The Gyrokinetic Approximation

An attempt at explaining the method based on Darboux Algorithm and Lie Transform

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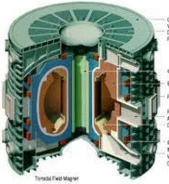
Part of PhD thesis of Mathieu Lutz

Charge particles submitted to Strong Magnetic Field

In Usual Coordinates : $(\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$

$$\mathbf{X}(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}(t; \mathbf{x}, \mathbf{v}, s)$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{q}{m}(\mathbf{E}(\mathbf{X}) + \mathbf{V} \times \mathbf{B}(\mathbf{X}))$$

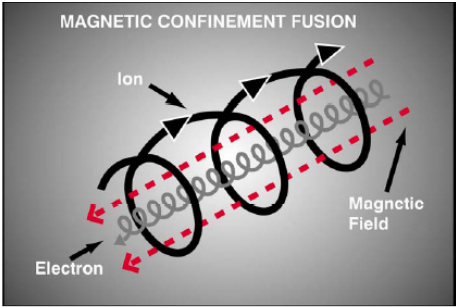


$$\mathbf{B} : \underbrace{\text{Strong Applied piece} + \text{Strong Self Induced piece}}_{\to \frac{1}{\varepsilon}\mathbf{B}} + \underbrace{\text{Self Induced Perturbations}}_{\text{Forgotten}}$$

$\mathbf{E}$: Self Induced piece

$$(\mathbf{E} = -\nabla\Phi, \ \mathbf{B} = \nabla \times \mathbf{A})$$

Helicoidal trajectories - Larmor Radius



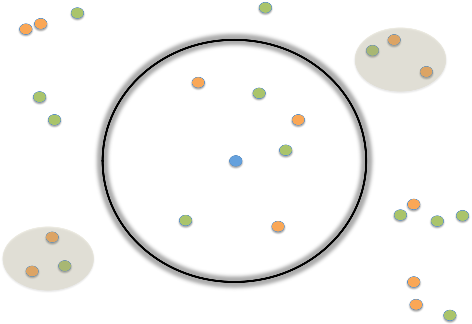
Source: S. Jardin's Lectures at Cemracs'10

In Tokamak:

Electron Larmor Radius $\sim 5 \cdot 10^{-4} m$

Ion Larmor Radius $\sim 10^{-2} m$

Debye Length



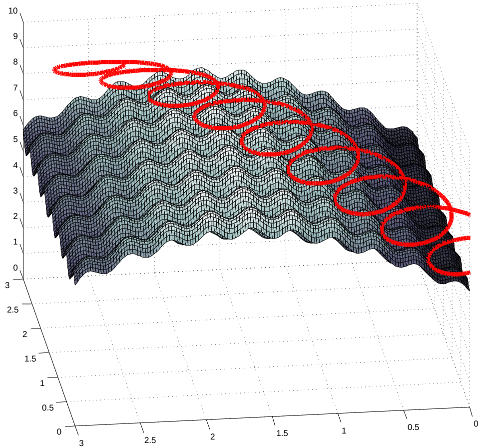
Particle pot. $\sim \dfrac{1}{r}$ (Green Funct. of Δ) $\Rightarrow -\Delta\Phi =$ Cst Ch.Density

Particle pot. $\sim \dfrac{e^{-r/\lambda}}{r}$ (Green Funct. of $\mathcal{D}$) $\Rightarrow -\mathcal{D}\Phi =$ Cst Ch.Density

In Tokamak: Debye Length : $\sim 10^{-4} m$

Symbolic representation of what Ions see

For Ions : Larmor radius $\sim 10^{-2} m >> $ Debye Length : $\sim 10^{-4} m$



Dimensionless Dynamical System

$$\varepsilon \sim \frac{\text{Ion Larmor Radius}}{\text{Tokamak size}} \sim \frac{10^{-2}m}{10m} \sim 10^{-3}$$

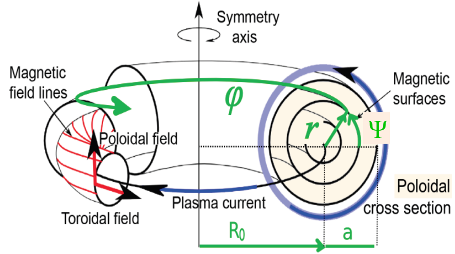
Debye Length linked variations (at size η) of the Electric Field and Potential:

$$\Phi(\mathbf{x}) = \Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta}) \quad \text{and} \quad \mathbf{E}(\mathbf{x}) = \mathbf{E}_0(\mathbf{x}) + \mathbf{E}_1(\frac{\mathbf{x}}{\eta})$$

$\left(\eta = \varepsilon^{1-\kappa},\ 0<\kappa<1, \eta >> \varepsilon\right)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{E}_0(\mathbf{X}) + \mathbf{E}_1(\frac{\mathbf{X}}{\eta}) + \mathbf{V} \times \frac{\mathbf{B}(\mathbf{X})}{\varepsilon}$$

Gyrokinetic model



Source: V. Grandgirard's Lectures at Cemracs'10

Coordinates: $(r, \psi, \varphi, w_{\|}, j, \gamma)$. Trajectory: $(r, \psi, \varphi, W_{\|}, J, \Gamma)$

$$\frac{\partial r}{\partial t} = (ED_r + MCD_r),\ \frac{\partial \psi}{\partial t} = \frac{W_{\|}}{q(r)R} + \frac{ED_\psi + MCD_\psi}{r},\ \frac{\partial \varphi}{\partial t} = \frac{W_{\|}}{R},$$

$$\frac{\partial W_{\|}}{\partial t} = (E_0 + \langle E_1 \rangle)_{\|} - \frac{J}{\varepsilon}\nabla_{\|}|\mathbf{B}| + \frac{W_{\|}}{|\mathbf{B}|} ED \cdot (\nabla|\mathbf{B}|)$$

$$\frac{\partial J}{\partial t} = 0,\ \frac{\partial \Gamma}{\partial t} = ?,\quad \nabla_{\|} = \frac{1}{R}\Big(\frac{\partial}{\partial \varphi} + \frac{1}{q(r)}\frac{\partial}{\partial \psi}\Big),\ R = R_0 + r\cos(\psi)$$

$$ED = \varepsilon \frac{\mathbf{B} \times (E_0 + \langle E_1 \rangle)}{|\mathbf{B}|^2},\ MCD = \varepsilon \frac{\mathbf{B} \times (\nabla|\mathbf{B}|)}{|\mathbf{B}|^3}\Big(W_{\|}^2 + J\frac{|\mathbf{B}|}{\varepsilon}\Big)$$

