The Geometrical Gyro-Kinetic Approximation

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Joint work with Mathieu Lutz

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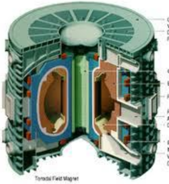
Charge particles submitted to Strong Magnetic Field

In Usual Coordinates : $(\mathbf{x},\mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$

$$\mathbf{X}(t;\mathbf{x},\mathbf{v},s), \mathbf{V}(t;\mathbf{x},\mathbf{v},s)$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{q}{m}(\mathbf{E}(\mathbf{X}) + \mathbf{V} \times \mathbf{B}(\mathbf{X}))$$

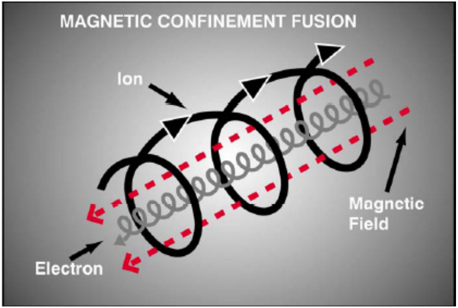


$$\mathbf{B} : \underbrace{\text{Strong Applied piece} + \text{Strong Self Induced piece}}_{\to \frac{1}{\varepsilon}\mathbf{B}} + \underbrace{\text{Self Induced Perturbations}}_{\text{Forgotten}}$$

$\mathbf{E}$: Self Induced piece

$(\mathbf{E} = -\nabla\Phi,\ \mathbf{B} = \nabla \times \mathbf{A})$

Helicoidal trajectories - Larmor Radius



Source: S. Jardin's Lectures at Cemracs'10

In Tokamak:

Electron Larmor Radius $\sim 5 \cdot 10^{-4} m$

Ion Larmor Radius $\sim 10^{-2} m$

Dimensionless Dynamical System

$$\varepsilon \sim \frac{\text{Ion Larmor Radius}}{\text{Tokamak size}} \sim \frac{10^{-2}m}{10m} \sim 10^{-3}$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{E}(\mathbf{X}) + \mathbf{V} \times \frac{\mathbf{B}(\mathbf{X})}{\varepsilon}$$

Simplifications

Skip **E**

Turn to dimension 2: $\mathbf{x} = (x_1, x_2)$, $\mathbf{v} = (v_1, v_2)$, with

$$\mathbf{B}(\mathbf{x}) = (0, 0, B(x_1, x_2))$$

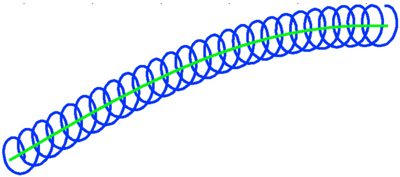
$$B > 1, \quad B(x_1, x_2) = \nabla \times \mathbf{A}(x_1, x_2) = \frac{\partial A_2}{\partial x_1}(x_1, x_2) - \frac{\partial A_1}{\partial x_2}(x_1, x_2)$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}, \qquad \mathbf{X}(0) = \mathbf{x}_0,$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon}\, B(\mathbf{X})\, {}^{\perp}\mathbf{V} = \frac{1}{\varepsilon}\, B(\mathbf{X}) \begin{pmatrix} V_2 \\ -V_1 \end{pmatrix}, \qquad \mathbf{V}(0) = \mathbf{v}_0$$

$$\frac{\partial}{\partial t} \begin{pmatrix} X_1 \\ X_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ \frac{1}{\varepsilon}\, B(\mathbf{X})\, V_2 \\ -\frac{1}{\varepsilon}\, B(\mathbf{X})\, V_1 \end{pmatrix}, \qquad \begin{pmatrix} X_1 \\ X_2 \\ V_1 \\ V_2 \end{pmatrix}(0) = \begin{pmatrix} x_{01} \\ x_{02} \\ v_{01} \\ v_{02} \end{pmatrix}$$

Gyrokinetic model



$$\frac{\partial \mathbf{Z}}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})} {}^{\perp}\nabla B(\mathbf{Z}), \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

$$\frac{\partial}{\partial t}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})}\begin{pmatrix} \dfrac{\partial B}{\partial x_2}(\mathbf{Z}) \\ -\dfrac{\partial B}{\partial x_1}(\mathbf{Z}) \end{pmatrix}, \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

for magnetic moment $\mathcal{J}$

What is hidden

$$\frac{\partial \mathbf{Z}}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})} {}^{\perp}\nabla B(\mathbf{Z}), \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

$$\frac{\partial \Gamma}{\partial t} = \frac{B(\mathbf{Z})}{\varepsilon} + \varepsilon \frac{\mathcal{J}}{2B(\mathbf{Z})^2}\left(B(\mathbf{Z})\nabla^2 B(\mathbf{Z}) - 3\left(\nabla B((\mathbf{Z}))\right)^2\right), \quad \Gamma(0) = \gamma_0$$

$$\frac{\partial \mathcal{J}}{\partial t} = 0, \qquad \mathcal{J}(0) = j_0$$

The Geometrical Gyro-Kinetic Approximation

Emmanuel Frénod

Key result

IF: In coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4)$, a Hamiltonian Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R}) \nabla_{\mathbf{r}} H(\mathbf{R}) \qquad \mathcal{P}(\mathbf{r}) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathtt{M}(\mathbf{r})$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right)$$

with

$$\frac{\partial H}{\partial r_3} = 0$$

$$\textbf{THEN:} \quad \frac{\partial \mathtt{M}}{\partial r_4} = 0 \qquad \textbf{AND:} \quad \frac{\partial R_4}{\partial t} = 0$$

(Trajectory $\mathbf{R} = (R_1, R_2, R_3, R_4)$)

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon} B(\mathbf{X})^{\perp}\mathbf{V}$$

1: Hamiltonian?

