Branching processes in random environment: sudden death versus slow extinction

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Let

$$Z := \{Z(n), n = 0, 1, ...,\}$$

be a branching process evolving in the random environment generated by a sequence of iid generating functions $f_0(s), f_1(s), ...$, and let

$$S_0 = 0, S_k = X_1 + ... + X_k, k \geq 1,$$

be the associated random walk with $X_i = \log f'_{i-1}(1)$.

Denote $\tau(m,n)$ the left-most point of minimum of $\{S_k, k \geq 0\}$ on the interval $[m,n]$, and let

$$T = \min\{k : Z(k) = 0\}.$$

Assuming that the random walk $\{S_k, k \geq 0\}$ satisfies the Doney condition

$$\mathbf{P}(S_n > 0) \to \rho \in (0,1), n \to \infty,$$

we prove (under the quenched and annealed approaches) conditional limit theorems, as $n \to \infty$, for the distribution of $Z(nt)$, $Z(\tau(0,nt))$, and $Z(\tau(nt,n))$, $t \in (0,1)$, given $T > n$ or $T = n$. It is shown that the form of the limit distributions essentially depends on the location of $\tau(n)$ with respect to the point nt.

We also present a number of conditional limit theorems describing the asymptotic behavior of the distribution of the number of particles in the critical branching processes in random environment given that the process does not extinct up to a moment $n \to \infty$. In particular, we show that if $t \in (0,1]$ is *fixed* then, given $Z(n) > 0$, the distribution of the random variable $Z(nt)e^{S_{\tau(nt)} - S_{nt}}$ converges in a certain sense to a proper distribution with no atom at zero. This means, roughly speaking, that if the process survives up to moment n then the size $Z(nt)$ of the population at moment nt is proportional to $e^{S_{nt} - S_{\tau(nt)}}$. Thus, contrary to the conditional limit theorems for the classical critical or supercritical branching processes in which the scaling functions for the population size increase with time linearly or exponentially, the (random) scaling function for the population size in the critical branching process in random environment is subject to large oscillations. This indicates that the population of a critical branching process in random environment passes through a number of bottlenecks at the moments close to the sequential points of minima of the associated random walk.

We also show that if the logarithm of the (random) expectation of the offspring number belongs to the domain of attraction of a non-gaussian stable law then the extinction of the process occurs owing to very unfavorable environment forcing the process, having at moment $T-1$ exponentially large population, to die out.