Key result

IF: In coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6)$, a Hamiltonian Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R}) \nabla_{\mathbf{r}} H(\mathbf{R}) \qquad \mathcal{P}(\mathbf{R}) = \left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & -1 & 0 \end{array} \right)$$

with

$$\frac{\partial H}{\partial r_6} = 0$$

$$\textbf{THEN:} \quad \frac{\partial \mathcal{M}}{\partial r_6} = 0 \qquad \textbf{AND:} \quad \frac{\partial R_5}{\partial t} = 0$$

(Trajectory $\mathbf{R} = (R_1, R_2, R_3, R_4, R_5, R_6)$)

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{E_0}(\mathbf{X}) + \mathbf{E_1}(\frac{\mathbf{X}}{\eta}) + \mathbf{V} \times \frac{\mathbf{B}(\mathbf{X})}{\varepsilon}$$

3 →

Cylindrical Coordinates

$(\mathbf{x}, v_{\parallel}, v_{\perp}, \theta)$

1: Hamiltonian? ↓ ; 2 ↑

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}^{\varepsilon}_{\eta} = \breve{H}^{\varepsilon}_{\eta}(\mathbf{q}, \mathbf{p})$:

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}} \breve{H}^{\varepsilon}_{\eta}$$
$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}} \breve{H}^{\varepsilon}_{\eta}$$

4: Darboux Method ↓

Darboux Almost Canonical Coordinates

$(\mathbf{y}, u_{\parallel}, k, \theta)$

5: Lie Method ↓

Lie Coordinates

$(\mathbf{z}, w_{\parallel}, j, \gamma)$

Panorama

Usual Coordinates
$(\mathbf{x},\mathbf{v})$

$H_\eta^\varepsilon(\mathbf{x},\mathbf{v}), \mathcal{P}_\eta^\varepsilon(\mathbf{x},\mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\eta^\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\eta^\varepsilon$$

3 →

Cylindrical Coordinates
$(\mathbf{x}, v_\parallel, v_\perp, \theta)$

$\widetilde{H}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta), \widetilde{\mathcal{P}}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta)$

1: Hamiltonian? ↓ 2 ↑

4: Darboux Method ↓

Canonical Coordinates
$(\mathbf{q},\mathbf{p})$

$\breve{H}_\eta^\varepsilon(\mathbf{q},\mathbf{p}), \breve{\mathcal{P}}_\eta^\varepsilon(\mathbf{q},\mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\eta^\varepsilon$$

$$\left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0\cdots 0 & 0 & 1 \\ 0\cdots 0 & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates
$(\mathbf{y}, u_\parallel, k, \theta)$

$\overline{H}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta), \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta)$

5: Lie Method ↓

Lie Coordinates
$(\mathbf{z}, w_\parallel, j, \gamma)$

$\widehat{H}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j), \widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$
$= \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$

The Canonical Coordinates

Usual Coordinates : $(\mathbf{x},\mathbf{v}) = (x_1,x_2,x_3,v_1,v_2,v_3)$
Trajectory : $(\mathbf{X}(t;\mathbf{x},\mathbf{v},s),\mathbf{V}(t;\mathbf{x},\mathbf{v},s))$ $\big((\mathbf{X},\mathbf{V}) = (X_1,X_2,X_3,V_1,V_2,V_3)\big)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{E}_0(\mathbf{X}) + \mathbf{E}_1(\frac{\mathbf{X}}{\eta}) + \frac{1}{\varepsilon}\mathbf{V}\times\mathbf{B}(\mathbf{X})$$

$$\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x})$$
$$\mathbf{E}_0(\mathbf{x}) + \mathbf{E}_1(\frac{\mathbf{x}}{\eta}) = -\nabla\Big[\Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta})\Big]$$

Canonical Coordinates : $(\mathbf{q},\mathbf{p}) = (q_1,q_2,q_3,p_1,p_2,p_3)$
Trajectory : $(\mathbf{Q}(t;\mathbf{q},\mathbf{p},s),\mathbf{P}(t;\mathbf{q},\mathbf{p},s))$ $\big((\mathbf{Q},\mathbf{P}) = (Q_1,Q_2,Q_3,P_1,P_2,P_3)\big)$

$$\mathbf{q} = \mathbf{x} \qquad \mathbf{x} = \mathbf{q} \qquad \mathbf{Q} = \mathbf{X} \qquad \mathbf{X} = \mathbf{Q}$$
$$\mathbf{p} = \mathbf{v} + \frac{\mathbf{A}(\mathbf{x})}{\varepsilon} \qquad \mathbf{v} = \mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon} \qquad \mathbf{P} = \mathbf{V} + \frac{\mathbf{A}(\mathbf{X})}{\varepsilon} \qquad \mathbf{V} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}}\breve{H}_\eta^\varepsilon$$

$$\breve{H}_\eta^\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\Big|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\Big|^2 + \Phi_0(\mathbf{q}) + \eta\Phi_1(\frac{\mathbf{q}}{\eta})$$
$$\mathcal{S} = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$$

Check of Canonical nature of Canonical Coordinates

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}}\breve{H}^{\varepsilon}_{\eta}, \quad \breve{H}^{\varepsilon}_{\eta}(\mathbf{q},\mathbf{p}) = \frac{1}{2}\Big|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\Big|^2 + \Phi_0(\mathbf{q}) + \eta\Phi_1(\frac{\mathbf{q}}{\eta})$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}}\breve{H}^{\varepsilon}_{\eta}(\mathbf{Q},\mathbf{P}) = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}}\breve{H}^{\varepsilon}_{\eta}(\mathbf{Q},\mathbf{P}) = \frac{(\nabla\mathbf{A}(\mathbf{Q}))^T}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) - \nabla\Big[\Phi_0(\mathbf{Q}) + \eta\Phi_1(\frac{\mathbf{Q}}{\eta})\Big]$$

$$(\nabla\mathbf{A})^T(\mathbf{p}-\mathbf{A}) = (\nabla\mathbf{A})(\mathbf{p}-\mathbf{A}) + (\mathbf{p}-\mathbf{A})\times(\nabla\times\mathbf{A})$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla\mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)\times\frac{\nabla\times\mathbf{A}(\mathbf{Q})}{\varepsilon} - \nabla\Big[\Phi_0(\mathbf{Q}) + \eta\Phi_1(\frac{\mathbf{Q}}{\eta})\Big]$$

Check of Canonical nature of Canonical Coord. - 2

$$\mathbf{X} = \mathbf{Q}$$
$$\mathbf{V} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) \times \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon} - \nabla\big[\Phi_0(\mathbf{Q}) + \eta\Phi_1(\frac{\mathbf{Q}}{\eta})\big]$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\frac{\partial \mathbf{Q}}{\partial t}\Big) = \frac{\partial\Big[\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big]}{\partial t} = \Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) \times \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon} - \nabla\big[\Phi_0(\mathbf{Q}) + \eta\Phi_1(\frac{\mathbf{Q}}{\eta})\big]$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{V} \times \frac{\nabla \times \mathbf{A}(\mathbf{X})}{\varepsilon} - \nabla\big[\Phi_0(\mathbf{X}) + \eta\Phi_1(\frac{\mathbf{X}}{\eta})\big]$$

