[\mathbf{A}]}(t,\mathbf{q},\dot{\mathbf{q}}), \quad \text{with} \quad (\mathbf{q},\dot{\mathbf{q}}) = (\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{v}). \tag{4.33}$$

Expression of $\breve{\mathcal{L}}^{[\mathbf{A}]}$ is the following:

$$\breve{\mathcal{L}}^{[\mathbf{A}]}(t,\mathbf{q},\dot{\mathbf{q}}) = \breve{L}_{\mathbf{q}}(\dot{\mathbf{q}}) + \{q\breve{\mathbf{A}}(t,\mathbf{q})\}(\dot{\mathbf{q}}) - q\breve{\Phi}(t,\mathbf{q}) = \frac{1}{2}m|\dot{\mathbf{q}}|^2 + q\breve{\mathbf{A}}(t,\mathbf{q})\cdot\dot{\mathbf{q}} - q\breve{\Phi}(t,\mathbf{q}), \tag{4.34}$$

$(\nabla_{\mathbf{q}}\breve{\mathcal{L}}^{[\mathbf{A}]})(t,\mathbf{q},\dot{\mathbf{q}})$ and $(\nabla_{\dot{\mathbf{q}}}\breve{\mathcal{L}}^{[\mathbf{A}]})(t,\mathbf{q},\dot{\mathbf{q}})$ are expression of $d_{\mathbf{x}}\widehat{\mathcal{L}_{\mathbf{v}}}^{[\mathbf{A}]}$ and $d_{\mathbf{v}}\bar{\mathcal{L}}_{\mathbf{x}}^{[\mathbf{A}]}$ within the coordinates and

$$(\nabla_{\dot{\mathbf{q}}}\breve{\mathcal{L}}^{[\mathbf{A}]})(t,\mathbf{q},\dot{\mathbf{q}}) = m\dot{\mathbf{q}} + q\breve{\mathbf{A}}(t,\mathbf{q}). \tag{4.35}$$

Hence, defining $\breve{\mathcal{V}}_{\mathbf{q}}^{[\mathbf{A}]}$ as the representation of $\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}$ within the coordinates, i.e.

$$\begin{aligned}(\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathfrak{B}\mathcal{V}^{[\mathbf{A}]}(\mathbf{x},\mathbf{m})) &= (\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m})) = (\mathcal{C}, d_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathcal{V}^{[\mathbf{A}]}(\mathbf{x},\mathbf{m}))\\ &= \mathfrak{B}\breve{\mathcal{V}}^{[\mathbf{A}]}(\mathbf{q},\mathbf{p}) = (\mathbf{q},\breve{\mathcal{V}}_{\mathbf{q}}^{[\mathbf{A}]}(\mathbf{p})) = (\mathbf{q},\breve{\mathcal{V}}^{[\mathbf{A}]}(\mathbf{q},\mathbf{p})), \quad \text{with} \quad (\mathbf{q},\mathbf{p}) = (\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m}),\end{aligned} \tag{4.36}$$

(4.2) yields

$$\breve{\mathcal{V}}_{\mathbf{q}}^{[\mathbf{A}]}(\mathbf{p}) = m^{-1}\big(\mathbf{p} - q\breve{\mathbf{A}}(t,\mathbf{q})\big), \tag{4.37}$$

which is the reverse of

$$\breve{\mathcal{M}}_{\mathbf{q}}^{[\mathbf{A}]}(\dot{\mathbf{q}}) = m\dot{\mathbf{q}} + q\breve{\mathbf{A}}(t,\mathbf{q}). \tag{4.38}$$

Finally, writing (4.31) within the coordinates gives

$$\breve{\mathcal{H}}^{[\mathbf{A}]}(t,\mathbf{q},\mathbf{p}) = \breve{L}_{\mathbf{q}}(\breve{\mathcal{V}}_{\mathbf{q}}^{[\mathbf{A}]}(\mathbf{p})) + q\Phi(t,\mathbf{x}) = \frac{1}{2m}\big|\mathbf{p} - q\breve{\mathbf{A}}(t,\mathbf{q})\big|^2 + q\Phi(t,\mathbf{q}). \tag{4.39}$$

Part II

Hamiltonian formulation of dynamics of a charged particle in an electromagnetic field

5 What does mean making "Hamiltonian Mechanics", or equivalently, "Symplectic Geometry"

Making "Symplectic Geometry" consists in considering a manifold provided with a closed non-degenerated differential 2-form which is called a "Symplectic Manifold".
If $\mathcal{X}$ is a manifold, its cotangent bundle $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ is naturally provided with a closed non-degenerated differential 2-form making of it a "Symplectic Manifold". Then making Hamiltonian Machanics consits in making Symplectic Geometry on $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$.

5.1 Symplectic structure of $\mathbf{T}^*\mathcal{X}$

5.1.1 Tangent and Cotangent Bundles

In the previous part, position space $\mathcal{X}$ was $\mathbb{R}^N$ with $N = 3$, 2 or 1. In most results and computations previously led, the fact that $\mathcal{X}$ was a flat Riemannian manifold was not necessary.
From now, $\mathcal{X}$ is considered as a differential manifold of dimension N, in any point $\mathbf{x}$ of $\mathcal{X}$ lays its tangent space $\mathbf{T_x}\mathcal{X}$ and stands its cotangent space $\mathbf{T^*_x}\mathcal{X}$. From them may be defined the "Tangent Bundle"

$$\mathbf{T}\mathcal{X} = \cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x}, \mathbf{T_x}\mathcal{X}), \tag{5.1}$$

and the "Cotangent Bundle"

$$\mathbf{T}^*\mathcal{X} = \cup_{\mathbf{x}\in\mathcal{X}}(\mathbf{x}, \mathbf{T^*_x}\mathcal{X}). \tag{5.2}$$

$\mathbf{T}\mathcal{X}$ and $\mathbf{T}^*\mathcal{X}$ may be seen as differentiable manifolds of dimension $2N$ using charts $(\mathcal{C}, d_{\mathbf{x}}\mathcal{C})$ defined by (3.14), (3.15) and (3.16) and $(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})$ defined by (3.14), (3.17) and (3.20).

As already noticed in the case $\mathcal{X} = \mathbb{R}^N$, in any $(\mathbf{x}, \mathbf{v})$ of $\mathbf{T}\mathcal{X}$, tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be identified with $\mathbf{T_x}\mathcal{X} \times \mathbf{T_x}\mathcal{X}$, meaning that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ may be written as (ν, υ) with $\nu \in \mathbf{T_x}\mathcal{X}$ and $\upsilon \in \mathbf{T_x}\mathcal{X}$. In a similar way, in any $(\mathbf{x}, \mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ may be identified with $\mathbf{T_x}\mathcal{X} \times \mathbf{T^*_x}\mathcal{X}$, and making this identification leads that any vector of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ writes (ν, π) with $\nu \in \mathbf{T_x}\mathcal{X}$ and $\pi \in \mathbf{T^*_x}\mathcal{X}$.

5.1.2 Natural differential 1-forme γ on $\mathbf{T}^*\mathcal{X}$

In any $(\mathbf{x}, \mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, to any (ν, π) of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ $(= \mathbf{T_x}\mathcal{X} \times \mathbf{T^*_x}\mathcal{X})$, it may be associated the following number in a natural way

$$\mathbf{m}(\nu) = \langle \mathbf{m}, \nu \rangle. \tag{5.3}$$

This defines a natural differential 1-form γ on $\mathbf{T}^*\mathcal{X}$ whose value $\{\gamma(\mathbf{x}, \mathbf{m})\}$ in any $(\mathbf{x}, \mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$ is given by

$$\{\gamma(\mathbf{x}, \mathbf{m})\}(\nu, \pi) = \mathbf{m}(\nu) = \langle \mathbf{m}, \nu \rangle, \tag{5.4}$$

for any (ν, π) of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$.
Expression of differential 1-form γ within the coordinate systems may be given. (It will be given is the case $N = 3$.) For this, considering a given $\mathbf{x}_0$ of $\mathcal{X}$, a neighborhood $\mathcal{B}(\mathbf{x}_0)$ of $\mathbf{x}_0$ and the charts $(\mathcal{C}, d_{\mathbf{x}}\mathcal{C})$ defined by (3.14) (with $\mathcal{B}(\mathbf{X}(t))$ replaced by $\mathcal{B}(\mathbf{x}_0)$), (3.15) and (3.16) (with $\mathcal{B}(\mathbf{X}(t))$ replaced by $\mathcal{B}(\mathbf{x}_0)$) and $(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})$ defined by (3.14),(with $\mathcal{B}(\mathbf{X}(t))$ replaced by $\mathcal{B}(\mathbf{x}_0)$), (3.17) and (3.20) (with $\mathcal{B}(\mathbf{X}(t))$ replaced by $\mathcal{B}(\mathbf{x}_0)$). Using this, for all $\mathbf{x}$ in $\mathcal{B}(\mathbf{x}_0)$ and all $\mathbf{m}$ in $\mathbf{T^*_x}\mathcal{X}$, $(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \mathbf{m}) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3$ and then

$$(\mathcal{C}, \mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x}, \gamma(\mathbf{x}, \mathbf{m})) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3, \tag{5.5}$$

or in short

$$\mathcal{D}_{\mathbf{x}}\mathcal{C}(\gamma(\mathbf{x}, \mathbf{m})) = \mathbf{p}_1 d\mathbf{q}_1 + \mathbf{p}_2 d\mathbf{q}_2 + \mathbf{p}_3 d\mathbf{q}_3, \tag{5.6}$$

and in shorter

$$(\mathcal{D}_{\mathbf{x}}\mathcal{C})(\gamma(\mathbf{x}, \mathbf{m})) = \mathbf{p}\ d\mathbf{q}, \tag{5.7}$$

which is only a notation saying

$$\big\{(\mathcal{D}_{\mathbf{x}}\mathcal{C})(\gamma(\mathbf{x},\mathbf{m}))\big\}((d_{(\mathbf{x},\mathbf{m})}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C}))(\nu,\pi)) = \big\{(\mathcal{D}_{\mathbf{x}}\mathcal{C})(\gamma(\mathbf{x},\mathbf{m}))\big\}((d_{\mathbf{x}}\mathcal{C})\nu,(\mathcal{D}_{\mathbf{x}}\mathcal{C})\pi))$$
$$= \big\{\mathbf{p}\ d\mathbf{q}\big\}(\breve{\nu},\breve{\pi}) = \big\{\mathbf{p}\ d\mathbf{q}\big\}(\breve{\nu}_1\partial_{\mathbf{q}_1}+\breve{\nu}_2\partial_{\mathbf{q}_2}+\breve{\nu}_3\partial_{\mathbf{q}_3}+\breve{\pi}_1 d\mathbf{q}_1+\breve{\pi}_2 d\mathbf{q}_2+\breve{\pi}_3 d\mathbf{q}_3)$$
$$= \mathbf{p}_1\breve{\nu}_1+\mathbf{p}_2\breve{\nu}_2+\mathbf{p}_3\breve{\nu}_3. \quad (5.8)$$

In other words the differential 1-form $\breve{\gamma}$ defined on $(\mathcal{C},d_{\mathbf{x}}\mathcal{C})\big(\cup_{\mathbf{x}\in\mathcal{B}(\mathbf{x}_0)}(\mathbf{x},\mathbf{T}_{\mathbf{x}}\mathcal{X})\big)\subset\mathbb{R}^N\times\mathbb{R}^N$ and representing γ, meaning

$$\{\gamma(\mathbf{x},\mathbf{m})\}(\nu,\pi) = \{\breve{\gamma}(\mathbf{q},\mathbf{p}))\}((d_{\mathbf{x}}\mathcal{C})\nu,(\mathcal{D}_{\mathbf{x}}\mathcal{C})\pi)),\ \text{with } (\mathbf{q},\mathbf{p}) = (\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m}), \quad (5.9)$$

is

$$\breve{\gamma}(\mathbf{q},\mathbf{p}) = \mathbf{p}\ d\mathbf{q}. \quad (5.10)$$

5.1.3 Natural differential 2-forme ω on $\mathbf{T}^*\mathcal{X}$

In any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, to any couple $((\nu,\pi),(\nu',\pi'))$ of $(\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big))^2$ $(=(\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X})^2)$, it may be associated the following number:

$$\pi(\nu')-\pi'(\nu) = \langle\pi,\nu'\rangle-\langle\pi',\nu\rangle, \quad (5.11)$$

