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# GEOMETRIC TWO-SCALE CONVERGENCE ON MANIFOLD AND APPLICATIONS TO THE VLASOV EQUATION.

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Abstract. We develop and we explain the two-scale convergence in the covariant formalism, i.e. using differential forms on a Riemannian manifold. For that purpose, we consider two manifolds $M$ and $Y$, the first one contains the positions and the second one the oscillations. We establish some convergence results working on geodesics on a manifold. Then, we apply this framework on examples.

1. **Introduction.** The two-scale convergence initiated by Nguetseng [7] and developed by Allaire [6], establishes convergence results for a sequence of functions $(u^\epsilon)_{\epsilon>0}$ containing oscillations of period $\epsilon$ and defined in an open domain $W$ of $\mathbb{R}^n$ to a function $U(\mathbf{x},\mathbf{y})$ on $W\times\mathbb{R}^n$ and periodic in the second variable $\mathbf{y}$.
The principle is:
On the open domain $W$, we fixe the period and we suppose that the solution of equation: $L^\epsilon u^\epsilon = f$ is $u^\epsilon$ where $L^\epsilon$ is a differential operator which induces oscillations of period $\epsilon$ and $f$ is a source term that does not depend on $\epsilon$, (we can also put boundary conditions). So, we will consider

- the space of functions $r$-integrable in $W$, denoted by $L^r(W)$ and defined such that the set of all measurable functions from $W$ to $\mathbb{R}$ or to $\mathbb{C}$ whose absolute value raised to the r-th power has finite integral, i.e
$$L^r(W)=\{f \text{ measurable such that } \|f\|_r=\left(\int_W \mid f\mid^r\right)^{\frac{1}{r}}<+\infty\},$$
for $1\leqslant r<+\infty$ and if $r=+\infty$ it is defined as the set of all measurable functions from $W$ to $\mathbb{R}$ or to $\mathbb{C}$ whose their essential supremum is finite,
- the space of functions $r$-integrable on $W$, $r$-integrable on $Y$ and $Y$-periodic and denoted by $L^r(W,L^r_{per}(Y))$ i.e
$$L^r(W,L^r_{per}(Y)=\{f \text{ measurable such that } \left(\int_W\int_Y \mid f\mid^r\right)^{\frac{1}{r}}<+\infty\},$$

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for $1 \leqslant r < +\infty$,
- and the space of functions of class $C^0$ with a compact support on $W$ and of class $C^0$ on $Y$ and $Y$-periodic denoted by $C_c^0(W, C_{per}^0(Y))$.

In the following, we will say that the sequence of functions $(u^\epsilon)_{\epsilon>0}$ in $L^r(W)$ for $r \in (1, +\infty]$ two-scale converges to a function $U$ in the space $L^r(W, L_{per}^r(Y))$ if for any functions $\psi$ in $C_c^0(W, C_{per}^0(Y))$, we have

$$\lim_{\epsilon \to 0} \int_W u^\epsilon(\mathbf{x})\psi(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})d\mathbf{x} = \int_Y \int_W U(\mathbf{x}, \mathbf{y})\psi(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y}\,. \tag{1}$$

We call $U$ the two-scale limit of $u^\epsilon$ in $L^r(W, L_{per}^r(Y))$. Nguetseng [7] and Allaire [6] established a two-scale convergence criterion which is very useful to establish an equation verified by the two-scale limit.
Since Physic's equations can be written using differential forms on manifold, we want to develop the two-scale convergence in this formalism, i.e. on manifold. We must use tools of differential geometry in covariant formalism. This point of view will allow us to work in a larger context, more adapted for differential equations.

First, we detail all important notions and all useful tools to establish geometric two-scale convergence. We can observe that differential forms are not always regular objects. So we remind some notions of functional analysis and we improve them to use them for differential forms. The functional analysis in the context of covariant formalism is developed by Scott [4, 3] and called $L^r$-cohomology. Then we adapt the two-scale convergence on manifold using the Birkhoff's theorem and we do asymptotic analysis using geometric objects (differential forms). This study was begun by Pak [2]. We explain notions of strong and weak convergence and in the last section, we apply this new point of view on Vlasov equation, in a context close to Frénod, Sonnendrücker [5] and Han-Kwan [13]. In this dimensionless Vlasov equation we observe the apparition of finite Larmor radius and oscillations with a period $\epsilon$. So, the dimensionless Vlasov equation can be written as $L^\epsilon u^\epsilon = 0$.

2. **Reminders on differential geometry.** In the following, we will consider that $M$ and $Y$ are two $n$-dimensional Riemannian manifolds.
For all points $p$ in $M$, there exists a tangent space to $M$ at $p$ denoted by $T_pM$. It represents the set of tangent vectors to $M$ at $p$. On tangent space $T_pM$, we have a scalar product $g_p$ (the Riemannian metric) represented by a symmetric matrix $g_p$, nondegenerate and positive definite, such that for all $u, v \in T_pM$, we have:

$$g_p(u, v) := u \cdot (g_p v),$$

where $\cdot$ is the scalar product. The tangent bundle of $M$ is the disjoint union of the tangent spaces to $M$:

$$TM := \bigsqcup_{p\in M} T_pM := \bigcup_{p \in M} \{p\} \times T_pM.$$

With the Riemannian metric we can define, on each tangent space $T_pM$, a Banach norm given by

$$\|v_p\| = g_p(v_p, v_p),$$

for all $p \in M$ and $v_p \in T_pM$. With the help of this norm, we can define the length of a piecewise $C^1$ curve $\gamma : [t_1, t_2] \to M$ joining $p_1$ to $p_2$ in $M$ by

$$L(p_1, p_2, \gamma) := \int_{p_1}^{p_2} \|\gamma'(t)\| dt.$$

We define also by

$$d(p_1,p_2) := \inf L(p_1,p_2,\gamma)$$

where the infimum is on all the curves which are piecewise $C^1$ and joining $p_1$ to $p_2$ in $M$. The curves minimizing the length $L(p_1,p_2,\gamma)$ are called geodesics and a geodesic which length equals $d(p_1,p_2)$ is called a minimizing geodesic. For $p \in M$ and $v_p \in T_pM$, there exists an unique geodesic $\gamma_{v_p}$ defined in the neighborhood of $p$ such that

$$\gamma_{v_p}(0) = p \ \text{ and } \ \gamma'_{v_p}(0) = v_p.$$

Let us denote by

$$V_0 = \{tv_p \in T_pM \mid v_p \in T_pM,\ \|v_p\| = 1 \text{ and } t \in [0, c(v_p)]\}, \tag{2}$$

and define on $V_0$ the exponential map

$$\begin{array}{rcccc} exp_p^M & : & V_0 & \to & M \\ & & v_p & \mapsto & \gamma_{v_p}(1), \end{array} \tag{3}$$

where

$$c(v_p) := \sup\{t > 0 : d(p, exp_p^M(t\,v_p) = t\}.$$

Notice that $V_0$ is the larger open set containing 0 in $T_pM$ such as if $v_p \in V_0$ then $\gamma_{v_p}$ is defined for $t = 1$. The geodesic verifying $\gamma_{v_p}(0) = p$ and $\gamma'_{v_p}(0) = v_p$ has the form

$$\{\gamma_{v_p}(t) = exp_p^M(tv_p),\ t \in [0, c(v_p)]\}.$$

$TM$ is a differential orientable manifold with dimension $2n$. A section of $TM$ is a map $f : M \to TM$ such that $\pi(f(p)) = p$ for all $p \in M$ where $\pi$ is the natural projection

$$\pi : TM \to M.$$

Let us also denote by $T_p^\star M$ the dual space of $T_pM$. It is also a vectorial space and its elements are called 1-forms. In the same way, we define the cotangent bundle of $M$ by $T^\star M$:

$$T^\star M := \bigsqcup_{p\in M} T_p^\star M := \bigcup_{p\in M} \{p\} \times T_p^\star M.$$

Using the exterior product $\wedge$, from $k$ elements $\mu_1, \dots, \mu_k$ of $T_p^\star M$, we can define the $k$-form $\mu_1 \wedge \dots \wedge \mu_k$ so the set of all the $k$-forms at $p$ by $\bigwedge^k(T_p^* M)$ and the bundle manifold

$$\bigwedge{}^k(T^*M) := \bigsqcup_{p\in M} \bigwedge{}^k(T_p^*M) := \bigcup_{p\in M} \{p\} \times \bigwedge{}^k(T_p^*M).$$

Since, as $TM$, $T^\star M$ and $\bigwedge^k(T^*M)$ for $k = 1, \dots, n$ are bundle manifolds with the natural projections $\pi^\star : T^\star M \to M$ and $\pi_k^\star : \bigwedge^k(T^*M) \to M$, we can define sections in $T^\star M$ and $\bigwedge^k(T^*M)$ as being the maps $f$ and $f_k$ from $M$ to $T^\star M$ and from $M$ to $\bigwedge^k(T^*M)$ respectively and such that $\pi^\star(f(p)) = p$ and $\pi_k^\star(f_k(p)) = p$ for all $p$ in $M$. And so a differential $k$-form on $M$ is a section of $\bigwedge^k(T^*M)$:

$$\omega^k : M \to \bigwedge{}^k(T^\star M),$$

and the set of differential $k$-forms on $M$ is denoted by $\Omega^k(M)$. In a local coordinate chart $(U, \varphi)$ such that $\varphi : U \subset \mathbb{R}^n \to \varphi(U) \subset M$, we denote by $(x_1, \dots, x_n)$ the local coordinate system of $p \in M$. All differential $k$-forms have the following expression in $U$

$$\sum_{i_1,\dots,i_k} \omega^k_{i_1,\dots,i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

where $\omega^k_{i_1,\dots,i_k}(x)$ are functions from $U$ to $\mathbb{R}$.
We also have some operators acting on differential forms as the exterior product, the exterior derivative, the pull-back, the push-forward, the interior product, the Lie derivative and the Hodge star operator. These operators are useful to translate equations in the language of differential geometry.

