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FIRST ORDER TWO-SCALE PARTICLE-IN-CELL NUMERICAL METHOD FOR VLASOV EQUATION

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Abstract. The aim of this work is to build an accurate numerical method for the simulation of the long time evolution of the Vlasov solution f^ε with an electric field $E^\varepsilon = E_0+\varepsilon E_1$ for small ε. To this purpose, we use the Two-Scale Convergence theory to determine a first order approximation $F+\varepsilon F_1$ of f^ε, then particle approximations to build an algorithm to obtain a numerical approximation of $F+\varepsilon F_1$.

Résumé. On cherche à construire une méthode numérique pour l'évolution en temps long de la solution f^ε de l'équation de Vlasov avec un champ électrique $E^\varepsilon = E_0+\varepsilon E_1$ pour ε petit. À cet effet, on utilise la théorie de la convergence à deux échelles pour obtenir une approximation d'ordre un $F+\varepsilon F_1$ de f^ε, puis une méthode particulaire pour construire l'algorithme d'approximation numérique de $F+\varepsilon F_1$.

Contents

1. Introduction

In this paper, we are interested in the solution f^ε to the following Vlasov problem set in a bi-dimensional phase space

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \dfrac{v}{\varepsilon}\dfrac{\partial f^\varepsilon}{\partial r} + \Big(E^\varepsilon - \dfrac{r}{\varepsilon}\Big)\dfrac{\partial f^\varepsilon}{\partial v} = 0, \\ f^\varepsilon(t=0,r,v) = f_0(r,v). \end{cases} \tag{1.1}$$

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where $f^\varepsilon : (t,r,v) \mapsto f^\varepsilon(t,r,v)$, for $t \in [0,T)$, $r \in \mathbb{R}^+$ and $v \in \mathbb{R}$. In (1.1), let

$$E^\varepsilon(t,r) = E_0\Big(t, \frac{t}{\varepsilon}, r\Big) + \varepsilon\, E_1\Big(t, \frac{t}{\varepsilon}, r\Big), \tag{1.2}$$

where $E_{i=0,1} : (t,\tau,r) \mapsto E_{i=0,1}(t,\tau,r)$ are regular and 2π-periodic with respect to τ. Our purpose is to build a numerical method which is efficient for simulating the long time evolution of f^ε in the limit $\varepsilon \to 0$.

In Section 2, we use Two-Scale Convergence theory to determine a first-order approximation of f^ε, that is $f^\varepsilon(t,r,v) \approx F(t,t/\varepsilon,r,v) + \varepsilon\, F_1(t,t/\varepsilon,r,v)$ where $F : (t,\tau,r,v) \mapsto F(t,\tau,r,v)$ and $F_1 : (t,\tau,r,v) \mapsto F(t,\tau,r,v)$ are 2π-periodic functions with respect to $\tau \in \mathbb{R}$.

We first remember in Section 2.1 that, for the zero order term, there exists a function $G : (t,q,u) \mapsto G(t,q,u)$ solution of an initial boundary condition partial differential problem such that $F(t,\tau,r,v) = G(t,\mathcal{R}^\tau(r,v))$, where $\mathcal{R}^\tau$ is the 2D-rotation of angle τ. In Section 2.2, we handle first order term and show that there exist two functions $G_1 : (t,q,u) \mapsto G_1(t,q,u)$ and $W : (t,\tau,q,u) \mapsto W(t,\tau,q,u)$ such that $F_1(t,\tau,r,v) = G_1(t,\mathcal{R}^\tau(r,v)) + W(t,\tau,\mathcal{R}^\tau(r,v))$ where W can be computed from $\nabla_{q,u}G$ and G_1 is yet the solution of an initial boundary condition partial differential problem.

In Section 3, we introduce a Particle-in-Cell based approximation to build a numerical algorithm which will allow us to solve numerically the problems satisfied by G and G_1 as well as to compute a numerical approximation of W.

2. On Two-Scale Convergence statements. Main Result

The concept of Two-Scale Convergence was introduced at the end of the 80's by Nguetseng [?, ?]. In 1992, Allaire gave a very understandable proof of this result [?]. Then several authors used this theory to build numerical methods called Two-Scale Numerical Methods (e.g. Frénod and Sonnendrücker [?], Frénod, Salvarani and Sonnendrücker [?], or Mouton [?]). For more details about Two-Scale Convergence theory, we refer to these authors and their references.

Let us here only recall the definition of the Two-Scale Convergence.

