MULTIVARIABLE LINK INVARIANTS ARISING FROM $\mathfrak{sl}(2|1)$ AND THE ALEXANDER POLYNOMIAL

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Abstract. In this paper we construct a multivariable link invariant arising from the quantum group associated to the special linear Lie superalgebra $\mathfrak{sl}(2|1)$. The usual quantum group invariant of links associated to (generic) representations of $\mathfrak{sl}(2|1)$ is trivial. However, we modify this construction and define a nontrivial link invariant. This new invariant can be thought of as a multivariable version of the Links-Gould invariant. We also show that after a variable reduction our invariant specializes to the Conway potential function, which is a refinement of the multivariable Alexander polynomial.

Introduction

There are deep connections between quantum group theory and low-dimensional topology. For example, every representation of a semisimple Lie algebra gives rise to a quantum group invariant of knots and, more generally, links. It is well known that similar invariants exist in the setting of Lie superalgebras. Most nontrivial invariants arising from Lie superalgebras are only invariants of long knots or (1,1)-tangles. This is true because, in many cases the super-dimension of a finite-dimensional module over a Lie superalgebra is zero. This implies that the corresponding deformed module has quantum dimension zero. The standard quantum link invariant arising from such a module is trivial. For this reason it can be difficult to construct nontrivial link invariants arising from Lie superalgebras. In this paper we construct a multivariable link invariant arising from representations over the quantum group associated to $\mathfrak{sl}(2|1)$.

Our construction uses the Reshetikhin-Turaev quantum group invariant. In particular, let F be the usual functor from the category of framed tangles colored by topologically free $U_h(\mathfrak{sl}(2|1))$-modules of finite rank to the category of $U_h(\mathfrak{sl}(2|1))$-modules (see [18]). In Section 2, we define a map d from the set of typical representations of $U_h(\mathfrak{sl}(2|1))$ to the ring $\mathbb{C}[[h]]$. If T_λ is a framed $(1,1)$-tangle colored by representations of $U_h(\mathfrak{sl}(2|1))$ such that the open string is colored by the deformed typical module $\widetilde{V}(\lambda)$ of weight λ, then $F(T_\lambda) = xId_{\widetilde{V}(\lambda)}$ for some x in $\mathbb{C}[[h]]$. We set $F'(T_\lambda) = x.d(\lambda)$.

Theorem 1. *The map F' induces a well defined invariant of framed links colored by at least one typical representation of $U_h(\mathfrak{sl}(2|1))$. In other words, if L is a framed link colored by $U_h(\mathfrak{sl}(2|1))$-modules at least one of which is typical and the closure of T_λ is equal to L then the map given by $L \mapsto F'(T_\lambda)$ is a well defined framed link invariant.*

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The proof of Theorem 1 is given in Section 2. We denote the framed link invariant of Theorem 1 by F'. This invariant can be thought of as a renormalization of the usual quantum invariant in a similar way as made by J. Murakami [15], Kashaev [12] and Degushi [3]. The construction here differs from theirs as we work with ribbon categories whereas their proofs use a Markov trace for the colored braid group.

Every irreducible topologically free representation of finite rank of $U_h(\mathfrak{sl}(2|1))$ has a highest weight $\lambda \in \Lambda \simeq \mathbb{N} \times \mathbb{C}$. Thus the isomorphism classes of such representations are indexed by the set $\mathbb{N} \times \mathbb{C}$. In Section 3 we will use F' and modules of highest weight of the form $(0, \alpha)$ to show that there exists an Alexander type multivariable link invariant M. When saying "Alexander type", we don't mean that $M(L)$ is determined by the homology of the maximal free Abelian covering of $S^3 \setminus L$ but, as the multivariable Alexander of links, it takes values in a ring naturally isomorphic with (a localization of) the group ring of $H_1(S^3 \setminus L)$ (but with coefficient in $\mathbb{Z}[q^{\pm 1}]$ instead of $\mathbb{Z}$) and both invariants vanish on disjoint union of links. In particular, we will prove the following theorem.

Theorem 2. *Let L' be a framed oriented link with n ordered components. Let L be the non-framed link which underlies L' (L is still oriented and has the same ordering on its components). There exists an Alexander type multivariable link invariant M with the following properties.*

1. *If $n = 1$ then $M(L)$ takes values in $(q_1 - q_1^{-1})^{-1}(q_1 q - q_1^{-1} q^{-1})^{-1}\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$.*
2. *If $n \geq 2$ then $M(L)$ takes values in $\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \dots, q_n^{\pm 1}]$.*
3. *If $(\alpha_1, \dots \alpha_n) \in (\mathbb{C} \setminus \{0, -1\})^n$ and the i-th component of L' is colored by the weight module of weight $(0, \alpha_i)$ then*
$$F'(L') = \mathrm{e}^{-\sum lk_{ij}(2\alpha_i\alpha_j + \alpha_i + \alpha_j)h/2} M(L)|_{q=\mathrm{e}^{h/2}, q_i = \mathrm{e}^{\alpha_i h/2}}$$

where lk_{ij} is the linking number of the i-th and j-th components of L.

As a consequence of the last point in Theorem 2, we get that up to a change of variable $(p - 1/p)(pq - 1/(pq))M(q, p, p \dots, p)$ is nothing but the Links-Gould invariant ([14]). The Links-Gould invariant is a two variable quantum group invariant arising from an explicit one-parameter family of representations of the general linear Lie superalgebra $\mathfrak{gl}(2|1)$. The invariant M can be thought of as a multivariable version of the Links-Gould invariant (see remark 5.1).

In Section 4, using techniques similar to Viro in [19] we extend F' to an invariant of colored oriented framed trivalent graphs. Using this extension, in Proposition 4.6 we give a complete set of skein relations to compute this invariant. With the use of these skein relations we are able to show that M specialize to the Conway potential function. (For a good history and a nice geometric construction the Conway potential function see [2].) In particular, in Section 4 we prove the following theorem.

Theorem 3. *Let ∇ be the Conway potential function of a link. Then*
$$\nabla(L)(t_1, ..., t_n)|_{t_k = q_k^2} = iM(L)(q, q_1, ..., q_n)|_{q=i}.$$

This generalize the proof of [5] that the Links-Gould invariant dominates the one-variable Alexander polynomial.

In subsequent papers the authors plan to use the framed link invariant F' to construct a quantum invariant of 3-manifolds. Because of the representation theory of

$\mathfrak{sl}(2|1)$, this invariant would be very different from 3-manifold invariant arising from Lie algebras. We also plan to generalize the constructions of the invariants F' and M to other Lie superalgebras. The corresponding $\mathfrak{sl}$-invariants should specialize to the multivariable invariants of [1] and to Kashaev's quantum dilogarithm invariants of links [12].

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1. Preliminaries

In the section we review background material that will be used in the following sections.

A *super-space* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{C}$. We denote the parity of a homogeneous element $x \in V$ by $\bar{x} \in \mathbb{Z}_2$. We say x is even (odd) if $x \in V_{\bar{0}}$ (resp. $x \in V_{\bar{1}}$). A *Lie superalgebra* is a super-space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a super-bracket $[,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ that preserves the $\mathbb{Z}_2$-grading, is super-antisymmetric ($[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$), and satisfies the super-Jacobi identity (see [11]). Throughout, all modules will be $\mathbb{Z}_2$-graded modules (module structures which preserve the $\mathbb{Z}_2$-grading, see [11]).

1.1. The Lie superalgebra $\mathfrak{sl}(2|1)$ and its weight modules.

In this subsection we define $\mathfrak{sl}(2|1)$ and discuss some properties of $\mathfrak{sl}(2|1)$-modules. Modules of $\mathfrak{sl}(2|1)$ are different in nature than modules over semi-simple Lie algebras. For example, the category of finite-dimensional $\mathfrak{sl}(2|1)$-modules is not semi-simple.