As by products : Poisson Matrix, Poisson Bracket, Change of Coordinates Formula

In any coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6)$, the Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R})\nabla_{\mathbf{r}} H(\mathbf{R}) = \{\mathbf{I}, H\}(\mathbf{R})$$

$$\{f, g\}(\mathbf{r}) = (\nabla_{\mathbf{r}} f(\mathbf{r})) \cdot (\mathcal{P}(\mathbf{r})(\nabla_{\mathbf{r}} g(\mathbf{r}))) \qquad (f \text{ and } g : \mathbb{R}^6 \to \mathbb{R})$$

$$(\{\mathbf{f}, g\}(\mathbf{r}))_i = (\nabla_{\mathbf{r}} \mathbf{f}_i(\mathbf{r})) \cdot (\mathcal{P}(\mathbf{r})(\nabla_{\mathbf{r}} g(\mathbf{r}))) \qquad (\mathbf{f} : \mathbb{R}^6 \to \mathbb{R}^6 \text{ and } g : \mathbb{R}^6 \to \mathbb{R})$$

$$\mathbf{I}(\mathbf{r}) = \mathbf{r}$$

Another coordinate system $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6)$ with $\tilde{\mathbf{r}} = \boldsymbol{\rho}(\mathbf{r})$, $\mathbf{r} = \tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}) = \boldsymbol{\rho}^{-1}(\tilde{\mathbf{r}})$

$$\frac{\partial \tilde{\mathbf{R}}}{\partial t} = \tilde{\mathcal{P}}(\tilde{\mathbf{R}})\nabla_{\tilde{\mathbf{r}}} \tilde{H}(\tilde{\mathbf{R}})$$

$$\tilde{H}(\tilde{\mathbf{r}}) = H(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}})) \qquad (\tilde{\mathcal{P}}(\tilde{\mathbf{r}}))_{ij} = \{\boldsymbol{\rho}_i, \boldsymbol{\rho}_j\}(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}))$$

Let us take stock

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H^\varepsilon_\eta(\mathbf{x}, \mathbf{v}), \mathcal{P}^\varepsilon_\eta(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}^\varepsilon_\eta \nabla_{\mathbf{x},\mathbf{v}} H^\varepsilon_\eta$$

1: Hamiltonian?

2

3

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}^\varepsilon_\eta(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}^\varepsilon_\eta(\mathbf{q}, \mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}^\varepsilon_\eta$$

Cylindrical Coordinates

$(\mathbf{x}, v_\parallel, v_\perp, \theta)$

$\widetilde{H}^\varepsilon_\eta(\mathbf{x}, v_\parallel, v_\perp, \theta), \widetilde{\mathcal{P}}^\varepsilon_\eta(\mathbf{x}, v_\parallel, v_\perp, \theta)$

4: Darboux Method

Darboux Almost Canonical Coordinates

$(\mathbf{y}, u_\parallel, k, \theta)$

$\overline{H}^\varepsilon_\eta(\mathbf{y}, u_\parallel, k, \theta), \overline{\mathcal{P}}^\varepsilon_\eta(\mathbf{y}, u_\parallel, k, \theta)$

$$\left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & -1 & 0 \end{array}\right)$$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, w_\parallel, j, \gamma)$

$\widehat{H}^\varepsilon_\eta(\mathbf{z}, w_\parallel, j), \widehat{\mathcal{P}}^\varepsilon_\eta(\mathbf{z}, w_\parallel, j, \gamma) = \overline{\mathcal{P}}^\varepsilon_\eta(\mathbf{z}, w_\parallel, j, \gamma)$

Hamiltonian Function and Poisson Matrix in Usual Coordinates

$$\boxed{H^\varepsilon_\eta(\mathbf{x},\mathbf{v}) = \frac{1}{2}\left|\mathbf{v}\right|^2 + \Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta})} \quad \left(\check{H}^\varepsilon_\eta(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p}-\frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2 + \Phi_0(\mathbf{q}) + \eta\Phi_1(\frac{\mathbf{q}}{\eta})\right)$$

$$\nabla_{\mathbf{x},\mathbf{v}} H^\varepsilon_\eta = \begin{pmatrix} \nabla\left[\Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta})\right] \\ \mathbf{v} \end{pmatrix}$$

$$\boxed{\mathcal{P}^\varepsilon_\eta(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 & I_3 \\ -I_3 & \frac{(\nabla A(\mathbf{x}))^T - (\nabla A(\mathbf{x}))}{\varepsilon} \end{pmatrix}}$$

Indeed:

$$\mathcal{P}^\varepsilon_\eta(\mathbf{x},\mathbf{v})\nabla_{\mathbf{x},\mathbf{v}} H^\varepsilon_\eta = \begin{pmatrix} \mathbf{v} \\ -\nabla\left[\Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta})\right] + \frac{(\nabla A(\mathbf{x}))^T - (\nabla A(\mathbf{x}))}{\varepsilon}\mathbf{v} \end{pmatrix}$$

To be compared with:

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{E}_0(\mathbf{X}) + \mathbf{E}_1(\frac{\mathbf{X}}{\eta}) + \frac{1}{\varepsilon}\mathbf{V}\times\mathbf{B}(\mathbf{X})$$

Formula giving Poisson Matrix

$$\mathcal{P}_\eta^\varepsilon(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 & I_3 \\ -I_3 & \frac{(\nabla A(\mathbf{x}))^T - (\nabla A(\mathbf{x}))}{\varepsilon} \end{pmatrix}$$

Change of coordinates: $\tilde{\mathbf{r}}$ and $\mathbf{r}$ with $\tilde{\mathbf{r}} = \rho(\mathbf{r})$ and $\mathbf{r} = \tilde{\rho}(\tilde{\mathbf{r}}) = \rho^{-1}(\tilde{\mathbf{r}})$

$$(\tilde{\mathcal{P}}(\tilde{\mathbf{r}}))_{ij} = \{\rho_i, \rho_j\}(\tilde{\rho}(\tilde{\mathbf{r}}))$$

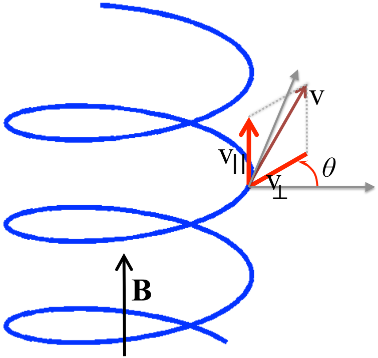
Here: $(\mathbf{x},\mathbf{v})$ and $(\mathbf{q},\mathbf{p})$ with $(\mathbf{x},\mathbf{v}) = \xi(\mathbf{q},\mathbf{p})$ and $(\mathbf{q},\mathbf{p}) = \pi(\mathbf{x},\mathbf{v})$