Definition 2.1. *A function $f^\varepsilon : (t,r,v) \in [0,T) \times \mathbb{R}^2 \mapsto f^\varepsilon(t,r,v)$ Two-Scale converges as $\varepsilon \to 0$ to a function $F : (t,\tau,r,v) \in [0,T) \times \mathbb{R} \times \mathbb{R}^2 \mapsto F(t,\tau,r,v)$ 2π-periodic in τ, if:*

$$\int_0^T\!\!\int_{\mathbb{R}^2} f^\varepsilon(t,r,v)\,[\phi]^\varepsilon(t,r,v)\,dr\,dv\,dt \to \int_0^T\!\!\int_0^{2\pi}\!\!\int_{\mathbb{R}^2} F(t,\tau,r,v)\,\phi(t,\tau,r,v)\,dr\,dv\,d\tau\,dt, \tag{2.1}$$

as $\varepsilon \to 0$, for all $\phi \in \mathcal{C}^0$ where $\mathcal{C}^i := C^i_c\Big([0,T); C^i_\#(\mathbb{R}; C^i_c(\mathbb{R}^2))\Big)$ for $i = 0,1$.

Remark 2.1 (Notations)**.** *In the above definition and in the following of the paper, we introduce the following general notations:*

(i) Subscript # in space definition stands for 2π-periodicity in τ.
(ii) Bracket $[\cdot]^\varepsilon$ stands for $[\phi]^\varepsilon(t,r,v) := \phi(t,t/\varepsilon,r,v)$ for any function ϕ depending of the four variables $(t,\tau,r,v) \in [0,T) \times \mathbb{R} \times \mathbb{R}^2$.

Moreover, in the whole section we will use the 2D-rotation of angle τ of a two-dimensional vector.

Definition 2.2. *For all $\tau \in [0,2\pi]$ and all $(r,v) \in \mathbb{R}^2$, we define the 2D-rotation $\mathcal{R}^\tau$ of angle τ by:*

$$\mathcal{R}^\tau(r,v) = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} r\,\cos\tau - v\,\sin\tau \\ r\,\sin\tau + v\,\cos\tau \end{pmatrix}. \tag{2.2}$$

Its inverse in then given by $\mathcal{R}^{-\tau}$ whose first component is denoted by $\mathcal{R}^{-\tau}_r$.

Remark 2.2. *Thanks to the above definition, we have that :*

$$\mathcal{R}^{\tau}(r,v)\cdot\begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} r \\ v \end{pmatrix}\cdot\mathcal{R}^{-\tau}(q,u). \tag{2.3}$$

Theorem 2.1. *Let $(f^{\varepsilon})_{\varepsilon}$ be a sequence of solutions to Vlasov problem (1.1) with the electric field E^{ε} given by (1.2). Therefore there exist two functions $F:(t,\tau,r,v)\in[0,T)\times\mathbb{R}\times\mathbb{R}^2\mapsto F(t,\tau,r,v)$ and $F_1:(t,\tau,r,v)\in[0,T)\times\mathbb{R}\times\mathbb{R}^2\mapsto F(t,\tau,r,v)$ 2π-periodic with respect to $\tau\in\mathbb{R}$ such that*

$$f^{\varepsilon}(t,r,v)\approx F\left(t,\frac{t}{\varepsilon},r,v\right)+\varepsilon\,F_1\left(t,\frac{t}{\varepsilon},r,v\right), \tag{2.4}$$

Moreover, regarding the zero order term, there exists a function $G:(t,q,u)\in[0,T)\times\mathbb{R}^2\mapsto G(t,q,u)$ such that

$$F(t,\tau,r,v)=G(t,\mathcal{R}^{\tau}(r,v)), \tag{2.5}$$

where $\mathcal{R}^{\tau}$ is the rotation defined in Definition 2.2 and G is solution to the problem

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,q,u)+\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mathcal{R}^{\tau}\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)d\tau\cdot\nabla_{q,u}G(t,q,u)=0, \\ G(t=0,q,u)=\dfrac{1}{2\pi}f_0(q,u). \end{cases} \tag{2.6}$$

Regarding the first order term, there exist two functions $G_1:(t,q,u)\in[0,T)\times\mathbb{R}^2\mapsto G_1(t,q,u)$ and $W:(t,\tau,q,u)\in[0,T)\times\mathbb{R}\times\mathbb{R}^2\mapsto W(t,\tau,q,u)$ such that

$$F_1(t,\tau,r,v)=G_1(t,\mathcal{R}^{\tau}(r,v))+W(t,\tau,\mathcal{R}^{\tau}(r,v)), \tag{2.7}$$

where on the one hand, W is given by

$$\begin{aligned} W(t,\tau,q,u) \;=\; & \bigg[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma \\ & -\int_0^{\tau}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\bigg]\cdot\nabla_{q,u}G(t,q,u), \end{aligned} \tag{2.8}$$

and on the other hand, G_1 is solution to the problem

$$\begin{cases} \dfrac{\partial G_1}{\partial t}(t,q,u)+\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mathcal{R}^{\tau}\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)d\tau\cdot\nabla_{q,u}G_1(t,q,u)= \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\bigg[\int_0^{\tau}\mathcal{R}^{\sigma}\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma \\ \qquad -\dfrac{\tau}{2\pi}\displaystyle\int_0^{2\pi}\mathcal{R}^{\sigma}\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\bigg]d\tau\cdot\nabla_{q,u}G(t,q,u) \\ +\bigg[\dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\int_0^{2\pi}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma \\ \quad -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\Big(\nabla_{q,u}\mathcal{R}^{\tau}\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\int_0^{\tau}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\Big)d\tau\bigg] \\ \qquad\qquad \cdot\nabla_{q,u}G(t,q,u)\,d\tau, \\ G_1(t=0,q,u)=0. \end{cases} \tag{2.9}$$