2

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon = \breve{H}_\varepsilon(\mathbf{q}, \mathbf{p})$:

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}} \breve{H}_\varepsilon$$
$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}} \breve{H}_\varepsilon$$

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

4: Darboux Method

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, v)$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

2

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{M}_{(\mathbf{r})}} & \begin{smallmatrix}0\\0\end{smallmatrix} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Canonical Coordinates

Usual Coordinates : $(\mathbf{x},\mathbf{v}) = (x_1, x_2, v_1, v_2)$
Trajectory : $(\mathbf{X}(t;\mathbf{x},\mathbf{v},s), \mathbf{V}(t;\mathbf{x},\mathbf{v},s))$ $\big((\mathbf{X},\mathbf{V}) = (X_1, X_2, V_1, V_2)\big)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V} \qquad B(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon}\, B(\mathbf{X})\,{}^{\perp}\mathbf{V}$$

Canonical Coordinates : $(\mathbf{q},\mathbf{p}) = (q_1, q_2, p_1, p_2)$
Trajectory : $(\mathbf{Q}(t;\mathbf{q},\mathbf{p},s), \mathbf{P}(t;\mathbf{q},\mathbf{p},s))$ $\big((\mathbf{Q},\mathbf{P}) = (Q_1, Q_2, P_1, P_2)\big)$

$$\mathbf{q} = \mathbf{x} \qquad \mathbf{x} = \mathbf{q} \qquad \mathbf{Q} = \mathbf{X} \qquad \mathbf{X} = \mathbf{Q}$$

$$\mathbf{p} = \mathbf{v} + \frac{\mathbf{A}(\mathbf{x})}{\varepsilon} \qquad \mathbf{v} = \mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon} \qquad \mathbf{P} = \mathbf{V} + \frac{\mathbf{A}(\mathbf{X})}{\varepsilon} \qquad \mathbf{V} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon \qquad \breve{H}_\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2$$

$$\mathcal{S} = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$$

Check of Canonical nature of Canonical Coordinates

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon, \qquad \breve{H}_\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}} \breve{H}_\varepsilon(\mathbf{Q},\mathbf{P}) = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}} \breve{H}_\varepsilon(\mathbf{Q},\mathbf{P}) = \frac{(\nabla \mathbf{A}(\mathbf{Q}))^T}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$

$$(\nabla \mathbf{A})^T(\mathbf{p}-\mathbf{A}) = (\nabla \mathbf{A})(\mathbf{p}-\mathbf{A}) + (\nabla \times \mathbf{A})\ {}^{\perp}(\mathbf{p}-\mathbf{A})$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon}\ {}^{\perp}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$

Check of Canonical nature of Canonical Coord. - 2

$$\mathbf{X} = \mathbf{Q}$$
$$\mathbf{V} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$
$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon}\,{}^{\perp}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$
$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\frac{\partial \mathbf{Q}}{\partial t}\Big) = \frac{\partial\Big[\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big]}{\partial t} = \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon}\,{}^{\perp}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$
$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\nabla \times \mathbf{A}(\mathbf{X})}{\varepsilon}\,{}^{\perp}\mathbf{V}$$

As by products : Poisson Matrix, Poisson Bracket, Change of Coordinates Formula

In any coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4)$, the Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R})\nabla_{\mathbf{r}} H(\mathbf{R}) = \{\mathbf{I}, H\}(\mathbf{R})$$

$\{f, g\}(\mathbf{r}) = (\nabla_{\mathbf{r}} f(\mathbf{r})) \cdot (\mathcal{P}(\mathbf{r})(\nabla_{\mathbf{r}} g(\mathbf{r})))$ $\qquad (f$ and $g : \mathbb{R}^4 \to \mathbb{R})$

$(\{\mathbf{f}, g\}(\mathbf{r}))_i = (\nabla_{\mathbf{r}} \mathbf{f}_i(\mathbf{r})) \cdot (\mathcal{P}(\mathbf{r})(\nabla_{\mathbf{r}} g(\mathbf{r})))$

$(\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R})$

$\mathbf{I}(\mathbf{r}) = \mathbf{r}$

Another coordinate system $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)$ with $\tilde{\mathbf{r}} = \boldsymbol{\rho}(\mathbf{r})$,
$\mathbf{r} = \tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}) = \boldsymbol{\rho}^{-1}(\tilde{\mathbf{r}})$

$$\frac{\partial \tilde{\mathbf{R}}}{\partial t} = \tilde{\mathcal{P}}(\tilde{\mathbf{R}})\nabla_{\tilde{\mathbf{r}}} \tilde{H}(\tilde{\mathbf{R}})$$

$$\tilde{H}(\tilde{\mathbf{r}}) = H(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}})) \qquad (\tilde{\mathcal{P}}(\tilde{\mathbf{r}}))_{ij} = \{\boldsymbol{\rho}_i, \boldsymbol{\rho}_j\}(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}))$$

Panorama

Usual Coordinates

$(\mathbf{x},\mathbf{v})$

$H_\varepsilon(\mathbf{x},\mathbf{v}),\, \mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

3

Polar Coordinates

$(\mathbf{x},\theta,v)$

$\widetilde{H}_\varepsilon(v),\, \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x},\theta,v)$

4: Darboux Method

1: Hamiltonian?

