LINK INVARIANT FOR THE SPINOR REPRESENTATION OF so_7

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ABSTRACT

Pulling back the weight system associated with the spinor representation of the Lie algebra $\mathfrak{so}_7$ by the universal Vassiliev-Kontsevich invariant yields a numerical link invariant with values in formal power series. Computing some skein relations satisfied by this invariant, I derive a recursive algorithm for its evaluation. The values of this invariant belong to the ring $\mathbb{Z}[W, W^{-1}]$ of Laurent polynomials in one variable.

Keywords: Vassiliev invariants of knots, Quantum invariants, Representation theory.

Introduction

This paper is a part of my thesis.

The universal Vassiliev invariant $Z_{\mathfrak{g}}(K)$ of a knot K associated with a Lie algebra $\mathfrak{g}$ takes values in formal power series in one variable with coefficients in the center of the enveloping algebra of $\mathfrak{g}$. We then get a numerical invariant by evaluating the trace of these elements on a representation of $\mathfrak{g}$. It is well known that the invariant obtained from the standard representation of the Lie superalgebra $\mathfrak{osp}(m, n)$ is given by a specialization of the two-variable Kauffman polynomial. The invariants associated to other representations of the group SO_n can be computed from the Kauffman polynomial of some cabling of the knot. This is not true for the spinor representations which are representations of the Lie algebra $\mathfrak{so}_n$ but not of the group SO_n (such representations exist because the Lie group SO_n is not simply connected). In this paper, we give an algorithm to compute the invariant associated to the spinor representation of $\mathfrak{so}_7$.

Consider the extended family of exceptional Lie superalgebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, the superalgebras $\mathfrak{g}(3)$, $\mathfrak{f}(4)$, and the superalgebras $\mathfrak{psl}(E_0)$, $\mathfrak{sl}(E_2)$, $\mathfrak{sl}(E_3)$, $\mathfrak{osp}(F_{-1})$, $\mathfrak{osp}(F_8)$ where E_n (respectively F_n) denotes a super space with super dimension n (respectively carrying a non degenerate symmetric bilinear form). Then, for L in this family, there exists a subalgebra l of L and a representation e of l such that we have an isomorphism of $(\mathfrak{sp}_2 \times l)$-modules

$$L \simeq \mathfrak{sp}_2 \oplus l \oplus v \otimes e$$

where v denotes the standard representation of $\mathfrak{sp}_2$.
The weight functions associated with the pairs (l, e) have common properties which allow to compute some skein relations satisfied by the associated link invariants. In general, this is not a complete system of relations (one can't deduce a direct way to compute these invariants). Nevertheless, applying this in the case $L = \mathfrak{f}(4)$ where $l = \mathfrak{so}_7$ and e is its spinor representation, the weight system associated with the pair $(\mathfrak{so}_7, \mathrm{spin}_7)$ has additional properties from which one can deduce other skein relations. We thus get a complete system of relations from which we can deduce an algorithm for the computation of the associated link invariant.

Remark that if one could compute the Kauffman polynomial of all the cablings of a given knot K, one could deduce the value of the universal invariant $Z_{\mathfrak{so}_n}(K)$ and, from this, the invariant obtained for the spinor representation of $\mathfrak{so}_n$. But practically this computation seems difficult.

One can also remark that spin_7 is the smallest spin representation whose associated quantum invariant can't be deduced from the classical polynomial invariants. Indeed, using isomorphisms between Lie algebras, the representation spin_3 of $\mathfrak{so}_3$ is isomorphic to the standard representation of $\mathfrak{sp}_2$, the representation $\mathrm{spin}_4^{\pm}$ of $\mathfrak{so}_4$ is isomorphic to the tensor product of the trivial representation by the standard representation of $\mathfrak{sp}_2 \times \mathfrak{sp}_2$, the representation spin_5 of $\mathfrak{so}_5$ is isomorphic to the standard representation of $\mathfrak{sp}_4$ and the representation $\mathrm{spin}_6^{\pm}$ of $\mathfrak{so}_6$ is isomorphic to the standard representation of $\mathfrak{sl}_4$ or to its dual (depending of the choice of the isomorphism between $\mathfrak{sl}_4$ and $\mathfrak{so}_6$). So the associated quantum invariants are equivalent to the Jones polynomial for spin_3 and $\mathrm{spin}_4^{\pm}$, to a specialization of the Kauffman polynomial for spin_5 and to a specialization of the Homfly polynomial for $\mathrm{spin}_6^{\pm}$. The standard representation of $\mathfrak{so}_8$ and its two spinor representations $\mathrm{spin}_8^{\pm}$ are permuted by the automorphisms of this Lie algebra. Therefore the quantum invariants associated to $\mathrm{spin}_8^{\pm}$ are specializations of the Kauffman polynomial.