- The *exterior product*: Let $\omega^k \in \Omega^k(M)$ and $\eta^l \in \Omega^l(M)$. The exterior product of $\omega^k$ and $\eta^k$ is a differential form $\omega^k \wedge \eta^l \in \Omega^{k+l}(M)$ which acts, for all points $p$ in $M$, and on $(k+l)$ vectors $\xi_1, \dots, \xi_{k+l}$ of $T_pM$ in the following way:

$$\left(\omega^k \wedge \eta^l\right)_p(\xi_1, \dots, \xi_{k+l}) := \sum_{\sigma \in S_{k+l}} \frac{(-1)^{sgn(\sigma)}}{k!\, l!} \omega^k_p\big(\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}\big)\, \eta^l_p\big(\xi_{\sigma(k+1)}, \dots, \xi_{\sigma(k+l)}\big),$$

  with $S_{k+l}$ the set of all permutations of $\{1, \dots, k+l\}$.
- The *exterior derivative* $d$ is a linear operator from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ and we have

$$\begin{aligned} d\omega^k_p(\xi_0, \dots, \xi_k) \quad &:= \quad \sum_{i=0}^{k} (-1)^i \xi_i \cdot \omega^k_p(\xi_0, \dots \hat{\xi_i}, \dots, \xi_k) \\ &+ \quad \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \omega^k_p([\xi_i, \xi_j], \dots, \hat{\xi_i}, \dots, \hat{\xi_j} \dots, \xi_k) \end{aligned}$$

  with $\omega^k_p$ a differential $k$-form on $M$ at $p$ and $[\xi_i, \xi_j] = \xi_i\xi_j - \xi_j\xi_i$ is the *Lie brackets*. The symbol $\hat{\xi_i}$ means that we take off the vector $\xi_i$.
- The *pull-back*: Let $f : Y \to M$ be a differentiable map and $\omega^k$ be a differential $k$-form on $M$. The pull-pack of $\omega^k$ by $f$, $f^\star(\omega^k)$ is the differential $k$-form on $Y$ defined by:

$$(f^\star(\omega^k))_q(\xi_1, \dots, \xi_k) := \omega^k_{f(q)}(f_\star\xi_1, \dots, f_\star\xi_k),$$

  where $q \in Y$, $\xi_1, \dots, \xi_k \in T_qY$ and $f_\star : TY \longrightarrow TM$ is the *push forward*, i.e. the map which associates to any vector $\xi$ of $T_qY$ the vector $df_q(\xi)$ in $T_{f(q)}M$.
- The *interior product* of a differential $k$-form $\omega^k$ on $M$ along the vector field $\tau$ on $M$, is a differential $(k-1)$-form $i_\tau\omega^k$ which acts for all $p \in M$ in the following way:

$$\forall\, \xi_1, \dots, \xi_{k-1} \in T_pM \quad i_\tau\omega^k_p(\xi_1, \dots, \xi_{k-1}) := \omega^k_p(\tau, \xi_1, \dots, \xi_{k-1}).$$

  $i_\tau$ is a linear operator from $\Omega^k(M)$ to $\Omega^{k-1}(M)$.
- The *Lie derivative* $\mathcal{L}$ *along the vector field* $\tau$ is linear operator from $\Omega^k(M)$ to $\Omega^k(M)$. For any differential $k$-forms $\omega^k$, for all points $p \in M$ and for all $\xi_1, \dots, \xi_k \in T_pM$:

$$\left(\mathcal{L}_\tau \omega^k\right)_p(\xi_1, \dots, \xi_k) = \left.\frac{d}{dt}\right|_{t=0} \omega^k_p\left((\phi^t_p)_\star(\xi_1), \dots, (\phi^t_p)_\star(\xi_k)\right),$$

  with $\phi^t_p$ the flow of the vector field $\tau \in T_pM$ and $(\phi^t_p)_\star$ the push forward such that $(\phi^t_p)_\star(\xi) = d\phi^t_p(\xi)$ for all $\xi \in T_pM$. We also rewrite the Lie derivative using the homotopy formula:

$$\mathcal{L}_\tau = d\, i_\tau + i_\tau\, d.$$

Before defining the Hodge star operator, we notice that the metric defines an inner product on $T^\star M$ as well as on $TM$. The map

$$\begin{array}{ccccc} \hat{g}_p & : & T_pM & \to & T_p^\star M \\ & & v_p & \mapsto & g(v_p, \cdot) \end{array}$$

defines an isomorphism from $T_pM$ to $T_p^\star M$, that says that we can identify the tangent $T_pM$ and the cotangent space $T^\star M$. We may extend this identification to all vector fields on $M$, $TM$ and the space of all differential forms on $M$, $T^\star M$. That means we can define an inner product $(\omega_p, \eta_p)$ for any two differential 1-forms $\omega$ and $\eta$ on $M$ at each point $p$ and so we have a function $(\omega, \eta)$ on $M$. We shall generalize it to the case of differential $k$-forms. For two elements of the form $\alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta_1 \wedge \cdots \wedge \beta_k$ $(\alpha_i, \beta_j \in T^\star M)$ the value of inner product is

$$(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = det((\alpha_i, \beta_j)_{i,j}).$$

In this way we have the inner product defined for any two differential $k$-forms on $M$. For example in the local coordinates of $p$, that gives

$$(dx_1 \wedge \cdots \wedge dx_n, dx_1 \wedge \cdots \wedge dx_n) = \mid g_p^{-1} \mid,$$

where $\mid g_p^{-1} \mid$ is the determinant of the inverse matrix associated with the metric $g_p$. In the local coordinate system of $p$, we can write the natural measure on $M$:

$$vol_p(g) = vol_p = \sqrt{(\mid g_p \mid)} dx_1 \wedge \cdots \wedge dx_n.$$

Now, we can define correctly the Hodge star operator and the co-exterior derivative.

- The *Hodge star* operator is a linear isomorphism $\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ which associates to each differential $k$-form $\omega^k$, a differential $(n-k)$-form $\omega^{n-k}$ (where $n$ is the dimension of $M$). It has the following property that at each point we have

  $$(\omega^k, \beta^k) vol_p(g) = \omega^k \wedge \star \beta^k,$$

  for $\omega^k$ and $\beta^k$ differential $k$-forms on $M$. And for an orthogonal basis $dx_1, \ldots, dx_n$ at a point $p$, we have that

  $$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \frac{\sqrt{|g_p|}}{(n-k)!} g_p^{i_1 l_1} \ldots g_p^{i_k l_k} \delta_{l_1 \ldots l_k l_{k+1} \ldots l_n} dx_{l_{k+1}} \wedge \cdots \wedge dx_{l_n},$$

  where $x$ is the local coordinate system of $p$, $g_p$ is the matrix associated to the metric in these chart, $|g_p|$ is the determinant of the Riemannian metric $g_p$, $g_p^{i_1 l_1}$ is the component of inverse matrix of $g_p$ and $\delta$ is the Levi-Civita permutation symbol.
  In the local coordinate system, for a differential $k$-form $\omega^k$ with the form

  $$\omega_p^k = \sum_{(i_1 < \ldots < i_k) \in \{1, \ldots, n\}} \omega_{i_1, \ldots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

  we have:

  $$\star\omega_p^k = \sum_{(j_1 < \ldots < j_{n-k}) \in \{1, \ldots, n\}} \sqrt{|g_p|} g_p^{i_1 l_1} \ldots g_p^{i_k l_k} \delta_{i_1 \ldots i_k l_{k+1} \ldots l_n} \omega_{i_1, \ldots, i_k}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}.$$

  The Hodge star operator has also the following property:

  $$\star \star \omega^k = (-1)^{k(n-k)} \omega^k.$$

- The *co-exterior derivative* $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ is a derivation operator defined by $d^* \omega^k := (-1)^{n(k-1)+1} \star d \star \omega^k$.

3. **Geometric tools: The $L^r$-cohomology.** The reader is referred to [1] to have more details about this section.

3.1. **Generalities.** To do $L^r$-cohomology, we must define what it means for differential $k$-forms to be $r$-integrable on a manifold. To do that, we remind some definitions.

We say that a differential $k$-form $\omega^k$ is *measurable* on $M$ if it is a measurable section of vectorial bundle $\bigwedge^k(T^\star M)$ and that it is *defined almost everywhere* on $M$ if there exists a domain $N \subset M$ with measure zero such that the function $\omega^k : M \to \bigwedge^k(T^\star M)$ is well-defined on the domain $M \setminus N$.
Using the definition of a differential $k$-form in local coordinate system, $\omega^k$ is *measurable* on $M$ if and only if for all charts covering $M$, the coefficients $\omega^k_{i_1,\dots,i_k}$ are measurable functions and *defined almost everywhere* if for all charts covering $M$ the coefficients $\omega^k_{i_1,\dots,i_k}$ are functions defined almost everywhere.