This is clearly a two-form on $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$, then formula (5.8) defines a differential 2-form ω on $\mathbf{T}^*\mathcal{X}$ whose value $\{\omega(\mathbf{x},\mathbf{m})\}$ in any $(\mathbf{x},\mathbf{m})$ of $\mathcal{X}$ is given by

$$\{\omega(\mathbf{x},\mathbf{m})\}((\nu,\pi),(\nu',\pi')) = \pi(\nu')-\pi'(\nu) = \langle\pi,\nu'\rangle-\langle\pi',\nu\rangle, \quad (5.12)$$

for any $((\nu,\pi),(\nu',\pi'))$ of $(\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big))^2$.
Expression of differential 2-form ω within the coordinates will be given in the case when $N=3$. For this chart $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$, already used lines above, will be used. Its differential $d_{(\mathbf{x},\mathbf{m})}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$, which is the induced chart on $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$, is also used. Using (as usual) identification $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big) = (\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$ and the fact that $d_{(\mathbf{x},\mathbf{m})}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C}) = (d_{\mathbf{x}}\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$,

$$d_{(\mathbf{x},\mathbf{m})}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\nu,\pi) = (d_{\mathbf{x}}\mathcal{C}(\nu),\mathcal{D}_{\mathbf{x}}\mathcal{C}(\pi)) = \breve{\nu}_1\partial_{\mathbf{q}_1}+\breve{\nu}_2\partial_{\mathbf{q}_2}+\breve{\nu}_3\partial_{\mathbf{q}_3}+\breve{\pi}_1 d\mathbf{q}_1+\breve{\pi}_2 d\mathbf{q}_2+\breve{\pi}_3 d\mathbf{q}_3, \quad (5.13)$$

$$d_{(\mathbf{x},\mathbf{m})}(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\nu',\pi') = (d_{\mathbf{x}}\mathcal{C}(\nu'),\mathcal{D}_{\mathbf{x}}\mathcal{C}(\pi')) = \breve{\nu}'_1\partial_{\mathbf{q}_1}+\breve{\nu}'_2\partial_{\mathbf{q}_2}+\breve{\nu}'_3\partial_{\mathbf{q}_3}+\breve{\pi}'_1 d\mathbf{q}_1+\breve{\pi}'_2 d\mathbf{q}_2+\breve{\pi}'_3 d\mathbf{q}_3, \quad (5.14)$$

and in view of (5.12),

$$\{\omega(\mathbf{x},\mathbf{m})\}((\nu,\pi),(\nu',\pi')) = \breve{\pi}_1\breve{\nu}'_1+\breve{\pi}_2\breve{\nu}'_2+\breve{\pi}_3\breve{\nu}'_3-\breve{\pi}'_1\breve{\nu}_1-\breve{\pi}'_2\breve{\nu}_2-\breve{\pi}'_3\breve{\nu}_3$$
$$= \begin{vmatrix}\breve{\pi}_1 & \breve{\nu}_1\\ \breve{\pi}'_1 & \breve{\nu}'_1\end{vmatrix}+\begin{vmatrix}\breve{\pi}_2 & \breve{\nu}_2\\ \breve{\pi}'_2 & \breve{\nu}'_2\end{vmatrix}+\begin{vmatrix}\breve{\pi}_3 & \breve{\nu}_3\\ \breve{\pi}'_3 & \breve{\nu}'_3\end{vmatrix}$$
$$= \begin{vmatrix}d\mathbf{p}_1((\breve{\nu},\breve{\pi})) & d\mathbf{q}_1((\breve{\nu},\breve{\pi}))\\ d\mathbf{p}_1((\breve{\nu}',\breve{\pi}')) & d\mathbf{q}_1((\breve{\nu}',\breve{\pi}'))\end{vmatrix}+\begin{vmatrix}d\mathbf{p}_2((\breve{\nu},\breve{\pi})) & d\mathbf{q}_2((\breve{\nu},\breve{\pi}))\\ d\mathbf{p}_2((\breve{\nu}',\breve{\pi}')) & d\mathbf{q}_2((\breve{\nu}',\breve{\pi}'))\end{vmatrix}+\begin{vmatrix}d\mathbf{p}_3((\breve{\nu},\breve{\pi})) & d\mathbf{q}_3((\breve{\nu},\breve{\pi}))\\ d\mathbf{p}_3((\breve{\nu}',\breve{\pi}')) & d\mathbf{q}_3((\breve{\nu}',\breve{\pi}'))\end{vmatrix}$$
$$= d\mathbf{p}_1\wedge d\mathbf{q}_1((\breve{\nu},\breve{\pi}),((\breve{\nu}',\breve{\pi}'))+d\mathbf{p}_2\wedge d\mathbf{q}_2((\breve{\nu},\breve{\pi}),(\breve{\nu}',\breve{\pi}'))+d\mathbf{p}_3\wedge d\mathbf{q}_3((\breve{\nu},\breve{\pi}),(\breve{\nu}',\breve{\pi}'))$$
$$= d\mathbf{p}\wedge d\mathbf{q}((\breve{\nu},\breve{\pi}),(\breve{\nu}',\breve{\pi}')), \quad (5.15)$$

with the convention that $d\mathbf{p}\wedge d\mathbf{q}$ is only a notation for what precedes.
In other words the differential 2-form $\breve{\omega}$ defined on $\cup_{\mathbf{x}\in\mathcal{B}(\mathbf{x}_0)}(\mathbf{x},\mathbf{T}_{\mathbf{x}}\mathcal{X})\subset\mathbb{R}^N\times\mathbb{R}^N$ and representing ω, meaning

$$\{\omega(\mathbf{x},\mathbf{m})\}((\nu,\pi),(\nu',\pi'))=\{\breve{\omega}(\mathbf{q},\mathbf{p}))\}\big(((d_{\mathbf{x}}\mathcal{C})\nu,(\mathcal{D}_{\mathbf{x}}\mathcal{C})\pi)),((d_{\mathbf{x}}\mathcal{C})\nu',(\mathcal{D}_{\mathbf{x}}\mathcal{C})\pi')\big),$$
$$\text{with } (\mathbf{q},\mathbf{p})=(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m}), \quad (5.16)$$

is

$$\breve{\omega}(\mathbf{q},\mathbf{p})=d\mathbf{p}\wedge d\mathbf{q}. \quad (5.17)$$

This last equality is a chart-dependant computation that shows the following chart-independant result:

$$\omega=d\gamma. \quad (5.18)$$

5.1.4 Other differential 1-forme on $\mathbf{T}^*\mathcal{X}$: $\gamma^{[\mathbf{A}]}$

If a regular time dependant differential 1-form $\mathbf{A}(t)$ is defined on $\mathcal{X}$, it is not forbidden to associate, in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, to any (ν,π) of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ $(=\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$ the following number

$$\{\gamma^{[\mathbf{A}]}(t,\mathbf{x}\}(\nu)=\mathbf{m}(\nu)+\{q\mathbf{A}(t,\mathbf{x})\}(\nu)=\langle\mathbf{m},\nu\rangle+\{q\mathbf{A}(t,\mathbf{x})\}(\nu). \quad (5.19)$$

This defines the differential 1-form $\gamma^{[\mathbf{A}]}$ on $\mathbf{T}^*\mathcal{X}$ whose expression in the coordinate systems is

$$(\mathcal{D}_{\mathbf{x}}\mathcal{C})(\gamma^{[\mathbf{A}]}(\mathbf{x},\mathbf{m}))=\breve{\gamma}^{[\mathbf{A}]}(\mathbf{q},\mathbf{p})=(\mathbf{p}_1+q\breve{\mathbf{A}}_1(t))d\mathbf{q}_1+(\mathbf{p}_2+q\breve{\mathbf{A}}_2(t))d\mathbf{q}_2+(\mathbf{p}_3+q\breve{\mathbf{A}}_3(t))d\mathbf{q}_3$$
$$=(\mathbf{p}+q\breve{\mathbf{A}}(t))\;d\mathbf{q}, \quad (5.20)$$

where $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m})=\mathbf{q}_1\mathbf{e}_{\mathbf{q}_1}+\mathbf{q}_2\mathbf{e}_{\mathbf{q}_2}+\mathbf{q}_3\mathbf{e}_{\mathbf{q}_3}+\mathbf{p}_1d\mathbf{q}_1+\mathbf{p}_2d\mathbf{q}_2+\mathbf{p}_3d\mathbf{q}_3$ and $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{A}(t,\mathbf{x}))=\mathbf{q}_1\mathbf{e}_{\mathbf{q}_1}+\mathbf{q}_2\mathbf{e}_{\mathbf{q}_2}+\mathbf{q}_3\mathbf{e}_{\mathbf{q}_3}+\breve{\mathbf{A}}_1(t,\mathbf{q})d\mathbf{q}_1+\breve{\mathbf{A}}_2(t,\mathbf{q})d\mathbf{q}_2+\breve{\mathbf{A}}_3(t,\mathbf{q})d\mathbf{q}_3$.

5.1.5 Intrinsic expression of differential 1-forms $\gamma^{[\mathbf{A}]}$ in terms of Pullback

It may be defined the following projection:

$$\begin{array}{rccc}\Pi^*: & \mathbf{T}^*\mathcal{X} & \to & \mathcal{X}\\ & (\mathbf{x},\mathbf{m}) & \mapsto & \mathbf{x}\ ,\end{array} \quad (5.21)$$

the Pushforward by Π^*: $d_{(\mathbf{x},\mathbf{m})}\Pi^*$ in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$:

$$\begin{array}{rccc}d_{(\mathbf{x},\mathbf{m})}\Pi^*: & \mathbf{T}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}^*\mathcal{X}\big) & \to & \mathbf{T}_{\mathbf{x}}\mathcal{X}\\ & (\nu,\pi) & \mapsto & \nu\ ,\end{array} \quad (5.22)$$

the Pullback by Π^*: $\Pi^*{}_{*(\mathbf{x},\mathbf{m})}$ in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$:

$$\begin{array}{rccc}\Pi^*{}_{*(\mathbf{x},\mathbf{m})}: & \mathbf{T}^*_{\mathbf{x}}\mathcal{X} & \to & \mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)\\ & \mu & \mapsto & \Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\mu)\ ,\end{array} \quad (5.23)$$

with, for any (ν,π) of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ $(=\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$

$$\langle\Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\mu),(\nu,\pi)\rangle=\{\Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\mu)\}(\nu,\pi)=\mu(\nu)=\langle\mu,\nu\rangle, \quad (5.24)$$

and the Pullback by Π^*

$$\begin{array}{cccc} \Pi^*{}_* : & \Lambda(\mathcal{X}) & \rightarrow & \Lambda\big(\mathbf{T}^*\mathcal{X}\big) \\ & \mathbf{A} & \mapsto & \Pi^*{}_*(\mathbf{A}) \ , \end{array} \tag{5.25}$$

defined by

$$(\Pi^*{}_*(\mathbf{A}))(\mathbf{x},\mathbf{m}) = \Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\mathbf{A}(\mathbf{x})). \tag{5.26}$$

As a mater of fact, in view of (5.19), it may be written:

$$\gamma^{[\mathbf{A}]}(t) = \gamma + \Pi^*{}_*(q\mathbf{A}(t)) = \Pi^*{}_*(\mathbf{m}) + \Pi^*{}_*(q\mathbf{A}(t)) \tag{5.27}$$

5.1.6 On Pullback of k-forms

In the previous paragraph, Pullbacks by Π^* of 1-forms of $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ to 1-forms of $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ in a given point $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$ and of differential 1-forms on $\mathcal{X}$ to differential 1-forms on $\mathbf{T}^*\mathcal{X}$ are defined. Those Pullback definitions may be extended to k-forms for any k.
In the following $\mathbf{T}^{*k}_{\mathbf{x}}\mathcal{X}$ stands for the set of k-forms on $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\Lambda^k(\mathcal{X})$ for the set of differential k-forms defined on $\mathcal{X}$.
Then Pullback $\Pi^*{}_{*(\mathbf{x},\mathbf{m})}$ in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$ may be extended in the following way:

$$\begin{array}{cccc} \Pi^*{}_{*(\mathbf{x},\mathbf{m})} : & \mathbf{T}^{*k}_{\mathbf{x}}\mathcal{X} & \rightarrow & \mathbf{T}^{*\,k}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big) \\ & \kappa & \mapsto & \Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\kappa) \ , \end{array} \tag{5.28}$$

where for any $((\nu_1,\pi_1),\ldots,(\nu_k,\pi_k))$ of $(\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big))^k$

$$\{\Pi^*{}_{*(\mathbf{x},\mathbf{v})}(\kappa)\}((\nu_1,\pi_1),\ldots,(\nu_k,\pi_k)) = \{\kappa\}(\nu_1,\ldots\nu_k). \tag{5.29}$$

In the same way, Pullback $\Pi^*{}_*$ may be extended as

$$\begin{array}{cccc} \Pi^*{}_* : & \Lambda^k(\mathcal{X}) & \rightarrow & \Lambda^k\big(\mathbf{T}^*\mathcal{X}\big) \\ & \mathbf{K} & \mapsto & \Pi^*{}_*(\mathbf{K}) \ , \end{array} \tag{5.30}$$

defined, in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$ as

$$(\Pi^*{}_*(\mathbf{K}))(\mathbf{x},\mathbf{m}) = \Pi^*{}_{*(\mathbf{x},\mathbf{m})}(\mathbf{K}(\mathbf{x})). \tag{5.31}$$