2

$$\left(\begin{array}{c|cc} \mathrm{M}_{(\mathbf{r})} & \begin{array}{c}0\\0\end{array} & \begin{array}{c}0\\0\end{array} \\ \hline \begin{array}{cc}0&0\\0&0\end{array} & \begin{array}{c}0\\-1\end{array} & \begin{array}{c}1\\0\end{array} \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y},\theta,k)$

$\overline{H}_\varepsilon(\mathbf{y},\theta,k),\, \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

Canonical Coordinates

$(\mathbf{q},\mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q},\mathbf{p}),\, \breve{\mathcal{P}}_\varepsilon(\mathbf{q},\mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

5: Lie Method

Lie Coordinates

$(\mathbf{z},\gamma,j)$

$\widehat{H}_\varepsilon(\mathbf{z},j),\, \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Hamiltonian Function and Poisson Matrix in Usual Coordinates

$$\boxed{H_\varepsilon(\mathbf{x},\mathbf{v}) = \frac{1}{2}\left|\mathbf{v}\right|^2} \qquad \left(\breve{H}_\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2\right)$$

$$\nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon = \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix}$$

$$\boxed{\mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{(\nabla \mathbf{A}(\mathbf{x}))^T - (\nabla \mathbf{A}(\mathbf{x}))}{\varepsilon} \end{pmatrix}}$$

Indeed:

$$\mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v})\nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon = \begin{pmatrix} \mathbf{v} \\ \frac{(\nabla \mathbf{A}(\mathbf{x}))^T - (\nabla \mathbf{A}(\mathbf{x}))}{\varepsilon}\mathbf{v} \end{pmatrix}$$

To be compared with:

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\nabla \times \mathbf{A}(\mathbf{X})}{\varepsilon}\,{}^\perp\mathbf{V} = \frac{1}{\varepsilon}\,B(\mathbf{X})\,{}^\perp\mathbf{V}$$

Formula giving Poisson Matrix

$$\mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{(\nabla A(\mathbf{x}))^T - (\nabla A(\mathbf{x}))}{\varepsilon} \end{pmatrix}$$

Change of coordinates: $\tilde{\mathbf{r}}$ and $\mathbf{r}$ with $\tilde{\mathbf{r}} = \rho(\mathbf{r})$ and $\mathbf{r} = \tilde{\rho}(\tilde{\mathbf{r}}) = \rho^{-1}(\tilde{\mathbf{r}})$

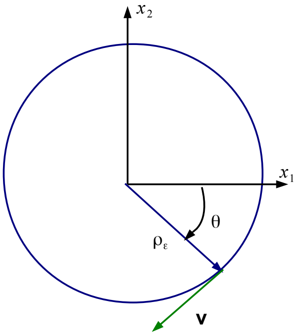
$$(\tilde{\mathcal{P}}(\tilde{\mathbf{r}}))_{ij} = \{\rho_i, \rho_j\}(\tilde{\rho}(\tilde{\mathbf{r}}))$$

Here: $(\mathbf{x},\mathbf{v})$ and $(\mathbf{q},\mathbf{p})$ with $(\mathbf{x},\mathbf{v}) = \xi(\mathbf{q},\mathbf{p})$ and $(\mathbf{q},\mathbf{p}) = \pi(\mathbf{x},\mathbf{v})$
For instance : $\xi_3(\mathbf{q},\mathbf{p}) = p_1 - \frac{A_1(\mathbf{q})}{\varepsilon}$ and $\xi_4(\mathbf{q},\mathbf{p}) = p_2 - \frac{A_2(\mathbf{q})}{\varepsilon}$

$$\nabla\xi_3(\mathbf{q},\mathbf{p}) = \begin{pmatrix} -\frac{1}{\varepsilon}\frac{\partial A_1}{\partial q_1}(\mathbf{q}) \\ -\frac{1}{\varepsilon}\frac{\partial A_1}{\partial q_2}(\mathbf{q}) \\ 1 \\ 0 \end{pmatrix} \quad \mathcal{S} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \quad \nabla\xi_4(\mathbf{q},\mathbf{p}) = \begin{pmatrix} -\frac{1}{\varepsilon}\frac{\partial A_1}{\partial q_1}(\mathbf{q}) \\ -\frac{1}{\varepsilon}\frac{\partial A_2}{\partial q_2}(\mathbf{q}) \\ 0 \\ 1 \end{pmatrix}$$

$$\{\xi_3,\xi_4\}(\pi(\mathbf{x},\mathbf{v})) = \frac{1}{\varepsilon}\Big(\frac{\partial A_2}{\partial q_1}(\mathbf{x}) - \frac{\partial A_1}{\partial q_2}(\mathbf{x})\Big)$$

Polar Coordinates (in velocity)