A differential $k$-form $\omega^k$ *has an $k$-dimensional compact support* $K$ in $M$ if for all point $p$ outside of $K$ and for all vectors $\xi_1,\dots,\xi_k \in T_pM$ we have $\omega^k_p(\xi_1,\dots,\xi_k) = 0$.
The integral of a differential $k$-form is defined as follows: Let $M^k$ be a differential $k$-dimensional manifold included in $M$, $(U_i \subset \mathbb{R}^k, \varphi_i)_{i\in I}$ be a set of charts covering $M^k$ and $\{\lambda_i\}_{i\in I}$ be a *partition of unity* subordinate to $\{\varphi(U_i)\}$ (i.e indexed over the same set $I$ such that *supp* $\lambda_i \subseteq \varphi(U_i)$). So all differential $k$-forms $\omega^k$ on $M^k$ can be written as

$$\omega^k = \sum_{i\in I} \lambda_i \omega^k.$$

So the integral over $M^k$ of $\omega^k$ is then defined as

$$\int_{M^k} \omega^k = \sum_{i\in I} \int_{\varphi_i(U_i)} \lambda_i \omega^k = \sum_{i\in I} \int_{U_i} \varphi_i^\star(\lambda_i \omega^k).$$

A differential $k$-form $\omega^k$ is said to be $C^s$ on $M$, if for all $p \in M$, $\omega^k_p \in \bigwedge^k(T^\star_p M)$ and $\varphi^\star(\omega^k)$ is a differential $k$-form $C^s$ on $U \subset \mathbb{R}^n$ for all charts $\varphi$ covering $M$. The set of differential $k$-forms $C^s$ on $M$ is denoted by $\Omega^k_s(M)$.
Now, we are ready to define differential $k$-forms $r$-integrable on a manifold.
We denote by $L^r(M, \bigwedge^k)$ the space of *$r$-integrable* differential $k$-forms. It is defined by

$$L^r(M, \textstyle\bigwedge^k) = \left\{\alpha \in \Omega^k(M) \text{ mesurable such that } \|\alpha\|_{L^r(M,\bigwedge^k)} < +\infty\right\},$$

for $1 \leqslant r \leqslant +\infty$, where

$$\|\alpha\|_{L^r(M,\bigwedge^k)} := \left(\int_M |\alpha|^r_p vol_p\right)^{\frac{1}{r}},$$

for $1 \leqslant r < \infty$ where

$$\|\alpha\|_{L^{+\infty}(M,\bigwedge^k)} := \sup_{p\in M} \text{ess} |\alpha|_p,$$

with

$$|\alpha|^r_p = \left(\star(\alpha^k_p \wedge \star\alpha^k_p)\right)^{\frac{r}{2}}.$$

In local coordinate system, we can observe that $|\alpha|_p^r$ corresponds to

$$|\alpha|_p^r = \left( \sum_{i_1,\dots,i_k} (\alpha_{i_1,\dots,i_k}(x))^2 \right)^{\frac{r}{2}} .$$

We can associate a scalar product with norm $\|\alpha\|_{L^2(M,\bigwedge^k)}$:

$$< \alpha, \beta >_{L^2(M,\bigwedge^k)} := \int_M \alpha \wedge \star\beta \,,$$

with $\alpha, \beta \in \Omega^k(M)$ being measurable. $\|\cdot\|$ and $< \cdot, \cdot >$ do not depend on charts on $M$. The function space $H^{1,d}(M, \bigwedge^k)$ is the completion of $\Omega^k(M)$ with the respect of the norm

$$\left(< \alpha, \alpha >_{L^2(M,\bigwedge^k)} + < d\alpha, d\alpha >_{L^2(M,\bigwedge^{k+1})}\right)^{\frac{1}{2}} , \tag{4}$$

and the function space $H^{1,\delta}(M, \bigwedge^k)$ is the completion of $\Omega^k(M)$ with the respect of the norm

$$\left(< \alpha, \alpha >_{L^2(M,\bigwedge^k)} + < \delta\alpha, \delta\alpha >_{L^2(M,\bigwedge^{k-1})}\right)^{\frac{1}{2}} . \tag{5}$$

If $M$ is a compact $n$-dimensionnal Riemannian manifold with boundary, we want to define the tangential component and the normal component. These notions are important because they permit to have the Green-Stokes formula and the integration by parts. For doing this, we observe that we can associate forms to vector fields (and vice-versa) and we can choose a local coordinate system such that the Riemannian metric has the form of 2-forms:

$$g_p^2 = \sum_{i,j} g_p^{i,j} dx^i \otimes dx^j ,$$

where $\otimes$ corresponds to the tensor product. For two vectors with the form $\sum_i a_i \frac{\partial}{\partial x^i}$ and $\sum_i b_i \frac{\partial}{\partial x^i}$ in the local coordinate system of $p$, we have

$$g_p^2(\sum_i a_i \frac{\partial}{\partial x^i}, \sum_j b_j \frac{\partial}{\partial x^j}) = \sum_{i,j} a_i g_p^{i,j} b_j ,$$

and

$$g_p^2(\sum_i a_i \frac{\partial}{\partial x^i}, \cdot) = \sum_{i,j} a_i g_p^{i,j} dx^j .$$

That means that $g_p^2$ takes a vector field $\mu_p$ to an 1-form denoted by $g_p^2(\mu_p) = \mu_p^\sharp$. We denote by $\mu$ the outgoing unit normal vector to $\partial M$ and by $\mu^\sharp$ the differential 1-form associated to $\mu$. With these, the tangential component of the differential $k$-form $\omega^k$ corresponds to $\mu^\sharp \wedge \omega^k$ and the normal component of $\omega^k$ is $i_\mu \omega^k$. We reformulate the Green-Stokes formula for the differential forms in two different ways:

- with the exterior derivative:

$$< d\omega^k, \alpha^{k+1} >_{L^2(M,\bigwedge^{k+1})} = < \omega^k, d^*\alpha^{k+1} >_{L^2(M,\bigwedge^k)} + < \nu^\sharp \wedge \omega^k, \alpha^{k+1} >_{L^2(\partial M,\bigwedge^{k+1})},$$

  with $\alpha^{k+1}$ a differential $(k+1)$-form,

- and with the co-exterior derivative

$$< d^* \omega^k, \psi^{k-1} >_{L^2(M,\bigwedge^{k-1})} = < \omega^k, d\,\psi^{k-1} >_{L^2(M,\bigwedge^k)} - < i_\nu \omega^k, \psi^{k-1} >_{L^2(M,\bigwedge^{k-1})},$$

  with $\psi^{k+1}$ a differential $(k-1)$-form.

Now we suppose that $\frac{1}{r} + \frac{1}{s} = 1$ with $1 \leqslant r, s \leqslant +\infty$.

**Proposition 1.** *1. (Hölder's Inegality) If $\alpha^k \in L^r(M, \bigwedge^k)$ and $\beta^k \in L^s(M, \bigwedge^k)$, so $\alpha^k \wedge \star\beta^k \in L^1(M, \bigwedge^n)$ and we have*

$$\|\alpha^k \wedge \star\beta^k\|_{L^1(M,\bigwedge^n)} \leqslant \|\alpha^k\|_{L^r(M,\bigwedge^k)} \|\beta^k\|_{L^s(M,\bigwedge^k)}.$$

*2. (Minkowski's inegality) If $\alpha^k, \beta^k \in L^r(M, \bigwedge^k)$ so $\alpha^k + \beta^k \in L^r(M, \bigwedge^k)$ and we have*

$$\|\alpha^k + \beta^k\|_{L^r(M,\bigwedge^k)} \leqslant \|\alpha^k\|_{L^r(M,\bigwedge^k)} + \|\beta^k\|_{L^r(M,\bigwedge^k)}.$$

*3. $L^r(M, \bigwedge^k)$ is a Banach space.*
*4. $L^2(M, \bigwedge^k)$ is a Hilbert space.*
*5. (Density) $C_c^2(M, \bigwedge^k)$ The set of differential k-forms of class $C^2$ with a compact support in $M$ are dense in $L^r(M, \bigwedge^k)$*

Now we can define the Sobolev's space in these formalism. Since we have three derivative operators, the exterior derivative $d$, the co-exterior derivative $d^*$ and the Laplacian $\Delta = d^*\, d + d\, d^*$, we have three types of Sobolev's space:

$$\begin{aligned}
W^{d,r}(M, \textstyle\bigwedge^k) &:= \{\omega \in L^r(M, \textstyle\bigwedge^k) : d\,\omega \in L^r(M, \textstyle\bigwedge^{k+1})\},\\
W^{d^*,r}(M, \textstyle\bigwedge^k) &:= \{\omega \in L^r(M, \textstyle\bigwedge^k) : d^*\,\omega \in L^r(M, \textstyle\bigwedge^{k-1})\},\\
W^{\Delta,r}(M, \textstyle\bigwedge^k) &:= \{\omega \in L^r(M, \textstyle\bigwedge^k) : \Delta\,\omega \in L^r(M, \textstyle\bigwedge^k)\},
\end{aligned}$$

with the following norms

$$\begin{aligned}
\|\omega\|_{W^{d,r}(M,\bigwedge^k)} &:= \left(\|\omega\|^r_{L^r(M,\bigwedge^k)} + \|d\,\omega\|^r_{L^r(M,\bigwedge^k)}\right)^{\frac{1}{r}},\\
\|\omega\|_{W^{d^*,r}(M,\bigwedge^k)} &:= \left(\|\omega\|^r_{L^r(M,\bigwedge^k)} + \|d^*\,\omega\|^r_{L^r(M,\bigwedge^k)}\right)^{\frac{1}{r}},\\
\|\omega\|_{W^{\Delta,r}(M,\bigwedge^k)} &:= \left(\|\omega\|^r_{L^r(M,\bigwedge^k)} + \|d\,\omega\|^r_{L^r(M,\bigwedge^k)} + \|d^*\,\omega\|^r_{L^r(M,\bigwedge^k)}\right)^{\frac{1}{r}}.
\end{aligned}$$

**Remark 1.** If $M$ is an open subset of $\mathbb{R}^n$, the usual Sobolev's space $W^{1,r}(M)$, is the intersection between $W^{d,r}(M, \bigwedge^k)$ and $W^{d^*,r}(M, \bigwedge^k)$.