5.1.7 Expression of Pullback of 1-forms and 2-forms within coordinate systems when $N=3$

Considering again a given $\mathbf{x}_0$ of $\mathcal{X}$, a neighborhood $\mathcal{B}(\mathbf{x}_0)$ of $\mathbf{x}_0$ and charts from $\mathcal{B}(\mathbf{x}_0)$ to $\mathbb{R}^N$ with associated frames $(\mathbf{e}_{\mathbf{q}_1},\mathbf{e}_{\mathbf{q}_2},\mathbf{e}_{\mathbf{q}_3})$ to locate position points, $(\partial_{\mathbf{q}_1},\partial_{\mathbf{q}_2},\partial_{\mathbf{q}_3})$ for velocity vectors, $(d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3)$ for differential 1-forms, (where at this level $d\mathbf{q}_1$ is defined as the mapping $\mathbf{v}\mapsto\dot{\mathbf{q}}_1$ from $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ to $\mathbb{R}^N$, etc.), $(d\mathbf{q}_3\wedge d\mathbf{q}_1, d\mathbf{q}_1\wedge d\mathbf{q}_2, d\mathbf{q}_2\wedge d\mathbf{q}_3)$ for differential 2-forms and $(d\mathbf{q}_1\wedge d\mathbf{q}_2\wedge d\mathbf{q}_3)$ for differential 3-forms; $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$ from $\cup_{\mathbf{x}\in\mathcal{B}(\mathbf{x}_0)}(\mathbf{x},\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$ to $\mathbb{R}^N\times\mathbb{R}^N$ with associated frames $(\mathbf{e}_{\mathbf{q}_1},\mathbf{e}_{\mathbf{q}_2},\mathbf{e}_{\mathbf{q}_3},d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3)$ to locate position-momentum in phase space, $(\partial_{\mathbf{q}_1},\partial_{\mathbf{q}_2},\partial_{\mathbf{q}_3},d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3)$ to locate vectors of tangent space $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$, $(d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3,d\mathbf{p}_1,d\mathbf{p}_2,d\mathbf{p}_3)$ for differential 1-forms and $(d\mathbf{q}_1\wedge d\mathbf{q}_2, d\mathbf{q}_1\wedge d\mathbf{q}_3, d\mathbf{q}_1\wedge d\mathbf{p}_1, d\mathbf{q}_1\wedge d\mathbf{p}_2, d\mathbf{q}_1\wedge d\mathbf{p}_3, d\mathbf{q}_2\wedge d\mathbf{q}_3, d\mathbf{q}_2\wedge d\mathbf{p}_1, d\mathbf{q}_2\wedge d\mathbf{p}_2, d\mathbf{q}_2\wedge d\mathbf{p}_3, d\mathbf{q}_3\wedge d\mathbf{p}_1, d\mathbf{q}_3\wedge d\mathbf{p}_2, d\mathbf{q}_3\wedge d\mathbf{p}_3, d\mathbf{p}_1\wedge d\mathbf{p}_2, d\mathbf{p}_1\wedge d\mathbf{p}_3, d\mathbf{p}_2\wedge d\mathbf{p}_3)$ for differential 2-forms.

Then considering a differential 1-form $\mathbf{A}$ on $\mathcal{X}$ and its expression $\breve{\mathbf{A}}_1(\mathbf{q})d\mathbf{q}_1+\breve{\mathbf{A}}_2(\mathbf{q})d\mathbf{q}_2+\breve{\mathbf{A}}_3(\mathbf{q})d\mathbf{q}_3$ within coordinate systems. Expression of $\Pi^*{}_*(\mathbf{A})$ is the following : $\breve{\mathbf{A}}_1(\mathbf{q})d\mathbf{q}_1 + \breve{\mathbf{A}}_2(\mathbf{q})d\mathbf{q}_2 + \breve{\mathbf{A}}_3(\mathbf{q})d\mathbf{q}_3 + 0d\mathbf{p}_1 + 0d\mathbf{p}_3 + 0d\mathbf{p}_3 = \breve{\mathbf{A}}_1(\mathbf{q})d\mathbf{q}_1 + \breve{\mathbf{A}}_2(\mathbf{q})d\mathbf{q}_2 + \breve{\mathbf{A}}_3(\mathbf{q})d\mathbf{q}_3$. In other words, despite $\mathbf{A}$ are not of the same nature, they have the same expression within coordinate systems.
This is the same for a differential 2-form $\mathbf{B}$ with expression $\breve{\mathbf{B}}_1d\mathbf{q}_3\wedge d\mathbf{q}_1+\breve{\mathbf{B}}_2d\mathbf{q}_2\wedge d\mathbf{q}_3+\breve{\mathbf{B}}_3d\mathbf{q}_3\wedge d\mathbf{q}_1$ and whom expression of $\Pi^*{}_*(\mathbf{B})$ is also $\breve{\mathbf{B}}_1d\mathbf{q}_3\wedge d\mathbf{q}_1+\breve{\mathbf{B}}_2d\mathbf{q}_2\wedge d\mathbf{q}_3+\breve{\mathbf{B}}_3d\mathbf{q}_3\wedge d\mathbf{q}_1$ having in mind that there are component with worth 0 on the other frame vectors.

5.1.8 Other differential 2-forme on $\mathbf{T}^*\mathcal{X}$: $\omega^{[\mathbf{A}]}$

In a natural way, the following differential 2-form may be defined on $\mathbf{T}^*\mathcal{X}$:

$$\omega^{[\mathbf{A}]} = d\gamma^{[\mathbf{A}]} = d\gamma + d(\Pi^*{}_*(q\mathbf{A})). \tag{5.32}$$

It may be shown that differential 2-form $\omega^{[\mathbf{A}]}$ has also the following expression:

$$\omega^{[\mathbf{A}]} = d\gamma^{[\mathbf{A}]} = d\gamma + (\Pi^*{}_*(q d\mathbf{A})). \tag{5.33}$$

To see this, the simplest way consists in leading a computation within coordinates. (This computation will be restricted to the case $N = 3$.) The point is to notice that the following computation concerning exterior derivative of $\mathbf{A}$ in coordinate system induced by $\mathcal{C}$

$$\begin{aligned} d(\breve{\mathbf{A}}_1(\mathbf{q})d\mathbf{q}_1+\breve{\mathbf{A}}_2(\mathbf{q})d\mathbf{q}_2+\breve{\mathbf{A}}_3(\mathbf{q})d\mathbf{q}_3) = \left(\frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1}(\mathbf{q})-\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2}(\mathbf{q})\right)d\mathbf{q}_1\wedge d\mathbf{q}_2 \\ +\left(\frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2}(\mathbf{q})-\frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3}(\mathbf{q})\right)d\mathbf{q}_2\wedge d\mathbf{q}_3+\left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3}(\mathbf{q})-\frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1}(\mathbf{q})\right)d\mathbf{q}_3\wedge d\mathbf{q}_1, \end{aligned} \tag{5.34}$$

remains valid when seen within coordinates induced by $(\mathcal{C},\mathcal{D}_\mathbf{x}\mathcal{C})$, with frame $(\mathbf{e}_{\mathbf{q}_1},\mathbf{e}_{\mathbf{q}_2},\mathbf{e}_{\mathbf{q}_3},d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3)$ on $(\mathcal{C},\mathcal{D}_\mathbf{x}\mathcal{C})\mathbf{T}^*\mathcal{X}$, with frame $(d\mathbf{q}_1,d\mathbf{q}_2,d\mathbf{q}_3,d\mathbf{p}_1,d\mathbf{p}_2,d\mathbf{p}_3)$ for differential 1-forms on $(\mathcal{C},\mathcal{D}_\mathbf{x}\mathcal{C})\mathbf{T}^*\mathcal{X}$ and with frame $(d\mathbf{q}_1\wedge d\mathbf{q}_2, d\mathbf{q}_1\wedge d\mathbf{q}_3, d\mathbf{q}_1\wedge d\mathbf{p}_1, d\mathbf{q}_1\wedge d\mathbf{p}_2, d\mathbf{q}_1\wedge d\mathbf{p}_3, d\mathbf{q}_2\wedge d\mathbf{q}_3, d\mathbf{q}_2\wedge d\mathbf{p}_1, d\mathbf{q}_2\wedge d\mathbf{p}_2, d\mathbf{q}_2\wedge d\mathbf{p}_3, d\mathbf{q}_3\wedge d\mathbf{p}_1, d\mathbf{q}_3\wedge d\mathbf{p}_2, d\mathbf{q}_3\wedge d\mathbf{p}_3, d\mathbf{p}_1\wedge d\mathbf{p}_2, d\mathbf{p}_1\wedge d\mathbf{p}_3, d\mathbf{p}_2\wedge d\mathbf{p}_3)$ for differential 2-forms on $(\mathcal{C},\mathcal{D}_\mathbf{x}\mathcal{C})\mathbf{T}^*\mathcal{X}$. In other words, the left hand side of (5.34) may be interpreted as the expression of the exterior derivative $d\mathbf{A}$ of $\mathbf{A}$ in the coordinates induced by $\mathcal{C}$ or as the expression $d\big(\Pi^*{}_*(\mathbf{A})\big)$ of $\Pi^*{}_*(\mathbf{A})$ in the coordinates induced by $(\mathcal{C},\mathcal{D}_\mathbf{x}\mathcal{C})$. In the same way, the right hand side of (5.34) which is the natural expression of $d\mathbf{A}$ is also the expression of $(\Pi^*{}_*(d\mathbf{A}))$.

From (5.33) and expression (5.11) of $\omega = d\gamma$ it may be obtained the following expression of differential 2-form $\omega^{[\mathbf{A}]}$:

$$\begin{aligned} \{\omega^{[\mathbf{A}]}(\mathbf{x},\mathbf{m})\}((\nu,\pi),(\nu',\pi')) = \langle\pi,\nu'\rangle - \langle\pi',\nu\rangle + \Big\{\Pi^*{}_{*(\mathbf{x},\mathbf{m})}(q d_\mathbf{x}\mathbf{A})\Big\}((\nu,\pi),(\nu',\pi')) \\ = \langle\pi,\nu'\rangle - \langle\pi',\nu\rangle + \big\{q d_\mathbf{x}\mathbf{A}\big\}(\nu,\nu'), \end{aligned} \tag{5.35}$$

where formula (5.30) was used to get expression of $\big\{\Pi^*{}_{*(\mathbf{x},\mathbf{m})}(q d_\mathbf{x}\mathbf{A})\big\}$

5.2 Differential 1-form representation by vector fields

5.2.1 Representations of a 1-forms by vectors

In any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, to any vector (ν,π) of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ $(=\mathbf{T}_\mathbf{x}\mathcal{X}\times\mathbf{T}^*_\mathbf{x}\mathcal{X})$ it may be associated 1-forms $I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)$ and $I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)$ of $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ with the help of ω or $\omega^{[\mathbf{A}]}$. They are defined by

$$\big\{I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)\big\}(\nu',\pi') = \big\langle I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi),(\nu',\pi')\big\rangle = \{\omega(\mathbf{x},\mathbf{m})\}((\nu',\pi'),(\nu,\pi)) \tag{5.36}$$

$$\big\{I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)\big\}(\nu',\pi') = \big\langle I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi),(\nu',\pi')\big\rangle = \{\omega^{[\mathbf{A}]}(\mathbf{x},\mathbf{m})\}((\nu',\pi'),(\nu,\pi)) \tag{5.37}$$

for any (ν',π') of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$. Linear applications $I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)$ and $I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)$ are one-to-one from $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ onto $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ and their respective reverse applications are denoted by $I_{(\mathbf{x},\mathbf{m})}$ and $I^{[A]}_{(\mathbf{x},\mathbf{m})}$.

Expressions of $I_{(\mathbf{x},\mathbf{m})}$ and $I^{[A]}_{(\mathbf{x},\mathbf{m})}$ are now going to be investigated. For this, something whose spirit was touched at the end of paragraph 4.4 will be done. Once identification of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ with $\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ is done, which consists in writing any element of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ as (ν,π) with $\nu\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$ and $\pi\in\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$, it is consistant to identify $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ with $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}$. This consists in saying that (μ,υ) of $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ with $\mu\in\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and $\upsilon\in\mathbf{T}_{\mathbf{x}}\mathcal{X}$ acts on any (ν',π') of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ by formula:

$$\big\{(\mu,\upsilon)\big\}(\nu',\pi') = \big\langle(\mu,\upsilon),(\nu',\pi')\big\rangle = \big\langle\mu,\nu'\big\rangle + \big\langle\pi',\upsilon\big\rangle. \tag{5.38}$$

Using expression (5.12) of $\{\omega(\mathbf{x},\mathbf{m})\}((\nu',\pi'),(\nu,\pi))$, equality (5.36) yields

$$\big\{I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)\big\}(\nu',\pi') = \ \langle\pi',\nu\rangle - \langle\pi,\nu'\rangle, \tag{5.39}$$

and, using (5.38), it may be gotten that (μ,υ) is the image of (ν,π) under $I^{-1}_{(\mathbf{x},\mathbf{m})}$ if

$$\mu = -\pi \text{ and } \upsilon = \nu, \tag{5.40}$$

and that (ν,π) is the image of (μ,υ) under $I_{(\mathbf{x},\mathbf{m})}$ if

$$\nu = \upsilon \text{ and } \pi = -\mu. \tag{5.41}$$

In other words

$$\begin{array}{rccc} I^{-1}_{(\mathbf{x},\mathbf{m})}: & \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X}) & \to & \mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T}^*_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}) \\ & (\nu,\pi) & \mapsto & I^{-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi) = (-\pi,\nu), \end{array} \tag{5.42}$$

$$\begin{array}{rccc} I_{(\mathbf{x},\mathbf{m})}: & \mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T}^*_{\mathbf{x}}\mathcal{X}\times\mathbf{T}_{\mathbf{x}}\mathcal{X}) & \to & \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X}) \\ & (\mu,\upsilon) & \mapsto & I_{(\mathbf{x},\mathbf{m})}(\mu,\upsilon) = (\upsilon,-\mu). \end{array} \tag{5.43}$$