Let $A = (a_{ij})$ be the 2×2 matrix given by $a_{11} = 2$, $a_{12} = a_{21} = -1$ and $a_{22} = 0$.

Definition 1.1. *Let $\mathfrak{sl}(2|1)$ be the Lie superalgebra generated by h_i, e_i, and f_i, $i = 1, 2$, where h_1, h_2, e_1 and f_1 are even, e_2 and f_2 are odd, and the generators satisfy the relations*

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij}e_j, \qquad [h_i, f_j] = -a_{ij}f_j \qquad [e_i, f_j] = \delta_{ij}h_i,$$

$$[e_2, e_2] = [f_2, f_2] = 0, \qquad [e_1, [e_1, e_2]] = [f_1, [f_1, f_2]] = 0.$$

Set $\Lambda = \mathbb{N} \times \mathbb{C}$. For every pair $(a_1, a_2) \in \mathbb{C}^2$ Kac [11] defined a highest weight $\mathfrak{sl}(2|1)$-module $V(a_1, a_2)$ with a highest weight vector v_0 having the property that $h_i.v_0 = a_i v_0$ and $e_i v_0 = 0$. We say $a = (a_1, a_2)$ is a weight and to simplify notation we will write $V(a)$ for $V(a_1, a_2)$. Kac showed that finite-dimensional irreducible $\mathfrak{sl}(2|1)$-modules are characterized up to isomorphism by the elements of Λ. Moreover, the weight modules $V(a)$, $a \in \Lambda$ are divided into two classes: typical and atypical.

There are many equivalent definitions for a weight module to be typical (see [10]). In the interest of space, we will give the following characterization of typical modules (which easily follows from Theorem 1 in [10]). A $\mathfrak{sl}(2|1)$ weight module $V(a_1, a_2)$ is *typical* if and only if $a_1 + a_2 + 1 \neq 0$ and $a_2 \neq 0$. If $V(a_1, a_2)$ is not typical we say it is *atypical*. The following lemma, due to Kac [10], is useful when working with typical modules.

Lemma 1.2. *Typical module are projective and injective in the category of finite-dimensional $\mathfrak{sl}(2|1)$-modules. In particular, a typical $V(a)$ splits in any finite-dimensional representation.*

Let $\mathcal{S}_1 \subset \Lambda$ be the set of atypical weights. Let $\mathcal{S}_2 \subset \Lambda^2$ be the set

$$\mathcal{S}_2 := (\mathcal{S}_1 \times \Lambda) \cup (\Lambda \times \mathcal{S}_1) \cup \{(a, b) \in \Lambda^2 : V(a) \otimes V(b) \text{ is not semi-simple}\}.$$

The character of a typical module is well known (see [10]), using these formulas and Lemma 1.2 we have the following results about the tensor product of two modules.

Lemma 1.3. *Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two weights such that $(a, b) \in \Lambda^2 \setminus \mathcal{S}_2$ and $b_1 = 0$. If $a_1 \neq 0$ then $V(a_1, a_2) \otimes V(0, b_2)$ is isomorphic to*

$$V(a_1, a_2 + b_2) \oplus V(a_1 + 1, a_2 + b_2) \oplus V(a_1 - 1, a_2 + b_2 + 1) \oplus V(a_1, a_2 + b_2 + 1). \quad (1)$$

If $a_1 = 0$ then $V(0, a_2) \otimes V(0, b_2)$ is isomorphic to the direct sum (1) *without the module $V(a_1 - 1, a_2 + b_2 + 1)$.*

Proof. Characters are living in the group-ring $\mathbb{Z}\mathfrak{h}^*$ of the space of weights $\mathfrak{h}^*$ (here and after, $\mathfrak{h} =< h_1, h_2 >$ is the Cartan subalgebra of $\mathfrak{sl}(2|1)$). A basis of $\mathfrak{h}^*$ is given by (ϵ_1, ϵ_2) whith $\epsilon_1(h_1) = \epsilon_2(h_2) = 1$, $\epsilon_1(h_2) = 0$ and $\epsilon_2(h_1) = -1$. Hence, if $(a_1, a_2) \in \Lambda$, the weight w defined by $w(h_i) = a_i$ is represented in $\mathbb{Z}\mathfrak{h}^*$ by the element $\mathrm{e}^{a_1\epsilon_1 + a_2(\epsilon_1+\epsilon_2)}$. In [10], Kac gives a general formula for the character of a typical module. In our context, the character of $V(a_1, a_2)$ is given by

$$\mathrm{ch}(V(a_1, a_2)) = (1 + \mathrm{e}^{\epsilon_1})(1 + \mathrm{e}^{\epsilon_2})\, \mathrm{e}^{a_1\epsilon_1 + a_2(\epsilon_1+\epsilon_2)}\, \frac{1 - r^{a_1+1}}{1 - r} \text{ whith } r = \mathrm{e}^{\epsilon_2 - \epsilon_1}. \quad (2)$$

Hence $\mathrm{ch}(V(a_1, a_2)).\, \mathrm{ch}(V(0, b_2)) = (1 + \mathrm{e}^{\epsilon_1})(1 + \mathrm{e}^{\epsilon_2})\, \mathrm{e}^{a_1\epsilon_1 + (a_2+b_2)(\epsilon_1+\epsilon_2)}\, X$ with

$$\begin{aligned} X &= (1 + \mathrm{e}^{\epsilon_1})(1 + \mathrm{e}^{\epsilon_2}) \frac{1 - r^{a_1+1}}{1 - r} \\ &= (1 + \mathrm{e}^{\epsilon_1+\epsilon_2} + e^{\epsilon_1}(1 + r)) \frac{1 - r^{a_1+1}}{1 - r} \\ &= \frac{1}{1 - r} \left((1 - r^{a_1+1}) + \mathrm{e}^{\epsilon_1+\epsilon_2}(1 - r^{a_1+1}) + \mathrm{e}^{\epsilon_1}(1 - r^{a_1+2}) + \mathrm{e}^{-\epsilon_1}\, \mathrm{e}^{\epsilon_1+\epsilon_2}(1 - r^{a_1}) \right) \end{aligned}$$

And using equation 2, we see that the product $\mathrm{ch}(V(a_1, a_2)).\, \mathrm{ch}(V(0, b_2))$ is the sum of the characters of the typical representations of (1). □

Remark 1.4. *Lemma 1.2 implies that if $(a, b) \in \Lambda^2$, the module $V(a) \otimes V(b)$ is always the direct sum of typical modules direct sum the semi-direct product of atypical modules.*

1.2. **The quantization $U_h(\mathfrak{sl}(2|1))$.** Let h be an indeterminate. Set $q = e^{h/2}$. We adopt the following notations:

$$q^z = \mathrm{e}^{zh/2} \quad \text{and} \quad \{z\} = q^z - q^{-z}.$$

Definition 1.5 ([6, 16]). *Let $U_h(\mathfrak{sl}(2|1))$ be the $\mathbb{C}[[h]]$-Hopf superalgebra generated by the elements h_i, E_i and F_i, $i = 1, 2$, subject to the relations:*

$$[h_i, h_j] = 0, \qquad [h_i, E_j] = a_{ij}E_j, \qquad [h_i, F_j] = -a_{ij}F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \qquad E_2^2 = F_2^2 = 0,$$

$$E_1^2E_2-(q+q^{-1})E_1E_2E_1+E_2E_1^2=0 \quad F_1^2F_2-(q+q^{-1})F_1F_2F_1+F_2F_1^2=0$$

where $[,]$ *is the super-commutator given by* $[x,y]=xy-(-1)^{\bar{x}\bar{y}}yx$. *All generators are even except for* E_2 *and* F_2 *which are odd. The coproduct, counit and antipode given by*

$$\begin{aligned}
\Delta(E_i) &= E_i\otimes 1+q^{-h_i}\otimes E_i, & \epsilon(E_i) &= 0 & S(E_i) &= -q^{h_i}E_i\\
\Delta(F_i) &= F_i\otimes q^{h_i}+1\otimes F_i, & \epsilon(F_i) &= 0 & S(F_i) &= -F_iq^{-h_i}\\
\Delta(h_i) &= h_i\otimes 1+1\otimes h_i, & \epsilon(h_i) &= 0 & S(h_i) &= -h_i
\end{aligned}$$