We can also define the Dirichlet boundary conditions by

$$\begin{aligned}
W_0^{d,r}(M, \textstyle\bigwedge^k) &:= \overline{\{\omega \in C_c^2(M, \textstyle\bigwedge^k) : \nu^\sharp \wedge \omega = 0 \quad \text{sur} \quad \partial M\}}^{\|\,\|_{W^{d,r}}},\\
W_0^{d^*,r}(M, \textstyle\bigwedge^k) &:= \overline{\{\omega \in C_c^2(M, \textstyle\bigwedge^k) : i_\nu \omega = 0 \quad \text{sur} \quad \partial M\}}^{\|\,\|_{W^{d^*,r}}}.
\end{aligned}$$

3.2. **The weak convergence.** For a differential $k$-form $\omega^k \in L^2(M, \bigwedge^k)$, we have a linear form on $C_c^2(M, \bigwedge^k)$:

$$< \omega^k, \psi^k >_{L^2(M,\bigwedge^k)} := \int_M \omega_p^k \wedge \star \psi_p^k,$$

with $\psi^k \in C_c^2(M, \bigwedge^k)$ and since $L^2(M, \bigwedge^k)$ is a Hilbert's space, we can define a weak convergence.

**Definition 3.1.** We say that a sequence of differential $k$-forms $(\omega_n^k)_{n\in\mathbb{N}} \in L^2(M, \bigwedge^k)$ weakly converges to a differential $k$-form $\omega^k \in L^2(M, \bigwedge^k)$ if and only if for a test differential $k$-form $\psi^k \in C_c^2(M, \bigwedge^k)$ we have

$$< \omega_n^k, \psi^k >_{L^2(M,\bigwedge^k)} \to < \omega^k, \psi^k >_{L^2(M,\bigwedge^k)},$$

when $n$ tends to infinity.

Moreover the test differential $k$-forms verify the Dirichlet's boundary conditions, so for all $\psi^{k+1} \in C_c^2(M, \bigwedge^{k+1})$ we can define the exterior derivative of $\alpha^k \in L^2(M, \bigwedge^k)$ in the weak sense i.e:

$$< d\,\omega^k, \psi^{k+1} >_{L^2(M,\bigwedge^{k+1})} = < \omega^k, d^\star \psi^{k+1} >_{L^2(M,\bigwedge^k)},$$

and its co-exterior derivative for all $\psi^{k-1} \in C_c^2(M, \bigwedge^{k-1})$:

$$< d^\star\,\omega^k, \psi^{k-1} >_{L^2(M,\bigwedge^{k-1})} = < \omega^k, d\,\psi^{k-1} >_{L^2(M,\bigwedge^k)}.$$

## 4. **The geometric two-scale convergence.**

4.1. **Introduction.** For the geometric two-scale convergence, we need to work with two $n$-dimensional Riemannian manifolds, denoted in the following by $M$ and $Y$. Later, we will suppose that $M$ is geodesically complete and possibly with boundary and that $Y$ is compact, without boundary and with ergodic geodesic flow. So we must define what is it to be a differential form on $M \times Y$. First, we see that, for $(p,q) \in M \times Y$, a Riemannian metric on $M \times Y$ can be defined as follows :

$$\begin{array}{rccc} g_{(p,q)}^{M\times Y} & : & T_{(p,q)}(M\times Y) \times T_{(p,q)}(M\times Y) & \to & \mathbb{R} \\ & & (u,v) & \mapsto & g_p^M(dP_{(p,q)}^M(u), dP_{(p,q)}^M(v)) \\ & & & & +g_q^Y(dP_{(p,q)}^Y(u), dP_{(p,q)}^Y(v)) \end{array}$$

where $P^M$ and $P^Y$ are the following natural projections

$$P^M : M \times Y \to M \text{ and } P^Y : M \times Y \to Y,$$

and so

$$dP_{(p,q)}^M : T_{(p,q)}(M\times Y) \to T_{P^M(p,q)}M \text{ and } dP_{(p,q)}^Y : T_{(p,q)}(M\times Y) \to T_{P(p,q)}^Y Y.$$

We deduce the identification

$$T_{(p,q)}(M\times Y) \cong T_pM \oplus T_qY,$$

and $T^\star_{(p,q)}(M\times Y)\cong T^\star_pM\oplus T^\star_qY$. With the previous projections $P^M$ and $P^Y$ we have that

$$\begin{aligned} T^\star(M\times Y) &:= \bigsqcup_{(p,q)\in M\times Y} T^\star_{(p,q)}(M\times Y) \\ &:= \bigcup_{(p,q)\in M\times Y}\{(p,q)\}\times T^\star_{(p,q)}(M\times Y) \\ &= \bigcup_{(p,q)\in M\times Y}\{(p,q)\}\times T^\star_pM\times T^\star_qY, \end{aligned}$$

We deduce that

$$\bigwedge\nolimits^k_{P^M} T^\star(M\times Y)=\bigcup_{(p,q)\in M\times Y}\{(p,q)\}\times\bigwedge\nolimits^k(T^\star_pM)$$

and so

$$\bigwedge\nolimits^k_{P^M} T^\star(M\times Y)\wedge\bigwedge\nolimits^l_{P^Y} T^\star(M\times Y)=\bigcup_{(p,q)\in M\times Y}\{(p,q)\}\times\bigwedge\nolimits^k(T^\star_pM)\times\bigwedge\nolimits^l(T^\star_qY).$$

Now we can define differential $(k,l)$-forms on $M\times Y$ who are $k$-form on $M$ and $l$-form on $Y$ as:

$$\omega^{k,l}_{(p,q)}:\underbrace{T_pM\wedge\ldots\wedge T_pM}_{k}\wedge\underbrace{T_qY\wedge\ldots\wedge T_qY}_{l}\longrightarrow\mathbb{R},$$

i.e. $\omega^{k,l}_{(p,q)}\in\bigwedge^k(T^\star_pM)\wedge\bigwedge^l(T^\star_qY)$ and so a differential $(k,l)$-form on $M\times Y$ is a map such that

$$\begin{array}{rccl}\omega^{k,l}: & M\times Y & \longrightarrow & \bigwedge^k_{P^M}(T^*(M\times Y))\wedge\bigwedge^l_{P^Y}(T^*(M\times Y)) \\ & (p,q) & \longmapsto & ((p,q),\omega^{k,l}_{(p,q)}).\end{array}$$

In a local coordinate chart $(U^M_i\times U^Y_j,\varphi^M_i,\varphi^Y_j)$ where $(U^M_i,\varphi^M_i)$ is a local coordinate chart on $M$ and $(U^Y_j,\varphi^Y_j)$ is a local coordinate chart on $Y$, a differential form $\omega^{k,l}$ on $M\times Y$ as the following expression

$$\omega^{k,l}_{(p,q)}=\sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_l}}\omega_{i_1,\ldots,i_k,j_1,\ldots,j_l}(x,y)dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_l},$$

for all $p\in M$ and $q\in Y$ and $x,y$ are respectively the local coordinate system of $p$ et $q$.

Now to introduce the space $L^r(M,\bigwedge^k L^s(Y,\bigwedge^l))$, the set of differential $(k,l)$-forms $M\times Y$, $r$-integrable on $M$ and $s$-integrable on $Y$, we denote by $\{\lambda_{i,j}\}_{(i,j)\in I\times J}$ a partition of unity subordinate to $\{\varphi^M_i(U^M_i)\times\varphi^Y_j(U^Y_j)\}$. So all differential $(k,l)$-forms $\omega^{k,l}$ on $M\times Y$ can be written as

$$\omega^{k,l}=\sum_{(i,j)\in I\times J}\lambda_{i,j}\omega^{k,l}.$$

We say that $\omega^{k,l}\in L^r(M,\bigwedge^k L^s(Y,\bigwedge^l))$ if all components of $(\varphi^M_i)^\star(\varphi^Y_j)^\star\lambda_{i,j}\omega^{k,l}=(\varphi^Y_j)^\star(\varphi^M_i)^\star\lambda_{i,j}\omega^{k,l}$ are in $L^r(U^M_i,L^s(U^Y_j))$ and

$$\sum_{(i,j)\in I\times J}\|(\varphi^Y_j)^\star(\varphi^M_i)^\star\lambda_{i,j}\omega^{k,l}\|_{L^r(U^M_i,L^s(U^Y_j))}<+\infty$$

Now if we observe (1) we see that we must define what means the evaluation in $\frac{x}{\epsilon}$ for differential forms on $M \times Y$. To explain this, we will use the geodesics on manifold $M$ and $Y$.