$(\mathbf{x}, \theta, v)$: $\quad v = |\mathbf{v}|, \quad \theta$ s.t. $\mathbf{v} = v \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix}$

Hamiltonian Function and Poisson Matrix in Polar Coordinates

$$\boxed{\widetilde{H}_\varepsilon(\mathbf{x},\theta,v) = \frac{v^2}{2}} \qquad \left(H_\varepsilon(\mathbf{x},\mathbf{v}) = \tfrac{1}{2}|\mathbf{v}|^2\right)$$

$$\boxed{\tilde{\mathcal{P}}_\varepsilon(\mathbf{x},\theta,v) = \begin{pmatrix} 0 & 0 & -\dfrac{\cos(\theta)}{v} & -\sin(\theta) \\ 0 & 0 & \dfrac{\sin(\theta)}{v} & -\cos(\theta) \\ \dfrac{\cos(\theta)}{v} & -\dfrac{\sin(\theta)}{v} & 0 & \dfrac{B(\mathbf{x})}{\varepsilon v} \\ \sin(\theta) & \cos(\theta) & -\dfrac{B(\mathbf{x})}{\varepsilon v} & 0 \end{pmatrix}}$$

Singularity in $v = 0$ and $\omega_\varepsilon = \dfrac{B(\mathbf{x})}{\varepsilon v} > 0$

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

2

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{M}_{(\mathbf{r})}} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \hline 0\ 0 & & 0 & 1 \\ 0\ 0 & & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Darboux Method Target

Find a Coordinate System $(\mathbf{y}, \theta, k)$ s.t. Poisson Matrix $(\overline{\mathcal{P}}_\varepsilon)$ shape:

$$\boxed{\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{\mathbb{M}}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)} \quad \text{or} \quad \boxed{\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{\mathbb{M}}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)}$$

$$(\mathbf{y}, \theta, k) = \boldsymbol{\Upsilon}(\mathbf{x}, \theta, v), \qquad (\mathbf{x}, \theta, v) = \boldsymbol{\xi}(\mathbf{y}, \theta, k), \qquad (\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1})$$

$$(\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k))_{ij} = \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\}(\boldsymbol{\xi}(\mathbf{y}, \theta, k)), \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\} = (\nabla \boldsymbol{\Upsilon}_i) \cdot (\widetilde{\mathcal{P}}_\varepsilon(\nabla \boldsymbol{\Upsilon}_j))$$

Needed:
$$\begin{array}{ll} \{\boldsymbol{\Upsilon}_4, \boldsymbol{\Upsilon}_3\} = -\frac{1}{\varepsilon} & (=\{\boldsymbol{\Upsilon}_k, \boldsymbol{\Upsilon}_\theta\} = \{k, \theta\}) \\ \{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_3\} = 0 & (=\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_\theta\} = \{y_1, \theta\}) \\ \{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_4\} = 0 & (=\{\boldsymbol{\Upsilon}_{y_1}, \boldsymbol{\Upsilon}_k\} = \{y_1, k\}) \\ \{\boldsymbol{\Upsilon}_2, \boldsymbol{\Upsilon}_3\} = 0 & (=\{\boldsymbol{\Upsilon}_{y_2}, \boldsymbol{\Upsilon}_\theta\} = \{y_2, \theta\}) \\ \{\boldsymbol{\Upsilon}_2, \boldsymbol{\Upsilon}_4\} = 0 & (=\{\boldsymbol{\Upsilon}_{y_2}, \boldsymbol{\Upsilon}_k\} = \{y_2, k\}) \end{array}$$

First equation processing

$\{\Upsilon_4,\Upsilon_3\} = -\frac{1}{\varepsilon} \quad (=\{\Upsilon_k,\Upsilon_\theta\}=\{k,\theta\})$
or

$$\{\Upsilon_3, \Upsilon_4\} = \frac{1}{\varepsilon} \quad (=\{\Upsilon_\theta \Upsilon_k,\}=\{\theta, k\}) \qquad (\bullet)$$

$$\boxed{\nabla\Upsilon_3(= \nabla\Upsilon_\theta) = (0,0,1,0)^T}$$

$\{\Upsilon_3, \Upsilon_4\} = (\nabla\Upsilon_3)\cdot(\widetilde{\mathcal{P}}^\varepsilon(\nabla\Upsilon_4))$: penultimate comp. of $(\widetilde{\mathcal{P}}^\varepsilon(\nabla\Upsilon_4))$
$(\bullet) \rightarrow$

$$\cos(\theta)\frac{\partial\Upsilon_4}{\partial x_1} - \sin(\theta)\frac{\partial\Upsilon_4}{\partial x_2} + \omega_\varepsilon\frac{\partial\Upsilon_4}{\partial v} = \frac{1}{\varepsilon} \qquad (\omega_\varepsilon(\mathbf{x}, v) = \frac{B(\mathbf{x})}{\varepsilon v})$$