Let $E'=E_1E_2-q^{-1}E_2E_1$ and $F'=F_2F_1-qF_1F_2$. Khoroshkin, Tolstoy [13] and Yamane [21] showed that $U_h(\mathfrak{sl}(2|1))$ has an explicit R-Matrix R. In particular, they showed that $R=\check{R}K$ where

$$\check{R}=exp_q(\{1\}E_1\otimes F_1)exp_q(-\{1\}E'\otimes F')exp_q(-\{1\}E_2\otimes F_2), \tag{3}$$

$$K=q^{-h_1\otimes h_2-h_2\otimes h_1-2h_2\otimes h_2} \tag{4}$$

and $exp_q(x):=\sum_{n=0}^{\infty}x^n/(n)_q!$, $(n)_q!:=(1)_q(2)_q...(n)_q$ and $(k)_q:=(1-q^k)/(1-q)$.

We say a $U_h(\mathfrak{sl}(2|1))$-module W is topologically free of finite rank if it is isomorphic as a $\mathbb{C}[[h]]$-module to $V[[h]]$, where V is a finite-dimensional $\mathfrak{sl}(2|1)$-module. Let $\mathcal{M}$ the category of topologically free of finite rank $U_h(\mathfrak{sl}(2|1))$-modules. A standard argument shows that $\mathcal{M}$ is a ribbon category (for details see [7]). Let V,W be objects of $\mathcal{M}$. We denote the braiding and twist morphisms of $\mathcal{M}$ as

$$c_{V,W}:V\otimes W\to W\otimes V, \qquad \theta_V:V\to V$$

respectively. We also denote the duality morphisms of $\mathcal{M}$ as

$$b_V:\mathbb{C}[[h]]\to V\otimes V^*, \qquad d'_V:V\otimes V^*\to\mathbb{C}[[h]]$$

Let $\mathcal{T}=Rib_{\mathcal{M}}$ be the ribbon category of ribbon graphs over $\mathcal{M}$ in the sense of Turaev (see [18]). The set of morphisms $\mathcal{T}((V_1,\dots,V_n),(W_1,\dots,W_m))$ is a space of formal linear combinations of ribbon graphs over $\mathcal{M}$. Let F be the usual ribbon functor from $\mathcal{T}$ to $\mathcal{M}$ (see [18]).

In [8], Geer shows that a $\mathfrak{sl}(2|1)$ weight module $V(a)$ can be deformed to a $U_h(\mathfrak{sl}(2|1))$ weight module $\widetilde{V}(a)$. In particular, Geer shows that the characters of $V(a)$ and $\widetilde{V}(a)$ are equal and that as a super-space $\widetilde{V}$ is equal to $V[[h]]$. In other words, the representation theory of $U_h(\mathfrak{sl}(2|1))$ is parallel to that of the Lie superalgebra $\mathfrak{sl}(2|1)$.

It is well known that the super-dimension of any typical representation of $\mathfrak{sl}(2|1)$ is zero. The discussion above implies that the quantum dimension of any deformed typical representation over $U_h(\mathfrak{sl}(2|1))$ is zero. It follows that the functor F is zero on all closed ribbon graph over with at least one color which is a deformed typical module. For this reason it can be difficult to construct nontrivial link invariants from $\mathfrak{sl}(2|1)$.

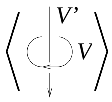
2. Proof of Theorem 1

In this section we prove Theorem 1. To this end, we compute the value of the long Hopf link colored by finite-dimensional representations. We use this to define a function d from the set of weights $\Lambda\setminus\mathcal{S}_1$ to $\mathbb{C}[[h]]$. Finally, we use d to show that F induces a well defined invariant of links.

Definition 2.1. *If $T \in \mathcal{T}(\widetilde{V}(a), \widetilde{V}(a))$ where $V(a)$ is an finite-dimensional irreducible $\mathfrak{sl}(2|1)$ weight module then $F(T) = x.\mathrm{Id}_{\widetilde{V}(a)} \in \mathrm{End}_{U_h(\mathfrak{sl}(2|1))}(\widetilde{V}(a))$ for an $x \in \mathbb{C}[[h]]$. We define the bracket of T to be*

$$< T >= x$$

For example, if V, V' are modules of $\mathcal{M}$ such that V' is irreducible, we define

$S'(V, V') =$ 

When $V = \widetilde{V}(a)$ and $V' = \widetilde{V}(b)$ are irreducible highest weight modules with weights a and b in Λ we write $S'(a, b)$ for $S'(V, V')$.

Proposition 2.2. *Let $a \in \Lambda \setminus \mathcal{S}_1$ and $b \in \Lambda$ then*

$$S'(a,b) = q^{-(2a_2+a_1+1)(2b_2+b_1+1)} \frac{\{(a_1+1)(b_1+1)\}}{\{b_1+1\}} \{b_2\}\{b_2+b_1+1\}$$

Proof. We will prove the proposition in three parts. In the first part we set up the notation, in the second we give two important facts and then finally we give a calculation which completes the proof.

A direct calculation (using the character) shows that $\widetilde{V}(a) = \widetilde{V}(a_1, a_2)$ has a basis of weight vectors $v_{\bar{j},i}, w_{\bar{j},i}$ for $j = 0, 1$ and $i = 0, ..., a_1$ where $v_{\bar{0},i}$, $w_{\bar{0},i}$ are even and $v_{\bar{1},i}, w_{\bar{1},i}$ are odd. The elements h_1 and h_2 act on these weight vectors as follows:

$$\begin{aligned}
h_1 v_{\bar{0},i} &= (a_1 - 2i) v_{\bar{0},i}, & h_2 v_{\bar{0},i} &= (a_2 + i) v_{\bar{0},i}, \\
h_1 v_{\bar{1},i} &= (a_1 - 2i + 1) v_{\bar{1},i}, & h_2 v_{\bar{1},i} &= (a_2 + i) v_{\bar{1},i}, \\
h_1 w_{\bar{0},i} &= (a_1 - 2i) w_{\bar{0},i}, & h_2 w_{\bar{0},i} &= (a_2 + i + 1) w_{\bar{0},i}, \\
h_1 w_{\bar{1},i} &= (a_1 - 2i - 1) w_{\bar{1},i}, & h_2 w_{\bar{1},i} &= (a_2 + i + 1) w_{\bar{1},i}.
\end{aligned} \tag{5}$$

Let v_b be a highest weight vector of $\widetilde{V}(b)$. Recall the R-matrix is of the form $R = \check{R}K$. From (5) we have

$$\begin{aligned}
K(v_b \otimes v_{\bar{0},i}) &= q^{-b_1(a_2+i) - b_2(a_1-2i) - 2b_2(a_2+i)} v_b \otimes v_{\bar{0},i}, \\
K(v_b \otimes v_{\bar{1},i}) &= q^{-b_1(a_2+i) - b_2(a_1-2i+1) - 2b_2(a_2+i)} v_b \otimes v_{\bar{1},i}, \\
K(v_b \otimes w_{\bar{0},i}) &= q^{-b_1(a_2+i+1) - b_2(a_1-2i) - 2b_2(a_2+i+1)} v_b \otimes w_{\bar{0},i}, \\
K(v_b \otimes w_{\bar{1},i}) &= q^{-b_1(a_2+i+1) - b_2(a_1-2i-1) - 2b_2(a_2+i+1)} v_b \otimes w_{\bar{1},i}.
\end{aligned} \tag{6}$$

and K is symmetric so acts by the same scalars on $v_{\bar{j},i} \otimes v_b$ and $w_{\bar{j},i} \otimes v_b$, respectively.