For instance : $\xi_5(\mathbf{q},\mathbf{p}) = p_2 - \frac{\mathbf{A_2}(\mathbf{q})}{\varepsilon}$ and $\xi_6(\mathbf{q},\mathbf{p}) = p_3 - \frac{\mathbf{A_3}(\mathbf{q})}{\varepsilon}$

$$\nabla\xi_5(\mathbf{q},\mathbf{p}) = \begin{pmatrix} \frac{1}{\varepsilon}\frac{\partial \mathbf{A_2}}{\partial q_1}(\mathbf{q}) \\ \frac{1}{\varepsilon}\frac{\partial \mathbf{A_2}}{\partial q_2}(\mathbf{q}) \\ \frac{1}{\varepsilon}\frac{\partial \mathbf{A_2}}{\partial q_3}(\mathbf{q}) \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathcal{S} = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \quad \nabla\xi_6(\mathbf{q},\mathbf{p}) = \begin{pmatrix} \frac{1}{\varepsilon}\frac{\partial \mathbf{A_3}}{\partial q_1}(\mathbf{q}) \\ \frac{1}{\varepsilon}\frac{\partial \mathbf{A_3}}{\partial q_2}(\mathbf{q}) \\ \frac{1}{\varepsilon}\frac{\partial \mathbf{A_3}}{\partial q_3}(\mathbf{q}) \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\{\xi_5, \xi_6\}(\pi(\mathbf{x},\mathbf{v})) = \frac{1}{\varepsilon}\Big(\frac{\partial \mathbf{A}_3}{\partial q_2}(\mathbf{x}) - \frac{\partial \mathbf{A}_2}{\partial q_3}(\mathbf{x})\Big)$$

Cylindrical Coordinates (in velocity)



$(\mathbf{x}, v_{||}, v_\perp, \theta)$ s.t.

$$v_{||} = \mathbf{v} \cdot \frac{\mathbf{B}}{|\mathbf{B}|}, \qquad v_\perp = \left| \mathbf{v} - \left(\mathbf{v} \cdot \frac{\mathbf{B}}{|\mathbf{B}|} \right) \frac{\mathbf{B}}{|\mathbf{B}|} \right|$$

$$\theta \text{ s.t. } \mathbf{v} - \left(\mathbf{v} \cdot \frac{\mathbf{B}}{|\mathbf{B}|} \right) \frac{\mathbf{B}}{|\mathbf{B}|} = v_\perp (\cos\theta, \sin\theta)$$

Hamiltonian Function and Poisson Matrix in Cylindrical Coordinates

$$\boxed{\widetilde{H}_\eta^\varepsilon(\mathbf{x}, v_\|, v_\perp, \theta) = \frac{1}{2}\left(v_\|^2 + v_\perp^2\right) + \Phi_0(\mathbf{x}) + \eta\Phi_1\left(\frac{\mathbf{x}}{\eta}\right)}$$

$$\left(\boldsymbol{H}_\eta^\varepsilon(\mathbf{x}, \mathbf{v}) = \frac{\mathbf{1}}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}} + \Phi_{\mathbf{0}}(\mathbf{x}) + \eta\Phi_{\mathbf{1}}\left(\frac{\mathbf{x}}{\eta}\right)\right)$$

When $\mathbf{B}(\mathbf{x}) = \begin{pmatrix} b(\mathbf{x}) \\ 0 \\ 0 \end{pmatrix}$

$$\widetilde{\mathcal{P}}_\eta^\varepsilon(\mathbf{x}, v_\|, v_\perp, \theta) = \begin{pmatrix} 0 & 0 & 0 & \frac{b(\mathbf{x})}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin(\theta) & -\frac{\cos(\theta)}{v_\perp} \\ 0 & 0 & 0 & 0 & -\cos(\theta) & \frac{\sin(\theta)}{v_\perp} \\ -\frac{b(\mathbf{x})}{\varepsilon} & 0 & 0 & 0 & \$\$ & \$ \\ 0 & \sin(\theta) & \cos(\theta) & -\$\$ & 0 & -\frac{b(\mathbf{x})}{\varepsilon v_\perp} \\ 0 & \frac{\cos(\theta)}{v_\perp} & -\frac{\sin(\theta)}{v_\perp} & -\$ & \frac{b(\mathbf{x})}{\varepsilon v_\perp} & 0 \end{pmatrix}$$

$$\left(\$ = \frac{v_\|}{\varepsilon}\left(\frac{\partial b}{\partial x_2}(\mathbf{x}) + \frac{\partial b}{\partial x_3}(\mathbf{x})\right),\ \$\$ = \frac{v_\|}{\varepsilon v_\perp}\left(\sin(\theta)\frac{\partial b}{\partial x_2}(\mathbf{x}) + \frac{1+\sin^2(\theta)}{\cos(\theta)}\frac{\partial b}{\partial x_3}(\mathbf{x})\right)\right)$$

Hamiltonian Function and Poisson Matrix in Cylindrical Coordinates (in more general cases)

$$(\widetilde{\mathcal{P}}_{\eta}^{\varepsilon}(\mathbf{x}, v_{\|}, v_{\perp}, \theta))_{56} = (\widetilde{\mathcal{P}}_{\eta}^{\varepsilon}(\mathbf{x}, v_{\|}, v_{\perp}, \theta))_{v_{\perp}\theta} = \frac{|\mathbf{B}(\mathbf{x})|}{\varepsilon v_{\perp}} > 0$$

Important for the Darboux Method.

$$\frac{|\mathbf{B}(\mathbf{x})|}{\varepsilon v_{\perp}} = \omega(\mathbf{x}, v_{\perp})$$

Let us take stock

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\eta^\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\eta^\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\eta^\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\eta^\varepsilon$$

1: Hamiltonian?