Using expression (5.35) of differential 2-form $\omega^{[\mathbf{A}]}$, equality (5.37) yields

$$\big\{I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi)\big\}(\nu',\pi') = \ \langle\pi',\nu\rangle - \langle\pi,\nu'\rangle + \big\{q d_{\mathbf{x}}\mathbf{A}\big\}(\nu',\nu). \tag{5.44}$$

Then having (5.38) in mind, it may be gotten that (μ,υ) is the image of (ν,π) under $I^{-1}_{(\mathbf{x},\mathbf{m})}$ if

$$\mu = -\pi + \big\{q d_{\mathbf{x}}\mathbf{A}\big\}(.,\nu) = -(\pi + \big\{q d_{\mathbf{x}}\mathbf{A}\big\}(\nu,.)) \quad \text{and} \quad \upsilon = \nu, \tag{5.45}$$

or

$$\nu = \upsilon \quad \text{and} \quad \pi = -(\mu + \big\{q d_{\mathbf{x}}\mathbf{A}\big\}(\upsilon,.)). \tag{5.46}$$

In section 2.6, for any $\mathbf{x} \in \mathcal{X}$ and any $\mathbf{v} \in \mathbf{T_x}\mathcal{X}$, it was introduced operator $\overset{\circ}{\iota}_{\mathbf{v}}$ (see (2.13)) as mapping $\mathbf{T_x^*}^k\mathcal{X}$ to $\mathbf{T}^{*\,k-1}_{(\mathbf{x},\mathbf{v})}\big(\mathbf{T}\mathcal{X}\big)$ and operator $\overset{\circ}{\iota}$ as mapping $\Lambda^k(\mathcal{X})$ to $\Lambda^{k-1}\big(\mathbf{T}\mathcal{X}\big)$.
In a close manner, for any $\mathbf{x} \in \mathcal{X}$ and any $\nu \in \mathbf{T_x}\mathcal{X}$ operator $\overset{*}{\iota}_\nu$:

$$\begin{array}{cccc} \overset{*}{\iota}_\nu : & \mathbf{T_x^*}^k\mathcal{X} & \to & \mathbf{T_x^*}^{k-1}\mathcal{X} \\ & \kappa & \mapsto & \overset{*}{\iota}_\nu \kappa \end{array} \tag{5.47}$$

may be defined by setting $\{\overset{*}{\iota}_\nu\kappa\}(\nu_1,\ldots,\nu_{k-1}) = \{\kappa\}(\nu,\nu_1,\ldots,\nu_{k-1})$. Using this notation, (5.40) reads

$$\mu = -(\pi + q\,\overset{*}{\iota}_\nu d_{\mathbf{x}}\mathbf{A}) \quad \text{and} \quad \upsilon = \nu, \tag{5.48}$$

and (5.46)

$$\nu = \upsilon \quad \text{and} \quad \pi = -(\mu + q\,\overset{*}{\iota}_\upsilon d_{\mathbf{x}}\mathbf{A}). \tag{5.49}$$

Then, the following definition of $I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}$ and $I^{[A]}_{(\mathbf{x},\mathbf{m})}$ are gotten:

$$\begin{array}{cccc} I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})} : & \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T_x}\mathcal{X}\times\mathbf{T_x^*}\mathcal{X}) & \to & \mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T_x^*}\mathcal{X}\times\mathbf{T_x}\mathcal{X}) \\ & (\nu,\pi) & \mapsto & I^{[A]\,-1}_{(\mathbf{x},\mathbf{m})}(\nu,\pi) = (-(\pi + q\,\overset{*}{\iota}_\nu d_{\mathbf{x}}\mathbf{A}),\nu), \end{array} \tag{5.50}$$

$$\begin{array}{cccc} I^{[A]}_{(\mathbf{x},\mathbf{m})} : & \mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T_x^*}\mathcal{X}\times\mathbf{T_x}\mathcal{X}) & \to & \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T_x}\mathcal{X}\times\mathbf{T_x^*}\mathcal{X}) \\ & (\mu,\upsilon) & \mapsto & I^{[A]}_{(\mathbf{x},\mathbf{m})}(\mu,\upsilon) = (\upsilon, -(\mu + q\,\overset{*}{\iota}_\upsilon d_{\mathbf{x}}\mathbf{A})). \end{array} \tag{5.51}$$

5.2.2 Representations of a differential 1-form by vector fields

Denoting by $\mathrm{V}(\mathbf{T}^*\mathcal{X})$ the space of regular vector fields on $\mathbf{T}^*\mathcal{X}$, an element $\mathcal{W}$ of this space writes $\mathcal{W} = \mathcal{W}(\mathbf{x},\mathbf{m}) = (\mathbf{W}(\mathbf{x},\mathbf{m}),\mathbf{N}(\mathbf{x},\mathbf{m}))$ where for any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$, $\mathbf{W}(\mathbf{x},\mathbf{m}) \in \mathbf{T_x}\mathcal{X}$ and $\mathbf{N}(\mathbf{x},\mathbf{m}) \in \mathbf{T_x^*}\mathcal{X}$. Any differential 1-form $\mathfrak{M}$ on $\mathbf{T}^*\mathcal{X}$ writes $\mathfrak{M} = \mathfrak{M}(\mathbf{x},\mathbf{m}) = (\tilde{\mathbf{N}}(\mathbf{x},\mathbf{m}),\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m}))$.
The following linear applications may be defined:

$$\begin{array}{cccc} I^{-1} : & \mathrm{V}(\mathbf{T}^*\mathcal{X}) & \to & \Lambda(\mathbf{T}^*\mathcal{X}) \\ & \mathcal{W} = (\mathbf{W},\mathbf{N}) & \mapsto & I^{-1}(\mathcal{W}) \end{array} \tag{5.52}$$

where $\big(I^{-1}(\mathcal{W})\big)(\mathbf{x},\mathbf{m}) = I^{-1}_{(\mathbf{x},\mathbf{m})}(\mathbf{W}(\mathbf{x},\mathbf{m}),\mathbf{N}(\mathbf{x},\mathbf{m})) = (-(\mathbf{N}(\mathbf{x},\mathbf{m})),\mathbf{W}(\mathbf{x},\mathbf{m}))$,

$$\begin{array}{cccc} I^{[A]\,-1} : & \mathrm{V}(\mathbf{T}^*\mathcal{X}) & \to & \Lambda(\mathbf{T}^*\mathcal{X}) \\ & \mathcal{W} = (\mathbf{W},\mathbf{N}) & \mapsto & I^{[A]\,-1}(\mathcal{W}), \end{array} \tag{5.53}$$

where $\big(I^{[A]\,-1}(\mathcal{W})\big)(\mathbf{x},\mathbf{m}) = I^{[A]}_{(\mathbf{x},\mathbf{m})}(\mathbf{W}(\mathbf{x},\mathbf{m}),\mathbf{N}(\mathbf{x},\mathbf{m})) = (-(\mathbf{N}(\mathbf{x},\mathbf{m}) + q\,\overset{*}{\iota}_{\mathbf{W}(\mathbf{x},\mathbf{m})} d_{\mathbf{x}}\mathbf{A}),\mathbf{W}(\mathbf{x},\mathbf{m}))$,

$$\begin{array}{cccc} I : & \Lambda(\mathbf{T}^*\mathcal{X}) & \to & \mathrm{V}(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{M} = (\tilde{\mathbf{N}},\tilde{\mathbf{W}}) & \mapsto & I(\mathfrak{M}), \end{array} \tag{5.54}$$

where $\big(I(\mathfrak{M})\big)(\mathbf{x},\mathbf{m}) = I_{(\mathbf{x},\mathbf{m})}(\tilde{\mathbf{N}}(\mathbf{x},\mathbf{m}),\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m})) = (\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m}), -(\tilde{\mathbf{N}}(\mathbf{x},\mathbf{m})),$
and

$$\begin{array}{rccc} I^{[A]}: & \Lambda(\mathbf{T}^*\mathcal{X}) & \to & \mathrm{V}(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{M} = (\tilde{\mathbf{N}},\tilde{\mathbf{W}}) & \mapsto & I^{[A]}(\mathfrak{M}), \end{array} \tag{5.55}$$

where $\big(I^{[A]}(\mathfrak{M})\big)(\mathbf{x},\mathbf{m}) = I^{[A]}_{(\mathbf{x},\mathbf{m})}(\tilde{\mathbf{N}}(\mathbf{x},\mathbf{m}),\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m})) = (\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m}), -(\tilde{\mathbf{N}}(\mathbf{x},\mathbf{m}) + q\,\overset{*}{\imath}_{\tilde{\mathbf{W}}(\mathbf{x},\mathbf{m})} d_{\mathbf{x}}\mathbf{A})).$

$I(\mathfrak{M})$ and $I^{[A]}(\mathfrak{M})$ are vector fields on $\mathbf{T}^*\mathcal{X}$ representing differential form $\mathfrak{M}$ on $\mathbf{T}^*\mathcal{X}$.

5.3 Remarks about notations

Operator $\overset{*}{\imath}_\nu : \mathbf{T}^{*\,k}_{\mathbf{x}}\mathcal{X} \to \mathbf{T}^{*\,k-1}_{\mathbf{x}}\mathcal{X}$ was just introduced by (5.47), in any $\mathbf{x}\in\mathcal{X}$ and for any $\nu \in \mathbf{T}_{\mathbf{x}}\mathcal{X}$.

Remark 5.1 This may also be done, in the same way, for $\overset{*}{\imath}_{(\nu,\pi)} : \mathbf{T}^{*\,k}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big) \to \mathbf{T}^{*\,k-1}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ in any $(\mathbf{x},\mathbf{m}) \in \mathbf{T}^*\mathcal{X}$ and for any $(\nu,\pi) \in \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$.

Remark 5.2 Operator $\overset{*}{\imath}_\nu$ is linked with interior product operator $\imath_{\mathbf{W}} : \Lambda^k(\mathcal{X}) \to \Lambda^{k-1}(\mathcal{X})$ defined for any regular vector field $\mathbf{W}$ of $\mathrm{V}(\mathcal{X})$ by

$$\big(\imath_{\mathbf{W}}\mathbf{K}\big)(\mathbf{x}) = \overset{*}{\imath}_{\mathbf{W}(\mathbf{x})}(\mathbf{K}(\mathbf{x})), \tag{5.56}$$

in any $\mathbf{x}$ of $\mathcal{X}$.

In subsection 2.6, it was built operators $\overset{\circ}{\imath}_\nu$ (for any ν of $\mathbf{T}_{\mathbf{x}}\mathcal{X}$) and $\overset{\circ}{\imath}$. They were defined as

$$\begin{array}{rccc} \overset{\circ}{\imath}_\nu : & \mathbf{T}^{*\,k}_{\mathbf{x}}\mathcal{X} & \to & \mathbf{T}^{*\,k-1}_{(\mathbf{x},\nu)}\big(\mathbf{T}\mathcal{X}\big) \\ & \mathbf{k} & \mapsto & \overset{\circ}{\imath}_\nu\mathbf{k}, \end{array} \tag{5.57}$$

with, $\big\{\overset{\circ}{\imath}_\nu\mathbf{k}\big\}((\nu_1,\pi_1),\dots,(\nu_{n-1},\pi_{n-1})) = \{\mathbf{k}\}(\nu,\nu_1,\dots,\nu_{n-1})$ and

$$\begin{array}{rccc} \overset{\circ}{\imath} : & \Lambda^k(\mathcal{X}) & \to & \Lambda^{k-1}(\mathbf{T}\mathcal{X}) \\ & \mathbf{K} & \mapsto & \overset{\circ}{\imath}\mathbf{K}, \end{array} \tag{5.58}$$

with $(\overset{\circ}{\imath}\mathbf{K})(\mathbf{x},\mathbf{v}) = \overset{\circ}{\imath}_{\mathbf{v}}(\mathbf{K}(\mathbf{x}))$.

