

# Uniform Law on the Unit Sphere of a Banach Space

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Abstract

We investigate the construction of a uniform probability law on the unit sphere of a finite-dimensional, real, rearrangement-invariant Banach space. We show that this construction is possible under a geometrical condition, and it comes from a probability on  $\mathbb{R}$ : there is a density  $f$ , on  $\mathbb{R}$ , such that if  $X_1, \dots, X_K$  are independent variables following this law, the law of  $\frac{(X_1, \dots, X_K)}{\|(X_1, \dots, X_K)\|}$  is the uniform law on the unit sphere of  $\mathbb{R}^K$ , equipped with this norm.

## I. Introduction

For practical purposes, one may want to generate a sample of points on the unit sphere of a Banach space. Here are two examples :

- Choosing a direction at random is useful for the search of solutions of PDE's : one starts at a given point and then moves to some direction. In this case, the norm is usually the Euclidean norm. See the acknowledgements at the end of the paper for our original motivation, in the framework of two contracts with IRSN (France).
- Choosing a proportion at random is useful, for instance in economics : a budget may depend on some goods, and one wants to study the variations of the budget, depending on various weights on the goods. In this case, the norm is usually the sum of absolute values of the coefficients, that is the  $l_1$  norm.

In all cases, one deals with a finite-dimensional real Banach space, that is the space  $\mathbb{R}^K$  equipped with some norm.

In general, one does not have any predefined wish or preference about the points which must be chosen : they need to be on the unit sphere, but no direction is privileged. This means that the law we want for our sample is the uniform law on the unit sphere.