We know give two facts. Let v be any weight vector of $\widetilde{V}(a)$.

Fact 1: $R(v_b \otimes v) = k(v_b \otimes v)$ where k is the element of $\mathbb{C}[[h]]$ given by the action of K on $v_b \otimes v$.

This fact follows from equation (3) and the property that $E_i v_b = 0$ for $i = 1, 2$.

Fact 2: All the pure tensors of the element $(\check{R} - 1)(v \otimes v_b) \in \widetilde{V}(a) \otimes \widetilde{V}(b)$ are of the form $v' \otimes w'$ where w' is a weight vector of $\widetilde{V}(b)$ and v' is a weight vector of $\widetilde{V}(a)$ whose weight is of strictly higher order than that of v.

Fact 2 is true because $E_i^n v$ (for $i = 1, 2$ and $n \in \mathbb{N}$) is zero or a weight vector whose weight is of strictly higher order than the weight of v.

We will now compute $S'(a, b)$ directly. Let V be an object of $\mathcal{M}$ and recall that the duality morphisms $b_V : \mathbb{C}[[h]] \to V \otimes V^*$ and $d'_V : V \otimes V^* \to \mathbb{C}[[h]]$ are defined as follows. The morphism b_V is the $\mathbb{C}[[h]]$-linear extension of the coevaluation map on the underlying $\mathfrak{sl}(2|1)$-module. In particular,

$$b_{\widetilde{V}(a)}(1) = \sum_{j=0}^{1} \sum_{i=0}^{a_1} (v_{\bar{j},i} \otimes v_{\bar{j},i}^* + w_{\bar{j},i} \otimes w_{\bar{j},i}^*) \tag{7}$$

As in the case of semi-simple Lie algebras we have

$$d'_{\widetilde{V}(a)}(v \otimes \alpha) = (-1)^{\bar{v}\bar{\alpha}} \alpha(e^{h<\mu,\rho>} v)$$

where v is a weight vector of $\widetilde{V}(a)$ of weight $\mu \in \mathfrak{h}^*$, ρ is the half sum of the positive even roots minus the half sum of the positive odd roots, and $<,>$ is the natural non-degenerate bilinear form on $\mathfrak{h}^*$ coming from the super-trace. From this a direct calculation shows

$$\begin{aligned} d'_{\widetilde{V}(a)}(v_{\bar{j},i} \otimes v_{\bar{j},i}^*) &= (-1)^{\bar{j}} e^{(-a_2-i)h} = (-1)^{\bar{j}} q^{-2a_2-2i} \\ d'_{\widetilde{V}(a)}(w_{\bar{j},i} \otimes w_{\bar{j},i}^*) &= (-1)^{\bar{j}} e^{(-a_2-i-1)h} = (-1)^{\bar{j}} q^{-2a_2-2i-2}. \end{aligned} \tag{8}$$

Consider the element $S \in \mathrm{End}_{U_h(\mathfrak{sl}(2|1))}(\widetilde{V}(b))$ given by

$$S = (Id_{\widetilde{V}(b)} \otimes d'_{\widetilde{V}(a)}) \circ (c_{\widetilde{V}(a), \widetilde{V}(b)} \otimes Id_{\widetilde{V}(a)^*}) \circ (c_{\widetilde{V}(b), \widetilde{V}(a)} \otimes Id_{\widetilde{V}(a)^*}) \circ (Id_{\widetilde{V}(b)} \otimes b_{\widetilde{V}(a)}).$$

To simplify notation set $S = (X_1)(X_2)(X_3)(X_4)$ where X_i is the corresponding morphism in the above formula. By definition $S(v_b) = S'(a, b) v_b$, so it suffices to compute $S(v_b)$.

$$\begin{aligned}
S(v_b) &= (X_1)(X_2)(X_3)\Big(v_b \otimes \sum_{j=0}^{1}\sum_{i=0}^{a_1}(v_{\bar{j},i}\otimes v^*_{\bar{j},i} + w_{\bar{j},i}\otimes w^*_{\bar{j},i})\Big)\\
&= (X_1)(X_2)\Big(\sum_{i=1}^{a_1}\big(q^{-b_1(a_2+i)-b_2(a_1-2i)-2b_2(a_2+i)}v_{\bar{0},i}\otimes v_b\otimes v^*_{\bar{0},i}\\
&\qquad + q^{-b_1(a_2+i+1)-b_2(a_1-2i)-2b_2(a_2+i+1)}w_{\bar{0},i}\otimes v_b\otimes w^*_{\bar{0},i}\\
&\qquad + q^{-b_1(a_2+i)-b_2(a_1-2i+1)-2b_2(a_2+i)}v_{\bar{1},i}\otimes v_b\otimes v^*_{\bar{1},i}\\
&\qquad + q^{-b_1(a_2+i+1)-b_2(a_1-2i-1)-2b_2(a_2+i+1)}w_{\bar{1},i}\otimes v_b\otimes w^*_{\bar{1},i})\Big)\\
&= (X_1)\Big(\sum_{i=1}^{a_1}\big(q^{-2b_1(a_2+i)-2b_2(a_1-2i)-4b_2(a_2+i)}v_b\otimes v_{\bar{0},i}\otimes v^*_{\bar{0},i}\\
&\qquad + q^{-2b_1(a_2+i+1)-2b_2(a_1-2i)-4b_2(a_2+i+1)}v_b\otimes w_{\bar{0},i}\otimes w^*_{\bar{0},i}\\
&\qquad + q^{-2b_1(a_2+i)-2b_2(a_1-2i+1)-4b_2(a_2+i)}v_b\otimes v_{\bar{1},i}\otimes v^*_{\bar{1},i}\\
&\qquad + q^{-2b_1(a_2+i+1)-2b_2(a_1-2i-1)-4b_2(a_2+i+1)}v_b\otimes w_{\bar{1},i}\otimes w^*_{\bar{1},i}\\
&\qquad + \sum_k w'_k\otimes v'_k\otimes z_k)\Big)\\
&= q^{-2b_1a_2-2b_2a_1-4b_2a_2-2a_2}\sum_{i=1}^{a_1}\Big(q^{-2b_1i+4b_2i-4b_2i-2i}+q^{-2b_1(i+1)+4b_2i-4b_2(i+1)-2i-2}\\
&\qquad - q^{-2b_1i-2b_2(-2i+1)-4b_2i-2i} - q^{-2b_1(i+1)-2b_2(-2i-1)-4b_2(i+1)-2i-2}\Big)v_b\\
&= \big(q^{-(2a_2+a_1+1)(2b_2+b_1+1)}\frac{\{(a_1+1)(b_1+1)\}}{\{b_1+1\}}\{b_2\}\{b_2+b_1+1\}\big)v_b
\end{aligned}$$

where $z_k = v^*_{\bar{j},i}$ or $w^*_{\bar{j},i}$ (for some $j=1,2$ and $i=0,...,a_1$), w'_k is a weight vector of $\widetilde{V}(b)$ and v'_k is a weight vector of $\widetilde{V}(a)$. From Fact 2 we have that $z_k(v'_k)=0$. Moreover, the first equality of the above equation follows from (7), the second from (6) and Fact 1, the third from (6) and Fact 2, and finally the fourth from (8). The key observation in this proof is that Facts 1 and 2 imply that in the above computation the only contribution of the action of the R-matrix comes from K. $\square$

We are lead to the following definition.