Remark 5.3 generalizing (2.15) to any differential k-form $\mathbf{K}$ in the following way: $\big(\imath_{\mathbf{W}}\mathbf{K}\big)(\mathbf{x}) = \big(\mathcal{D}_{(\mathbf{x},\mathbf{W}(\mathbf{x}))}\Pi\big)\big(\overset{\circ}{\imath}\mathbf{K}\big)$, the following formula may be deduced

$$\overset{*}{\imath}_{\mathbf{W}(\mathbf{x})}(\mathbf{K}(\mathbf{x})) = \big(\mathcal{D}_{(\mathbf{x},\mathbf{W}(\mathbf{x}))}\Pi\big)\big(\overset{\circ}{\imath}\mathbf{K}\big). \tag{5.59}$$

Remark 5.4 Using the interior product of differential forms by vector fields on $\mathbf{T}^*\mathcal{X}$, (5.36) and (5.37) say nothing but that the following expressions of I^{-1} and ${I^{[A]}}^{-1}$ may be given:

$$I^{-1}(\mathcal{W}) = I^{-1}(\mathbf{W},\mathbf{N}) = \imath_{\mathcal{W}}\omega = \imath_{(\mathbf{W},\mathbf{N})}\omega, \tag{5.60}$$

$${I^{[A]}}^{-1}(\mathcal{W}) = {I^{[A]}}^{-1}(\mathbf{W},\mathbf{N}) = \imath_{\mathcal{W}}\omega^{[\mathbf{A}]} = \imath_{(\mathbf{W},\mathbf{N})}\omega^{[\mathbf{A}]}. \tag{5.61}$$

Remark 5.5 With the remark just made, it can be stated that for any differential 1-form $\mathfrak{M}$ on $\mathbf{T}^*\mathcal{X}$, $I(\mathfrak{M})$ and $I^{[A]}(\mathfrak{M})$ satisfy

$$\mathfrak{M} = i_{(I(\mathfrak{M}))}\omega \quad \text{and} \quad \mathfrak{M} = i_{(I^{[A]}(\mathfrak{M}))}\omega^{[\mathbf{A}]}. \tag{5.62}$$

5.4 Volume forms

5.4.1 Building volume forms $\omega^{\wedge N}$ and $\omega^{[\mathbf{A}]^{\wedge N}}$

Remembering that $\omega^{\wedge 2} = \omega \wedge \omega$ is defined by

$$\begin{aligned}\omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_4,\pi_4)\big) &= 2\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\\ - 2\omega\big((\nu_1,\pi_1),(\nu_3,\pi_3)\big)\,\omega\big((\nu_2,\pi_2),(\nu_4,\pi_4)\big) &+ 2\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\,\omega\big((\nu_2,\pi_2),(\nu_3,\pi_3)\big),\end{aligned} \tag{5.63}$$

and $\omega^{\wedge 3} = \omega \wedge \omega \wedge \omega$ by

$$\begin{aligned}\omega \wedge \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),&(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\\
&= \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_4,\pi_4)\big)\,\omega\big((\nu_5,\pi_5),(\nu_6,\pi_6)\big)\big)\\
&- \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_5,\pi_5)\big)\,\omega\big((\nu_4,\pi_4),(\nu_6,\pi_6)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_3,\pi_3),(\nu_6,\pi_6)\big)\big)\\
&- \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_1,\pi_1),(\nu_6,\pi_6)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_6,\pi_6)\big)\,\omega\big((\nu_4,\pi_4),(\nu_5,\pi_5)\big)\big)\\
&- \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_4,\pi_4),(\nu_6,\pi_6)\big)\,\omega\big((\nu_3,\pi_3),(\nu_5,\pi_5)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_6,\pi_6)\big)\,\omega\big((\nu_2,\pi_2),(\nu_5,\pi_5)\big)\big)\\
&- \omega \wedge \omega\big((\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_6,\pi_6)\big)\,\omega\big((\nu_1,\pi_1),(\nu_5,\pi_5)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\big)\\
&- \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_3,\pi_3),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\,\omega\big((\nu_2,\pi_2),(\nu_4,\pi_4)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\,\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_1,\pi_1),(\nu_4,\pi_4),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\,\omega\big((\nu_2,\pi_2),(\nu_3,\pi_3)\big)\big)\\
&+ \omega \wedge \omega\big((\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)\,\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2)\big)\big),\end{aligned} \tag{5.64}$$

or by

$$
\begin{aligned}
\omega\wedge\omega\wedge\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2),(\nu_3,\pi_3),(\nu_4,\pi_4),(\nu_5,\pi_5),(\nu_6,\pi_6)\big)&\\
=6\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\,\omega\big((\nu_5,\pi_5),(\nu_6,\pi_6)\big)&\\
-4\omega\big((\nu_1,\pi_1),(\nu_3,\pi_3)\big)\,\omega\big((\nu_2,\pi_2),(\nu_4,\pi_4)\big)\,\omega\big((\nu_5,\pi_5),(\nu_6,\pi_6)\big)&\\
+6\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\,\omega\big((\nu_2,\pi_2),(\nu_3,\pi_3)\big)\,\omega\big((\nu_5,\pi_5),(\nu_6,\pi_6)\big)&\\
-6\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2)\big)\,\omega\big((\nu_3,\pi_3),(\nu_5,\pi_5)\big)\,\omega\big((\nu_4,\pi_4),(\nu_6,\pi_6)\big)&\\
+4\omega\big((\nu_1,\pi_1),(\nu_3,\pi_3)\big)\,\omega\big((\nu_2,\pi_2),(\nu_5,\pi_5)\big)\,\omega\big((\nu_4,\pi_4),(\nu_6,\pi_6)\big)&\\
-6\omega\big((\nu_1,\pi_1),(\nu_5,\pi_5)\big)\,\omega\big((\nu_2,\pi_2),(\nu_3,\pi_3)\big)\,\omega\big((\nu_4,\pi_4),(\nu_6,\pi_6)\big)&\\
+6\omega\big((\nu_1,\pi_1),(\nu_2,\pi_2)\big)\,\omega\big((\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_3,\pi_3),(\nu_6,\pi_6)\big)&\\
-6\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\,\omega\big((\nu_2,\pi_2),(\nu_5,\pi_5)\big)\,\omega\big((\nu_3,\pi_3),(\nu_6,\pi_6)\big)&\\
+6\omega\big((\nu_1,\pi_1),(\nu_5,\pi_5)\big)\,\omega\big((\nu_2,\pi_2),(\nu_4,\pi_4)\big)\,\omega\big((\nu_3,\pi_3),(\nu_6,\pi_6)\big)&\\
-4\omega\big((\nu_1,\pi_1),(\nu_3,\pi_3)\big)\,\omega\big((\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)&\\
+4\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\,\omega\big((\nu_3,\pi_3),(\nu_5,\pi_5)\big)\,\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)&\\
-4\omega\big((\nu_1,\pi_1),(\nu_5,\pi_5)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\,\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)&\\
+6\omega\big((\nu_2,\pi_2),(\nu_3,\pi_3)\big)\,\omega\big((\nu_4,\pi_4),(\nu_5,\pi_5)\big)\,\omega\big((\nu_1,\pi_1),(\nu_6,\pi_6)\big)&\\
-6\omega\big((\nu_2,\pi_2),(\nu_4,\pi_4)\big)\,\omega\big((\nu_3,\pi_3),(\nu_5,\pi_5)\big)\,\omega\big((\nu_1,\pi_1),(\nu_6,\pi_6)\big)&\\
+6\omega\big((\nu_2,\pi_2),(\nu_5,\pi_5)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\,\omega\big((\nu_1,\pi_1),(\nu_6,\pi_6)\big)&\\
+2\omega\big((\nu_1,\pi_1),(\nu_4,\pi_4)\big)\,\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)\,\omega\big((\nu_3,\pi_3),(\nu_5,\pi_5)\big)&\\
-2\omega\big((\nu_2,\pi_2),(\nu_6,\pi_6)\big)\,\omega\big((\nu_3,\pi_3),(\nu_4,\pi_4)\big)\,\omega\big((\nu_1,\pi_1),(\nu_5,\pi_5)\big)&,
\end{aligned}
\tag{5.65}
$$

clearly,

$$
\omega^{\wedge N}=\underbrace{\omega\wedge\cdots\wedge\omega}_{N\text{ times}}=(d\gamma)^{\wedge N}=\underbrace{d\gamma\wedge\cdots\wedge d\gamma}_{N\text{ times}},
\tag{5.66}
$$

and

$$
{\omega^{[\mathbf{A}]}}^{\wedge N}=\underbrace{\omega^{[\mathbf{A}]}\wedge\cdots\wedge\omega^{[\mathbf{A}]}}_{N\text{ times}}=\big(d\gamma^{[\mathbf{A}]}\big)^{\wedge N}=\underbrace{d\gamma^{[\mathbf{A}]}\wedge\cdots\wedge d\gamma^{[\mathbf{A}]}}_{N\text{ times}},
\tag{5.67}
$$

are volume forms (i.e. non-degenerated differential $2N$-forms) on $\mathbf{T}^*\mathcal{X}$.

5.4.2 $\omega^{\wedge N}$ and ${\omega^{[\mathbf{A}]}}^{\wedge N}$ are the same volume form

In fact, the two volume forms previously built $\omega^{\wedge N}$ and ${\omega^{[\mathbf{A}]}}^{\wedge N}$ are the same one. The easiest way to see this fact consists in making the computation within the coordinates system. In fact, in view of equalities (5.18) and (5.33), it needs to be proven that

$$
\big(d\gamma+(\Pi^*_*(q\,d\mathbf{A}))\big)^{\wedge N}=(d\gamma)^{\wedge N}.
\tag{5.68}
$$

This will be done is the cases $N=1,2$ and 3.

When $N=1$, it is obvious since $d\mathbf{A}$ is 0.

When $N = 2$,

$$(d\gamma + (\Pi^*_*(qd\mathbf{A}))) \wedge (d\gamma + (\Pi^*_*(qd\mathbf{A})))$$
$$= \left(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} \right) d\mathbf{q}_1 \wedge d\mathbf{q}_2 \right) \wedge$$
$$\left(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} \right) d\mathbf{q}_1 \wedge d\mathbf{q}_2 \right)$$
$$= 0 + d\mathbf{p}_1 \wedge d\mathbf{q}_1 \wedge d\mathbf{p}_2 \wedge d\mathbf{q}_2 + 0 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 \wedge d\mathbf{p}_1 \wedge d\mathbf{q}_1 + 0 + 0 + 0 + 0 + 0$$
$$= 2d\mathbf{p}_1 \wedge d\mathbf{q}_1 \wedge d\mathbf{p}_2 \wedge d\mathbf{q}_2 = \big(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2\big) \wedge \big(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2\big)$$
$$= (d\gamma) \wedge (d\gamma). \quad (5.69)$$

When $N = 3$,

$$(d\gamma + (\Pi^*_*(qd\mathbf{A}))) \wedge (d\gamma + (\Pi^*_*(qd\mathbf{A}))) \wedge (d\gamma + (\Pi^*_*(qd\mathbf{A})))$$
$$= \Bigg(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 + d\mathbf{p}_3 \wedge d\mathbf{q}_3$$
$$+ \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} \right) d\mathbf{q}_1 \wedge d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3} \right) d\mathbf{q}_2 \wedge d\mathbf{q}_3 + \left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3} - \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} \right) d\mathbf{q}_3 \wedge d\mathbf{q}_1, \Bigg) \wedge$$
$$\Bigg(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 + d\mathbf{p}_3 \wedge d\mathbf{q}_3$$
$$+ \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} \right) d\mathbf{q}_1 \wedge d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3} \right) d\mathbf{q}_2 \wedge d\mathbf{q}_3 + \left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3} - \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} \right) d\mathbf{q}_3 \wedge d\mathbf{q}_1, \Bigg) \wedge$$
$$\Bigg(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 + d\mathbf{p}_3 \wedge d\mathbf{q}_3$$
$$+ \left(\frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_1} - \frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_2} \right) d\mathbf{q}_1 \wedge d\mathbf{q}_2 + \left(\frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_2} - \frac{\partial \breve{\mathbf{A}}_2}{\partial \mathbf{q}_3} \right) d\mathbf{q}_2 \wedge d\mathbf{q}_3 + \left(\frac{\partial \breve{\mathbf{A}}_1}{\partial \mathbf{q}_3} - \frac{\partial \breve{\mathbf{A}}_3}{\partial \mathbf{q}_1} \right) d\mathbf{q}_3 \wedge d\mathbf{q}_1, \Bigg)$$
$$= 3 \text{ times } \Big((6 \text{ times } 0) + 0 + 0 + d\mathbf{p}_1 \wedge d\mathbf{q}_1 \wedge d\mathbf{p}_2 \wedge d\mathbf{q}_2 \wedge d\mathbf{p}_3 \wedge d\mathbf{q}_3 + (3 \text{ times } 0) + 0$$
$$+ d\mathbf{p}_1 \wedge d\mathbf{q}_1 \wedge d\mathbf{p}_3 \wedge d\mathbf{q}_3 \wedge d\mathbf{p}_2 \wedge d\mathbf{q}_2 + 0 + (3 \text{ times } 0) + (3 \times 6 \text{ times } 0) \Big) + 108 \text{ times } 0$$
$$= \big(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 \wedge d\mathbf{p}_3 \wedge d\mathbf{q}_3\big) \wedge \big(d\mathbf{p}_1 \wedge d\mathbf{q}_1 + d\mathbf{p}_2 \wedge d\mathbf{q}_2 \wedge d\mathbf{p}_3 \wedge d\mathbf{q}_3\big) = (d\gamma) \wedge (d\gamma). \quad (5.70)$$

In fact the following formula holds true:

$$\omega^{[\mathbf{A}]^{\wedge N}} = \big(d\gamma + (\Pi^*_*(qd\mathbf{A}))\big)^{\wedge N} = \omega^{\wedge N} = (d\gamma)^{\wedge N} = N\, d\mathbf{p}_1 \wedge d\mathbf{q}_1 \wedge d\mathbf{p}_2 \wedge d\mathbf{q}_2 \wedge \cdots \wedge d\mathbf{p}_N \wedge d\mathbf{q}_N. \quad (5.71)$$

5.5 Vector fields and differential $(2N-1)$-forms

With the help of the volume form $\omega^{\wedge N}$, to any vector field $\mathcal{W} = (\mathbf{W}, \mathbf{N})$ of $\mathrm{V}(\mathbf{T}^*\mathcal{X})$ it may be associated the following differential $(2N-1)$-form: $i_{\mathcal{W}}(\omega^{\wedge N})$. The inner product in game here is the one in $\mathbf{T}^*\mathcal{X}$. Then, in any $(\mathbf{x},\mathbf{m})$ of $\mathbf{T}^*\mathcal{X}$ the value of $i_{\mathcal{W}}(\omega^{\wedge N})(\mathbf{x},\mathbf{m})$ is given by

$$
\begin{aligned}
\big\{i_{\mathcal{W}}(\omega^{\wedge N})(\mathbf{x},\mathbf{m})\big\}&\big((\nu_1,\pi_1),\dots,(\nu_{2N-1},\pi_{2N-1})\big)\\
&= \big\{(\omega^{\wedge N})(\mathbf{x},\mathbf{m})\big\}\big(\mathcal{W}(\mathbf{x},\mathbf{m}),(\nu_1,\pi_1),\dots,(\nu_{2N-1},\pi_{2N-1})\big)\\
&= \big\{(\omega^{\wedge N})(\mathbf{x},\mathbf{m})\big\}\big((\mathbf{W}(\mathbf{x},\mathbf{m}),\mathbf{N}(\mathbf{x},\mathbf{m})),(\nu_1,\pi_1),\dots,(\nu_{2N-1},\pi_{2N-1})\big), \qquad (5.72)
\end{aligned}
$$

for any $2N-1$ vectors $(\nu_1,\pi_1),\dots,(\nu_{2N-1},\pi_{2N-1})$ of $\mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)(=\mathbf{T}_{\mathbf{x}}\mathcal{X}\times\mathbf{T}^*_{\mathbf{x}}\mathcal{X})$.