Let $p_0 \in M$, $q_0 \in Y$ and an isomorphism $j$ be such that $T_{p_0}M \overset{j}{\cong} T_{q_0}Y$. We see in (2) page 3 that for all $p \in \exp_{p_0}^M(V_0)$, there exists $v \in V_0 \subset T_{p_0}M$ such that

$$p = \exp_{p_0}^M(v),$$

where $\exp_{p_0}^M$ is defined by (3).

The geometric two-scale convergence results from Birkhoff's theorem (1931) [9], Hopf's theorem (1939) [11] and the Mautner's Theorem (1957) [10]. Birkhoff's theorem [9] says that for all probability space $(\chi, \mu)$ and an ergodic flow $\phi^t$, we have for $f \in L^r(\chi, \mu)$,

$$\frac{1}{T}\int_0^T f(\phi^t)dt \xrightarrow[+\infty]{T} \int_M f(x)d\mu\,.$$

For our concerns, the geodesic flow must be ergodic on $Y$ to develop geometric two-scale convergence issues. The Hopf's theorem [11] and the Mautner's theorem [10] give the conditions for the geodesic flow to be ergodic. The Hopf's theorem [11] stipulates that in a compact Riemannian manifold with a finite volume and with a negative curvature the geodesic flow is ergodic. Mautner showed that in symmetric Riemannian manifold the geodesic flows are also ergodic.

Torus, projective spaces, hyperbolic spaces, Heisenberg's space, $Sl(n, \mathbb{R})/SO(n, \mathbb{R})$, the symmetric space of quaternion-Kähler are examples of symmetric Riemanian manifold. If the manifold $Y$ satisfies these conditions then it is geodesically complete and so the Hopf-Rinow's theorem says there exists $v \in V_0 \subset T_{p_0}M$ that all $q$ in $Y$ can be written as

$$q = \exp_{q_0}^Y(j(v)),$$

where $\exp_{q_0}^Y$ is the exponential map on $Y$ (see (3) page 3 for its definition). To use Birkhoff's theorem with an ergodic flow, we suppose that $M$ and $Y$ are $n$-dimensional Riemannian manifolds, moreover $M$ is assumed to be geodesically complete and possibly with boundary and $Y$ is assumed to be compact, without boundary and with ergodic geodesic flow i.e. verify the Mautner's condition or Hopf's condition. With the properties of $M$ and $Y$, for any given $p_0 \in M$ and $q_0 \in Y$, we define for all $p \in M$, $p^\epsilon \in Y$ as

$$p^\epsilon = \exp_{q_0}^Y\left(\frac{1}{\epsilon}\, j(v)\right),$$

where $v \in V_0$ is such that $p = \exp_{p_0}^M(v)$. Once we have introduced this notation, we easily see that if $\psi^{k,0}_{(p,q)}$ is a differential $k$-form on $M$ and a differential 0-form on $Y$ at point $(p, q)$ in $M \times Y$ with enough regularity, then $\psi^{k,0}_{(p,p^\epsilon)}$ is a differential $k$-form on $M$.

We have defined all the context and all tools to do geometric two-scale convergence in the covariant formalism.

4.2. **The geometric two-scale convergence.**

**Definition 4.1.** For $(\alpha_\epsilon^k)_{\epsilon>0}$ a sequence of differential $k$-forms in $L^r(M,\bigwedge^k)$, we say that it converges to *the two-scale limit* $\alpha_0^k \in L^r(M,\bigwedge^k L^r(Y,\bigwedge^0))$ if for all differential $k$-form $\psi^k \in C_c^2(M,\bigwedge^k \Omega^0(Y))$, we have

$$< \alpha_\epsilon^k, \psi^k_{(p,p^\epsilon)} >_{L^r(M,\bigwedge^k)} \longrightarrow < \alpha_0^k, \psi^k >_{L^r(M,\bigwedge^k L^r(Y,\bigwedge^0))},$$

when $\epsilon$ tends to 0. $\alpha_0^k$ is called the two-scale limit of $\alpha_\epsilon^k$ in $L^r(M,\bigwedge^k L^r(Y,\bigwedge^0))$. We also say that $\alpha_\epsilon^k$ *two-scale converges strongly* to $\alpha_0^k$ if

$$\lim_{\epsilon\to 0} \|\alpha_\epsilon^k - (\alpha_0^k)_{(p,p^\epsilon)}\|_{L^r(M,\bigwedge^k)} = 0.$$

We have also the following proposition [1, 2]:

**Proposition 2.** *We suppose that $M$ and $Y$ are $n$-dimensional Riemannian manifolds, and $M$ is complete possibly with boundary and $Y$ is compact and verify the Mautner's condition or the Hopf's condition. For $\psi^k \in L^2(M,\bigwedge^k \Omega^0(Y,\bigwedge^0))$, we have*

$$\lim_{\epsilon\to 0} \|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} = \|\psi^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.$$

*Proof.* $Y$ is a compact Riemannian manifold so there exists a finite open $(U_i)_{1\leqslant i\leqslant m}$ covering of $Y$. We denote by

$$(\psi_n^k)_{(p,q)} = \sum_{j_1\leqslant\cdots\leqslant j_k} \sum_i \psi_{j_1,\ldots,j_k}(x,y_i)\chi_i(y) dx_{j_1}\wedge\cdots\wedge dx_{j_k}$$

for $x, y$ the local coordinates of $p$ and $q$, $y_i \in U_i$ and $\chi_i$ the characteristic function on $U_i$. With the help of the dominate convergence theorem [6], $\psi_n^k$ converges to $\psi^k$ when $n$ tends to infinity and so we have

$$\begin{aligned}
\|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} - \|\psi^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} &= \|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} \\
&- \|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} \\
&+ \|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} \\
&- \|\psi_n^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} \\
&+ \|\psi_n^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} \\
&- \|\psi^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.
\end{aligned}$$

First, we show that

$$\lim_{\epsilon\to 0} \|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} = \|\psi_n^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.$$

For the left hand side of the equality, we get

$$\lim_{\epsilon\to 0} \|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} = \lim_{\epsilon\to 0} \sqrt{\sum_{j_1\leqslant\cdots\leqslant j_k} \sum_i \int_M (\psi_{j_1,\ldots,j_k}(x,y_i)\chi_i(x^\epsilon))^2 vol_x}.$$

Since $Y$ is geodesically complete and since on $Y$ the geodesic flow is ergodic, we have $\chi_i(x^\epsilon)$ converges weakly to $vol(U_i)$ . Hence

$$\begin{aligned}\lim_{\epsilon\to 0}\|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} &= \sqrt{\sum_{j_1\leqslant\cdots\leqslant j_k}\sum_i\int_M(\psi_{j_1,\dots,j_k}(x,y_i)vol(U_i))^2 vol_x} \\ &= \sqrt{\sum_{j_1\leqslant\cdots\leqslant j_k}\sum_i\int_M\int_Y(\psi_{j_1,\dots,j_k}(x,y_i)\chi_i(y))^2 vol_x\, vol_y} \\ &= \|\psi_n^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.\end{aligned}$$

That implies

$$\begin{aligned}\lim_{\epsilon\to 0}\|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} = & \lim_{\epsilon\to 0}\|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} - \|(\psi_n^k)_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} \\ & +\|\psi_n^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.\end{aligned}$$

Then when $n$ tends to infinity we found that

$$\lim_{\epsilon\to 0}\|\psi^k_{(p,p^\epsilon)}\|_{L^2(M,\bigwedge^k)} - \|\psi^k\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} = 0.$$

□

With the help of this proposition, we formulate a geometric two-scale convergence with one parameter with the same conditions for $M$ and $Y$.

**Theorem 4.2.** *For $(\alpha^k_\epsilon)$ a bounded sequence in $L^2([0,+\infty),L^2(M,\bigwedge^k))$, there exists a subsequence $(\alpha^k_{\epsilon_j})$ of $(\alpha^k_\epsilon)$ and a differential form*

$$\alpha_0^k \in L^2([0,+\infty),L^2(M,\bigwedge{}^k L^2(Y,\bigwedge{}^0)))$$

*such that for all*

$$\psi^k \in L^2([0,+\infty),C^2_c(M,\bigwedge{}^k\,\Omega^0(Y))),$$

*we have*

$$\lim_{\epsilon_j\to 0}\int_0^\infty <\alpha^k_{\epsilon_j}(t),\psi^k_{(p,p^\epsilon)}(t)>_{L^2(M,\bigwedge^k)}dt = \int_0^\infty <\alpha^k_0(t),\psi^k(t)>_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}dt\,.$$

*Proof.* We denote by

$$\mathcal{F}^\epsilon(\psi^k) := \int_0^\infty <\alpha^k_\epsilon(t),\psi^k_{(p,p^\epsilon)}(t)>_{L^2(M,\bigwedge^k)}dt,$$

and since $(\alpha^k_\epsilon)$ is a bounded sequence there exists a positive real $c$ such that

$$\|\alpha^k_\epsilon\|_{L^2([0,+\infty),L^2(M,\bigwedge^k))} \leqslant c.$$

We obtain

$$\begin{aligned}|\mathcal{F}^\epsilon(\psi^k)| &\leqslant \int_0^\infty \|\alpha^k_\epsilon\|_{L^2(M,\bigwedge^k)}\|\psi^k_{(p,p^\epsilon)}(t)\|_{L^2(M,\bigwedge^k)}dt \\ &\leqslant c\int_0^\infty\|\psi^k(t)\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}dt\end{aligned}$$

and so $\mathcal{F}^\epsilon \in \Big(L^2([0,+\infty),L^2(M,\bigwedge^k))\Big)'$ and there exists a subsequence such that $\mathcal{F}^{\epsilon_j}$ converges to $\mathcal{F}^0 \in \Big(L^2([0,+\infty),L^2(M,\bigwedge^k))\Big)'$. Moreover,

$$\|\psi^k_{(p,p^\epsilon)}(t)\|_{L^2(M,\bigwedge^k)} \leqslant \|\psi^k(t)\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))},$$

using the dominate convergence theorem, so we deduce

$$\begin{aligned}\mathcal{F}^0(\psi^k)=\lim_{\epsilon_j\to 0}|\mathcal{F}^\epsilon(\psi^k)| &\leqslant c\int_0^\infty \lim_{\epsilon_j\to 0}\sup\|\psi^k_{(p,p^\epsilon)}(t)\|_{L^2(M,\bigwedge^k)}\\ &= c\int_0^\infty \|\psi^k(t)\|_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}.\end{aligned}$$