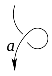
Definition 2.3. *If $a\in\Lambda\setminus\mathcal{S}_1$, we set*

$$d(a) = \frac{\{a_1+1\}}{\{1\}\{a_2\}\{a_2+a_1+1\}}$$

so that one has the symmetry

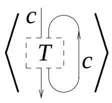
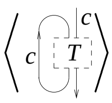
$$\forall (a,b)\in(\Lambda\setminus\mathcal{S}_1)^2,\quad d(b)S'(a,b) = d(a)S'(b,a). \tag{9}$$

Lemma 2.4. *Let $a=(a_1,a_2)\in\Lambda\setminus\mathcal{S}_1$. Then the value of the twist $\theta_{\widetilde{V}(a)}$ is $q^{-2a_2(1+a_1+a_2)}$. In other words,*

$\Big\langle$  $\Big\rangle = q^{-2a_2(1+a_1+a_2)}.$

Proof. The proof follow from (6), (8) and Fact 1 in the proof of Proposition 2.2. □

Lemma 2.5. *Let c be the weight $(0,1)$. Set $V_0 := V(c)$. Then we have*

 $=$ 

for all $T \in \mathcal{T}\big((\widetilde{V}_0, \widetilde{V}_0), (\widetilde{V}_0, \widetilde{V}_0)\big)$.

Proof. Set $E = \mathrm{End}_{U_h(\mathfrak{sl}(2|1))}(\widetilde{V}_0 \otimes \widetilde{V}_0)$. Consider the following linear forms on E:

$$tr_L(f) = (d_{\widetilde{V}_0} \otimes Id_{\widetilde{V}_0}) \circ (Id_{\widetilde{V}_0^*} \otimes f) \circ (b'_{\widetilde{V}_0} \otimes Id_{\widetilde{V}_0}) \in \mathrm{End}_{U_h(\mathfrak{sl}(2|1))}(\widetilde{V}_0) \cong \mathbb{C}[[h]],$$

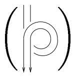
$$tr_R(f) = (Id_{\widetilde{V}_0} \otimes d'_{\widetilde{V}_0}) \circ (f \otimes Id_{\widetilde{V}_0^*}) \circ (Id_{\widetilde{V}_0} \otimes b_{\widetilde{V}_0}) \in \mathrm{End}_{U_h(\mathfrak{sl}(2|1))}(\widetilde{V}_0) \cong \mathbb{C}[[h]].$$

To complete the proof it suffices to show that for any $T \in \mathcal{T}((\widetilde{V}_0, \widetilde{V}_0), (\widetilde{V}_0, \widetilde{V}_0))$ we have

$$(tr_L \circ F)(T) = (tr_R \circ F)(T) \tag{10}$$

where F is the functor described in subsection 1.2.

From Lemma 1.3 we have that $\widetilde{V}_0 \otimes \widetilde{V}_0 \cong \widetilde{V}(0,2) \oplus \widetilde{V}(1,2) \oplus \widetilde{V}(0,3)$. Any element of E acts as a scalar on each summand and so the dimension of E is 3. We will now give a basis for E.

Consider the twist $\theta_{\widetilde{V}_0 \otimes \widetilde{V}_0} = F$  which is an element of E. From Lemma 2.4 we have

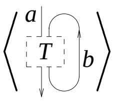
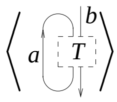
$$\theta_{\widetilde{V}(0,2)} = q^{-12} \qquad \theta_{\widetilde{V}(1,2)} = q^{-16} \qquad \theta_{\widetilde{V}(0,3)} = q^{-24}.$$

Since

$$\begin{vmatrix} 1 & q^{-12} & q^{-24} \\ 1 & q^{-16} & q^{-32} \\ 1 & q^{-24} & q^{-48} \end{vmatrix} = (q^{-24} - q^{-16})(q^{-24} - q^{-12})(q^{-16} - q^{-12}) \neq 0$$

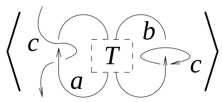
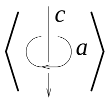
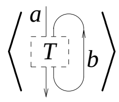
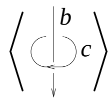
we have that $(Id_{\widetilde{V}_0 \otimes \widetilde{V}_0}, \theta_{\widetilde{V}_0 \otimes \widetilde{V}_0}, \theta^2_{\widetilde{V}_0 \otimes \widetilde{V}_0})$ form a basis of $E \otimes \mathbb{C}[[h]][h^{-1}]$. But tr_L and tr_R clearly have the same values on this basis and thus equality (10) holds as $\mathbb{C}[[h]]$ is an integral domain. □

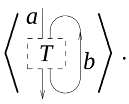
Lemma 2.6. *Let (a,b) be any pair of weights belonging to $(\Lambda \setminus \mathcal{S}_1)^2$. Then we have*

$d(a)$  $= d(b)$ 

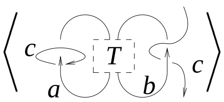
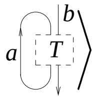
for all $T \in \mathcal{T}\Big(\big(\widetilde{V}(a), \widetilde{V}(b)\big), \big(\widetilde{V}(a), \widetilde{V}(b)\big)\Big)$.

Proof. Let $T \in \mathcal{T}\Big(\big(\widetilde{V}(a), \widetilde{V}(b)\big), \big(\widetilde{V}(a), \widetilde{V}(b)\big)\Big)$. Let c be the weight $(0,1)$. By definition we have

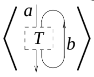
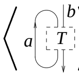
 $=$   

$= S'(a,c)S'(c,b)$ . (11)

Similarly,

$$\left\langle \right.$$  $$\left.\right\rangle = S'(b,c)S'(c,a) \left\langle \right.$$  $$\left.\right\rangle . \tag{12}$$

From Lemma 2.5 we have that the left sides of equations (11) and (12) are equal. Thus, the results follows from relation (9). □

Proof of Theorem 1. Any closed ribbon graph $L \in \mathcal{T}(\emptyset, \emptyset)$ over $\mathcal{M}$ with at least one edge colored by a typical module $\widetilde{V}(a)$ can be represented as the closure of $T_a \in \mathcal{T}(\widetilde{V}(a), \widetilde{V}(a))$. We set $F'(L) = d(a) < T_a >$. If L can also be represented as the closure of $T_b \in \mathcal{T}(\widetilde{V}(b), \widetilde{V}(b))$ for some typical weight b then there exits $T \in \mathcal{T}\Big(\big(\widetilde{V}(a), \widetilde{V}(b)\big), \big(\widetilde{V}(a), \widetilde{V}(b)\big)\Big)$ such that $T_a = \Big\langle$  $\Big\rangle$ and $T_b = \Big\langle$  $\Big\rangle$ so by Lemma 2.6 the definition of $F'(L)$ does not depend on the choice of T_a. □

3. The multi-variable Alexander type invariant

Let $\rho_a : U_h(\mathfrak{sl}(2|1)) \to \mathrm{End}(V)$ be the representation associated to the module $\widetilde{V}(a) \simeq V$. When $a_2 \in \mathbb{C} \setminus \{0, -1\}$ the representation $\rho_{(0,a_2)}$ is four-dimensional. We fix a super vector space V of dimension four with basis $B = (v_1, v_2, v_3, v_4)$ corresponding to the weight vectors $(v_{\bar{0},0}, w_{\bar{1},0}, w_{\bar{0},0}, v_{\bar{1},0})$ of the proof of Proposition 2.2. The super-grading of V is given by $\overline{v_i} = \overline{i+1}$.