$(\Upsilon_4(\mathbf{x},\theta,v) = k)$

Method of Characteristics

Method of Characteristics - 1: exact solution

$$\cos(\theta)\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_1} - \sin(\theta)\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_2} + \omega_\varepsilon \frac{\partial \boldsymbol{\Upsilon}_4}{\partial v} = \frac{1}{\varepsilon}, \quad \left(\omega_\varepsilon(\mathbf{x}, v) = \frac{B(\mathbf{x})}{\varepsilon v}\right)$$

$$\frac{\partial \boldsymbol{\Upsilon}_4}{\partial v} + \varepsilon\frac{v\cos(\theta)}{B(\mathbf{x})}\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_1} - \varepsilon\frac{v\sin(\theta)}{B(\mathbf{x})}\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_2} = \frac{v}{B(\mathbf{x})}$$

$$\boldsymbol{\Upsilon}_{4|v=0} = 0 \quad \left(\text{In fact } \boldsymbol{\Upsilon}_{\mathbf{4}|v=\nu} = 0 \text{ for } \nu \to 0\right)$$

$$\mathcal{X}_1(\theta; v; \mathbf{x}, u) \text{ s.t. } \frac{\partial \mathcal{X}_1}{\partial v} = \varepsilon\frac{v\cos(\theta)}{B(\mathcal{X}_1, \mathcal{X}_2)}, \qquad \mathcal{X}_1(\theta; u; \mathbf{x}, u) = x_1$$

$$\mathcal{X}_2(\theta; v; \mathbf{x}, u) \text{ s.t. } \frac{\partial \mathcal{X}_2}{\partial v} = -\varepsilon\frac{v\sin(\theta)}{B(\mathcal{X}_1, \mathcal{X}_2)}, \qquad \mathcal{X}_2(\theta; u; \mathbf{x}, u) = x_2$$

$$\boldsymbol{\Upsilon}_4(\mathbf{x}, \theta, v) = \boldsymbol{\Upsilon}_4(\boldsymbol{\mathcal{X}}(\theta; 0; \mathbf{x}, v), \theta, 0) + \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, v))} ds$$
$$= \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, v))} ds$$

Gives explicit expression of k in terms of $(\mathbf{x}, \theta, v)$

Method of Characteristics - 2: Asymptotic expansion

$$\boldsymbol{\mathcal{X}}(\theta; v; \mathbf{x}, u) = \begin{pmatrix} \mathcal{X}_1(\theta; v; \mathbf{x}, u) \\ \mathcal{X}_2(\theta; v; \mathbf{x}, u) \end{pmatrix}$$

$\frac{\partial \boldsymbol{\mathcal{X}}}{\partial v} = \varepsilon v_\times$something $\Rightarrow \boldsymbol{\mathcal{X}}$ close to $\mathbf{x} \Rightarrow \boldsymbol{\mathcal{X}} = \mathbf{x} + \varepsilon v \boldsymbol{\mathcal{X}}^1 + \varepsilon^2 v^2 \boldsymbol{\mathcal{X}}^2 + \dots$

$$\frac{\partial \boldsymbol{\mathcal{X}}}{\partial v} = v\ \varepsilon \mathbf{F}(\boldsymbol{\mathcal{X}}, v, \theta), \boldsymbol{\mathcal{X}}(\theta; u, \mathbf{x}, u) = \mathbf{x}, \ (\mathbf{F}(\mathbf{x}, \theta) = \begin{pmatrix} \dfrac{\cos(\theta)}{B(x_1, x_2)} \\ -\dfrac{\sin(\theta)}{B(x_1, x_2)} \end{pmatrix})$$

$$\boldsymbol{\mathcal{X}} = \mathbf{x} + v\varepsilon \mathbf{F}(\mathbf{x}, \theta) + v^2 \mathcal{L}_{\varepsilon \mathbf{F}}(\varepsilon \mathbf{F})(\mathbf{x}, \theta) + v^3 \mathcal{L}^2_{\varepsilon \mathbf{F}}(\varepsilon \mathbf{F})(\mathbf{x}, \theta) + \dots$$

$\mathcal{L}(\mathbf{F})$ linked to Lie derivative
Indentifying with:

$$\boldsymbol{\mathcal{X}} = \mathbf{x} + \varepsilon v \boldsymbol{\mathcal{X}}^1 + \varepsilon^2 v^2 \boldsymbol{\mathcal{X}}^2 + \dots$$

gives

Method of Characteristics - 3: Asymptotic expansion

$$\boldsymbol{\mathcal{X}}^1 = \mathbf{F}(\mathbf{x},\theta), \boldsymbol{\mathcal{X}}^2 = \mathcal{L}_{\mathbf{F}}(\mathbf{F})(\mathbf{x},\theta), \dots$$

$$\boldsymbol{\Upsilon}_4(\mathbf{x},\theta,v) = \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta;v;\mathbf{x},u))} ds = \int_0^v \frac{s}{B(\mathbf{x})} ds + \varepsilon \int_0^v s^2\, \mathcal{T}^1(\frac{1}{B(\mathbf{x})}) \cdot \boldsymbol{\mathcal{X}}^1 ds + + \varepsilon^2 \int_0^v s^3 \left(\mathcal{T}^2(\frac{1}{B(\mathbf{x})}) \cdot \boldsymbol{\mathcal{X}}^1 + \mathcal{T}^1(\frac{1}{B(\mathbf{x})}) \cdot \boldsymbol{\mathcal{X}}^2\right) ds + \dots = \frac{v^2}{2B(\mathbf{x})} + \dots$$

($\mathcal{T}^i$ linked with the Taylor expansion coefficients)