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1. The Weight System $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$

1.1. *Definition of diagrams*

Let X_1 and X_2 be finite set. We call bicoloured (X_1, X_2)-diagram a finite uni-trivalent graph K, with the following data:

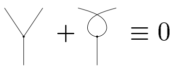
(i) For each trivalent vertex x of K, a cyclic order on the three oriented edges incident with x.
(ii) An isomorphism that identifies the univalent vertices with $X_1 \amalg X_2$.
(iii) A map c from the set of all edges of K to the set of colours $\{1, 2\}$ such that $c^{-1}(\{2\})$ forms a non oriented curve with boundary X_2.

We represent such a diagram by a graph immersed in the plane in such a way that the cyclic order at each vertex is given by the orientation of the plane. We draw the edges of the second colour as thick lines.
We define the degree of a bicoloured (X_1, X_2)-diagram to be half the number of the vertices which are not in X_2.

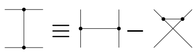
1.2. *Definition of modules of diagrams*

Let $\mathcal{A}(X_1, X_2)$ denote the quotient of the $\mathbb{Q}$-vector space with basis the (X_1, X_2)-diagrams by the relations (AS), (IHX) and (STU):

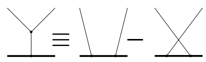
(i) If two diagrams are the same except for the cyclic order of one of their vertices, then one is minus the other (relation called (AS) for antisymmetry).



(ii) The relation (IHX) deals with three diagrams which differ only in the neighbourhood of an edge.



(iii) The relation (STU) which is a variation of (IHX) with edges of the second colour.



1.3. *Categories of diagrams*

Let $[n]$ be the set $[n] = \{1, 2, \ldots, n\}$. Let $\overline{\mathcal{B}}$ and $\mathcal{B}$ denote the $\mathbb{C}$-linear monoidal categories defined by

- $\mathrm{Obj}(\overline{\mathcal{B}}) = \mathrm{Obj}(\mathcal{B}) = \{[n], n \in \mathbb{N}\}$
- $\overline{\mathcal{B}}([p],[q]) = \mathcal{A}([p] \amalg [q]) \qquad \mathcal{B}([p],[q]) = \mathcal{A}(\emptyset, [p] \amalg [q])$
- The composition of a diagram from $[p]$ to $[q]$ with a diagram from $[q]$ to $[r]$ is given by gluing the two diagrams along $[q]$ if the colours of the individual vertices in the two sets $[q]$ match, and it is zero otherwise.
- The tensor product of two object is $[p] \otimes [q] = [p+q]$ and the tensor product of two diagrams is given by their disjoint union.

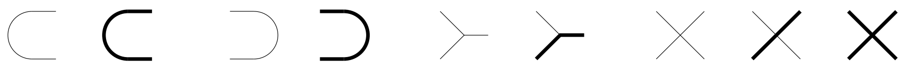
The morphisms in the category $\mathcal{B}$ are thus given by linear combinations of diagrams whose univalent vertices are all of the second colour. We identify $\mathcal{B}([p],[q])$ with a subspace of $\overline{\mathcal{B}}([p],[q])$. In the case where $p = q$ the inclusion map $\mathcal{B}([p],[p]) \subset \overline{\mathcal{B}}([p],[p])$ respect the composition but it does not send the identity of $\mathcal{B}([p],[p])$ on the identity of $\overline{\mathcal{B}}([p],[p])$.

1.4. *Weight functions*

Let L denote a quadratic Lie superalgebra (i.e. given with a supersymmetric invariant non degenerate bilinear form $<.,.>_L$) and let $\Omega \in L \otimes L$ be the associated Casimir element.
For E a self-dual L-module with a supersymmetric invariant non degenerate bilinear form $<.,.>_E$ and $\pi \in E \otimes E$ the associated Casimir element one has:

Lemma 1. There exists an unique $\mathbb{C}$-linear monoidal functor $\overline{\Phi}_{L,E}$ from $\overline{\mathcal{B}}$ to the category Mod_L of the representations of L such that:

1 $\overline{\Phi}_{L,E}([1]) = L \oplus E$.
2 Its values on the elementary morphisms



are given by:

(i) The Casimir of L: $\Omega \in L^{\otimes 2} \hookrightarrow \mathrm{Mod}_L(\mathbb{C}, (L \oplus E)^{\otimes 2})$
(ii) The Casimir of E: $\pi \in E^{\otimes 2} \hookrightarrow \mathrm{Mod}_L(\mathbb{C}, (L \oplus E)^{\otimes 2})$
(iii) The bilinear form $<.,.>_L$: $L^{\otimes 2} \longrightarrow \mathbb{C}$ seen in $\mathrm{Mod}_L((L \oplus E)^{\otimes 2}, \mathbb{C})$
(iv) The bilinear form $<.,.>_E$: $E^{\otimes 2} \longrightarrow \mathbb{C}$ seen in $\mathrm{Mod}_L((L \oplus E)^{\otimes 2}, \mathbb{C})$
(v) The Lie bracket seen as a map in $\mathrm{Mod}_L((L \oplus E)^{\otimes 2}, (L \oplus E))$
(vi) The map $L \otimes E \longrightarrow E$ given by the action of L on E
(vii) The three symmetry operators: $\begin{array}{ccc} X \otimes Y & \longrightarrow & Y \otimes X \\ x \otimes y & \mapsto & y \otimes x \end{array}$

Similarly, we denote by $\Phi_{L,E} : \mathcal{B} \longrightarrow \mathrm{Mod}_L$ the $\mathbb{C}$-linear monoidal functor which sends $[1]$ to E and which is defined on morphisms by $\Phi_{L,E}(K) = i^*(p_*(\overline{\Phi}_{L,E}(K)))$ where p denotes the canonical projection from $L \oplus E$ to E and i stands for the canonical inclusion from E to $L \oplus E$.

1.5. *Representations of* $\mathfrak{so}_7$

Let us denote by $\Gamma(p,q,r)$ the representation of $\mathfrak{so}_7$ with highest weight $p\omega_1 + q\omega_2 + r\omega_3$ where ω_1 (respectively ω_2, respectively ω_3) is the highest weight of the standard representation v (respectively of the adjoint reprsentation l, respectively of the spinor representation s). In the category of representations of $\mathfrak{so}_7$, one has:

$$\begin{array}{cc} \Lambda^2 s = v \oplus l & S^2 s = \mathbb{C} \oplus \Gamma(0,0,2) \\ \Lambda^3 s = s \oplus \Gamma(1,0,1) & S^3 s = s \oplus \Gamma(0,0,3) \\ s \otimes v = s \oplus \Gamma(1,0,1) & s \otimes l = s \oplus \Gamma(1,0,1) \oplus \Gamma(0,1,1) \end{array}$$
$$s \otimes \Gamma(0,0,2) = s \oplus \Gamma(1,0,1) \oplus \Gamma(0,1,1) \oplus \Gamma(0,0,3)$$

We take for the Casimir Ω forty times the one which is associated with the Killing form. It acts on these representations by the following scalars:

Representation	dimension	action of the Casimir
$v = \Gamma(1,0,0)$	7	24
$l = \Gamma(0,1,0)$	21	40
$s = \Gamma(0,0,1)$	8	21
$\Gamma(0,0,2)$	35	48
$\Gamma(1,0,1)$	48	49
$\Gamma(0,1,1)$	112	69
$\Gamma(0,0,3)$	112	81

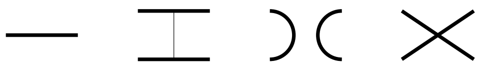
Thus

$$\mathrm{Mod}_L(s^{\otimes 2}, s^{\otimes 2}) \simeq \mathbb{C}^4 \ \text{(as an algebra)}$$
$$\mathrm{Mod}_L(s^{\otimes 3}, s^{\otimes 3}) \simeq M_4(\mathbb{C}) \times M_3(\mathbb{C}) \times M_2(\mathbb{C}) \times \mathbb{C}$$

We denote by $\mathcal{A}_3$ the algebra $\mathcal{B}([3],[3])$ and by $\mathcal{A}'_3$ the graded algebra