Remark 2.3. *Note that the expression of* W *in* (2.8) *as well as problems* (2.6) *and* (2.9) *do not depend on* $1/\varepsilon$*-frequency oscillations any more. The proof of Theorem 2.1 is given in the two following sections.*

2.1. **Zero order approximation**

Here, we briefly recall arguments developped in [**?**]. Let us consider the Vlasov equation for f^ε in problem (1.1). First, we multiply by f^ε and integrate over $r \in \mathbb{R}^+$ and $v \in \mathbb{R}$ to obtain:

$$\frac{1}{2}\frac{d\left(\|f^\varepsilon(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)}\right)}{dt} = 0, \tag{2.10}$$

which leads to the following estimate:

$$\|f^\varepsilon\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} \leq C. \tag{2.11}$$

Then it is known that, up to a subsequence, $(f^\varepsilon)_\varepsilon$ Two-Scale converges to some $F : (t,\tau,r,v) \mapsto F(t,\tau,r,v)$ with $F \in L^\infty\left([0,T); L^\infty_\#(\mathbb{R}, L^2(\mathbb{R}^2))\right)$. Now, we multiply Vlasov equation in (1.1) by $[\phi]^\varepsilon$ with $\phi \in \mathcal{C}^1$, integrate over $t \in [0,T)$ and $(r,v) \in \mathbb{R}^2$ and then integrate by parts. Thus we show that F satisfies in a weak sense the following equation

$$\frac{\partial F}{\partial \tau} + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} F = 0, \tag{2.12}$$

where we note that ε is not involved anymore. From (2.12), we deduce that F is constant along the characteristics associated to this advection equation, and deduce from Theorem 4.2 in [**?**] that there exists a function $G : (t,q,u) \in [0,T) \times \mathbb{R}^2 \mapsto G(t,q,u)$ such that (2.5) is satisfied with G is solution to problem (2.6) of Theorem 2.1.

Remark 2.4. *Let* $G : (t,q,u) \in [0,T) \times \mathbb{R}^2 \mapsto G(t,q,u)$ *be a given function and define the family* $\big(F^\varepsilon\big)_{\varepsilon>0}$ *by*

$$[F]^\varepsilon(t,r,v) = G\big(t, \mathcal{R}^{t/\varepsilon}(r,v)\big). \tag{2.13}$$

Then

(i) $\nabla_{r,v}[F]^\varepsilon(t,r,v) = \mathcal{R}^{-t/\varepsilon}\big(\nabla_{q,u} G\big(t,\mathcal{R}^{t/\varepsilon}(r,v)\big)\big)$.

(ii) $\dfrac{\partial [F]^\varepsilon}{\partial t}(t,r,v) + \dfrac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}[F]^\varepsilon(t,r,v) = \dfrac{\partial G}{\partial t}\big(t,\mathcal{R}^{t/\varepsilon}(r,v)\big)$.

2.2. **First order approximation**

We consider the zero order Two-Scale approximation F of f^ε introduced in section 2.1. Let us define $[F]^\varepsilon$ as in Remark 2.1(ii). Then thanks to Remark 2.4(ii)

$$\frac{\partial [F]^\varepsilon}{\partial t}(t,r,v) + \frac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}[F]^\varepsilon(t,r,v) = \frac{\partial G}{\partial t}\left(t,\mathcal{R}^{t/\varepsilon}(r,v)\right). \tag{2.14}$$

For the sake of simplicity in computations below, we introduce the notation $E^\varepsilon = E_0(t,t/\varepsilon,r) + \varepsilon\, E_1(t,t/\varepsilon,r)$ and omit the dependency of $[F]^\varepsilon$ and its partial derivatives in (t,r,v). First, we add the term $E^\varepsilon\, \partial[F]^\varepsilon/\partial v$ on both sides of equation (2.14) to get (2.15). Then we rewrite the right-hand-side to show up the term $\nabla_{r,v}[F]^\varepsilon$ and get (2.16). Using Remark 2.4(i) then Remark 2.2, we obtain respectively (2.17) then (2.18).