2

3

Cylindrical Coordinates

$(\mathbf{x}, v_\parallel, v_\perp, \theta)$

$\widetilde{H}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta), \widetilde{\mathcal{P}}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta)$

4: Darboux Method

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\eta^\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\eta^\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\eta^\varepsilon$$

$$\left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, u_\parallel, k, \theta)$

$\overline{H}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta), \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta)$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, w_\parallel, j, \gamma)$

$\widehat{H}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j), \widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$
$= \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$

Darboux Method Target

Find a Coordinate System $(\mathbf{y}, u_{\|}, k, \theta)$ s.t. Poisson Matrix $(\overline{\mathcal{P}}^{\varepsilon}_{\eta})$ shape:

$$\left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0\cdots 0 & 0 & 1 \\ 0\cdots 0 & -1 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0\cdots 0 & 0 & \frac{1}{\varepsilon} \\ 0\cdots 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)$$

$$(\mathbf{y}, u_{\|}, k, \theta) = \boldsymbol{\Upsilon}(\mathbf{x}, v_{\|}, v_{\perp}, \theta),\ (\mathbf{x}, v_{\|}, v_{\perp}, \theta) = \boldsymbol{\xi}(\mathbf{y}, u_{\|}, k, \theta),\ (\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1})$$

$$(\overline{\mathcal{P}}^{\varepsilon}_{\eta}(\mathbf{y}, v_{\|}, k, \theta))_{ij} = \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\}(\boldsymbol{\xi}(\mathbf{y}, v_{\|}, k, \theta)), \{\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j\} = (\nabla\boldsymbol{\Upsilon}_i)\cdot(\widetilde{\mathcal{P}}^{\varepsilon}(\nabla\boldsymbol{\Upsilon}_j))$$

Needed: $\{\boldsymbol{\Upsilon}_5,\boldsymbol{\Upsilon}_6\} = \frac{1}{\varepsilon} \quad (=\{\boldsymbol{\Upsilon}_k,\boldsymbol{\Upsilon}_\theta\}=\{k,\theta\})$

$\{\boldsymbol{\Upsilon}_1,\boldsymbol{\Upsilon}_5\} = 0(=\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_k\}=\{y_1,k\}),\ \{\boldsymbol{\Upsilon}_1,\boldsymbol{\Upsilon}_6\} = 0(=\{\boldsymbol{\Upsilon}_{y_1},\boldsymbol{\Upsilon}_\theta\}=\{y_1,\theta\})$

$\{\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}_5\} = 0(=\{\boldsymbol{\Upsilon}_{y_2},\boldsymbol{\Upsilon}_k\}=\{y_2,k\}),\ \{\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}_6\} = 0(=\{\boldsymbol{\Upsilon}_{y_2},\boldsymbol{\Upsilon}_\theta\}=\{y_2,\theta\})$

$\{\boldsymbol{\Upsilon}_3,\boldsymbol{\Upsilon}_5\} = 0(=\{\boldsymbol{\Upsilon}_{y_3},\boldsymbol{\Upsilon}_k\}=\{y_3,k\}),\ \{\boldsymbol{\Upsilon}_3,\boldsymbol{\Upsilon}_6\} = 0(=\{\boldsymbol{\Upsilon}_{y_3},\boldsymbol{\Upsilon}_\theta\}=\{y_3,\theta\})$

$\{\boldsymbol{\Upsilon}_4,\boldsymbol{\Upsilon}_5\} = 0(=\{\boldsymbol{\Upsilon}_{y_4},\boldsymbol{\Upsilon}_k\}=\{v_{\|},k\}),\ \{\boldsymbol{\Upsilon}_4,\boldsymbol{\Upsilon}_6\} = 0(=\{\boldsymbol{\Upsilon}_{v_{\|}},\boldsymbol{\Upsilon}_\theta\}=\{v_{\|},\theta\})$

First equation processing

$\{\Upsilon_5,\Upsilon_6\} = \frac{1}{\varepsilon} \quad (=\{\Upsilon_k,\Upsilon_\theta\}=\{k,\theta\})$
or

$$\{\Upsilon_6, \Upsilon_5\} = -\frac{1}{\varepsilon} \quad (=\{\Upsilon_\theta \Upsilon_k,\}=\{\theta, k\}) \qquad (\bullet)$$

$$\boxed{\nabla\Upsilon_6(= \nabla\Upsilon_\theta) = (0,0,0,0,0,1)^T}$$

$\{\Upsilon_6, \Upsilon_5\} = (\nabla\Upsilon_6) \cdot (\widetilde{\mathcal{P}}^\varepsilon(\nabla\Upsilon_5))$: last component of $(\widetilde{\mathcal{P}}^\varepsilon(\nabla\Upsilon_5))$
$(\bullet) \rightarrow$

$$F_1 \frac{\partial\Upsilon_5}{\partial x_1} + F_2 \frac{\partial\Upsilon_5}{\partial x_2} + F_3 \frac{\partial\Upsilon_5}{\partial x_3} + F_{||} \frac{\partial\Upsilon_5}{\partial v_{||}} + \omega \frac{\partial\Upsilon_5}{\partial v_\perp} = -\frac{1}{\varepsilon}$$

F_n and $\omega > 0$: functions of $(\mathbf{x}, v_{||}, v_\perp, \theta)$ $\qquad (\Upsilon_5(\mathbf{x}, v_{||}, v_\perp, \theta) = k)$

Method of Characteristics

Method of Characteristics - 1

$$F_1\frac{\partial \Upsilon_5}{\partial x_1}+F_2\frac{\partial \Upsilon_5}{\partial x_2}+F_3\frac{\partial \Upsilon_5}{\partial x_3}+F_{\|}\frac{\partial \Upsilon_5}{\partial v_{\|}}+\omega\frac{\partial \Upsilon_5}{\partial v_{\perp}}=-\frac{1}{\varepsilon},\quad \left(\omega(\mathbf{x},v_\perp)=\frac{|\mathbf{B}(\mathbf{x})|}{\varepsilon v_\perp}\right)$$

$$\frac{\partial \Upsilon_5}{\partial v_\perp}+\varepsilon\frac{v_\perp F_1}{|\mathbf{B}|}\frac{\partial \Upsilon_5}{\partial x_1}+\varepsilon\frac{v_\perp F_2}{|\mathbf{B}|}\frac{\partial \Upsilon_5}{\partial x_2}+\varepsilon\frac{v_\perp F_3}{|\mathbf{B}|}\frac{\partial \Upsilon_5}{\partial x_3}+\varepsilon\frac{v_\perp F_{\|}}{|\mathbf{B}|}\frac{\partial \Upsilon_5}{\partial v_{\|}}=\frac{v_\perp}{|\mathbf{B}|}$$

$$\Upsilon_{5|v_\perp=\nu}=0 \qquad \text{for a small } \nu$$

$$\mathcal{X}_1(v_\perp;\mathbf{x},v_{\|},u_\perp) \text{ s.t. } \frac{\partial \mathcal{X}_1}{\partial v_\perp}=\varepsilon\frac{v_\perp F_1(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{V}_{\|},v_\perp,\theta)}{|\mathbf{B}|(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)} \quad \mathcal{X}_1(u_\perp)=x_1$$