With the help of Riesz theorem, we see that we have

$$\alpha_0^k\in L^2([0,+\infty),L^2(M,\bigwedge{}^k L^2(Y,\bigwedge{}^0))),$$

such that

$$\mathcal{F}^0(\psi^k)=\int_0^\infty <\alpha_0^k(t),\psi^k(t)>_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}\,dt,$$

for all $\psi^k\in L^2([0,+\infty),C_c^2(M,\bigwedge^k\Omega^0(Y)))$ and where

$$\begin{aligned}\lim_{\epsilon_j\to 0}\int_0^\infty <\alpha^k_{\epsilon_j}(t),\psi^k_{(p,p^\epsilon)}(t)>_{L^2(M,\bigwedge^k)}\,dt &= \lim_{\epsilon_j\to 0}\mathcal{F}^{\epsilon_j}(\psi^k)\\ &= \mathcal{F}^0(\psi^k)\\ &=\int_0^\infty <\alpha_0^k(t),\psi^k(t)>_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))}\,dt.\end{aligned}$$

□

In the following, we can not apply directly the previous theorem because we want the two-scale convergence in time and not in space as E. Frénod and E. Sonnendrücker did for Vlasov-Poisson equations [8, 5]. So we must adapt this theorem. By hypotheses, $Y$ is a compact, symmetric with the same dimension of the time manifold i.e with dimension 1. That says that $Y$ must be correspond to a ring $S^1$. And so we obtain the following theorem:

**Theorem 4.3.** *For $(\alpha^k_\epsilon)$ a bounded sequence in $L^2([0,T),\bigwedge^0 L^2(M,\bigwedge^k))$, where $T$ can be equal to $+\infty$. There exists a subsequence $(\alpha^k_{\epsilon_j})$ of $(\alpha^k_\epsilon)$ and $\alpha_0^k\in L^2([0,T)\times S^1,\bigwedge^0 L^2(M,\bigwedge^k))$, such that for all differential forms $\psi^k\in L^2([0,T)\times S^1,\bigwedge^0 C^2_c(M,\bigwedge^k))$ we have*

$$\lim_{\epsilon_j\to 0}\int_0^T <\alpha^k_{\epsilon_j}(t),\psi^k(t,t^\epsilon)>_{L^2(M,\bigwedge^k)}\,dt=\int_0^T\int_{S^1}<\alpha_0^k(t,s),\psi^k(t,s)>_{L^2(M,\bigwedge^k)}\,ds\,dt\,,$$

*with $t^\epsilon=\exp_t^{S^1}(\frac{1}{\epsilon}j(v))$ for $v\in T_t[0,T)$, and $j$ an isomorphism such that there exists $q_0\in S^1$ with $T_t[0,T)\overset{j}{\cong}T_{q_0}S^1$.*

For the classic two-scale convergence, we also have a theorem on the derivative of a sequence of differential $k$-forms which it can also adapt for the two-scale convergence.

**Proposition 3.** *For a bounded sequence $(\alpha^k_\epsilon)\in L^2(M,\bigwedge^k)$ such that $(d\alpha^k_\epsilon)$ is also bounded in $L^2(M,\bigwedge^{k+1})$, there exists a subsequence $(\alpha^k_{\epsilon_j})$ of $(\alpha^k_\epsilon)$ such that $(\alpha^k_{\epsilon_j})$ two-scale converges to $\alpha^k_0$ in $L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))$ and $(d\alpha^k_{\epsilon_j})$ two-scale converges to $d\alpha^k_0+d_Y\alpha^{k+1}_1$ in $L^2(M,\bigwedge^{k+1}L^2(Y,\bigwedge^0))$ for $\alpha^k_0$ in $L^2(M,\bigwedge^k H^{1,d}(Y,\bigwedge^0))$ (see (4) page 7 ) and $\alpha^{k+1}_1$ in $L^2(M,\bigwedge^{k+1}H^{1,d}(Y,\bigwedge^0))$. Moreover, $\alpha^k_0$ is in ker$(d_Y)$ (the kernel of the exterior derivative on $Y$, $d_Y$).*

*Proof.* Let $(\alpha^k_{\epsilon_j})$ be a subsequence of $(\alpha^k_\epsilon)$ and $(d\alpha^k_{\epsilon_j})$ a subsequence of $(d\alpha^k_\epsilon)$ such that

$$\begin{array}{rcl} <\alpha^k_{\epsilon_j}, \psi^k_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)} & \longrightarrow & <\alpha^k_0, \psi^k >_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} \\ <d\alpha^k_{\epsilon_j}, \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^{k+1})} & \longrightarrow & <\eta^{k+1}, \phi^{k+1} >_{L^2(M,\bigwedge^{k+1} L^2(Y,\bigwedge^0))} . \end{array}$$

with $\psi^k \in C^2_c(M, \bigwedge^k C^2(Y, \bigwedge^0)), \phi^{k+1} \in C^2_c(M, \bigwedge^{k+1} C^2(Y, \bigwedge^0))$, and $\alpha^k_0 \in L^2(M, \bigwedge^k L^2(Y, \bigwedge^0))$ and $\eta^{k+1} \in L^2(M, \bigwedge^{k+1} L^2(Y, \bigwedge^0))$. Since

$$\begin{array}{rcl} <d\alpha^k_{\epsilon_j}, \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^{k+1})} & = & <\alpha^k_{\epsilon_j}, d^\star_M \phi^{k+1}_{(p,p_\epsilon)} >_{L^2(M,\bigwedge^k)} \\ & & +\dfrac{1}{\epsilon} <\alpha^k_{\epsilon_j}, d^\star_Y \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)}, \end{array}$$

where $d^\star_M$ et $d^\star_Y$ corresponds respectively to the co-exterior derivative on $M$ and on $Y$. So, we have

$$\begin{array}{rcl} <\alpha^k_{\epsilon_j}, d^\star_M \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)} & = & <d\alpha^k_{\epsilon_j}, \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^{k+1})} \\ & & -\dfrac{1}{\epsilon} <\alpha^k_{\epsilon_j}, d^\star_Y \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)} . \end{array}$$

But for $\phi^{k+1} \in C^2_c(M, \bigwedge^{k+1} C^2(Y, \bigwedge^0))$ such that $d^\star_Y \phi^{k+1} = 0$, we have

$$<\alpha^k_{\epsilon_j}, d^\star_Y \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)} = 0.$$

We can deduce that

$$\begin{array}{rcl} <\alpha^k_0, d^\star_M \phi^{k+1} >_{L^2(M,\bigwedge^k L^2(Y,\bigwedge^0))} & = & \lim\limits_{\epsilon_j\to 0} <\alpha^k_{\epsilon_j}, d^\star_M \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^k)} \\ & = & \lim\limits_{\epsilon_j\to 0} <d\alpha^k_{\epsilon_j}, \phi^{k+1}_{(p,p^\epsilon)} >_{L^2(M,\bigwedge^{k+1})} \\ & = & <\eta^{k+1}, \phi^{k+1} >_{L^2(M,\bigwedge^{k+1} L^2(Y,\bigwedge^0))}, \end{array}$$

for all $\phi^{k+1} \in C^2_c(M, \bigwedge^{k+1} C^2(Y, \bigwedge^0))$ such that $d^\star_Y \phi^{k+1} = 0$, so

$$<\eta^{k+1} - d\alpha^k_0, \phi^{k+1} >_{L^2(M,\bigwedge^{k+1} L^2(Y,\bigwedge^0))} = 0,$$

because $d_Y \alpha^k_0 = 0$. So, since for $\alpha^{k+1}_1 \in L^2(M, \bigwedge^{k+1} H^{1,d}(Y, \bigwedge^0))$ we have

$$<d_Y \alpha^{k+1}_1, \phi^{k+1} >_{L^2(M,\bigwedge^{k+1} L^2(Y,\bigwedge^0))} = 0,$$

so there exists $\alpha^{k+1}_1 \in L^2(M, \bigwedge^{k+1} H^{1,d}(Y, \bigwedge^0))$ as

$$\eta - d\alpha^0 = d_Y \alpha^{k+1}_1.$$

□

## 5. Examples on Vlasov equation.

**5.1. With a strong magnetic field.** To apply the two-scale convergence to Vlasov equation, we use the articles [8, 5]. For $M$ a 3-dimensional space, the phase space decoupled of time has the form

$$P = T^\star M = \{(x, v) | x \in M, v \in T^\star_x M\}.$$

In this space the Vlasov equation reads

$$\frac{\partial f^6_\epsilon}{\partial t}(t) + \mathcal{L}_{\mathcal{T}^\epsilon} f^6_\epsilon(t) = 0\,,$$

and

$$f^6_\epsilon(t=0) = f_0,$$

with $\tau^\epsilon$ a vector field equal to

$$v\frac{\partial}{\partial x} + \left(i_{\frac{\partial}{\partial x}}\mathbf{E}^1_\epsilon + i_{\frac{\partial}{\partial x}} i_{v\frac{\partial}{\partial x}}\left(\mathbf{B}^2_\epsilon + \frac{\mathcal{M}}{\epsilon}\right)\right)\frac{\partial}{\partial v},$$

where $\mathcal{M}$ is a constant differential 2-form, $f^6_\epsilon(t)$ a differential volume form i.e a differential 6-form on $P$, more precisely $f^6_\epsilon \in L^2([0,T), \bigwedge^0 L^2(P, \bigwedge^6))$. Moreover, we suppose that