$$\begin{aligned}\frac{\partial [F]^\varepsilon}{\partial t} &+ \frac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix}\cdot\nabla_{r,v}F^\varepsilon + E^\varepsilon\,\frac{\partial [F]^\varepsilon}{\partial v} = \\ &= E^\varepsilon\,\frac{\partial [F]^\varepsilon}{\partial v} + \frac{\partial G}{\partial t}\Big(t,\mathcal{R}^{t/\varepsilon}(r,v)\Big), \qquad (2.15)\\ &= \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix}\cdot\nabla_{r,v}[F]^\varepsilon + \frac{\partial G}{\partial t}\Big(t,\mathcal{R}^{t/\varepsilon}(r,v)\Big), \qquad (2.16)\\ &= \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix}\cdot\mathcal{R}^{-t/\varepsilon}\Big(\nabla_{q,u}G\big(t,\mathcal{R}^{t/\varepsilon}(r,v)\big)\Big) + \frac{\partial G}{\partial t}\Big(t,\mathcal{R}^{t/\varepsilon}(r,v)\Big), \qquad (2.17)\\ &= \mathcal{R}^{t/\varepsilon}(0,E^\varepsilon)\cdot\nabla_{q,u}G\big(t,\mathcal{R}^{t/\varepsilon}(r,v)\big) + \frac{\partial G}{\partial t}\Big(t,\mathcal{R}^{t/\varepsilon}(r,v)\Big). \qquad (2.18)\end{aligned}$$

To handle the right-hand-side of (2.18), we introduce the function Υ^ε such that

$$\Upsilon^\varepsilon\Big(t,\tau,r,v\Big) := \mathcal{R}^\tau(0,E^\varepsilon)\cdot\nabla_{q,u}G\big(t,\mathcal{R}^\tau(r,v)\big) + \frac{\partial G}{\partial t}\,(t,\mathcal{R}^\tau(r,v)). \qquad (2.19)$$

Then defining $[\Upsilon^\varepsilon]^\varepsilon$ as in Remark 2.1(ii), we can subtract (2.18) from (1.1) and multiply the result by $1/\varepsilon$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{f^\varepsilon-[F]^\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix}\cdot\nabla_{r,v}\left(\frac{f^\varepsilon-[F]^\varepsilon}{\varepsilon}\right) + E^\varepsilon\frac{\partial}{\partial v}\left(\frac{f^\varepsilon-[F]^\varepsilon}{\varepsilon}\right) = -\frac{1}{\varepsilon}[\Upsilon^\varepsilon]^\varepsilon. \qquad (2.20)$$

Now, let a function $W^\varepsilon : (t,\tau,q,u)\in[0,T)\times\mathbb{R}\times\mathbb{R}^2 \mapsto W^\varepsilon(t,\tau,q,u)$ be such that

$$\frac{\partial W^\varepsilon}{\partial \tau}(t,\tau,\mathcal{R}^\tau(r,v)) = -\Upsilon^\varepsilon(t,\tau,r,v). \qquad (2.21)$$

On the one hand, if we substitute Υ^ε by its expression in (2.19) and then consider equation (2.21) in variables $(q,u)=\mathcal{R}^\tau(r,v)$, we obtain

$$\frac{\partial W^\varepsilon}{\partial \tau}(t,\tau,q,u) = -\mathcal{R}^\tau\Big(0,E^\varepsilon\left(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\right)\Big)\cdot\nabla_{q,u}G(t,q,u) - \frac{\partial G}{\partial t}(t,q,u). \qquad (2.22)$$

We substitute the time derivative of G in the right-hand-side of (2.22) by its expression in (2.6). Integrating the result with respect to the second variable from 0 to τ gives

$$\begin{aligned}W^\varepsilon(t,\tau,q,u) &= \left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\ &\qquad \left.-\int_0^\tau \mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\,d\sigma\right]\cdot\nabla_{q,u}G(t,q,u) \qquad (2.23)\\ &\qquad -\varepsilon\int_0^\tau \mathcal{R}^\tau\big(0,E^1\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\,d\sigma\cdot\nabla_{q,u}G(t,q,u).\end{aligned}$$

When ε goes to 0 in (2.23), we obtain the limit W of W^ε defined by (2.8) in Theorem 2.1.

On the other hand, let us introduce $\widetilde{W}^\varepsilon$ define by $\widetilde{W}^\varepsilon(t,\tau,r,v) := W^\varepsilon(t,\tau,\mathcal{R}^\tau(r,v))$. Therefore, (2.21) is equivalent to

$$\frac{\partial \widetilde{W}^\varepsilon}{\partial \tau}\Big(t,\tau,r,v\Big) + \begin{pmatrix} v \\ -r \end{pmatrix}\cdot\nabla_{r,v}\widetilde{W}^\varepsilon\Big(t,\tau,r,v\Big) = -\Upsilon^\varepsilon(t,\tau,r,v). \qquad (2.24)$$

Using Remark 2.1(ii), we note that $\partial_t[\widetilde{W}^\varepsilon]^\varepsilon = [\partial_t \widetilde{W}^\varepsilon]^\varepsilon + [\partial_\tau \widetilde{W}^\varepsilon]^\varepsilon/\varepsilon$ and $\partial_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon = [\partial_{r,v}\widetilde{W}^\varepsilon]^\varepsilon$ so that considering (2.24) in $\tau = t/\varepsilon$ and multiply the result by $1/\varepsilon$ leads to

$$\frac{\partial[\widetilde{W}^\varepsilon]^\varepsilon}{\partial t} - \left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon = -\frac{1}{\varepsilon}[\Upsilon^\varepsilon]^\varepsilon. \tag{2.25}$$