$$\mathcal{X}_2(v_\perp;\mathbf{x},v_{\|},u_\perp) \text{ s.t. } \frac{\partial \mathcal{X}_2}{\partial v_\perp}=\varepsilon\frac{v_\perp F_2(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{V}_{\|},v_\perp,\theta)}{|\mathbf{B}|(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)} \quad \mathcal{X}_2(u_\perp)=x_2$$

$$\mathcal{X}_3(v_\perp;\mathbf{x},v_{\|},u_\perp) \text{ s.t. } \frac{\partial \mathcal{X}_3}{\partial v_\perp}=\varepsilon\frac{v_\perp F_3(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{V}_{\|},v_\perp,\theta)}{|\mathbf{B}|(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)} \quad \mathcal{X}_3(u_\perp)=x_3$$

$$\mathcal{V}_{\|}(v_\perp;\mathbf{x},v_{\|},u_\perp) \text{ s.t. } \frac{\partial \mathcal{V}_{\|}}{\partial v_\perp}=\varepsilon\frac{v_\perp F_{\|}(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{V}_{\|},v_\perp,\theta)}{|\mathbf{B}|(\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3)} \quad \mathcal{V}_{\|}(u_\perp)=v_{\|}$$

Method of Characteristics - 2: exact solution

$$\boldsymbol{\Upsilon}_5(\mathbf{x}, v_{\|}, v_{\perp}, \theta) =$$

$$\boldsymbol{\Upsilon}_5(\boldsymbol{\mathcal{X}}(\nu; \mathbf{x}, v_{\|}, v_{\perp}), \mathcal{V}_{\|}(\nu; \mathbf{x}, v_{\|}, v_{\perp}), \nu, \theta) + \int_{\nu}^{v_{\perp}} \frac{s}{|\mathbf{B}(\boldsymbol{\mathcal{X}}(s; \mathbf{x}, v_{\|}, v_{\perp}))|} ds$$

$$= \int_{\nu}^{v_{\perp}} \frac{s}{|\mathbf{B}(\boldsymbol{\mathcal{X}}(s; \mathbf{x}, v_{\|}, v_{\perp}))|} ds$$

$$\left.\begin{array}{l} \dfrac{\partial \boldsymbol{\mathcal{X}}}{\partial v_{\perp}} = \varepsilon v_{\perp} \times \text{something} \\ \dfrac{\partial \mathcal{V}_{\|}}{\partial v_{\perp}} = \varepsilon v_{\perp} \times \text{something} \end{array}\right. \Rightarrow \begin{array}{l} \boldsymbol{\mathcal{X}} \text{ close to } \mathbf{x} \\ \mathcal{V}_{\|} \text{ close to } v_{\|} \end{array} \Rightarrow$$

$$\boldsymbol{\mathcal{X}} = \mathbf{x} + \varepsilon v_{\perp} \boldsymbol{\mathcal{X}}^1 + \varepsilon^2 v_{\perp}^2 \boldsymbol{\mathcal{X}}^2 + \dots$$

$$\mathcal{V}_{\|} = v_{\|} + \varepsilon v_{\perp} \mathcal{V}_{\|}^1 + \varepsilon^2 v_{\perp}^2 \mathcal{V}_{\|}^2 + \dots$$

Method of Characteristics - 3: Asymptotic expansion

Yet: $\begin{pmatrix} \boldsymbol{\mathcal{X}} \\ \mathcal{V}_{||} \end{pmatrix} (v_\perp; \mathbf{x}, v_{||}, u_\perp)$

$$\frac{\partial \begin{pmatrix} \boldsymbol{\mathcal{X}} \\ \mathcal{V}_{||} \end{pmatrix}}{\partial v_\perp} = \varepsilon \mathbf{F}(\boldsymbol{\mathcal{X}}_1, \mathcal{V}_{||}, v_\perp, \theta), \begin{pmatrix} \boldsymbol{\mathcal{X}} \\ \mathcal{V}_{||} \end{pmatrix} (u_\perp) = \begin{pmatrix} \mathbf{x} \\ v_{||} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\mathcal{X}} \\ \mathcal{V}_{||} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ v_{||} \end{pmatrix} + v_\perp \varepsilon \mathbf{F}(\mathbf{x}, v_{||}, v_\perp, \theta) + v_\perp^2 \mathcal{L}_{\varepsilon \mathbf{F}}(\varepsilon \mathbf{F})(\mathbf{x}_1, v_{||}, v_\perp, \theta) + v_\perp^3 \mathcal{L}^2_{\varepsilon \mathbf{F}}(\varepsilon \mathbf{F})(\mathbf{x}_1, v_{||}, v_\perp, \theta) + \dots$$

$\mathcal{L}(\mathbf{F})$ linked to Lie derivative
Indentifying with:

$$\boldsymbol{\mathcal{X}} = \mathbf{x} + \varepsilon v_\perp \boldsymbol{\mathcal{X}}^1 + \varepsilon^2 v_\perp^2 \boldsymbol{\mathcal{X}}^2 + \dots$$
$$\mathcal{V}_{||} = v_{||} + \varepsilon v_\perp \mathcal{V}_{||}^1 + \varepsilon^2 v_\perp^2 \mathcal{V}_{||}^2 + \dots$$

gives

Method of Characteristics - 3: Asymptotic expansion

$$\begin{pmatrix} \boldsymbol{\mathcal{X}}^1 \\ \mathcal{V}_\parallel^1 \end{pmatrix} = \mathbf{F}(\mathbf{x}, v_\parallel, v_\perp, \theta), \begin{pmatrix} \boldsymbol{\mathcal{X}}^2 \\ \mathcal{V}_\parallel^2 \end{pmatrix} = \mathcal{L}_{\mathbf{F}}(\mathbf{F})(\mathbf{x}, v_\parallel, v_\perp, \theta), \dots$$

$$\Upsilon_5(\mathbf{x}, v_\parallel, v_\perp, \theta) = \int_\nu^{v_\perp} \frac{s}{|\mathbf{B}(\boldsymbol{\mathcal{X}}(s; \mathbf{x}, v_\parallel, v_\perp))|} ds =$$

$$\int_\nu^{v_\perp} \frac{s}{|\mathbf{B}(\mathbf{x})|} ds + \varepsilon \int_\nu^{v_\perp} s^2\, \mathcal{T}^1(\frac{1}{|\mathbf{B}(\mathbf{x})|}) \cdot \boldsymbol{\mathcal{X}}^1 ds +$$

$$+ \varepsilon^2 \int_\nu^{v_\perp} s^3 \left(\mathcal{T}^2(\frac{1}{|\mathbf{B}(\mathbf{x})|}) \cdot \boldsymbol{\mathcal{X}}^1 + \mathcal{T}^1(\frac{1}{|\mathbf{B}(\mathbf{x})|}) \cdot \boldsymbol{\mathcal{X}}^2 \right) ds + \dots$$

$$\frac{(v_\perp - \nu)^2}{2|\mathbf{B}(\mathbf{x})|} + \dots$$

($\mathcal{T}^i$ linked to the Taylor expansion coefficients)