- $f^6_\epsilon(0)$ is bounded in $L^2(P, \bigwedge^6)$,

$$\tag{6}$$

- $\mathbf{E}^1_\epsilon \in L^2([0,T), \bigwedge^0 L^2(M, \bigwedge 1))$ strongly converges to
$$\mathbf{E}^1 \text{ in } L^2([0,T), \bigwedge{}^0 L^2(M, \bigwedge{}^1)), \tag{7}$$
- $\mathbf{B}^2_\epsilon \in L^2([0,T), \bigwedge^0 L^2(M, \bigwedge^2))$ strongly converges to
$$\mathbf{B}^2 \text{ in } L^2([0,T), \bigwedge{}^0 L^2(M, \bigwedge{}^2)). \tag{8}$$

With these assumptions, we have the conservation of the norm of $f^6_\epsilon$ in time, i.e there exists a constant $c \geqslant 0$ such that

$$\| f^6_\epsilon \|_{L^2([0,T), \bigwedge^0 L^2(P, \bigwedge^6))} \leqslant c.$$

To prove this, we must do the wedge product between $\star f^6_\epsilon$ and the Vlasov equation and then we integrate it over $P$. We obtain that the time derivative of the norm of $f^6_\epsilon$ in $L^2([0,T), \bigwedge^0 L^2(P, \bigwedge^6))$ is equal to 0. This result allows to apply theorem 4.3 page 14: there exists a subsequence of $f^6_\epsilon$, (still denoted by $f^6_\epsilon$) and a differential 6-form $F^6 \in L^2([0,T) \times S^1, \bigwedge^0 L^2(P, \bigwedge^6))$ such that for all $\psi^6 \in L^2([0,T) \times S^1, \bigwedge^0 C^2_c(P, \bigwedge^6))$, we have

$$\lim_{\epsilon \to 0} < f^6_\epsilon, \psi^6 >_{L^2([0,T), \bigwedge^0 L^2(P, \bigwedge^6))} dt =< F^6, \psi^6 >_{L^2([0,T)\times S^1, \bigwedge^0 L^2(P, \bigwedge^6))} .$$

After the calculation of the scalar product between the Vlasov equation and $(\psi^6_\epsilon)_q = \psi^6_q(t, t^\epsilon)$ on the space $[0,T) \times P$ ($t^\epsilon$ is described page 14) we deduce that

$$< f^6_\epsilon, \frac{\partial \psi^6_\epsilon}{\partial t} + \frac{1}{\epsilon}\frac{\partial \psi^6_\epsilon}{\partial s} + \mathcal{L}_{\tau^\epsilon}\psi^6_\epsilon >_{L^2([0,T), \bigwedge^0 L^2(P, \bigwedge^6))} = - < f^6_\epsilon(0), \psi^6_\epsilon(0) >_{L^2(P, \bigwedge^6)} . \tag{9}$$

To obtain the two-scale limit, we multiply equation (9) by $\epsilon$, then passing to the limit

$$< {}^6F, \frac{\partial \psi^6}{\partial s} + \mathcal{L}_{\left(i_{\frac{\partial}{\partial x}} i_{v\frac{\partial}{\partial x}}\mathcal{M}\right)\frac{\partial}{\partial v}} \psi^6 >_{L^2([0,T)\times S^1, \bigwedge^0 L^2(P, \bigwedge^6))} = 0,$$

with $F^6 \in L^2([0,T) \times S^1, \bigwedge^0 L^2(P, \bigwedge^6))$. That is why we have

$$\frac{\partial F^6}{\partial s} + \mathcal{L}_{\left(i_{\frac{\partial}{\partial x}} i_{v\frac{\partial}{\partial x}}\mathcal{M}\right)\frac{\partial}{\partial v}} F^6 = 0 \text{ in } \left(L^2([0,T) \times S^1, \bigwedge{}^0 L^2(P, \bigwedge{}^6))\right)'. \tag{10}$$

This means that $F^6$ is a constant along the characteristics. The characteristics are helices around the magnetic field $\mathcal{M}$. A transformation $\varphi$ which keeps invariant the projection of the velocity $v$ on $\mathcal{M}$ and makes a rotation with an angle $s$ for the projection on the orthogonal plan to $\mathcal{M}$ writes:

$$\varphi(v,s)=\begin{pmatrix} v_1 \\ v_2\cos(s)-v_3\sin(s) \\ v_2\sin(s)+v_3\cos(s)\end{pmatrix}.$$

Taking into account the periodicity condition, Frénod and Sonnendrücker [12] show the following lemma

**Lemma 5.1.** *$F^6\in L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))$ satisfies*

$$\frac{\partial F^6}{\partial s}+\mathcal{L}_{\left(i_{\frac{\partial}{\partial x}}i_{v\frac{\partial}{\partial x}}\mathcal{M}\right)\frac{\partial}{\partial v}}F^6=0$$

*in $\left(L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))\right)'$, if and only if there exists*

$$G^6\in L^2([0,T),\bigwedge{}^0 L^2(P,\bigwedge{}^6))$$

*such that $F_q^6(t,s)=(\varphi^\star G^6)_q(t,s)$.*

We know that $F^6\in L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))$ and $G^6\in L^2([0,T),\bigwedge^0 L^2(P,\bigwedge^6))$ verify the previous lemma. Now, we will set out the following well-posed problem for $G^6$:

$$\frac{\partial G^6}{\partial t}+\mathcal{L}_{u_\parallel\frac{\partial}{\partial x}+\left(i_{\frac{\partial}{\partial x}}\mathbf{E}_\parallel^1+i_{\frac{\partial}{\partial x}}i_{u\frac{\partial}{\partial x}}\mathbf{B}_\parallel^2\right)\frac{\partial}{\partial u}}G^6=0, \tag{11}$$

with $G^6(0)=\frac{F^6(0)}{vol(S^1)}=\frac{F^6(0)}{2\pi}$ and $u_\parallel=u_1\mathbf{e^1}$ the projection in the direction of the magnetic field $\mathcal{M}$. The solution of this equation is unique.
To obtain (11), for every $\phi^6\in L^2([0,T),\bigwedge^0 L^2(P,\bigwedge^6))$ we define $\psi^6\in L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))$ by $\psi_q^6(t,s)=(\varphi^\star\phi^6)_q(t,s)$. $\psi^6$ satisfied

$$\frac{\partial\,\psi^6}{\partial s}+\mathcal{L}_{\left(i_{\frac{\partial}{\partial x}}i_{v\frac{\partial}{\partial x}}\mathcal{M}\right)\frac{\partial}{\partial v}}\psi^6=0.$$

For a 6-form test $\psi_\epsilon^6$ and using the formula (9) page 16, we can do the two-scale limit using the assumptions that $\mathbf{E}_\epsilon^1$ and $\mathbf{B}_\epsilon^2$ strongly converge and so:

$$<F^6,\frac{\partial\,\psi^6}{\partial t}+\mathcal{L}_\tau\,\psi^6>_{L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))}=-<F^6(0),\psi^6(0,0,\cdot,\cdot)>_{L^2(P,\bigwedge^6)}, \tag{12}$$

with

$$\tau=v\frac{\partial}{\partial x}+\left(i_{\frac{\partial}{\partial x}}\mathbf{E}^1+i_{\frac{\partial}{\partial x}}i_{v\frac{\partial}{\partial x}}\mathbf{B}^2\right)\frac{\partial}{\partial v}.$$