Now, we subtract (2.25) from (2.20) and subtract from the both sides of the result the term $E^\varepsilon\, \partial_v[\widetilde{W}^\varepsilon]^\varepsilon$ to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right) &+ \frac{1}{\varepsilon}\begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}\left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right) \\ &+ E^\varepsilon \frac{\partial}{\partial v}\left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right) \\ &= -\left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t}\right]^\varepsilon - E^\varepsilon \frac{\partial [\widetilde{W}^\varepsilon]^\varepsilon}{\partial v}. \end{aligned} \tag{2.26}$$

We want to obtain an a priori estimate in order to prove convergence result. Therefore we multiply (2.26) by $\dfrac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon$ and integrate by part. This leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right\|^2_{L^2(\mathbb{R}^2)}(t) \\ &\quad\le \left\|\left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t}\right]^\varepsilon + \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot \nabla_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon\right\|_{L^2(\mathbb{R}^2)}(t)\ \left\|\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right\|_{L^2(\mathbb{R}^2)}(t). \end{aligned} \tag{2.27}$$

If the first factor on the right-hand-side of (2.27) is bounded, by Gronwall Lemma, we obtain the following estimate

$$\left\|\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right\|_{L^\infty([0,T);L^2(\mathbb{R}^2))} \le C. \tag{2.28}$$

From this estimate, we deduce the following Two-Scale convergence result

$$\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \longrightarrow F_1 - \widetilde{W}, \tag{2.29}$$

where $\widetilde{W} = W(t, \tau, \mathcal{R}^\tau(r, v))$ and W given by (2.8).

Now, we want to compute F_1. Returning to (2.26), we multiply by $\phi \in \mathcal{C}^1$ (see Definition 2.1), integrate over $t \in [0, T)$ and $(r, v) \in \mathbb{R}^2$ and then integrate by parts. We obtain

$$\begin{aligned} &-\int_0^T\!\!\int_{\mathbb{R}^2}\left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right)\left(\left[\frac{\partial\phi}{\partial t}\right]^\varepsilon + E^\varepsilon\left[\frac{\partial\phi}{\partial v}\right]^\varepsilon\right) dr\, dv\, dt \\ &-\frac{1}{\varepsilon}\int_0^T\!\!\int_{\mathbb{R}^2}\left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right)\left(\left[\frac{\partial\phi}{\partial\tau}\right]^\varepsilon + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot [\nabla_{r,v}\phi]^\varepsilon\right) dr\, dv\, dt \\ &\quad = -\int_0^T\!\!\int_{\mathbb{R}^2}\left(\left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t}\right]^\varepsilon - \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot \left[\nabla_{r,v}\widetilde{W}^\varepsilon\right]^\varepsilon\right)[\phi]^\varepsilon\, dr\, dv\, dt. \end{aligned} \tag{2.30}$$

Multiplying (2.30) by ε and passing to the limit using the Two-Scale convergence (2.29), we obtain

$$\frac{\partial}{\partial\tau}\left(F_1 - \widetilde{W}\right) + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}\left(F_1 - \widetilde{W}\right) = 0, \tag{2.31}$$

in a weak sense. Thus, $F_1 - \widetilde{W}$ is constant along the characteristics so that there exists a function $G_1 : (t,q,u) \in [0,T) \times \mathbb{R}^2 \mapsto G_1(t,q,u)$ such that

$$F_1(t,\tau,r,v) - \widetilde{W}(t,\tau,r,v) = G_1(t, \mathcal{R}^\tau(r,v)), \tag{2.32}$$

and therefore, thanks to the definition of $\widetilde{W}$, we obtain the decomposition (2.7) of F_1 in Theorem 2.1. Now, let $\gamma : (t,q,u) \in [0,T) \times \mathbb{R}^2 \mapsto \gamma(t,q,u)$ a function in $C_c^1\left([0,T); C_c^1(\mathbb{R}^2)\right)$ such that

$$\phi(t,\tau,r,v) = \gamma(t, \mathcal{R}^\tau(r,v)). \tag{2.33}$$

We consider (2.30) for this function ϕ. Then, the second term on the left hand side cancels and we obtain

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{R}^2} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon\right)\left(\left[\frac{\partial\phi}{\partial t}\right]^\varepsilon + E^\varepsilon\left[\frac{\partial\phi}{\partial v}\right]^\varepsilon\right) dr\,dv\,dt \\ = -\int_0^T\!\!\int_{\mathbb{R}^2}\left(\left[\frac{\partial\widetilde{W}^\varepsilon}{\partial t}\right]^\varepsilon - \begin{pmatrix}0\\E^\varepsilon\end{pmatrix}\cdot\left[\nabla_{r,v}\widetilde{W}^\varepsilon\right]^\varepsilon\right)[\phi]^\varepsilon\,dr\,dv\,dt.\end{aligned} \tag{2.34}$$