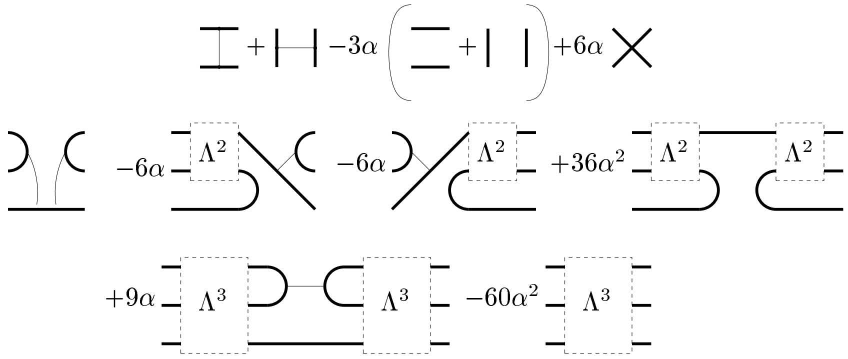
$$\mathcal{A}'_3 = \mathrm{Mod}_L(s^{\otimes 3}, s^{\otimes 3}) \otimes \mathbb{C}[\alpha] \simeq M_4(\mathbb{C}[\alpha]) \times M_3(\mathbb{C}[\alpha]) \times M_2(\mathbb{C}[\alpha]) \times \mathbb{C}[\alpha]$$

where degree$(\alpha) = 1$.
The weight function $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$ associated with the choice of a Casimir equal to $\alpha\Omega$ induces a morphism of graded algebras from $\mathcal{A}_3 = \mathcal{B}([3],[3])$ to $\mathcal{A}'_3$. We denote respectively by i, r, a and τ the following four diagrams:



as morphisms in $\mathcal{B}([1],[1])$ (respectively $\mathcal{B}([2],[2])$) for which the source is on the left and the target on the right.

Theorem 1. The weight function $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$ is zero on the following elements:



Here, the box Λ^2 (respectively Λ^3) stands for the linear combination of diagrams of $\mathcal{B}([2],[2])$ (respectively $\mathcal{B}([3],[3])$) given by $\frac{1}{2}\sum_{\sigma\in\mathfrak{S}_2}\epsilon(\sigma)\sigma\ (=\frac{1}{2}(1-\tau))$ (respectively $\frac{1}{6}\sum_{\sigma\in\mathfrak{S}_3}\epsilon(\sigma)\sigma$).
Furthermore, up to an isomorphism of $\mathcal{A}'_3$ the elements $r'_{1,2}=\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}(r\otimes 1)$, $a'_{1,2}=\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}(a\otimes 1)$, $\tau'_{1,2}=\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}(\tau\otimes 1)$ and $\tau'_{2,3}=\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}(1\otimes\tau)$ are given by:

$$r'_{1,2}=\left(\begin{bmatrix}21\alpha & 10\alpha & 8\alpha & 0\\ 0 & \alpha & -4\alpha & 0\\ 0 & 0 & 9\alpha & 6\alpha\\ 0 & 0 & 0 & -3\alpha\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}\alpha & -20\alpha & -12\alpha\\ 0 & 9\alpha & 6\alpha\\ 0 & 0 & -3\alpha\end{bmatrix},\begin{bmatrix}-3\alpha & -2\alpha\\ 0 & \alpha\end{bmatrix},\begin{bmatrix}-3\alpha\end{bmatrix}\right)$$

$$\tau'_{1,2}=\left(\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & -1 & 0 & -1\\ 0 & 0 & -1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & -1\\ 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix},\begin{bmatrix}1\end{bmatrix}\right)$$

$$\tau'_{2,3}=\left(\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & -1 & 0 & 0\\ -1 & 0 & -1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix},\begin{bmatrix}1 & 0 & 0\\ -1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix},\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}1\end{bmatrix}\right)$$

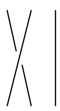
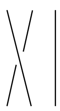
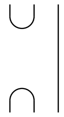
$$a'_{1,2}=\left(\begin{bmatrix}8 & 4 & 4 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0\end{bmatrix}\right)$$

Proof. We only justify the existence of the above relations for $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$ and the method we use to determinate the matrices of the elementary diagrams:
The first three relations can be interpreted in the following way: the representation spin_7 is eight dimensional and the bilinear form on $\mathfrak{so}_7$ induced by the trace on spin_7 (respectively the Killing form) is eight times (respectively forty times) the one we took for reference (associated with Ω). The last two relations are consequences of

the fact that the spaces $\mathrm{Mod}_L(S^2s, S^2s)$ and $\mathrm{Mod}_L(\Lambda^3s, \Lambda^3s)$ are two dimensional. We next use these relations to find a system of generators of $\mathcal{A}_3$ modulo the kernel of $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$, and then to compute the multiplication table for these generators. This allows us to give, up to an automorphism of $\mathcal{A}'_3$, explicit matrices for these elements. $\square$