It is interesting to compute expression of $i_{\mathcal{W}}(\omega^{\wedge N})$ in frame $((d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N\wedge d\mathbf{q}_N),(d\mathbf{p}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N\wedge d\mathbf{q}_N),\dots,(d\mathbf{p}_1\wedge d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{q}_N),(d\mathbf{p}_1\wedge d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N))$ which a natural frame to express $2N-1$ differential forms once charts $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$ and $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})$ are set. from (5.71), it is gotten:

$$
\big\{(\omega^{\wedge N})(\mathbf{x},\mathbf{m})\big\}\big((\mathbf{W},\mathbf{N}),(\nu_1,\pi_1),\dots,(\nu_{2N-1},\pi_{2N-1})\big)
$$

$$
= N\begin{vmatrix}
d\mathbf{p}_1(\breve{\mathbf{W}},\breve{\mathbf{N}}) & d\mathbf{q}_1(\breve{\mathbf{W}},\breve{\mathbf{N}}) & d\mathbf{p}_2(\breve{\mathbf{W}},\breve{\mathbf{N}}) & \dots & d\mathbf{p}_N(\mathbf{W},\mathbf{N}) & d\mathbf{q}_N(\mathbf{W},\mathbf{N})\\
d\mathbf{p}_1(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{q}_1(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{p}_2(\breve{\nu}_1,\breve{\pi}_1) & \dots & d\mathbf{p}_N(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{q}_N(\breve{\nu}_1,\breve{\pi}_1)\\
d\mathbf{p}_1(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{p}_1(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{p}_2(\breve{\nu}_2,\breve{\pi}_2) & \dots & d\mathbf{p}_N(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{q}_N(\breve{\nu}_2,\breve{\pi}_2)\\
\vdots & \vdots & & & & \\
d\mathbf{p}_1(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{q}_1(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{p}_2(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & \dots & d\mathbf{p}_N(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{q}_N(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1})
\end{vmatrix}
$$

$$
= N\begin{vmatrix}
\breve{\mathbf{N}}_1 & \breve{\mathbf{W}}_1 & \breve{\mathbf{N}}_2 & \dots & \breve{\mathbf{N}}_N & \breve{\mathbf{W}}_N\\
d\mathbf{p}_1(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{q}_1(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{p}_2(\breve{\nu}_1,\breve{\pi}_1) & \dots & d\mathbf{p}_N(\breve{\nu}_1,\breve{\pi}_1) & d\mathbf{q}_N(\breve{\nu}_1,\breve{\pi}_1)\\
d\mathbf{p}_1(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{p}_1(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{p}_2(\breve{\nu}_2,\breve{\pi}_2) & \dots & d\mathbf{p}_N(\breve{\nu}_2,\breve{\pi}_2) & d\mathbf{q}_N(\breve{\nu}_2,\breve{\pi}_2)\\
\vdots & \vdots & & & & \\
d\mathbf{p}_1(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{q}_1(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{p}_2(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & \dots & d\mathbf{p}_N(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1}) & d\mathbf{q}_N(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1})
\end{vmatrix}
$$

$$
\begin{aligned}
= N\big\{&\breve{\mathbf{N}}_1(d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N\wedge d\mathbf{q}_N)+\breve{\mathbf{W}}_1(d\mathbf{p}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N\wedge d\mathbf{q}_N)+\dots\\
&+\breve{\mathbf{N}}_N(d\mathbf{p}_1\wedge d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{q}_N)+\breve{\mathbf{W}}_N(d\mathbf{p}_1\wedge d\mathbf{q}_1\wedge d\mathbf{p}_2\wedge d\mathbf{q}_2\wedge\dots\wedge d\mathbf{p}_N))\big\}\\
&\qquad\qquad((\breve{\nu}_1,\breve{\pi}_1),\dots(\breve{\nu}_{2N-1},\breve{\pi}_{2N-1})). \qquad (5.73)
\end{aligned}
$$

This computation shows that the components of $i_{\mathcal{W}}(\omega^{\wedge N})$ in a natural frame for differential $(2N-1)$-forms are the coordinates of $\mathcal{W}=(\mathbf{W},\mathbf{N})$ in a natural frame for vector fields.

As a consequence, application

$$
\begin{array}{rcl}
\mathrm{V}(\mathbf{T}^*\mathcal{X}) & \to & \Lambda^{2N-1}(\mathbf{T}^*\mathcal{X})\\
\mathcal{W}=(\mathbf{W},\mathbf{N}) & \mapsto & i_{\mathcal{W}}(\omega^{\wedge N})
\end{array} \qquad (5.74)
$$

is linear, one-to-one and onto.

5.6 Divergence of a vector field

By definition, the divergence $\mathrm{Div}(\mathcal{W})$ of any vector field $\mathcal{W}=(\mathbf{W},\mathbf{N})$ of $\mathrm{V}(\mathbf{T}^*\mathcal{X})$ is the function (or differential 0-form) on $\mathbf{T}^*\mathcal{X}$ which is such that

$$
d(i_{\mathcal{W}}(\omega^{\wedge N})) = \mathrm{Div}(\mathcal{W})\,\omega^{\wedge N}. \qquad (5.75)
$$

Using Cartan's Formula, which yields

$$\mathit{l}_{\mathcal{W}}(\omega^{\wedge N}) = \mathit{i}_{\mathcal{W}}(\mathit{d}(\omega^{\wedge N})) + \mathit{d}(\mathit{i}_{\mathcal{W}}(\omega^{\wedge N})), \tag{5.76}$$

and that $\mathit{d}(\omega^{\wedge N}) = 0$, (5.75) also reads

$$\mathit{l}_{\mathcal{W}}(\omega^{\wedge N}) = \mathrm{Div}(\mathcal{W})\,\omega^{\wedge N}. \tag{5.77}$$

Beside this, since by definition

$$\{\mathit{l}_{\mathcal{W}}(\omega^{\wedge N})(\mathbf{x},\mathbf{m})\} = \frac{\partial(\{g^s_{*(\mathbf{x},\mathbf{m})}\}((\omega^{\wedge N})(g^s(\mathbf{x},\mathbf{m})))}{\partial s}(0), \tag{5.78}$$

where g^s is the flow associated with $\mathcal{W}$ and where $g^s_{*(\mathbf{x},\mathbf{m})}$ is the Pullback in $(\mathbf{x},\mathbf{m})$ by flow g^s .

Then, if a vector field $\mathcal{W} = (\mathbf{W},\mathbf{N})$ of $\mathrm{V}(\mathbf{T}^*\mathcal{X})$ is such that $\mathrm{Div}(\mathcal{W}) = 0$, its associated flow preserves volume form $\omega^{\wedge N}$.

5.7 Pure Geometrical Operators transforming differential 1-forms into differential $(2N-1)$-forms

From a differential 1-form a vector field may be defined. Then from this vector field a differential $(2N-1)$-form may be defined. This process is linear and reversible: From a differential $(2N-1)$-form a vector field may be defined. Then from this vector field a differential 1-form may defined.
The two evoked processes only use the natural symplectic structure of $\mathbf{T}^*\mathcal{X}$. Then those operators transforming differential 1-forms into differential $(2N-1)$-forms (and vice versa) are purely geometric.
The precise definitions of those operators are now given.

$$\begin{array}{rccc} * : & \Lambda(\mathbf{T}^*\mathcal{X}) & \rightarrow & \Lambda^{2N-1}(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{M} = (\tilde{\mathbf{N}},\tilde{\mathbf{W}}) & \mapsto & *\mathfrak{M} = \mathit{i}_{(I(\mathfrak{M}))}(\omega^{\wedge N}), \end{array} \tag{5.79}$$

$$\begin{array}{rccc} *^{[\mathbf{A}]} : & \Lambda(\mathbf{T}^*\mathcal{X}) & \rightarrow & \Lambda^{2N-1}(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{M} = (\tilde{\mathbf{N}},\tilde{\mathbf{W}}) & \mapsto & *^{[\mathbf{A}]}\mathfrak{M} = \mathit{i}_{(I^{[A]}(\mathfrak{M}))}(\omega^{\wedge N}), \end{array} \tag{5.80}$$

where the application mapping vector fields onto differential $(2N-1)$-forms is given in (5.74) and where I and $I^{[A]}$ are defined by (5.54) and (5.55). As $*$ and $*^{[\mathbf{A}]}$ are clearly one-to-one and onto, their reverse applications may be considered, with the same notation

$$\begin{array}{rccc} * : & \Lambda^{2N-1}(\mathbf{T}^*\mathcal{X}) & \rightarrow & \Lambda(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{S} & \mapsto & *\mathfrak{S}, \end{array} \tag{5.81}$$

$$\begin{array}{rccc} *^{[\mathbf{A}]} : & \Lambda^{2N-1}(\mathbf{T}^*\mathcal{X}) & \rightarrow & \Lambda(\mathbf{T}^*\mathcal{X}) \\ & \mathfrak{S} & \mapsto & *^{[\mathbf{A}]}\mathfrak{S}. \end{array} \tag{5.82}$$

5.8 Vector Field associated with a function

If $\mathcal{H}$ is a function (or a differential 0-form) defined on $\mathbf{T}^*\mathcal{X}$, its exterior derivative $d\mathcal{H} = d\mathcal{H}$ may be computed. This is a differential 1-form defined on $\mathbf{T}^*\mathcal{X}$ with which vector fields $I(d\mathcal{H})$ and $I^{[A]}(d\mathcal{H})$, with I defined by (5.54) and $I^{[A]}$ by (5.55), may be associated.
To get an expression of all above evoked object, it has to be noticed that function $\mathcal{H}$ may be considered as a function of $(\mathbf{x},\mathbf{m}) \in \mathbf{T}^*\mathcal{X}$ with $\mathbf{x} \in \mathcal{X}$ and $\mathbf{m} \in \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$. Then, as it was done in (4.4), (4.5) the following functions may be considered

$$\begin{aligned} \widehat{\mathcal{H}}_{\mathbf{m}} : \mathcal{X} &\to \mathbb{R} \\ \mathbf{x} &\mapsto \mathcal{H}(t,\mathbf{x},\mathbf{m}), \end{aligned} \tag{5.83}$$

$$\begin{aligned} \bar{\mathcal{H}}_{\mathbf{x}} : \mathbf{T}^*_{\mathbf{x}}\mathcal{X} &\to \mathbb{R} \\ \mathbf{m} &\mapsto \mathcal{H}(t,\mathbf{x},\mathbf{m}), \end{aligned} \tag{5.84}$$

and in any $(\mathbf{x},\mathbf{m}) \in \mathbf{T}^*\mathcal{X}$ for any $(\nu,\pi) \in \mathbf{T}_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$, $\{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}\}(\nu,\pi) = \{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}\}(\nu,\pi) = \big\{d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}\big\}(\nu) + \big\{d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}}\big\}(\pi)$. In this writing, $d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}$ is clearly naturally an element of $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ but $d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}}$ is not naturally in $\mathbf{T}_{\mathbf{x}}\mathcal{X}$. Nevertheless, it is not a big issue to consider element $(d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}$ of $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ defined by $\big\{d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}}\big\}(\pi) = \big\langle \pi, (d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}\big\rangle$ for any π in $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$. Considering the identification of $\mathbf{T}^*_{(\mathbf{x},\mathbf{m})}\big(\mathbf{T}^*\mathcal{X}\big)$ by $\mathbf{T}^*_{\mathbf{x}}\mathcal{X} \times \mathbf{T}_{\mathbf{x}}\mathcal{X}$, equality $\{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}\}(\nu,\pi) = \{d_{(\mathbf{x},\mathbf{m})}\mathcal{H}\}(\nu,\pi) = \big((d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}, d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}\big)$ can be considered. Hence, in view of (5.43) and (5.51), the values of $I(d\mathcal{H})$ and $I^{[A]}(d\mathcal{H})$ in any $(\mathbf{x},\mathbf{m})$ is