Thanks to the Two-Scale convergence, we pass to the limit as $\varepsilon \to 0$ and integrate over $\tau \in [0, 2\pi]$ which leads to

$$\begin{aligned}\int_0^T\!\!\int_0^{2\pi}\!\!\int_{\mathbb{R}^2} G_1\left(t,\mathcal{R}^\tau(r,v)\right)\left(\frac{\partial\phi}{\partial t} + \begin{pmatrix}0\\E^0\end{pmatrix}\cdot\nabla_{r,v}\phi\right) dr\,dv\,d\tau\,dt \\ = \int_0^T\!\!\int_0^{2\pi}\!\!\int_{\mathbb{R}^2}\left(\frac{\partial\widetilde{W}}{\partial t} - \begin{pmatrix}0\\E^0\end{pmatrix}\cdot\nabla_{r,v}\widetilde{W}\right)\phi\,dr\,dv\,d\tau\,dt.\end{aligned} \tag{2.35}$$

Then substituting ϕ by its expression in terms of γ and changing the variables (r,v) in $(q,u) = \mathcal{R}^\tau(r,v)$ that is $(r,v) = \mathcal{R}^{-\tau}(q,u)$ gives

$$\begin{aligned}\int_0^T\!\!\int_0^{2\pi}\!\!\int_{\mathbb{R}^2} G_1(t,q,u)\,\Bigg(&\frac{\partial\gamma}{\partial t}(t,q,u) \\ &+\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\cdot\nabla_{q,u}\gamma(t,q,u)\Bigg)dq\,du\,d\tau\,dt \\ = \int_0^T\!\!\int_0^{2\pi}\!\!\int_{\mathbb{R}^2}\Bigg(&\frac{\partial W}{\partial t}(t,\tau,q,u) \\ &+\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\cdot\nabla_{q,u}W(t,\tau,q,u)\Bigg)\gamma(t,q,u)\,dq\,du\,d\tau\,dt.\end{aligned} \tag{2.36}$$

We finally deduce that G_1 satisfies, in a weak sense, the following problem

$$\begin{cases}\dfrac{\partial G_1}{\partial t}(t,q,u) + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)d\tau\cdot\nabla_{q,u}G_1(t,q,u) = \\ -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left(\dfrac{\partial W}{\partial t}(t,\tau,q,u) + \mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\cdot\nabla_{q,u}W(t,\tau,q,u)\right)d\tau \\ G_1(t=0,q,u) = 0.\end{cases} \tag{2.37}$$

To handle the right hand side of (2.37), we need to compute the partial derivatives of W. First from (2.8), we have that

$$\begin{aligned}\frac{\partial W}{\partial t}(t,\tau,q,u) &= \left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\mathcal{R}^\sigma\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\right]\cdot\nabla_{q,u}G(t,q,u)\\ &+\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\cdot\nabla_{q,u}\frac{\partial G}{\partial t}(t,q,u),\end{aligned}\tag{2.38}$$

so that we can then replace $\partial_t G$ by its expression in (2.6) to get

$$\begin{aligned}\frac{\partial W}{\partial t}(t,\tau,q,u) &= \left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\mathcal{R}^\sigma\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\right]\cdot\nabla_{q,u}G(t,q,u)\\ &+\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\cdot\\ &\quad\cdot\left[\left[\frac{1}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\nabla_{q,u}G(t,q,u)\right.\\ &\quad\left.+H_{q,u}G(t,q,u)\,\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right],\end{aligned}\tag{2.39}$$

where $H_{q,u}$ is the Hessian matrix with respect to variable q an u. Next, from (2.8), we deduce that

$$\begin{aligned}\nabla_{q,u}W(t,\tau,q,u) &= \left[\frac{\tau}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\cdot\nabla_{q,u}G(t,q,u)\\ &+H_{q,u}G(t,q,u)\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\ &\quad\left.-\int_0^\tau\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right].\end{aligned}\tag{2.40}$$

We consider the scalar product of (2.40) by vector $\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)$, sum the result with (2.39) and then integrate over $\tau\in[0,2\pi]$. Using Fubini's Theorem for the terms involving the

Hessian matrix of G which is symmetric yields

$$
\begin{aligned}
&\int_0^{2\pi}\left(\frac{\partial W}{\partial t}(t,\tau,q,u)+\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\cdot\nabla_{q,u}W(t,\tau,q,u)\right)d\tau\\
&=\int_0^{2\pi}\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\left(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\right)d\sigma\right.\\
&\qquad\left.-\int_0^{\tau}\mathcal{R}^\sigma\left(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\right)d\sigma\right]\cdot\nabla_{q,u}G(t,q,u)\,d\tau\\
&+\int_0^{2\pi}\left[\left[\frac{\tau}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\right.\\
&\qquad\left.-\int_0^{\tau}\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\\
&\qquad-\frac{1}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\\
&\qquad\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\
&\qquad\left.\left.-\int_0^{\tau}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\right]\cdot\nabla_{q,u}G(t,q,u)\,d\tau\\
&-\int_0^{2\pi}H_{q,u}G(t,q,u)\left[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\
&\qquad\left.-\int_0^{\tau}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right]\cdot\\
&\qquad\cdot\left[\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}^\sigma\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\right.\\
&\qquad\left.-\mathcal{R}^\tau\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\right]d\tau.
\end{aligned}
\tag{2.41}
$$