Lemma 3.1. *Up to equivalence, the matrices of the four-dimensional representation $\rho_{(0,a_2)}$ ($a_2 \in \mathbb{C} \setminus \{0,-1\}$) in the basis B are:*

$$\underset{\rho_{(0,a_2)}(h_1)}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \qquad \underset{\rho_{(0,a_2)}(E_1)}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}} \qquad \underset{\rho_{(0,a_2)}(F_1)}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\underset{\rho_{(0,a_2)}(h_2)}{\begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_2+1 & 0 & 0 \\ 0 & 0 & a_2+1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}} \quad \underset{\rho_{(0,a_2)}(E_2)}{\begin{pmatrix} 0 & 0 & 0 & q^{-a_2}\frac{\{a_2\}}{\{1\}} \\ 0 & 0 & \frac{\{a_2+1\}}{\{1\}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{\rho_{(0,a_2)}(F_2)}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q^{a_2} & 0 & 0 & 0 \end{pmatrix}}$$

Proof. One can easily check that the relation of Definition 1.5 are satisfied and that the highest weight vector is v_1 whose weight is $(0, a_2)$. □

Lemma 3.2. *There exists an $R(x,y,z) \in \mathrm{GL}(16; \mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}])$ such that, for all typical weights $a = (0, a_2)$ and $b = (0, b_2)$ the action of the R-matrix on $\widetilde{V}(a) \otimes \widetilde{V}(b)$ with respect to the basis $B \times B$ is given by $q^{-2a_2b_2}R(q, q^{a_2}, q^{b_2})$.*

Proof. With the use of Lemma 3.1 and equations (3) and (4) the lemma follows from a direct calculation. □

Proof of Theorem 2. Choose n complex numbers $\alpha_1, \ldots \alpha_n$ such that $(1, \alpha_1, \ldots, \alpha_n)$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{C}$. Then the ring map $\phi : \mathbb{Z}[q^{\pm1}, q_1^{\pm1}, \ldots, q_n^{\pm1}] \to \mathbb{C}[[h]]$ defined by $\phi(q) = \mathrm{e}^{h/2}$ and $\phi(q_i) = e^{\alpha_i h/2}$ is injective since the family $\{\phi(q^{k_0} q_1^{k_1} \cdots q_n^{k_n}) : (k_0, \ldots, k_n) \in \mathbb{Z}^{n+1}\}$ is free. Let L' be any framed link with n ordered components. Color the i-th component of L' with the weight $(0,\alpha_i)$. Then from the definition of F' and Lemmas 2.4 and 3.2 we have

$$F'(L') \in \frac{\mathrm{e}^{-\sum lk_{ij}\alpha_i\alpha_j h}}{\{\alpha_k\}\{\alpha_k+1\}}\,\mathrm{Im}(\phi) \tag{13}$$

where $k \in \{1\cdots n\}$ is the number of the component opened to compute $F'(L')$ (the denominator comes from $d(0,\alpha_k)$).

Let L be the non-framed link which underlies L'. If $n=1$ define

$$M(L) := (q_1 - q_1^{-1})^{-1}(q_1 q - q_1^{-1}q^{-1})^{-1}\phi^{-1}\left(\mathrm{e}^{lk_{11}(\alpha_1^2+\alpha_1)h}\{\alpha_1\}\{\alpha_1+1\}F'(L')\right).$$

The first half of the theorem follow from this definition.

Next we will show that if $n \geq 2$ then $F'(L') \in \mathrm{e}^{-\sum lk_{ij}\alpha_i\alpha_j h}\,\mathrm{Im}(\phi)$. For $i=1,2$ let T_i be a $(1,1)$-tangles whose closure is L' and whose open strand is the i-th component of L'. From Theorem 1 we have $F'(T_1) = F'(T_2)$. Then equation (13) implies the existence of Laurent polynomials P_1 and P_2 such that

$$\mathrm{e}^{\sum lk_{ij}\alpha_i\alpha_j h}F'(L') = d(0,\alpha_1)\phi(P_1) = d(0,\alpha_2)\phi(P_2).$$

It follows that $(q_2 - 1/q_2)(q_2 q - 1/(q_2 q))P_1 = (q_1 - 1/q_1)(q_1 q - 1/(q_1 q))P_2$. Since $R = \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$ is an unique factorization domain we have that $(q_1 - 1/q_1)(q_1 q - 1/(q_1 q))$ divides P_1. Therefore we can conclude that

$$\mathrm{e}^{\sum lk_{ij}\alpha_i\alpha_j h}F'(L') = \phi\big(P_1/[(q_1 - 1/q_1)(q_1 q - 1/(q_1 q))]\big).$$

For $n \geq 2$ we are now able to define

$$M(L) := \phi^{-1}(\mathrm{e}^{\sum lk_{ij}(2\alpha_i\alpha_j+\alpha_i+\alpha_j)h/2}F'(L'))$$

where the additional correction is needed (see lemma 2.4) to make M framing independent (i.e. a link invariant).

Because of Lemma 3.2, $M(L)$ is independent of the choice $\alpha = (\alpha_1, \ldots \alpha_n)$ lying in the dense subset of $\mathbb{C}^n$ defined by the condition: $(1, \alpha_1, \ldots \alpha_n)$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{C}$. Now the two maps $F'(L')$ and $\phi(M(L))$ depend continuously of α so the relation between F' and M in Theorem 2 is valid for any $(\alpha_1, \ldots \alpha_n) \in (\mathbb{C} \setminus \{0,-1\})^n$. $\square$

4. Skein relations

In this section we will give a complete set of skein relations for F' and use this to show that after a variable reduction M is the Conway potential function. To this end, we extend F and F' to invariants of colored oriented framed trivalent graphs. For a detailed account of similar extensions see the work of Viro [19].

We will now define some normalized elementary $U_h(\mathfrak{sl}(2|1))$-module morphisms. Consider the following element of $V \otimes V$

$$\bar{b} = v_1 \otimes v_3 + q^{-1}v_3 \otimes v_1 + v_4 \otimes v_2 - q^{-1}v_2 \otimes v_4.$$

Remark that all typical representations $\widetilde{V}(0,a)$ have the same underlying vector-space V and the same basis B.

Lemma 4.1. *The map* $\bar{b}_a : \mathbb{C}[[h]] \to \widetilde{V}(0,a) \otimes \widetilde{V}_{(0,-1-a)}$ *that sends* 1 *to* $q^{-a}\bar{b}$ *is* $U_h(\mathfrak{sl}(2|1))$*-invariant and satisfies:*

$$(\mathrm{Id}_{\widetilde{V}(0,-1-a)} \otimes \theta_{\widetilde{V}(0,a)}) \circ c_{\widetilde{V}(0,a),\widetilde{V}(0,-1-a)} \circ \bar{b}_a = \bar{b}_{-1-a}$$

We use these morphisms to identify $\left(\widetilde{V}(0,a)\right)^* = \widetilde{V}(0,-1-a)$.