2. The Universal Vassiliev-Kontsevich Invariant

Let $R = \mathbb{C}[[\alpha]]$ be the ring of formal power series in α with complex coefficients and $K = \mathbb{C}[\alpha^{-1}][[\alpha]]$ its fraction field of Laurent series.
We use the notion of q-tangles introduced in [13].
Let $\mathcal{T}_3$ denote the monoid generated by the five following elementary q-tangles, which are unoriented and given with the parentheses $((**)*)$:

$R_{1,2} =$  , $R_{1,2}^{-1} =$  , $R_{2,3} =$  , $R_{2,3}^{-1} =$  , $A_{1,2} =$ 

Remark that $R_{1,2}$ and $R_{2,3}$ generate together with their inverses the Artin group $B_3 \subset \mathcal{T}_3$ (the braid group with three strands).
In the following, if A is a graded $\mathbb{C}$-algebra, we denote by $\widehat{A}$ the completion of A for the graduation. The universal Vassiliev-Kontsevich invariant (cf [13]) is map on q-tangles which when restricted to $\mathcal{T}_3$ takes values in $\widehat{\mathcal{A}_3}$. Let Z denote this restriction. Then one has

$$Z(R_{1,2}) = \tau_{1,2}\exp(-\frac{1}{2}r_{1,2}) \quad \text{and} \quad Z(R_{2,3}) = \phi^{-1}\tau_{2,3}\exp(-\frac{1}{2}r_{2,3})\phi$$

where $\phi \in \widehat{\mathcal{A}_3}$ is an associator.
Let $Z_{\mathrm{spin}_7} : \mathcal{T}_3 \longrightarrow \widehat{\mathcal{A}'_3}$ be the composition of Z with $\Phi_{\mathfrak{so}_7,\mathrm{spin}_7}$ and set

$$R'_{i,j} = Z_{\mathrm{spin}_7}(R_{i,j}) \quad \text{and} \quad A'_{i,j} = Z_{\mathrm{spin}_7}(A_{i,j}).$$

We again call Z_{spin_7} the invariant of unoriented framed links with values in R.

3. Skein Relations for Z_{spin_7}

Restricting Z_{spin_7} to $B_3 \subset \mathcal{T}_3$ and projecting onto the factors of $\widehat{\mathcal{A}'_3}$ gives four linear representations of B_3, say $\zeta_i : B_3 \longrightarrow \mathrm{GL}_i(R)$ for $i \in \{1, 2, 3, 4\}$.

Theorem 2. The representations ζ_i are simple.

Proof. The proof is based on the knowledge of $R'_{1,2} = \exp(-\frac{\alpha}{2}r'_{1,2})\tau'_{1,2}$ and of $R'_{2,3}$ in degree zero (i.e. modulo α): indeed the associator being equal to 1 in degree zero we have $R'_{2,3} \equiv \tau'_{2,3}$ modulo (α). First remark that ${R'_{1,2}}^2 \equiv {R'_{2,3}}^2 \equiv 1$ modulo (α).

So these representations of B_3 modulo (α) factor through the permutation group $\mathfrak{S}_3$. One can easily check that ζ_2 modulo (α) is the standard representation of $\mathfrak{S}_3$ (the simple representation associated with the Young diagram $[2,1]$).
For $i=3$ and $i=4$, $\zeta_i(R_{1,2})$ has distinct eigenvalues. So, we can compute a basis (v_1,v_2,v_3,v_4) of R^4 (respectively (v_1,v_2,v_3) of R^3) in which $\zeta_i(R_{1,2})$ is diagonal. Therefore if a subspace V of R^4 (respectively R^3) is stable by $\zeta_i(B_3)$, it splits as a direct sum of one dimensional subspaces $\bigoplus_{j\in J} R.v_j$. Now one computes $\zeta_i(R_{2,3})$ modulo (α) in the basis (v_i):

$$\frac{1}{8}\begin{bmatrix} 1 & 9 & 3 & -15 \\ 3 & -5 & 1 & -5 \\ 7 & 7 & 1 & 35 \\ -1 & -1 & 1 & 3 \end{bmatrix} \qquad \frac{1}{4}\begin{bmatrix} -2 & -2 & 10 \\ -1 & -3 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

This shows that the existence of a non-trivial sub-B_3-module is impossible. □

Let us denote by W the element $e^{-\frac{1}{2}\alpha}\in R$.