For both last factors on the right hand side of (2.41), we remark that the second one is the exact derivative of the first one with respect to τ. Since the Hessian matrix is symmetric and does not depend on τ, we can take it out of the integral. Thus, the remaining integrand is the exact derivative of a quadratic form with respect to variable τ, so that integrating over a period in τ gives 0. Moreover, we can perform an integration by parts in the first part of the second term on the right hand side of (2.41). We obtain

$$
\begin{aligned}
&\int_0^{2\pi}\Big(\frac{\partial W}{\partial t}(t,\tau,q,u)+\mathcal{R}^{\tau}\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\cdot\nabla_{q,u}W(t,\tau,q,u)\Big)\,d\tau\\
&=\int_0^{2\pi}\Bigg[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^{\sigma}\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\\
&\qquad-\int_0^{\tau}\mathcal{R}^{\sigma}\Big(0,\frac{\partial E^0}{\partial t}\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\Big)\,d\sigma\Bigg]\cdot\nabla_{q,u}G(t,q,u)\,d\tau\\
&-\int_0^{2\pi}\Bigg[\Bigg[\frac{1}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\\
&\qquad-\nabla_{q,u}\mathcal{R}^{\tau}\big(0,E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\big)\Bigg]\int_0^{\tau}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\\
&\qquad-\frac{1}{2\pi}\int_0^{2\pi}\nabla_{q,u}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\\
&\qquad\Bigg[\frac{\tau}{2\pi}\int_0^{2\pi}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\\
&\qquad-\int_0^{\tau}\mathcal{R}^{\sigma}\big(0,E^0\big(t,\sigma,\mathcal{R}_r^{-\sigma}(q,u)\big)\big)\,d\sigma\Bigg]\Bigg]\cdot\nabla_{q,u}G(t,q,u)\,d\tau.
\end{aligned}
\tag{2.42}
$$

In the second term on the right hand side of (2.42), we remark that first and last products cancel. Moreover, $\tau/2\pi$ is the only factor depending of τ in the third product, integration of which over $[0,2\pi]$ gives π. Substituting the right-hand side of equation in problem (2.37) by its expression in (2.42) finally leds to problem (2.9)

3. Numerical algorithm

To build our numerical algorithm, we use a Particle-in-Cell (PIC) method, that deals with macroparticles rather than directly with the distribution function.

Theorem 2.1 tells that, provided that problem (2.6) is solved and space derivatives of G can be computed, then we can compute W from (2.8) and source term in problem (2.9). Then it remains to solve problem (2.9) to obtain G_1. From G, G_1 and W, we finally get the Two-Scale first order approximation of f^ε thanks to (2.4), (2.5) and (2.7). Thus, the main steps of the algorithm are the following.

Main steps of the algorithm

(i) Compute a particle approximation of G, that is push the macroparticles with respect to the advection operator associated to the partial differential equation in problem (2.6) as in [**?**].
(ii) Compute an approximation of the gradient $\nabla_{q,u}G$ from the particle approximation of G which is compatible with the desired particle approximation of W.
(iii) Compute a particle approximation of W from equation (2.8).
(iv) Compute a particle approximation of the source term of the partial differential equation in problem (2.9), and then a particle approximation of G_1 by solving problem (2.9) with the same advection operator than in problem (2.6).

To deal with the first step, we introduce the following particle approximation of function G

$$G(t,q,u)=\sum_{k=1}^{N_p}\omega_k\,\delta(q=Q_k(t))\,\delta(u=U_k(t)), \tag{3.1}$$

where δ is the Dirac mass, N_p is the number of macroparticles and $(Q_k(t), U_k(t))$ is the position in phase space of macroparticle k which moves along a characteristic curve of the equation in (2.6). Hence our problem is reduced to compute the location of macroparticles at the next time step from their positions at the previous time step, as the solution of the following dynamical system

$$\frac{d}{dt}\begin{pmatrix} Q_k \\ U_k \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}^\tau\big(0, E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(Q_k,U_k)\big)\big)d\tau. \tag{3.2}$$

The main difficulty to solve coupled problem (3.2) lies in the fact that we need to approximate the integral term. A solution to this problem is described in section 5.1 of [?].

Once the macroparticles are pushed along the characteristics, we can compute G at the next time step. Applying the *ad hoc* rotation, we then obtain a particle approximation of F from (2.5). Finally, evaluating this approximation in $\tau = t/\varepsilon$ gives the zero order approximation of f^ε.