With this identification, the evaluation map $d_{\widetilde{V}(0,a)} = \left(\widetilde{V}(0,a)\right)^* \otimes \widetilde{V}(0,a) \to \mathbb{C}[[h]]$ induces a family of bilinear maps: $\bar{d}_a : \widetilde{V}(0,-1-a) \otimes \widetilde{V}(0,a) \to \mathbb{C}[[h]]$ whose matrix in B is given by

$$q^a \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \end{pmatrix}.$$

We also consider two families of morphisms (related to the so called quantum Clebsch-Gordan coefficients):

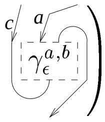
$$\gamma_+^{a,b} : \widetilde{V}(0,a+b) \to \widetilde{V}(0,a) \otimes \widetilde{V}(0,b), \quad \gamma_-^{a,b} : \widetilde{V}(0,a+b+1) \to \widetilde{V}(0,a) \otimes \widetilde{V}(0,b)$$

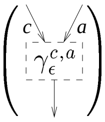
given by:

$$\begin{aligned}
\gamma_+^{a,b}(v_1) &= v_1 \otimes v_1 \\
\gamma_+^{a,b}(v_2) &= q^{-a} v_1 \otimes v_2 + v_2 \otimes v_1 \\
\gamma_+^{a,b}(v_3) &= q^{-a} v_1 \otimes v_3 - q^b v_2 \otimes v_4 + q^b v_3 \otimes v_1 + q^{b+1} v_4 \otimes v_2 \\
\gamma_+^{a,b}(v_4) &= q^{-a} v_1 \otimes v_4 + v_4 \otimes v_1
\end{aligned}$$

$$\begin{aligned}
\gamma_-^{a,b}(v_1) &= q^{b+1}\{a\}\{a+1\} v_1 \otimes v_3 + q^b \{a+1\}\{b+1\} v_2 \otimes v_4 \\
&\quad + q^{-a-1}\{b\}\{b+1\} v_3 \otimes v_1 - q^{b+1}\{a+1\}\{b+1\} v_4 \otimes v_2 \\
\gamma_-^{a,b}(v_2) &= \{a+1\}\{a+b+1\} v_2 \otimes v_3 + q^{-a-1}\{b+1\}\{a+b+1\} v_3 \otimes v_2 \\
\gamma_-^{a,b}(v_3) &= \{a+b+1\}\{a+b+2\} v_3 \otimes v_3 \\
\gamma_-^{a,b}(v_4) &= q^{-a-1}\{b+1\}\{a+b+1\} v_3 \otimes v_4 + \{a+1\}\{a+b+1\} v_4 \otimes v_3
\end{aligned}$$

Lemma 4.2. *The map $\gamma_+^{a,b}$ (resp. $\gamma_-^{a,b}$) is $U_h(\mathfrak{sl}(2|1))$-invariant and for $c = -1 - a - b$ (resp. $c = -2 - a - b$), it satisfies:*

F (

) $= F$ (

)

This allow us to define F' for a certain class of uni-trivalent ribbon graphs:

Definition 4.3. *A trivalent tangle T is a framed smooth embedding of a uni-trivalent graph with oriented trivalent vertices in $\mathbb{R}^3$. Here the framing is given by a continuous vector field along the image of T nowhere tangeant to the image of T. We impose that the three tangeant vectors at a trivalent vertex are coplanar and two of them must not be on the same half-line. An orientation of a trivalent vertex x is a cyclic order on the three edges going to x. In the embedding T this ordering must be given by the framing. (At a trivalent vertex x , the framing gives an orientation of the affine plane tangeant to the image of T. This orientation must be compatible with the cyclic order of the tangeant of the three edges going to x).*
We denote by $\mathcal{E}(T)$ the set of oriented edges of T.

- *T is marked if it is given with a map from its trivalent vertices to the set $\{+,-\} \simeq \{\pm 1\}$.*

- *T is colored if it is marked and if it is given with a map $f : \mathcal{E}(T) \to \mathbb{C}$ such that for each oriented edge $\overrightarrow{e} \in \mathcal{E}(T)$, one has $f(\overrightarrow{e}) = -1 - f(\overleftarrow{e})$.*
- *A coloring of T is admissible if at each trivalent vertex marked with "+" (resp. "−"), the sum of the values of f at the three incoming edges is -1 (resp. -2).*
- *T is typical if it is colored with an admissible coloring for which f takes values in $\mathbb{C} \setminus \{0, -1\}$.*
- *T is closed if it has no univalent vertices.*

To any ribbon graph G over $\mathcal{M}$ with edges colored by typical modules $\widetilde{V}(0, a)$ and coupons marked by the morphisms $\gamma_+^{a,b}$ and $\gamma_-^{a,b}$ one can associate a typical trivalent tangle T. Lemma 4.1 and 4.2 says that $F(G)$ and $F'(G)$ depend only of T. We still denote by F and F' the induced maps on typical trivalent tangles.

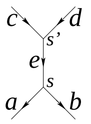
Remark 4.4. *Let T be a closed typical trivalent tangle. We suppose* $\dim(H_1(T)) > 1$ *i.e. T is not a knot. Because of the nice expression of $\bar{b}$, $\bar{d}$, γ_+, γ_- and of the R-matrix, we can make the following observation: As for links, $F'(T)$ of a typical trivalent tangles has the form $q^{p_1}P_2$ where p_1 is a degree 2 polynomial in $\{f(e) : e \in \mathcal{E}(T)\}$ and P_2 is a Laurent polynomial in $\{q, q^{f(e)} : e \in \mathcal{E}(T)\}$. Unfortunately, there is no canonical correction for the term q^{p_1} and thus the Laurent polynomial P_2 is only defined up to an unit.*
For unclosed typical trivalent tangles we have that the coordinates of $F(T)$ live in $\mathbb{Z}[q^{\pm 1}, q^{f(e)}, q^{f(e)f(e')} : e, e' \in \mathcal{E}(T)]$.

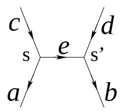
Proposition 4.5. *The set of admissible colorings of a marked trivalent tangle T is either empty or an affine space over $H_1(T, \mathbb{C})$. In the later case, the non typical colorings are a finite reunion of codimension 1 subspace. Furthermore, $F(T)$ is a continuous function of the coloring. Thus one can use this to compute $F(T)$ by perturbing the coloring: If f is the coloring of T and $c \in H_1(T, \mathbb{C})$, let T_ϵ be the marked trivalent tangle T colored by $f + \epsilon c$ then*

$$F(T) = \lim_{\epsilon \to 0} F(T_\epsilon).$$

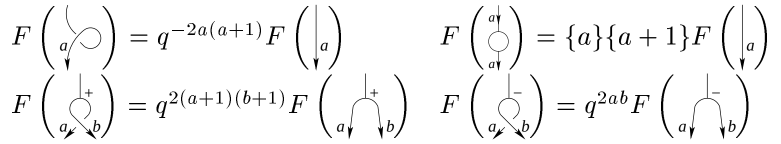
Proof. Let E be the set of oriented edges of T. The difference of two admissible coloring of T (as two elements of $\mathbb{C}^E$) is then equivalent to a 1-cycle well defined up to a boundary (the relation $f(\overrightarrow{e}) = -1 - f(\overleftarrow{e})$). The continuity is a consequence of Remark 4.4 (In fact, the coordinates of $F(T_\epsilon)$ have the form $q^{k\epsilon^2 + \epsilon l(f)} P(q^\epsilon)$ where $k \in \mathbb{Z}$, l is a linear map of $\{f(e) : e \in \mathcal{E}(T)\}$ and $P \in \mathbb{Z}[q^{\pm 1}, q^{f(e)}, q^{f(e)f(e')} : e, e' \in \mathcal{E}(T)][q^{\pm \epsilon}]$ is such that $P(1)$ is the corresponding coordinate of $F(T)$). $\square$

For $s, s' \in \{\pm 1\}$, $a, b, c, d, e \in \mathbb{C}$, satisfying $a + b - e = -\frac{s+1}{2}$ and $c + d - e = \frac{s'-1}{2}$ (resp. $a - c + e = -\frac{s+1}{2}$ and $d - b + e = \frac{s'-1}{2}$) we set:

$\boldsymbol{I}_{ab}^{cd}(ss') =$ 

and resp. $\boldsymbol{H}_{ab}^{cd}(ss') =$ 

Proposition 4.6. *F and F' satisfy the following skein relations. Furthermore up to the perturbation principle of proposition 4.5, this set of skein relations is complete for closed typical trivalent tangles. Indeed, we sketch in the proof an algorithm that* *allows to recursively compute the value of F' on any closed typical trivalent tangle using these relations and proposition 4.5.*