Corollary 1. The representation Z_{spin_7} of B_3 in $\mathcal{A}'_3$ is given up to an automorphism of $\mathcal{A}'_3$ by

$$R'_{1,2} = \left(\begin{bmatrix} W^{21} & -W(1+\varepsilon W^4+W^8) & -W^9(1+\varepsilon W^4+W^8) & W^{-3} \\ 0 & -W & -(1+\varepsilon W^4)W^9 & W^{-3} \\ 0 & 0 & -W^9 & W^{-3} \\ 0 & 0 & 0 & W^{-3} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -W & -W^{-11}(W^{20}-1) & -W^9 \\ 0 & -W^9 & -W^9 \\ 0 & 0 & W^{-3} \end{bmatrix}, \begin{bmatrix} -W & -W \\ 0 & W^{-3} \end{bmatrix}, [W^{-3}] \right)$$

$$R'_{2,3} = \left(\begin{bmatrix} W^{-3} & 0 & 0 & 0 \\ W^9 & -W^9 & 0 & 0 \\ -\varepsilon W^{-3} & W^{-3}(W^4+\varepsilon) & -W & 0 \\ -\varepsilon W^9 & \varepsilon W^9(W^8+\varepsilon W^4+1) & -W^{13}(W^8+\varepsilon W^4+1) & W^{21} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} W^{-3} & 0 & 0 \\ W^9 & -W^9 & 0 \\ -W^9 & W^{-11}(W^{20}-1) & -W \end{bmatrix}, \begin{bmatrix} W^{-3} & 0 \\ -W^{-3} & -W \end{bmatrix}, [W^{-3}] \right)$$

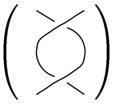
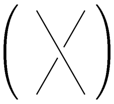
where $\varepsilon=-1$. Furthermore, $Z_{\mathrm{spin}_7}(A_{1,2})$ is given in the same basis by:

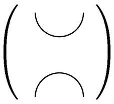
$$A'_{1,2} = \left(\frac{\Delta}{(W^{20}+1)(W^4+1)} \begin{bmatrix} (W^{20}+1)(W^4+1) & -(W^{12}+1) & -W^4(W^{16}+1) & \frac{W^{12}}{W^{12}+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, [0] \right)$$

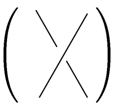
$$\text{where} \qquad \Delta = W^{-18}(W^{20}+1)(W^{12}+1)(W^4+1)$$

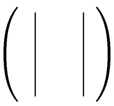
Proof. We use theorem 2 and the classification of small simple representations of B_3 made in [17] to justify the form of the above expressions for $R'_{1,2}$ and $R'_{2,3}$:
In [17], it is shown that there exists up to conjugation a unique simple representation of the group B_3 in $GL_n(K)$, where K is an algebraically closed field and $n \leq 3$, provided the list $(\lambda_1, \dots, \lambda_n) \in K^n$ of the eigenvalues of the half twist is fixed. This gives the result for the last three factors.
Again in [17], it is shown that there exists up to conjugation two simple representations of the group B_3 in $GL_4(K)$ depending on the choice of a square root of $\frac{\lambda_2\lambda_3}{\lambda_1\lambda_4}$. So the representation ζ_4 of B_3 is determined by the choice of a square root εW^{-4} of $\frac{\lambda_2\lambda_3}{\lambda_1\lambda_4} = W^{-8}$. We have kept this ε in the expression of $R'_{2,3}$ and we only have to show that $\varepsilon = -1$.
To see this, look at the representation ζ_4 modulo (α) of $\mathfrak{S}_3$. It is given by the first factor of $\tau'_{1,2}$ and $\tau'_{2,3}$ (see Theorem 1). One can easily check that this representation splits as the sum of the three simple representations of $\mathfrak{S}_3$ (i.e. the one dimensional trivial representation with Young diagram [3], the one dimensional alternating representation with Young diagram [1,1,1] and the two dimensional standard representation with Young diagram [2,1]). The same is true if we put $\varepsilon = -1$ in the above expressions for $R'_{1,2}$ and $R'_{2,3}$. On the other hand, for $\varepsilon = 1$, ζ_4 modulo (α) would be the sum of two times the trivial representation of $\mathfrak{S}_3$ and two times the alternating one. This shows that the representation ζ_4 is equivalent to that obtained with $\varepsilon = -1$.
Next $Z_{\mathrm{spin}_7}(A_{1,2})$ is sent to $\varphi a'_{1,2} = 8\varphi p_E$ for some $\varphi \in R^*$, where p_E stands for the projection onto $E \subset s^{\otimes 3}$, the eigenspace of $R'_{1,2}$ associated to the eigenvalue W^{21}. But the identity $A_{1,2}R_{1,2}R_{2,3}A_{1,2} = A_{1,2}$ in $\mathcal{T}_3$ gives $a'_{1,2}R'_{1,2}R'_{2,3}a'_{1,2} = \varphi^{-1}a'_{1,2}$ and allow us to compute $\varphi = \frac{1}{8}\Delta$. $\square$