Using the lemma 5.1 and since

$$\mathcal{L}_\tau\varphi^\star(\phi^6)=\varphi^\star(\mathcal{L}_{(\varphi^\star)^{-1}(\tau)}\phi^6),$$

and

$$\frac{\partial\varphi^\star(\phi^6)}{\partial t}=\varphi^\star(\frac{\partial\phi^6}{\partial t}),$$

we can simplify (12) by

$$<G^6,\frac{\partial\,\phi^6}{\partial t}+\mathcal{L}_{(\varphi^\star)^{-1}(\tau)}\,\phi^6>_{L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))}=-<G^6(0),\phi^6(0,0,\cdot,\cdot)>_{L^2(P,\bigwedge^6)},$$

with
$$(\varphi^\star)^{-1}(\tau)=\tilde{u}\frac{\partial}{\partial x}+\left(i_{\frac{\partial}{\partial x}}\,\mathbf{E}^1+i_{\frac{\partial}{\partial x}}i_{\tilde{u}\frac{\partial}{\partial x}}\,\mathbf{B}^2\right)\frac{\partial}{\partial\tilde{u}},$$
and where
$$\tilde{u}=\left(\begin{array}{c}u_1\\ u_2\cos(s)+u_3\sin(s)\\ -u_2\sin(s)+u_3\cos(s)\end{array}\right)\text{ and }\frac{\partial}{\partial\tilde{u}}=\left(\begin{array}{c}\frac{\partial}{\partial u_1}\\ \cos(s)\frac{\partial}{\partial u_2}+\sin(s)\frac{\partial}{\partial u_3}\\ -\sin(s)\frac{\partial}{\partial u_2}+\cos(s)\frac{\partial}{\partial u_3}\end{array}\right).$$
Moreover, $G^6$ and $\phi^6$ do not depend of $s$ so we can integrate it over $S^1$ and we obtain
$$<G^6,\frac{\partial\,\phi^6}{\partial t}+\mathcal{L}_{\tilde{\tau}}\,\phi^6>_{L^2([0,T),\bigwedge^0L^2(P,\bigwedge^6))}=-\frac{1}{2\pi}<G^6(0),\phi^6(0,0,\cdot,\cdot)>_{L^2(P,\bigwedge^6)},$$
with
$$\tilde{\tau}=u_1\cdot\frac{\partial}{\partial x}+\left((i_{\frac{\partial}{\partial x}}\,\mathbf{E}^1)_1+i_{\frac{\partial}{\partial x}}i_{\tilde{u}\frac{\partial}{\partial x}}(B_xdy\wedge dz)\right)\frac{\partial}{\partial u}.$$
So we deduce the following theorem

**Theorem 5.2.** *Under the assumptions (6-7-8), we have that*
$$f_\epsilon^6\textit{ two-scale converges to }F^6\in L^2([0,T)\times S^1,\bigwedge{}^0L^2(P,\bigwedge{}^6)).$$
*Moreover there exists* $G^6\in L^2([0,T),\bigwedge^0L^2(P,\bigwedge^6))$ *such that* $F^6_q(t,s)=(\varphi^\star\,G^6)_q(t,s)$ *and* $G^6$ *verify the following equation*
$$\frac{\partial\,G^6}{\partial t}+\mathcal{L}_{\tilde{\tau}}\,G^6=0,$$
*where*
$$\tilde{\tau}=u_1\cdot\frac{\partial}{\partial x}+\left((i_{\frac{\partial}{\partial x}}\,\mathbf{E}^1)_1+i_{\frac{\partial}{\partial x}}i_{\tilde{u}\frac{\partial}{\partial x}}(B_xdy\wedge dz)\right)\frac{\partial}{\partial u},$$
*and with*
$$G^6(0)=\frac{F^6(0)}{vol(S^1)}=\frac{F^6(0)}{2\pi}$$

5.2. **With strong magnetic and electric field.** The Vlasov equation with a strong magnetic and electric field has the form:
$$\frac{\partial\,f_\epsilon^6}{\partial t}(t)+\mathcal{L}_{\tau^\epsilon}\,f_\epsilon^6(t)=0,$$
and
$$f_\epsilon^6(t=0)=f_0,$$
with $\tau^\epsilon$ the vector field is equal to
$$v\frac{\partial}{\partial x}+i_{\frac{\partial}{\partial x}}\left(\mathbf{E}^1_\epsilon+\frac{\mathcal{N}}{\epsilon}+i_{v\frac{\partial}{\partial x}}\left(\mathbf{B}^2_\epsilon+\frac{\mathcal{M}}{\epsilon}\right)\right)\frac{\partial}{\partial v},$$
where $\mathcal{M}$ is a constant differential 2-form, $\mathcal{N}$ a constant differential 1-form and $f_\epsilon^6(t)$ a volume form on $P$, more precisely
$$f_\epsilon^6\in L^2([0,T),\bigwedge{}^0L^2(P,\bigwedge{}^6)).$$
We suppose the same initial conditions than in the previous section.
In the same way, we have the existence of a subsequence of $f_\epsilon^6$, also denoted by $f_\epsilon^6$ and a differential 6-form $F^6\in L^2([0,T)\times S^1,\bigwedge^0L^2(P,\bigwedge^6))$ such that for all $\psi^6\in L^2([0,T)\times S^1,\bigwedge^0C^2_c(P,\bigwedge^6))$, we have
$$\lim_{\epsilon\to0}<f_\epsilon^6,\psi^6>_{L^2([0,T),\bigwedge^0L^2(P,\bigwedge^6))}\,dt=<F^6,\psi^6>_{L^2([0,T)\times S^1,\bigwedge^0L^2(P,\bigwedge^6))}.$$

The scalar product between the Vlasov equation and $(\psi_\epsilon^6)_q = \psi_q^6(t,t^\epsilon)$, the multiplication by $\epsilon$ and then the passage to the limit give the following constraint equation:

$$\frac{\partial F^6}{\partial s} + \mathcal{L}_{i_{\frac{\partial}{\partial x}}\left(\mathcal{N}+i_{v\frac{\partial}{\partial x}}\mathcal{M}\right)\frac{\partial}{\partial v}} F^6 = 0$$

in $\left(L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))\right)'$. This equation means that $F^6$ is a constant along the characteristics :

$$\frac{dV}{ds} = i_{\frac{\partial}{\partial x}}\mathcal{N} + i_{\frac{\partial}{\partial x}} i_{v\frac{\partial}{\partial x}}\mathcal{M}.$$

So for a transformation

$$\varphi(v,s) = \begin{pmatrix} v_1 \\ v_2\cos(s) - (v_3+1)\sin(s) \\ v_2\sin(s) + (v_3+1)\cos(s) - 1 \end{pmatrix},$$

there exists $G^6 \in L^2([0,T),\bigwedge^0 L^2(P,\bigwedge^6))$ such that $F_q^6(t,s) = (\varphi^\star G^6)_q(t,s)$. Moreover for a differential 6-form $\psi_\epsilon^6$, we do the two-scale limit. Then, since $\mathbf{E}_\epsilon^1$ and $\mathbf{B}_\epsilon^2$ strongly converge, we deduce that:

$$< F^6, \frac{\partial \psi^6}{\partial t} + \mathcal{L}_\tau \psi^6 >_{L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))} = - < F^6(0), \psi^6(0,0,\cdot,\cdot) >_{L^2(P,\bigwedge^6)},$$

with

$$\tau = v\frac{\partial}{\partial x} + \left(i_{\frac{\partial}{\partial x}}\mathbf{E}^1 + i_{\frac{\partial}{\partial x}} i_{v\frac{\partial}{\partial x}}\mathbf{B}^2\right)\frac{\partial}{\partial v}.$$

To obtain the equation satisfied by $G^6$, we use the previous equality and we inject the expression of ${}^6F$ and of ${}^6\psi$ and so we obtain

$$\begin{aligned} < \varphi^\star(G^6), \frac{\partial \varphi^\star(\phi^6)}{\partial t} + \mathcal{L}_\tau \varphi^\star(\phi^6) >_{L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))} = \\ - < \varphi^\star(G^6(0)), \varphi^\star(\phi^6)(0,0,\cdot,\cdot) >_{L^2(P,\bigwedge^6)} . \end{aligned}$$

That gives us

$$< G^6, \frac{\partial \phi^6}{\partial t} + \mathcal{L}_{(\varphi^\star)^{-1}(\tau)}\phi^6 >_{L^2([0,T)\times S^1,\bigwedge^0 L^2(P,\bigwedge^6))} = - < G^6(0), \phi^6(0,0,\cdot,\cdot) >_{L^2(P,\bigwedge^6)},$$

with

$$(\varphi^\star)^{-1}(\tau) = \tilde{u}\frac{\partial}{\partial x} + \left(i_{\frac{\partial}{\partial x_1}}\mathbf{E}^1 - i_{\frac{\partial}{\partial x_1}} i_{\frac{\partial}{\partial x_3}}\mathbf{B}^2 + i_{\frac{\partial}{\partial x}} i_{\tilde{u}\frac{\partial}{\partial x}}\mathbf{B}_1^2 dy\wedge dz\right)\frac{\partial}{\partial \tilde{u}},$$

where $\tilde{u} = \begin{pmatrix} u_1 \\ 0 \\ -1 \end{pmatrix}$ and

$$\tilde{u}\frac{\partial}{\partial x} = u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + (u_3+1)\frac{\partial}{\partial u_3},$$

and

$$G^6(0) = \frac{1}{2\pi}F^6(0).$$

So we deduce the following theorem

**Theorem 5.3.** *Under the assumptions (6-7-8), we have that*

$$f_\epsilon^6 \text{ two-scale converges to } F^6 \in L^2([0,T)\times S^1, \bigwedge{}^0 L^2(P,\bigwedge{}^6)).$$

*Moreover there exists* $G^6 \in L^2([0,T), \bigwedge^0 L^2(P,\bigwedge^6))$ *such that* $F_q^6(t,s) = (\varphi^\star G^6)_q(t,s)$ *and* $G^6$ *verify the following equation*

$$\frac{\partial G^6}{\partial t} + \mathcal{L}_{(\varphi^\star)^{-1}(\tau)}, G^6 = 0,$$

*and with*

$$G^6(0) = \frac{F^6(0)}{vol(S^1)} = \frac{F^6(0)}{2\pi}$$

For $\mathcal{M} = dy\wedge dz$ and $\mathcal{N} = dy$, we notice that $-i_{\frac{\partial}{\partial y}\mathcal{N}\frac{\partial}{\partial y}}\mathcal{M} = -dz$ is a drift effect and this term is found in the centre-guide approximation when we use the same hypotheses on the magnetic and electric field.

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