Gives new variable k as an expansion in ε

On other equations - Poisson Matrix in Darboux Coordinates

$\{\Upsilon_5,\Upsilon_6\} = \frac{1}{\varepsilon} \quad (=\{\Upsilon_k,\Upsilon_\theta\}=\{k,\theta\})$

Processed. Gave k

$\{\Upsilon_1,\Upsilon_5\} = 0(=\{\Upsilon_{y_1},\Upsilon_k\}=\{y_1,k\}),\ \{\Upsilon_1,\Upsilon_6\} = 0(=\{\Upsilon_{y_1},\Upsilon_\theta\}=\{y_1,\theta\})$
$\{\Upsilon_2,\Upsilon_5\} = 0(=\{\Upsilon_{y_2},\Upsilon_k\}=\{y_2,k\}),\ \{\Upsilon_2,\Upsilon_6\} = 0(=\{\Upsilon_{y_2},\Upsilon_\theta\}=\{y_2,\theta\})$
$\{\Upsilon_3,\Upsilon_5\} = 0(=\{\Upsilon_{y_3},\Upsilon_k\}=\{y_3,k\}),\ \{\Upsilon_3,\Upsilon_6\} = 0(=\{\Upsilon_{y_3},\Upsilon_\theta\}=\{y_3,\theta\})$
$\{\Upsilon_4,\Upsilon_5\} = 0(=\{\Upsilon_{y_4},\Upsilon_k\}=\{v_\parallel,k\}),\ \{\Upsilon_4,\Upsilon_6\} = 0(=\{\Upsilon_{v_\parallel},\Upsilon_\theta\}=\{v_\parallel,\theta\})$

To be Processed.

Will give $\mathbf{y}, u_\parallel, k$ as expansions in ε i.e.

$$\Upsilon = \Upsilon^0 + \varepsilon\Upsilon^1 + \varepsilon^2\Upsilon^2 + \dots$$

Other terms of new Poisson matrix $\overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta)$:

$(\widehat{\mathcal{P}}_\eta^\varepsilon)_{12} = \{\Upsilon_1,\Upsilon_2\}(=\{\Upsilon_{y_1},\Upsilon_{y_2}\}=\{y_1,y_2\}),$

$(\widehat{\mathcal{P}}_\eta^\varepsilon)_{13} = \{\Upsilon_1,\Upsilon_3\}(=\{\Upsilon_{y_1},\Upsilon_{y_3}\}=\{y_1,y_3\}),$

...

Hamiltonian Function in Darboux Coordinates

We know:

- $\widetilde{H}^{\varepsilon}_{\eta}(\mathbf{x}, v_{\|}, v_{\perp}, \theta) = \frac{1}{2}\left(v_{\|}^2 + v_{\perp}^2\right) + \Phi_0(\mathbf{x}) + \eta\Phi_1(\frac{\mathbf{x}}{\eta})$
- $\overline{H}^{\varepsilon}_{\eta}(\mathbf{y}, u_{\|}, k, \theta) = \widetilde{H}^{\varepsilon}_{\eta}(\boldsymbol{\xi}(\mathbf{y}, u_{\|}, k, \theta))$ with $\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1}$
- $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^0 + \varepsilon\boldsymbol{\Upsilon}^1 + \varepsilon^2\boldsymbol{\Upsilon}^2 + \dots$

We do :

- $\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \varepsilon\boldsymbol{\xi}^1 + \varepsilon^2\boldsymbol{\xi}^2 + \dots$
- $\widetilde{H}^{\varepsilon}_{\eta}(\boldsymbol{\xi}^0 + \varepsilon\boldsymbol{\xi}^1 + \varepsilon^2\boldsymbol{\xi}^2 + \dots) = \widetilde{H}^{\varepsilon}_{\eta}(\boldsymbol{\xi}^0) + \varepsilon\mathcal{T}^1(\widetilde{H}^{\varepsilon}_{\eta})(\boldsymbol{\xi}^0)\cdot\boldsymbol{\xi}^1 + \dots$

$$\overline{H}^{\varepsilon}_{\eta}(\mathbf{y}, u_{\|}, k, \theta) = \frac{u_{\|}^2}{2} + |\mathbf{B}(\mathbf{y})|k + \Phi_0(\mathbf{y}) + \eta\Phi_1(\frac{\mathbf{y}}{\eta}) + \varepsilon\overline{H}^1(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon^2\overline{H}^2(\mathbf{y}, u_{\|}, k, \theta)$$

First term : Independent of θ

Let us take stock

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\eta^\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\eta^\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\eta^\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\eta^\varepsilon$$

1: Hamiltonian? (↓)

2 (↑)

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\eta^\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\eta^\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\eta^\varepsilon$$

3 (→)

Cylindrical Coordinates

$(\mathbf{x}, v_\parallel, v_\perp, \theta)$

$\widetilde{H}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta), \widetilde{\mathcal{P}}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta)$

4: Darboux Method (↓)

$$\begin{pmatrix} \mathcal{M} & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 \cdots 0 \\ 0 \cdots 0 \end{matrix} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, u_\parallel, k, \theta)$

$\overline{H}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta), \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta)$

5: Lie Method (↓)

Lie Coordinates

$(\mathbf{z}, w_\parallel, j, \gamma)$

$\widehat{H}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j), \widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma) = \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$

Lie Transform based Method Target

We have $\overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\|, k, \theta)$ Nice shape. But:

$$\overline{H}_\eta^\varepsilon(\mathbf{y}, u_\|, k, \theta) = \overline{H}^0(\mathbf{y}, u_\|, k, \not\theta) + \varepsilon \overline{H}^1(\mathbf{y}, u_\|, k, \theta) + \varepsilon^2 \overline{H}^2(\mathbf{y}, u_\|, k, \theta) + \dots$$

depends on θ.