$$I_{(\mathbf{x},\mathbf{m})}(d_{(\mathbf{x},\mathbf{m})}\mathcal{H}) = I_{(\mathbf{x},\mathbf{m})}(d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}, (d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}) = ((d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}, -d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}), \tag{5.85}$$

$$I^{[A]}_{(\mathbf{x},\mathbf{m})}(d_{(\mathbf{x},\mathbf{p})}\mathcal{H}) = I^{[A]}_{(\mathbf{x},\mathbf{m})}(d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}, (d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}) = ((d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%}, -(d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}} + q\,\overset{*}{\iota}_{((d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}})^{\%})} d_{\mathbf{x}}\mathbf{A}). \tag{5.86}$$

when translated into the coordinate systems, those formulas read

$$\breve{I}_{(\mathbf{q},\mathbf{p})}(\nabla_{(\mathbf{q},\mathbf{p})}\breve{\mathcal{H}}) = \breve{I}_{(\mathbf{q},\mathbf{p})}(\nabla_{\mathbf{q}}\breve{\mathcal{H}}, \nabla_{\mathbf{p}}\breve{\mathcal{H}}) = (\nabla_{\mathbf{p}}\breve{\mathcal{H}}, -\nabla_{\mathbf{q}}\breve{\mathcal{H}}), \tag{5.87}$$

$$\breve{I}^{[A]}_{(\mathbf{q},\mathbf{p})}(\nabla_{(\mathbf{q},\mathbf{p})}\breve{\mathcal{H}}) = \breve{I}^{[A]}_{(\mathbf{q},\mathbf{p})}(\nabla_{\mathbf{q}}\breve{\mathcal{H}}, \nabla_{\mathbf{p}}\breve{\mathcal{H}}) = (\nabla_{\mathbf{p}}\breve{\mathcal{H}}, -(\nabla_{\mathbf{q}}\breve{\mathcal{H}} + q\,\overset{*}{\iota}_{(\nabla_{\mathbf{p}}\breve{\mathcal{H}})} d_{\mathbf{x}}\mathbf{A})). \tag{5.88}$$

6 Hamiltonian Formulation I

A first Hamiltonian formulation of dynamics of a charge particle in an electromagnetic field consists in noticing, that with very small accommodations formulas (3.1) - (3.8), allowing the electromagnetic field computation, (3.67) - (3.73), defining the Lagrange Function, and (4.1) - (4.21) leading to the particle dynamics after using the Legendre's Transform, are valid if $\mathcal{X}$ is a manifold on which forms with zero exterior derivatives are exterior derivatives. (In order to claim this in a completely convincing way, it would be nice to define Hodge Operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, which has not been done yet.)

Then, considering, a position space $\mathcal{X}$, its tangent bundle $\mathbf{T}\mathcal{X}$ and its cotangent bundle $\mathbf{T}^*\mathcal{X}$ the particle dynamics is given as being its initial position and momentum transported by the flow of vector field $I(d\mathcal{H}^{[\mathbf{A}]})$ where Hamiltonian function $\mathcal{H}^{[\mathbf{A}]}$ is defined by (4.3), via the Legendre's Transform, from Lagrange Function defined by (3.67).

The dynamical system for $(\mathbf{M},\mathbf{X})$, where momentum trajectory is linked with velocity trajectory $\mathbf{V}$ by the following relation $\mathbf{V} = \mathcal{V}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{M})$ or $\mathbf{M} = \mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{V})$ which is a sub-product of the Legendre's

Trasform, is given by equations (4.19) and (4.21), which can be rewritten as

$$\frac{\partial \mathbf{M}}{\partial t} = -d_{\mathbf{X}(t)} \widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{M}}, \tag{6.1}$$

$$\frac{\partial \mathbf{X}}{\partial t} = \left(d_{\mathbf{M}} \bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{X}} \right)^{\%}, \tag{6.2}$$

or within the coordinate systems by (4.23) and (4.24) or (4.25) and (4.26).

7 Hamiltonian Formulation II

7.1 The second Hamiltonian Formulation

In this section $\mathcal{L}$ stands for the Lagrange Function defined by (3.67) with $\mathbf{A} = 0$, i.e.

$$\mathcal{L}(t, \mathbf{x}, \mathbf{v}) = \bar{L}_{\mathbf{x}}(\mathbf{v}) - q\Phi(t, \mathbf{x}) = \frac{1}{2} m |\mathbf{v}|^2 - q\Phi(t, \mathbf{x}), \tag{7.1}$$

With $\mathcal{L}$ are associated functions $\bar{\mathcal{L}}_{\mathbf{v}}$ and $\widehat{\mathcal{L}}_{\mathbf{x}}$ by (3.69) and (3.70) (of course with $\mathbf{A} = 0$ again). From $\mathcal{L}$ it may be built Hamiltonian Function $\mathcal{H}$ by the mean of Legendre's Transform. For this, formula (4.2), with $\mathcal{L}^{[\mathbf{A}]}$ replaced by $\mathcal{L}$, leads to a $\mathcal{L}\mathcal{H}$ function and (4.3) to the definition of $\mathcal{V}_{\mathbf{x}}$, i.e.

$$d_{(\mathcal{V}_{\mathbf{x}}(\mathbf{m}))} \bar{\mathcal{L}}_{\mathbf{x}} = \mathbf{m}. \tag{7.2}$$

Interpreting (4.16) with $\mathbf{A} = 0$ yields the fact that $\mathcal{V}_{\mathbf{x}}$ is the reverse function of $\mathcal{M}_{\mathbf{x}}$ defined by (3.11) with $\mathbf{A} = 0$, or equivalently by (2.4). Finally, (4.3) with $\mathbf{A} = 0$, or

$$\mathcal{H}(t, \mathbf{x}, \mathbf{m}) = \mathcal{L}\mathcal{H}(t, \mathbf{x}, \mathcal{V}_{\mathbf{x}}(\mathbf{m}), \mathbf{m}) = \langle \mathbf{m}, \mathcal{V}_{\mathbf{x}}(\mathbf{m}) \rangle - \mathcal{L}(t, \mathbf{x}, \mathcal{V}_{\mathbf{x}}(\mathbf{m})). \tag{7.3}$$

defines the Hamiltonian Function associated with $\mathcal{L}$.

Considering here again, a position space $\mathcal{X}$, its tangent bundle $\mathbf{T}\mathcal{X}$ and its cotangent bundle $\mathbf{T}^*\mathcal{X}$, it will be shown that the particle dynamics is given as being its initial position and momentum transported by the flow of vector field $\left(I^{[A]}(d\mathcal{H}) - q\Pi^*_*(\frac{\partial \mathbf{A}}{\partial t}) \right)$ where Hamiltonian function $\mathcal{H}$ is defined just above and pullback Π^*_* is defined by (5.30).

Another way to say this consists in claiming that the dynamical system for $(\tilde{\mathbf{M}}, \mathbf{X})$, where momentum trajectory is linked with velocity trajectory $\mathbf{V}$ by the following relation $\mathbf{V} = \mathcal{V}_{\mathbf{x}}(\tilde{\mathbf{M}})$ or $\tilde{\mathbf{M}} = \mathcal{M}_{\mathbf{x}}(\mathbf{V})$ (which is a sub-product of the Legendre's Transform), is given by

$$\frac{\partial \tilde{\mathbf{M}}}{\partial t} = -\left(d_{\mathbf{X}} \widehat{\mathcal{H}}_{\tilde{\mathbf{M}}} + q \overset{*}{\iota}_{((d_{\tilde{\mathbf{M}}} \bar{\mathcal{H}}_{\mathbf{X}})^{\%})} d_{\mathbf{X}} \mathbf{A} \right) - q \frac{\partial \mathbf{A}}{\partial t}(t, \mathbf{X}), \tag{7.4}$$

$$\frac{\partial \mathbf{X}}{\partial t} = \left(d_{\tilde{\mathbf{M}}} \bar{\mathcal{H}}_{\mathbf{X}} \right)^{\%}. \tag{7.5}$$

7.2 Equivalence of the two Hamiltonian formulation

7.2.1 Link between the differentials of the two Hamiltonian Functions

There exists a link between Hamiltonian Functions $\mathcal{H}$ and $\mathcal{H}^{[\mathbf{A}]}$, where $\mathcal{H}^{[\mathbf{A}]}$ is defined by (4.3) from Lagrange Function (3.67), which is not so difficult to set out. First, it has been set out (see (3.11)):

$$\mathcal{M}^{[\mathbf{A}]}_{\mathbf{x}}(\mathbf{v}) = \mathcal{M}_{\mathbf{x}}(\mathbf{v}) + q\mathbf{A}(t, \mathbf{x}). \tag{7.6}$$

Then, from (4.2), and since $\mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{v}) = \mathcal{L}(t,\mathbf{x},\mathbf{v}) + q\mathbf{A}(t,\mathbf{x})$, it may be deduced:

$$d_{(\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}))}\bar{\mathcal{L}}_{\mathbf{x}} = \mathbf{m} - q\mathbf{A}(t,\mathbf{x}). \tag{7.7}$$

Comparing (7.7) and (7.3) simply leads to

$$\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}) = \mathcal{V}_{\mathbf{x}}(\mathbf{m} - q\mathbf{A}(t,\mathbf{x})). \tag{7.8}$$

Now, starting from (4.3), a direct computation gives:

$$\begin{aligned}\mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m}+q\mathbf{A}(t,\mathbf{x})) &= \langle \mathbf{m}+q\mathbf{A}(t,\mathbf{x}), \mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}+q\mathbf{A}(t,\mathbf{x}))\rangle - \mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}^{[\mathbf{A}]}(\mathbf{m}+q\mathbf{A}(t,\mathbf{x})))\\ &= \langle \mathbf{m}+q\mathbf{A}(t,\mathbf{x}), \mathcal{V}_{\mathbf{x}}(\mathbf{m})\rangle - \mathcal{L}^{[\mathbf{A}]}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}(\mathbf{m}))\\ &= \langle \mathbf{m}+q\mathbf{A}(t,\mathbf{x}), \mathcal{V}_{\mathbf{x}}(\mathbf{m})\rangle - \mathcal{L}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}(\mathbf{m})) - \langle q\mathbf{A}(t,\mathbf{x}), \mathcal{V}_{\mathbf{x}}(\mathbf{m})\rangle = \langle \mathbf{m}, \mathcal{V}_{\mathbf{x}}(\mathbf{m})\rangle - \mathcal{L}(t,\mathbf{x},\mathcal{V}_{\mathbf{x}}(\mathbf{m}))\\ &= \mathcal{H}(t,\mathbf{x},\mathbf{m}),\end{aligned} \tag{7.9}$$

using at the end (7.3). Hence

$$\mathcal{H}(t,\mathbf{x},\mathbf{m}) = \mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m}+q\mathbf{A}(t,\mathbf{x})) \quad \text{and} \quad \mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m}) = \mathcal{H}(t,\mathbf{x},\mathbf{m}-q\mathbf{A}(t,\mathbf{x})). \tag{7.10}$$

From this last equality,

$$d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}}^{[\mathbf{A}]} = d_{(\mathbf{m}-q\mathbf{A}(t,\mathbf{x}))}\bar{\mathcal{H}}_{\mathbf{x}}, \tag{7.11}$$

$$d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}^{[\mathbf{A}]} = d_{\mathbf{x}}\widehat{\mathcal{H}}_{(\mathbf{m}-q\mathbf{A}(t,\mathbf{x}))} - q(\mathcal{D}_{(\mathbf{x},(d_{(\mathbf{m}-q\mathbf{A}(t,\mathbf{x}))}\bar{\mathcal{H}}_{\mathbf{x}})^{\%})}\Pi)\Big(d(\mathring{\imath}\breve{\mathbf{A}})\Big), \tag{7.12}$$

where operator $\mathcal{D}_{(\mathbf{x},\mathbf{v})}\Pi : \mathbf{T}^*_{(\mathbf{x},\mathbf{v})}(\mathbf{T}\mathcal{X}) \to \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ is defined by a straightforward accommodation of (2.28) to the case when $\mathcal{X}$ is a regular manifold.
Equality (7.11) is obvious. Equality is a bit more complicated to get. To get it, in the case when $N=3$, something of the spirit of what was done in subsection 3.2 has to be set out. This is what it is done now.