Gives new variable k as an expansion in ε

On other equations - Poisson Matrix in Darboux Coordinates

$\{\Upsilon_4, \Upsilon_3\} = -\frac{1}{\varepsilon} \quad (=\{\Upsilon_k, \Upsilon_\theta\}=\{k, \theta\})$

Processed. Gave k

$\{\Upsilon_1, \Upsilon_3\} = 0\,(=\{\Upsilon_{y_1}, \Upsilon_\theta\}=\{y_1, \theta\}),\ \{\Upsilon_1, \Upsilon_4\} = 0\,(=\{\Upsilon_{y_1}, \Upsilon_k\}=\{y_1, k\})$
$\{\Upsilon_2, \Upsilon_3\} = 0\,(=\{\Upsilon_{y_2}, \Upsilon_\theta\}=\{y_2, \theta\}),\ \{\Upsilon_2, \Upsilon_4\} = 0\,(=\{\Upsilon_{y_2}, \Upsilon_k\}=\{y_2, k\})$

To be Processed.

Will give $\mathbf{y}$ and k in terms of $(\mathbf{x}, \theta, v)$ and expansions in ε:

$$\Upsilon = \Upsilon^0 + \varepsilon\Upsilon^1 + \varepsilon^2\Upsilon^2 + \dots$$

Hence: $(\mathbf{y}, \theta, k)$ gotten
Last term of new Poisson matrix $\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k)$:
$(\overline{\mathcal{P}}_\varepsilon)_{12} = -(\overline{\mathcal{P}}_\varepsilon)_{21} = \{\Upsilon_1, \Upsilon_2\}(=\{\Upsilon_{y_1}, \Upsilon_{y_2}\}=\{y_1, y_2\}),$

$$\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 \\ \frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix}$$

Hamiltonian Function in Darboux Coordinates

We know:

- $\widetilde{H}_\varepsilon(\mathbf{x}, \theta, v) = \dfrac{v^2}{2}$
- $\overline{H}_\varepsilon(\mathbf{y}, \theta, k) = \widetilde{H}_\varepsilon(\boldsymbol{\xi}(\mathbf{y}, \theta, k))$ with $\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1}$
- $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^0 + \varepsilon \boldsymbol{\Upsilon}^1 + \varepsilon^2 \boldsymbol{\Upsilon}^2 + \dots$

We do :

- $\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \varepsilon \boldsymbol{\xi}^1 + \varepsilon^2 \boldsymbol{\xi}^2 + \dots$
- $\widetilde{H}_\varepsilon(\boldsymbol{\xi}^0 + \varepsilon \boldsymbol{\xi}^1 + \varepsilon^2 \boldsymbol{\xi}^2 + \dots) = \widetilde{H}_\varepsilon(\boldsymbol{\xi}^0) + \varepsilon \mathcal{T}^1(\widetilde{H}_\varepsilon)(\boldsymbol{\xi}^0) \cdot \boldsymbol{\xi}^1 + \dots$

$$\overline{H}_\varepsilon(\mathbf{y}, \theta, k) = B(\mathbf{y})k + \varepsilon \overline{H}^1(\mathbf{y}, \theta, k) + \varepsilon^2 \overline{H}^2(\mathbf{y}, \theta, k)$$

First term : Independent of θ

Let us take stock

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

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Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

1: Hamiltonian?

2

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

4: Darboux Method

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{M}_{(\mathbf{r})}} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Lie Transform based Method Target - 1

We have $\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k)$: nice shape. But:

$$\overline{H}_\varepsilon(\mathbf{y},\theta,k) = \overline{H}^0(\mathbf{y},\cancel{\theta},k) + \varepsilon\overline{H}^1(\mathbf{y},\theta,k) + \varepsilon^2\overline{H}^2(\mathbf{y},\theta,k) + \dots$$

depends on θ.

Key result $\longleftarrow$ $\theta-$independent Hamiltonian Function.
Target: Change of coordinates

$$(\mathbf{y},\theta,k) \mapsto (\mathbf{z},\gamma,j) = \boldsymbol{\zeta}(\mathbf{y},\theta,k)$$

leaving $\overline{\mathcal{P}}_\varepsilon$ unchanged,

$$(\widehat{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j) = \overline{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j), \overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k) = \widehat{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k))$$

$\varepsilon-$parametrized, close to identity, i.e.:

$$\boldsymbol{\zeta}(\mathbf{y},\theta,k) = [\boldsymbol{\zeta}(\mathcal{E})](\mathbf{y},\theta,k) = (\mathbf{y},\theta,k) + \varepsilon\boldsymbol{\zeta}^1(\mathbf{y},\theta,k) + \varepsilon^2\boldsymbol{\zeta}^2(\mathbf{y},\theta,k) + \dots$$

$$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) = \overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \widehat{H}^0(\mathbf{z},j) + \varepsilon\widehat{H}^1(\mathbf{z},j) + \varepsilon^2\widehat{H}^2(\mathbf{z},j) + \dots$$

$(\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$

Lie Transform based Method Target - 2

We have $\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, u_\|, k, \theta)$: nice shape. But:

$$\overline{H}_\varepsilon(\mathbf{y}, \theta, k) = \overline{H}^0(\mathbf{y}, \cancel{\theta}, k) + \varepsilon\overline{H}^1(\mathbf{y}, \theta, k) + \varepsilon^2\overline{H}^2(\mathbf{y}, \theta, k) + \dots$$

depends on θ.