Key result $\longleftarrow$ $\theta-$independent Hamiltonian Function.
Target: Change of coordinates

$$(\mathbf{y}, u_\|, k, \theta) \mapsto (\mathbf{z}, w_\|, j, \gamma) = \boldsymbol{\zeta}(\mathbf{y}, u_\|, k, \theta)$$

leaving $\overline{\mathcal{P}}_\eta^\varepsilon$ unchanged (i.e. symplectic),

$$(\widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\|, j, \gamma) = \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\|, j, \gamma), \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\|, k, \theta) = \widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\|, k, \theta))$$

$\varepsilon-$parametrized, close to identity, i.e.:

$$\boldsymbol{\zeta}(\mathcal{E}, \mathbf{y}, u_\|, k, \theta) = (\mathbf{y}, u_\|, k, \theta) + \varepsilon \boldsymbol{\zeta}^1(\mathbf{y}, u_\|, k, \theta) + \varepsilon^2 \boldsymbol{\zeta}^2(\mathbf{y}, u_\|, k, \theta) + \dots$$

The way to do

Take ζ as the flow of a Hamiltonian Vector Field. i.e.:
Hamiltonian Function:

$$G(\varepsilon, \mathbf{y}, u_{\|}, k, \theta) = G^0(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon G^1(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon^2 G^2(\mathbf{y}, u_{\|}, k, \theta) + \dots$$

s.t. $\zeta(\varepsilon, \mathbf{y}, u_{\|}, k, \theta)$ solution to:

$$\frac{\partial \zeta}{\partial \varepsilon} = \overline{\mathcal{P}}_\eta^\varepsilon \nabla G, \quad \zeta(\varepsilon = 0) = (\mathbf{y}, u_{\|}, k, \theta)$$

Insures: Poisson Matrix $\overline{\mathcal{P}}_\eta^\varepsilon$ left unchanged

New target: G^0, G^1, G^2 s.t.:

$$\widehat{H}_\eta^\varepsilon(\mathbf{z}, w_{\|}, j) = \overline{H}_\eta^\varepsilon(\boldsymbol{\lambda}(\mathbf{z}, w_{\|}, j, \gamma)) \qquad (\boldsymbol{\lambda} = \zeta^{-1}),$$

A result

For: Two Hamiltonian Functions in two systems of coordinates:

$$\overline{H}^{\varepsilon}_{\eta}(\mathbf{y}, u_{\|}, k, \theta) = \overline{H}^{0}(\mathbf{y}, u_{\|}, k) + \varepsilon\overline{H}^{1}(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon^{2}\overline{H}^{2}(\mathbf{y}, u_{\|}, k, \theta) + \dots$$

$$\widehat{H}^{\varepsilon}_{\eta}(\mathbf{z}, w_{\|}, j) = \widehat{H}^{0}(\mathbf{z}, w_{\|}, j) + \varepsilon\widehat{H}^{1}(\mathbf{z}, w_{\|}, j) + \varepsilon^{2}\widehat{H}^{2}(\mathbf{z}, w_{\|}, j) + \dots$$

with $\widehat{H}^{0}(\mathbf{z}, w_{\|}, j) = \overline{H}^{0}(\mathbf{z}, w_{\|}, j)$

There exists:

$$G(\varepsilon, \mathbf{y}, u_{\|}, k, \theta) = G^{0}(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon G^{1}(\mathbf{y}, u_{\|}, k, \theta) + \varepsilon^{2} G^{2}(\mathbf{y}, u_{\|}, k, \theta) + \dots$$

$$\frac{\partial \boldsymbol{\zeta}}{\partial \varepsilon} = \overline{\mathcal{P}}^{\varepsilon}_{\eta} \nabla G, \quad \boldsymbol{\zeta}(\varepsilon = 0) = (\mathbf{y}, u_{\|}, k, \theta)$$

s.t: $\widehat{H}^{\varepsilon}_{\eta}(\mathbf{z}, w_{\|}, j) = \overline{H}^{\varepsilon}_{\eta}(\boldsymbol{\lambda}(\mathbf{z}, w_{\|}, j, \gamma)) \qquad (\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$

And: The Hamiltonian $g(\mathbf{z}, w_{\|}, j, \gamma)$ of $\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1}$ s.t.

$$g(\mathbf{z}, w_{\|}, j, \gamma) = g^{0}(\mathbf{z}, w_{\|}, j, \gamma) + \varepsilon g^{1}(\mathbf{z}, w_{\|}, j, \gamma) + \varepsilon^{2} g^{2}(\mathbf{z}, w_{\|}, j, \gamma) + \dots$$

$$\left\{g^{0}, \overline{H}^{0}\right\} = \mathcal{O}_{0}(\overline{H}^{0}), \left\{g^{1}, \overline{H}^{0}\right\} = \mathcal{O}_{1}(\overline{H}^{0}, \overline{H}^{1}, g^{0}), \dots$$

The Lie Transform based Method

Fix : $\widehat{H}^1(\mathbf{z}, w_{||}, j), \widehat{H}^2(\mathbf{z}, w_{||}, j), \dots$
Solve : $\left\{g^0, \overline{H}^0\right\} = \mathcal{O}_0(\overline{H}^0), \left\{g^1, \overline{H}^0\right\} = \mathcal{O}_1(\overline{H}^0, \overline{H}^1, g^0), \dots$
Get : $\boldsymbol{\lambda}$ and $\boldsymbol{\zeta}$
Compute : $(\mathbf{z}, w_{||}, j, \gamma) = \boldsymbol{\zeta}(\mathbf{y}, u_{||}, k, \theta)$
(We have : $\widehat{H}^\varepsilon_\eta(\mathbf{z}, w_{||}, j) = \overline{H}^\varepsilon_\eta(\boldsymbol{\lambda}(\mathbf{z}, w_{||}, j, \gamma)))$

At the end of the day

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\eta^\varepsilon(\mathbf{x},\mathbf{v}), \mathcal{P}_\eta^\varepsilon(\mathbf{x},\mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\eta^\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\eta^\varepsilon$$

3 →

Cylindrical Coordinates

$(\mathbf{x}, v_\parallel, v_\perp, \theta)$

$\widetilde{H}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta), \widetilde{\mathcal{P}}_\eta^\varepsilon(\mathbf{x}, v_\parallel, v_\perp, \theta)$

1: Hamiltonian? ↓

2 ↑

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\eta^\varepsilon(\mathbf{q},\mathbf{p}), \breve{\mathcal{P}}_\eta^\varepsilon(\mathbf{q},\mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\eta^\varepsilon$$

4: Darboux Method ↓

$$\left(\begin{array}{c|cc} \mathcal{M} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0\cdots 0 & 0 & 1 \\ 0\cdots 0 & -1 & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, u_\parallel, k, \theta)$

$\overline{H}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta), \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{y}, u_\parallel, k, \theta)$

5: Lie Method ↓

Lie Coordinates

$(\mathbf{z}, w_\parallel, j, \gamma)$

$\widehat{H}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j), \widehat{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma) = \overline{\mathcal{P}}_\eta^\varepsilon(\mathbf{z}, w_\parallel, j, \gamma)$

Hence

In Lie Coordinates: Trajectories $(\mathbf{Z}, W_{\|}, J, \Gamma)$

$$\frac{\partial \Gamma}{\partial t} = \text{Something complicated}$$
$$\frac{\partial J}{\partial t} = 0$$
$$\frac{\partial \begin{pmatrix} \mathbf{Z} \\ W_{\|} \end{pmatrix}}{\partial t} = \text{Something independent of } \Gamma$$