For the second step of the algorithm, we need to recover an approximation of the space derivatives of G thanks to the particle approximation of G. Therefore, a regularization of approximation (3.1) is needed. To this purpose, we introduce a regular function γ^α with support included in the intervall $[0,\alpha]$, and such that the function $\gamma_k^\alpha : (q,u) \mapsto \gamma^\alpha\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big)$ has an integral over q and u with worth 1. Thus a regularization of (3.1) is given by

$$G(t,q,u) = \sum_{k=1}^{N_p} \omega_k\, \gamma^\alpha\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big)\,, \tag{3.3}$$

Thus we can compute numerical approximations of the space derivatives of G.

Now we can handle the third step of the algorithm. Indeed, in (2.8), we can compute an approximation of the right hand side which has the following shape

$$\sum_{k=1}^{N_p} \beta_k(t,\tau,q,u)\,(\gamma^\alpha)'\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big)\,. \tag{3.4}$$

In order to get W, we need to distribute the above approximation over the macroparticles. Let the particle approximation of W be given by

$$W(t,\tau,q,u) = \sum_{k=1}^{N_p} \widetilde{\beta}_k(t,\tau)\,\delta(q = Q_k(t))\,\delta(u = U_k(t)), \tag{3.5}$$

which can be regularized by

$$W(t,\tau,q,u) = \sum_{k=1}^{N_p} \widetilde{\beta}_k(t,\tau)\,\gamma^\alpha\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big)\,. \tag{3.6}$$

Then we want that all the integrals of the sums in (3.4) and (3.6) over each subdomain of the phase space are almost equals. From the numerical point of view, we want to verify this proximity only for the support C_i of functions γ_i^α that is

$$\int_{C_i}\sum_{k=1}^{N_p} \widetilde{\beta}_k(t,\tau)\,\gamma_k^\alpha(q,u)\,dq\,du = \int_{C_i}\sum_{k=1}^{N_p} \beta_k(t,\tau,q,u)\,\gamma_k^\alpha(q,u)\,dq\,du, \tag{3.7}$$

which writes again

$$\sum_{k=1}^{N_p} \widetilde{\beta}_k(t,\tau) \int_{C_i} \gamma_k^\alpha(q,u)\, dq\, du = \int_{C_i} \sum_{k=1}^{N_p} \beta_k(t,\tau,q,u)\, \gamma_k^\alpha(q,u)\, dq\, du. \tag{3.8}$$

Therefore we can determine the coefficient $\widetilde{\beta}_k$ of the particle approximation of W up to the solution the linear system (3.8).

Finally, we consider the last step of the algorithm. We want to build a particle approximation of G_1 from problem (2.9). We first note that in this problem, the advection operator is exactly the same than in problem (2.6). Therefore the macroparticles to approximate G_1 can be chosen as being localized exactly at the same phase space locations as the ones used to approximate G. Thus G_1 is approximated by

$$G_1(t,q,u) = \sum_{k=1}^{N_p} \omega_k^1(t)\, \delta(q = Q_k(t))\, \delta(u = U_k(t)), \tag{3.9}$$

where the weights ω_k^1 now depend on time t, which allows to take into account the effect of the source term on the right-hand side of the equation in (2.9). Once more, we can regularize this approximation by

$$G_1(t,q,u) = \sum_{k=1}^{N_p} \omega_k^1(t)\, \gamma^\alpha\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big). \tag{3.10}$$

Then we handle the right hand side of equation in problem (2.9) exactly as for W. Indeed, approximation of this term has the following shape

$$\sum_{k=1}^{N_p} \eta_k(t,\tau,q,u)\, (\gamma^\alpha)'\Big(\sqrt{(q-Q_k(t))^2 + (u-U_k(t))^2}\Big). \tag{3.11}$$

Therefore in order to distribute this approximation on the macroparticles, we want to determine $\widetilde{\eta}_k$ such that

$$\int_{C_i} \sum_{k=1}^{N_p} \widetilde{\eta}_k(t,\tau)\, \gamma_k^\alpha(q,u)\, dq\, du = \int_{C_i} \sum_{k=1}^{N_p} \eta_k(t,\tau,q,u)\, \gamma_k^\alpha(q,u)\, dq\, du, \tag{3.12}$$

which writes again

$$\sum_{k=1}^{N_p} \widetilde{\eta}_k(t,\tau) \int_{C_i} \gamma_k^\alpha(q,u)\, dq\, du = \int_{C_i} \sum_{k=1}^{N_p} \eta_k(t,\tau,q,u)\, \gamma_k^\alpha(q,u)\, dq\, du. \tag{3.13}$$

Therefore we can determine the coefficient $\widetilde{\eta}_k$ up to the solution the linear system (3.13). Since G_1 is solution to problem (2.9), we finally have that

$$\frac{d\omega_k^1}{dt} = \widetilde{\eta}_k, \tag{3.14}$$

that needs to be numerically solved to get the full approximation of G_1.

References