Key result $\longleftarrow$ $\theta-$independent Hamiltonian Function.
Target: Change of coordinates

$$(\mathbf{y}, \theta, k) \mapsto (\mathbf{z}, \gamma, j) = \boldsymbol{\zeta}(\mathbf{y}, \theta, k)$$

leaving $\overline{\mathcal{P}}_\varepsilon$ almost unchanged (up to order N in ε)

$(\widehat{\mathcal{P}}_\varepsilon(\mathbf{z}, \gamma, j) = \overline{\mathcal{P}}_\varepsilon(\mathbf{z}, \gamma, j) + \varepsilon^N$Something$, \overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) = \widehat{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) + \varepsilon^N$Something$)$

$\varepsilon-$parametrized, close to identity, i.e.:

$$\boldsymbol{\zeta}(\mathbf{y}, \theta, k) = [\boldsymbol{\zeta}(\mathcal{E})](\mathbf{y}, \theta, k) = (\mathbf{y}, \theta, k) + \varepsilon\boldsymbol{\zeta}^1(\mathbf{y}, \theta, k) + \varepsilon^2\boldsymbol{\zeta}^2(\mathbf{y}, \theta, k) + \dots$$

$$\widehat{H}_\varepsilon(\mathbf{z}, \gamma, j) = \overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z}, \gamma, j)) = \widehat{H}^0(\mathbf{z}, j) + \varepsilon\widehat{H}^1(\mathbf{z}, j) + \varepsilon^2\widehat{H}^2(\mathbf{z}, j)$$
$$+ \varepsilon^N\widehat{H}^N(\mathbf{z}, j) + \varepsilon^{N+1}\widehat{H}^{N+1}(\mathbf{z}, \theta, j)$$

$(\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$

A remark

$$\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}} = \varepsilon\bar{\mathcal{P}}_{\varepsilon}\nabla\bar{f}; \quad \left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon f}\right)^n\cdot\{g,h\} = \sum_{k=0}^{n} C_n^k\left\{\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon f}\right)^k\cdot g, \left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon f}\right)^{n-k}\cdot h\right\}$$

$$ij \geq N \Rightarrow \left(\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot\{g,h\}\right) =$$

$$\left\{\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot g, \left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot h\right\} + \varepsilon^N\text{Something}$$

$$\vartheta^{i,j}_{\varepsilon,\bar{f}}(\mathbf{y},\theta,k) = \left(\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot\begin{pmatrix}\bar{\mathbf{r}}_1\\ \bar{\mathbf{r}}_2\\ \bar{\mathbf{r}}_3\\ \bar{\mathbf{r}}_4\end{pmatrix}\right)(\mathbf{y},\theta,k)$$

$$\bar{\mathbf{r}}_1 : (\mathbf{y},\theta,k)\mapsto y_1, \bar{\mathbf{r}}_2 : (\mathbf{y},\theta,k)\mapsto y_2, \bar{\mathbf{r}}_3 : (\mathbf{y},\theta,k)\mapsto \theta, \bar{\mathbf{r}}_4 : (\mathbf{y},\theta,k)\mapsto k$$

Consequence of the remark

For $g_1, \dots, g_N$, $\alpha_i = \min\{k \in \mathbb{N} \text{ s.t. } ki \geq N\}\ (= \mathbb{E}(\frac{N}{i}) + 1)$

$$\boldsymbol{\zeta} = \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,-\bar{g}_1} \circ \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,-\bar{g}_2} \circ \dots \circ \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,-\bar{g}_N}, \qquad (\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$$

$$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) =$$

$$\overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \left(\sum_{n=0}^{\alpha_1} \frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_1}\right)^n\right)\cdot\left(\sum_{n=0}^{\alpha_2} \frac{\varepsilon^{2n}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_2}\right)^n\right)\cdot\dots\cdot \left(\sum_{n=0}^{\alpha_N} \frac{\varepsilon^{Nn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{g}_N}\right)^n\right)\cdot\ \overline{H}_\varepsilon(\mathbf{z},\gamma,j) + \varepsilon^{N+1}\text{Something},$$

Since for i, j s.t. $ij \geq N$ $\left(\left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot\{g,h\}\right) =$
$\left\{\left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot g, \left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right)\cdot h\right\} + \varepsilon^N\text{Something}$

$$(\hat{\mathcal{P}}_\varepsilon(\mathbf{z},\theta,j))_{k,l} = \{\zeta_k,\zeta_l\}(\boldsymbol{\lambda}(\mathbf{z},\theta,j)) = (\bar{\mathcal{P}}_\varepsilon(\mathbf{z},\theta,j))_{k,l} + \varepsilon^{N-1}\text{Something}$$

The game to play

Build $g_1, \dots, g_N$ s.t.