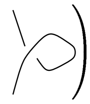
Corollary 2. The invariant Z_{spin_7} of framed unoriented links with values in R has the following properties:

Z_{spin_7}  $= W^7\, Z_{\mathrm{spin}_7}$ 

$+W^4(W^{24}-1)(1+W^4)^{-1} Z_{\mathrm{spin}_7}$ 

$-W^9(1+W^{-8}-W^{-12}) Z_{\mathrm{spin}_7}$  (3.1)

$+W^{-2}(1+W^8-W^{12}) Z_{\mathrm{spin}_7}$ 

$Z_{\mathrm{spin}_7}\Bigg($  $\Bigg) = W^{-21} Z_{\mathrm{spin}_7}\Bigg($  $\Bigg)$ (3.2)

$$Z_{\mathrm{spin}_7}\left(\bigcirc \amalg L\right) = \frac{(1+W^4)(1+W^{12})(1+W^{20})}{W^{18}} Z_{\mathrm{spin}_7}(L) \tag{3.3}$$

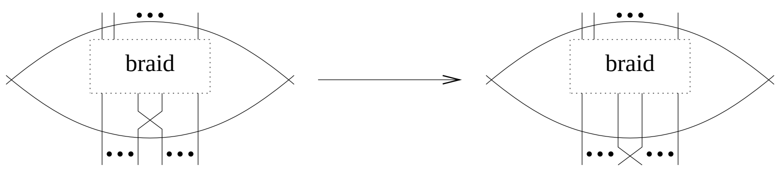
Furthermore, it satisfies the skein relation in Figure 1.
Also one has $Z_{\mathrm{spin}_7}(L \amalg L') = Z_{\mathrm{spin}_7}(L) Z_{\mathrm{spin}_7}(L')$. And if one calls M the involution of R obtained by sending α to $-\alpha$, then the image by Z_{spin_7} of the mirror image of an link L is given by $M_*(Z_{\mathrm{spin}_7}(L))$. In particular, one gets other valid skein relations by inverting all the crossings and replacing W with $M_*(W) = W^{-1}$.

Remark: In [21], the eigenvalues of the half twist for the spinor representation of the quantum group $\mathcal{U}_q\mathfrak{so}_{2n+1}$ are computed. Relations 3.3, 3.2 and 3.1 could be deduced from their work. In theory, one could also use the R-matrix to establish the relation in Figure 1 but in practice, this seems difficult because the relation lives in a 262144 dimensional vector space.

4. An Algorithm for Z_{spin_7}

Theorem 3. The skein relations established above give an algorithmic way to compute Z_{spin_7}. Furthermore this link invariant takes its values in $\mathbb{Z}[W, W^{-1}]$.

Proof. The algorithm works by induction on the number of crossings of the planar projection of the link.
Let G be a planar projection of a link L (G is a 4-valent graph with the information of over or under-crossing at each vertex). We call an eye a region of the plane delimited by two segments which don't intersect except at their extremities. An eye is minimal if it doesn't contain another one. If G doesn't contain any eye then the invariant of L can be computed using the relations (3.3) and (3.2). Observe that if in the interior of a minimal eye the graph is not isotopic to a braid, then it can be reduced using the relations (3.3), (3.2) and removing from its interior the parts of the graph which are not connected to it.
If the interior of a minimal eye is empty use a type II Reidemeister move or the relation (3.1) to reduce the number of crossings in G. If the interior is a braid which can be written as the product $\sigma_1 \ldots \sigma_n$ of n elementary braids then use a type III Reidemeister move or the relation in Figure 1 to get some graphs with less crossings and a graph with the same number of crossings but in which an eye contains a braid isotopic to the smaller product $\sigma_2 \ldots \sigma_n$:



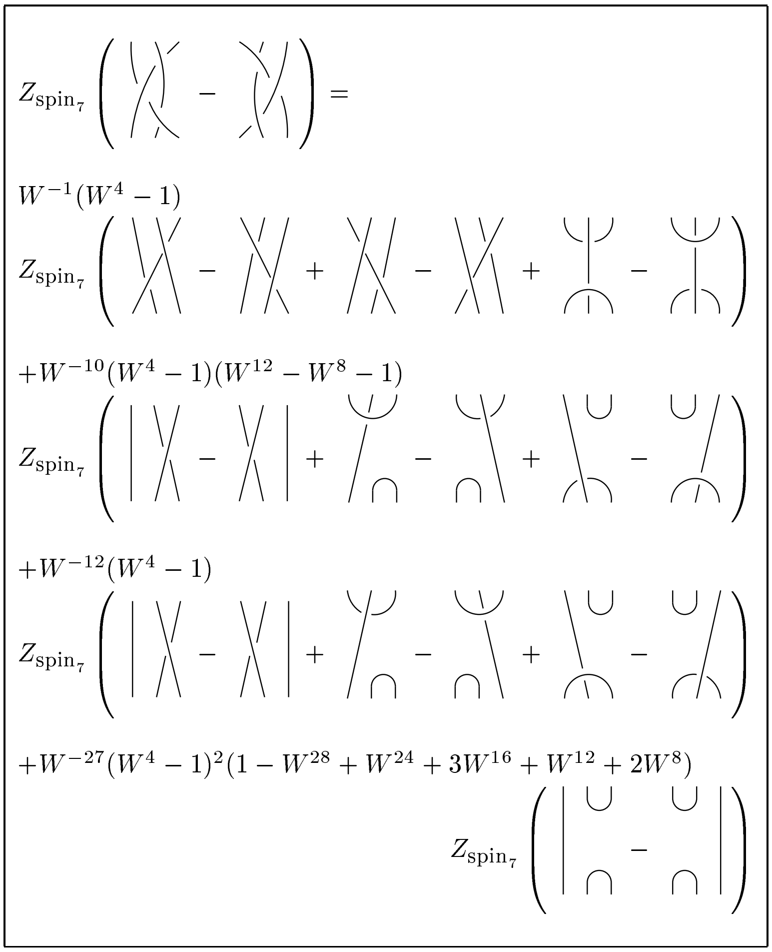
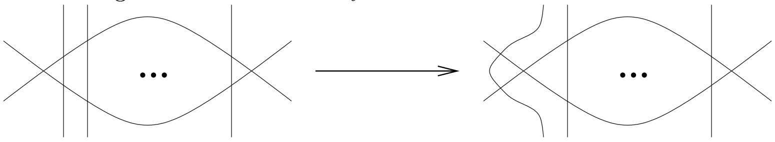


Fig. 1. Skein relation for Z_{spin_7}

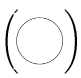
Repeating this process n times leads us to a graph with an eye that contains k vertical strands. Next, again by using a type III Reidemeister move or the relation in Figure 1, we get some graph with less crossings and a graph with the same number of crossings but in which an eye contains $k-1$ vertical strands:

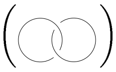


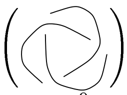
Repeating k times this process to the graph leads to the case of an empty eye. Then use a type II Reidemeister move or the relation (3.1) to reduce the number of crossings in the graph G. $\square$

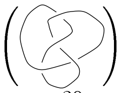
5. Examples

Here are the Z_{spin_7} invariants of some simple links. (In fact the relations (3.3) to (3.1) are sufficient to compute these examples):

$\Delta = Z_{\text{spin}_7}\big($  $\big) = \dfrac{(1+W^4)(1+W^{12})(1+W^{20})}{W^{18}}$

$Z_{\text{spin}_7}\big($  $\big) = \dfrac{\Delta}{W^{24}}(1+W^{24})(1+W^{16})(1+W^{8})$

$Z_{\text{spin}_7}\big($  $\big) = \Delta\big(W^{27}+W^{19}-W^{15}+W^{11}-W^{7}+2W^{3}-2W^{-1}+2W^{-5}$
$-2W^{-9}+2W^{-13}-2W^{-17}+2W^{-21}-2W^{-25}+W^{-29}-2W^{-33}+W^{-37}$
$-W^{-41}+W^{-45}\big)$

$Z_{\text{spin}_7}\big($  $\big) = \Delta\big(W^{48}-W^{44}+2W^{40}-3W^{36}+3W^{32}-4W^{28}+6W^{24}$
$-6W^{20}+7W^{16}-8W^{12}+8W^{8}-9W^{4}+9-9W^{-4}+8W^{-8}-8W^{-12}+7W^{-16}$
$-6W^{-20}+6W^{-24}-4W^{-28}+3W^{-32}-3W^{-36}+2W^{-40}-W^{-44}+W^{-48}\big)$

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