Bundlization $\mathfrak{B}\mathbf{A}$ of $\mathbf{A}$ is first considered:

$$\begin{aligned}\mathfrak{B}\mathbf{A} :\ & \mathcal{X} \to \mathbf{T}^*\mathcal{X}\\ & \mathbf{x} \mapsto (\mathbf{x},\mathbf{A}(\mathbf{x})).\end{aligned} \tag{7.13}$$

Its expression within the coordinates $\mathfrak{B}\breve{\mathbf{A}}$, which is such that $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathfrak{B}\breve{\mathbf{A}}(\mathbf{x})) = \mathfrak{B}\breve{\mathbf{A}}(\mathbf{q}) = (\mathbf{q},\breve{\mathbf{A}}(\mathbf{q}))$ with $\mathbf{q} = \mathcal{C}(\mathbf{x})$, is

$$\mathfrak{B}\breve{\mathbf{A}}(\mathbf{q},\mathbf{p}) = \mathbf{q}_1\mathbf{e}_{\mathbf{q}_1} + \mathbf{q}_2\mathbf{e}_{\mathbf{q}_2} + \mathbf{q}_3\mathbf{e}_{\mathbf{q}_3} + \breve{\mathbf{A}}_1(\mathbf{q})d\mathbf{q}_1 + \breve{\mathbf{A}}_2(\mathbf{q})d\mathbf{q}_2 + \breve{\mathbf{A}}_3(\mathbf{q})d\mathbf{q}_3. \tag{7.14}$$

The differential $d_{\mathbf{x}}(\mathfrak{B}\mathbf{A})$ of $\mathfrak{B}\mathbf{A}$ in any $\mathbf{x}$ of $\mathcal{X}$ is represented by differential $d_{\mathbf{q}}(\mathfrak{B}\breve{\mathbf{A}})$ in the coordinate systems, but also by $\nabla_{\mathbf{q}}(\mathfrak{B}\breve{\mathbf{A}})$ which expression is:

$$\nabla_{\mathbf{q}}(\mathfrak{B}\breve{\mathbf{A}}) = \begin{pmatrix} I_{\mathbb{R}^N} \\ \begin{pmatrix} \frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_1} & \frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2} & \frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3} \\ \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1} & \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_2} & \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3} \\ \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1} & \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2} & \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_3} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} I_{\mathbb{R}^N} \\ \nabla_{\mathbf{q}}\breve{\mathbf{A}} \end{pmatrix}. \tag{7.15}$$

Then $\breve{\mathcal{H}}$ and $\breve{\mathcal{H}}^{[\mathbf{A}]}$ are the expressions of $\mathcal{H}$ and $\mathcal{H}^{[\mathbf{A}]}$ in the coordinates. In other words $\mathcal{H}(t,\mathbf{x},\mathbf{m}) = \breve{\mathcal{H}}(t,\mathbf{q},\mathbf{p})$ and $\mathcal{H}^{[\mathbf{A}]}(t,\mathbf{x},\mathbf{m}) = \breve{\mathcal{H}}^{[\mathbf{A}]}(t,\mathbf{q},\mathbf{p})$ where $(\mathcal{C},\mathcal{D}_{\mathbf{x}}\mathcal{C})(\mathbf{x},\mathbf{m}) = (\mathbf{q},\mathbf{p})$. Writing (7.10) in the coordinates gives:

$$\breve{\mathcal{H}}^{[\mathbf{A}]}(t,\mathbf{q},\mathbf{p}) = \breve{\mathcal{H}}(t,\mathbf{q},\mathbf{p}-q\breve{\mathbf{A}}(t,\mathbf{q})). \tag{7.16}$$

Hence, $\nabla_{\mathbf{q}}\breve{\mathcal{H}}^{[\mathbf{A}]}$ which represents $d_{\mathbf{x}}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{m}}$ in the coordinates and $\nabla_{\mathbf{q}}\breve{\mathcal{H}}$ which represents $d_{\mathbf{x}}\widehat{\mathcal{H}}_{\mathbf{m}}$ and $\nabla_{\mathbf{p}}\breve{\mathcal{H}}$ which represents $d_{\mathbf{m}}\bar{\mathcal{H}}_{\mathbf{x}}$ are linked by

$$\nabla_{\mathbf{q}}\breve{\mathcal{H}}^{[\mathbf{A}]}(t,\mathbf{q},\mathbf{p}) = \nabla_{\mathbf{q}}\breve{\mathcal{H}}(t,\mathbf{q},\mathbf{p}-q\breve{\mathbf{A}}(t,\mathbf{q})) - q(\nabla_{\mathbf{q}}\breve{\mathbf{A}}(t,\mathbf{q}))^T\big(\nabla_{\mathbf{p}}\breve{\mathcal{H}}(t,\mathbf{q},\mathbf{p}-q\breve{\mathbf{A}}(t,\mathbf{q}))\big). \tag{7.17}$$

having a look on (3.54) and on computations leading to it, it is gotten that

$$\begin{aligned}(\mathcal{D}_{(\mathbf{q},\breve{\nu})}\Pi)\Big(d(\mathring{\iota}\breve{\mathbf{A}})\Big) &= \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_1}\breve{\nu}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_1}\breve{\nu}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_1}\breve{\nu}_3\right)d\mathbf{q}_1 \\ &+ \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_2}\breve{\nu}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_2}\breve{\nu}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_2}\breve{\nu}_3\right)d\mathbf{q}_2 + \left(\frac{\partial\breve{\mathbf{A}}_1}{\partial\mathbf{q}_3}\breve{\nu}_1 + \frac{\partial\breve{\mathbf{A}}_2}{\partial\mathbf{q}_3}\breve{\nu}_2 + \frac{\partial\breve{\mathbf{A}}_3}{\partial\mathbf{q}_3}\breve{\nu}_3\right)d\mathbf{q}_3,\end{aligned} \tag{7.18}$$

and then it can be deduced that $(\nabla_{\mathbf{q}}\breve{\mathbf{A}})^T\breve{\nu}$ is $(\mathcal{D}_{(\mathbf{q},\breve{\nu})}\Pi)\big(d(\mathring{\iota}\breve{\mathbf{A}})\big)$ and then is nothing but an expression of $(\mathcal{D}_{(\mathbf{x},\nu)}\Pi)\big(d(\mathring{\iota}\mathbf{A})\big)$ in the coordinate systems. Hence (7.17) is nothing but an expression of (7.12) in the coordinate systems.

7.2.2 Link between the differentials of the two momentum trajectories

Since in a given point of the phase-space trajectory $(\mathbf{X},\mathbf{M})$, $\mathbf{M} = \mathcal{M}^{[\mathbf{A}]}_{\mathbf{X}}(\mathbf{V})$ and $\mathbf{V} = \mathcal{V}_{\mathbf{X}}(\tilde{\mathbf{M}})$ it is deduced $\mathbf{M} = \mathcal{M}^{[\mathbf{A}]}_{\mathbf{X}}(\mathcal{V}_{\mathbf{X}}(\tilde{\mathbf{M}}))$. Because of (7.6), it is finally gotten: $\mathbf{M} = \mathcal{M}_{\mathbf{X}}(\mathcal{V}_{\mathbf{X}}(\tilde{\mathbf{M}})) + q\mathbf{A}(t,\mathbf{X}) = \tilde{\mathbf{M}} + q\mathbf{A}(t,\mathbf{X})$ and

$$\frac{\partial\mathbf{M}}{\partial t} = \frac{\partial\tilde{\mathbf{M}}}{\partial t} + \{q\,(\mathcal{D}_{(\mathbf{X},\mathbf{V})}\Pi)\big(\mathring{\iota}(d\mathbf{A})\big)\} + \{q\,(\mathcal{D}_{(\mathbf{X},\mathbf{V})}\Pi)\big(d(\mathring{\iota}\mathbf{A})\big)\} + \Big\{q\frac{\partial\mathbf{A}}{\partial t}(t,\mathbf{X})\Big\}, \tag{7.19}$$

which is gotten by a computation in the coordinate systems of the same type as the one yielding (3.55). Now using (5.59) and that

$$\mathbf{V} = \frac{\partial\mathbf{X}}{\partial t} = \Big(d_{\mathbf{M}}\bar{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{X}}\Big)^{\%} = \Big(d_{\tilde{\mathbf{M}}}\bar{\mathcal{H}}_{\mathbf{X}}\Big)^{\%}, \tag{7.20}$$

which gives (7.5), equation (7.19) reads

$$\frac{\partial\mathbf{M}}{\partial t} = \frac{\partial\tilde{\mathbf{M}}}{\partial t} + q\,\overset{*}{\iota}_{((d_{\tilde{\mathbf{M}}}\bar{\mathcal{H}}_{\mathbf{X}})^{\%})}d_{\mathbf{X}}\mathbf{A} + q\,(\mathcal{D}_{(\mathbf{X},\mathbf{V})}\Pi)\big(d(\mathring{\iota}\mathbf{A})\big) + q\frac{\partial\mathbf{A}}{\partial t}(t,\mathbf{X}). \tag{7.21}$$

using finally (6.1) and (7.12),

$$\begin{aligned}\frac{\partial\mathbf{M}}{\partial t} = -d_{\mathbf{X}}\widehat{\mathcal{H}}^{[\mathbf{A}]}_{\mathbf{M}} &= -d_{\mathbf{X}}\widehat{\mathcal{H}}_{(\mathbf{M}-q\mathbf{A}(t,\mathbf{X}))} + q(\mathcal{D}_{(\mathbf{X},(d_{(\mathbf{M}-q\mathbf{A}(t,\mathbf{X}))}\bar{\mathcal{H}}_{\mathbf{X}})^{\%})}\Pi)\Big(d(\mathring{\iota}\mathbf{A})\Big) \\ &= -d_{\mathbf{X}}\widehat{\mathcal{H}}_{\tilde{\mathbf{M}}} + q(\mathcal{D}_{(\mathbf{X},(d_{\tilde{\mathbf{M}}}\bar{\mathcal{H}}_{\mathbf{X}})^{\%})}\Pi)\Big(d(\mathring{\iota}\mathbf{A})\Big)\end{aligned} \tag{7.22}$$

equation for $\tilde{\mathbf{M}}$:

$$\frac{\partial\tilde{\mathbf{M}}}{\partial t} = -\Big(d_{\mathbf{X}}\widehat{\mathcal{H}}_{\tilde{\mathbf{M}}} + q(\mathcal{D}_{(\mathbf{X},(d_{\tilde{\mathbf{M}}}\bar{\mathcal{H}}_{\mathbf{X}})^{\%})}\Pi)\Big(d(\mathring{\iota}\mathbf{A})\Big)\Big) - q\frac{\partial\mathbf{A}}{\partial t}(t,\mathbf{X}), \tag{7.23}$$

which is (7.4).

8 Hodge Operators ε and μ

Classically, Hodge Operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are linked with the metric on $\mathcal{X}$. Essentially they allow to give the way electromagnetic waves, or electromagnetic energy, propagate. Hence, they are linked with the notion of geodesic, and this last notion is induced by the metric.
Here, no metric is explicitly considered (which is not absolutely true but which makes up the framework in which the present Part of the document is built), but the Lagrange Function

$$\begin{array}{rrl} L: & \mathbf{T}\mathcal{X} & \to \mathbb{R} \\ & (\mathbf{x},\mathbf{v}) & \mapsto \bar{L}_{\mathbf{x}}(\mathbf{v}), \end{array} \tag{8.1}$$

with no force indicates a kind of propagation principle when no force acts on an object.
This is the point of view which is adopted here and that will be resumed to be.

8.1 Non degenerated scalar product on 1-forms induced by the Lagrange Function

From Lagrange Function defined by (8.1), a Hamiltonian Function may be built. Doing this leads to consider (4.1) with $\mathcal{L}^{[\mathbf{A}]}$ replaced by L and to define of $\mathcal{V}_{\mathbf{x}}$ by

$$d_{(\mathcal{V}_{\mathbf{x}}(\mathbf{m}))}\bar{L}_{\mathbf{x}} = \mathbf{m}. \tag{8.2}$$

In other words, in any $\mathbf{x}$ of $\mathcal{X}$, there is a mapping which is defined as follows

$$\begin{array}{rccc} \mathcal{V}_{\mathbf{x}}: & \mathbf{T}^*_{\mathbf{x}}\mathcal{X} & \to & \mathbf{T}_{\mathbf{x}}\mathcal{X} \\ & \mathbf{m} & \mapsto & \mathcal{V}_{\mathbf{x}}(\mathbf{m}), \end{array} \tag{8.3}$$

Hence a non degenerated scalar product $g_{\mathbf{x}}(.,.)$ on 1-forms (or on $\mathbf{T}^*_{\mathbf{x}}\mathcal{X}$) is defined in any $\mathbf{x}$ of $\mathcal{X}$ by

$$g_{\mathbf{x}}(\pi,\mu) = \frac{1}{2}\Big(\langle \pi, \mathcal{V}_{\mathbf{x}}(\mu)\rangle + \langle \mu, \mathcal{V}_{\mathbf{x}}(\pi)\rangle\Big) \tag{8.4}$$

for any $\pi \in \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and any $\mu \in \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$.

8.2 Metric on $\mathcal{X}$ induced by the Lagrange Function

from the non degenerated scalar product defined by (8.3) and (8.4) a non degenerated scalar on $\mathbf{T}_{\mathbf{x}}\mathcal{X}$ in any $\mathbf{x}$ of $\mathcal{X}$ may be defined by:

$$g_{\mathbf{x}}(\nu,\upsilon) = \frac{1}{2}\Big(\langle \mathcal{M}_{\mathbf{x}}(\pi), \mathcal{V}_{\mathbf{x}}(\mathcal{M}_{\mathbf{x}}(\upsilon))\rangle + \langle \mathcal{M}_{\mathbf{x}}(\upsilon), \mathcal{V}_{\mathbf{x}}(\mathcal{M}_{\mathbf{x}}(\nu))\rangle\Big) \tag{8.5}$$

for any $\pi \in \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$ and any $\mu \in \mathbf{T}^*_{\mathbf{x}}\mathcal{X}$.