$$\widehat{H}^0(\mathbf{z},j)+\varepsilon\widehat{H}^1(\mathbf{z},j)+\varepsilon^2\widehat{H}^2(\mathbf{z},j)+\varepsilon^N\widehat{H}^N(\mathbf{z},j)+\varepsilon^{N+1}\widehat{H}^{N+1}(\mathbf{z},\theta,j) =$$

$$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) =$$

$$\overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \left(\sum_{n=0}^{\alpha_1}\frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^n\right)\cdot\left(\sum_{n=0}^{\alpha_2}\frac{\varepsilon^{2n}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\right)^n\right)\cdot\dots\cdot$$

$$\left(\sum_{n=0}^{\alpha_N}\frac{\varepsilon^{Nn}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_N}\right)^n\right)\cdot\ \overline{H}_\varepsilon(\mathbf{z},\gamma,j)+\varepsilon^{N+1}\text{Something}$$

$$= \left(\sum_{n=0}^{\alpha_1}\frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^n\right)\cdot\left(\sum_{n=0}^{\alpha_2}\frac{\varepsilon^{2n}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\right)^n\right)\cdot\dots\cdot$$

$$\left(\sum_{n=0}^{\alpha_N}\frac{\varepsilon^{Nn}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_N}\right)^n\right)\cdot\left(\overline{H}^0(\mathbf{z},\cancel{\gamma},j)+\varepsilon\overline{H}^1(\mathbf{z},\gamma,j)+\varepsilon^2\overline{H}^2(\mathbf{z},\gamma,j)+\dots\right)$$

$$+\varepsilon^{N+1}\text{Something},$$

If you play the game ...

... with:

$$\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 \\ \frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix} = \frac{1}{\varepsilon}\bar{\mathcal{T}}_0 + \varepsilon\bar{\mathcal{T}}_2(\mathbf{y})$$

$$\bar{\mathcal{T}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}; \quad \bar{\mathcal{T}}_2(\mathbf{y}) = \begin{bmatrix} 0 & \frac{-1}{B(\mathbf{y})} & 0 & 0 \\ \frac{1}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{H}^0(\mathbf{z},j) = \overline{H}^0(\mathbf{z},\cancel{\gamma},j),$$
$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_1\right)\cdot\nabla\overline{H}_0 = \overline{H}^1 - \widehat{H}^1$$
$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_2\right)\cdot\nabla\overline{H}_0 = \mathcal{V}_2(\overline{H}^1,\overline{H}^2,g_1) - \widehat{H}^2$$

etc.

... you have

$$\widehat{H}^0(\mathbf{z},j) = \overline{H}^0(\mathbf{z},\cancel{\gamma},j),$$

$$\widehat{H}_1(\mathbf{z},j) = -\frac{1}{2\pi}\int_0^{2\pi} \overline{H}^1\, d\gamma,$$

$$\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_1\right)\cdot \nabla \overline{H}_0 = \overline{H}^1 - \frac{1}{2\pi}\int_0^{2\pi} \overline{H}^1\, d\gamma$$

$$\widehat{H}_2(\mathbf{z},j) = -\frac{1}{2\pi}\int_0^{2\pi} \mathcal{V}_2(\overline{H}^1,\overline{H}^2,g_1)d\gamma$$

$$\left(\bar{\mathcal{T}}_0 \nabla \bar{g}_2\right)\cdot \nabla \overline{H}_0 = \mathcal{V}_2(\overline{H}^1,\overline{H}^2,g_1) - \frac{1}{2\pi}\int_0^{2\pi} \mathcal{V}_2(\overline{H}^1,\overline{H}^2,g_1)d\gamma$$

etc.

At the end of the day

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

2

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{M}_{(\mathbf{r})}} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \hline 0\;0 & & 0 & 1 \\ 0\;0 & & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Hence

In Lie Coordinates: Trajectories $(\mathbf{Z}, \Gamma, \mathcal{J})$

$$\frac{\partial \mathbf{Z}}{\partial t} = \text{Something independent of } \Gamma + \varepsilon^{N+1}\text{Remainder}(\mathbf{Z}, \Gamma, \mathcal{J})$$
$$\frac{\partial \Gamma}{\partial t} = \text{Something complicated}$$
$$\frac{\partial \mathcal{J}}{\partial t} = \varepsilon^{N-1}\text{Something}(\mathbf{Z}, \Gamma, \mathcal{J})$$

$(\mathbf{Z}^T, \Gamma^T, \mathcal{J}^T)$:

$$\frac{\partial \mathbf{Z}^T}{\partial t} = \text{Something independent of } \Gamma$$
$$\frac{\partial \Gamma^T}{\partial t} = \text{Something complicated}$$
$$\frac{\partial \mathcal{J}^T}{\partial t} = 0$$

$$|(\mathbf{Z}^T, \Gamma^T, \mathcal{J}^T)(t) - (\mathbf{Z}, \Gamma, \mathcal{J})(t)| \leq C\varepsilon^{N-1}$$

Implementing with N=3

$$\frac{\partial \mathbf{Z}^T}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B\left(\mathbf{Z}^T\right)}\,{}^{\perp}\nabla B\left(\mathbf{Z}^T\right),$$

$$\frac{\partial \Gamma^T}{\partial t} = \frac{B\left(\mathbf{Z}^T\right)}{\varepsilon} + \varepsilon \frac{\mathcal{J}^T}{2\left(B\left(\mathbf{Z}^T\right)\right)^2}\left(B\left(\mathbf{Z}^T\right)\nabla^2 B\left(\mathbf{Z}^T\right) - 3\left(\nabla B\left(\mathbf{Z}^T\right)\right)^2\right)$$

$$\frac{\partial \mathcal{J}^T}{\partial t} = 0,$$

Thank for your attention