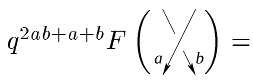


(14)

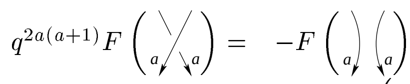
$$F\left(\boldsymbol{I}_{ab}^{cd}(++)\right) = F\left(\boldsymbol{H}_{ab}^{cd}(++)\right) \tag{15}$$

$$\{c-a\}\{c-a+1\}F\left(\boldsymbol{I}_{ab}^{cd}(--)\right) = \{a+b\}\{a+b+1\}F\left(\boldsymbol{H}_{ab}^{cd}(--)\right) \tag{16}$$

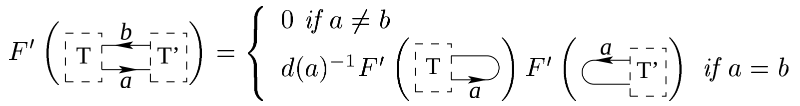
$$i_1 F\left(\boldsymbol{I}_{ab}^{cd}(-+)\right) + i_2 F\left(\boldsymbol{I}_{ab}^{cd}(+-)\right) = h_1 F\left(\boldsymbol{H}_{ab}^{cd}(-+)\right) + h_2 F\left(\boldsymbol{H}_{ab}^{cd}(+-)\right) \tag{17}$$



$$= \frac{1}{\{a+b+1\}}\left(\frac{q^b}{\{b\}} F\left(\boldsymbol{I}_{ab}^{ba}(-+)\right) + \frac{q^{-1-a}}{\{a+1\}} F\left(\boldsymbol{I}_{ab}^{ba}(+-)\right)\right) - \frac{1}{\{b\}\{a+1\}} F\left(\boldsymbol{H}_{ab}^{ba}(+-)\right) \tag{18}$$



$$+\frac{1}{\{2a+1\}}\left(\frac{q^a}{\{a\}} F\left(\boldsymbol{I}_{aa}^{aa}(-+)\right) + \frac{q^{-1-a}}{\{a+1\}} F\left(\boldsymbol{I}_{aa}^{aa}(+-)\right)\right) \tag{19}$$



(20)

where
$$\begin{aligned} i_1 &= \{d-b\}\{c+1\}\{d+1\} \\ i_2 &= -\{d-b\}\{a\}\{b\} \\ h_1 &= -\{a+b+1\}\{b\}\{d+1\} \\ h_2 &= \{a+b+1\}\{a\}\{c+1\}. \end{aligned}$$

Proof. In general the existence of such relations is a basic consequence of the representation theory of $U_h(\mathfrak{sl}(2|1))$. In this case the relations follow from lemma 1.3. We use a computer to find the coefficients.

We now give a sketch of an algorithm to compute F' with these relations: Consider a regular planar projection of a closed typical trivalent tangle.

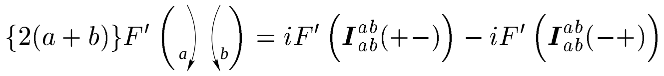
(1) One can convert it to a linear combination of planar graph using relations (18) and (19).
(2) Then the digons can be reduced by the second relation (14).
(3) We say that an edge e is critical if its neighborhood is of the form $\boldsymbol{I}_{ab}^{cd}$ with $c=a$ or $d=b$. We ignore for a moment the problem of critical edges for which the relations (15), (16) and (17) can't be applied.
(4) Consider the smallest n-gon. If it has two consecutive vertices marked with the same sign, it can be reduced to a $(n-1)$-gon using relation (15) or (16). then we reapply the process from step 2.
(5) If the smallest n-gon has its vertices alternatively marked $+$ and $-$ then use relation (17) to obtain two $(n-1)$-gon and a n-gon as in step 4.
(6) If during step 4 and 5 we deal with a critical edge then either it can be changed to a non critical edge using the perturbation principle, or it can be reduced by relation (20).

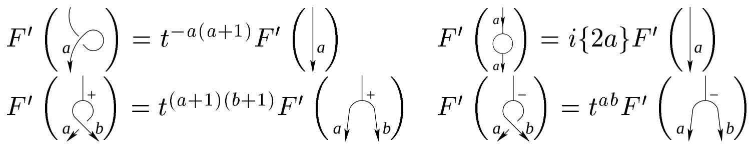
□

In the rest of this section, we show that the specialization $q = i$ of M is essentially the Conway function:
For $\alpha \in \mathbb{C}$, let $t^\alpha = q^{2\alpha} = \mathrm{e}^{\alpha h}$ and take $h = i\pi$. Then for $k \in \mathbb{Z}$ we have

$$q = i \qquad \{2k\} = 0 \qquad \{2k+1\} = (-1)^k 2i \qquad \{\alpha+2\} = -\{\alpha\}$$
$$\text{and} \qquad \{\alpha\}\{\alpha+1\} = i\{2\alpha\} = i(t^\alpha - t^{-\alpha}).$$

Proposition 4.7. *The specialization $h = i\pi$ of F' satisfies the following skein relations :*


(21)

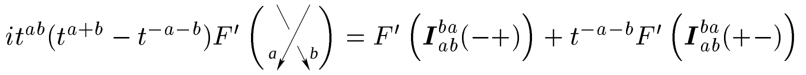

(22)

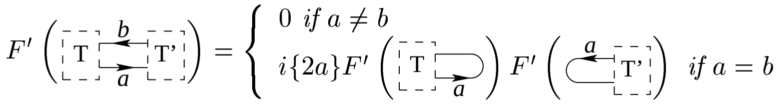
$$F'\left(\boldsymbol{I}^{cd}_{ab}(++)\right) = F'\left(\boldsymbol{H}^{cd}_{ab}(++)\right) \tag{23}$$

$$\{2(c-a)\}F'\left(\boldsymbol{I}^{cd}_{ab}(--)\right) = \{2(a+b)\}F'\left(\boldsymbol{H}^{cd}_{ab}(--)\right) \tag{24}$$

$$\{2(a+b)\}F'\left(\boldsymbol{H}^{cd}_{ab}(+-)\right) = \{2d\}F'\left(\boldsymbol{I}^{cd}_{ab}(-+)\right) - \{2b\}F'\left(\boldsymbol{I}^{cd}_{ab}(+-)\right) \tag{25}$$

$$\{2(a+b)\}F'\left(\boldsymbol{H}^{cd}_{ab}(-+)\right) = \{2a\}F'\left(\boldsymbol{I}^{cd}_{ab}(+-)\right) - \{2c\}F'\left(\boldsymbol{I}^{cd}_{ab}(-+)\right) \tag{26}$$


(27)


(28)

Proof. Relations (22)-(24) and (28) are specializations of relations of Proposition 4.6 at $h = i\pi$. Some new relations are only valid for F' i.e. for closed trivalent tangles: Relations (25), (26) and (27) are obtained from relation (21) by composing it with an $\boldsymbol{H}$ or with the braiding.

We now prove the relation (21): Set $E = \mathcal{T}((\widetilde{V}(0,a), \widetilde{V}(0,b)), (\widetilde{V}(0,a), \widetilde{V}(0,b)))$ and let p be the element of E given by

$$p = \{a+b\}\{a+b+1\}\{a+b+2\}\,\mathrm{Id} - \{a+b\}\boldsymbol{I}^{ab}_{ab}(+-) - \{a+b+2\}\boldsymbol{I}^{ab}_{ab}(-+)$$

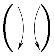
Assume that $a+b \notin \mathbb{Z}$. If T is any element of E, denote the closure (trace in $\mathcal{T}$) of T by $\widehat{T}$. Relation (21) just says that for the specialization $h = i\pi$, one has $F'(\widehat{T \circ p})_{|h=i\pi} = 0$ for all $T \in E$. Now $F(p)$ is just λp_2 where $\lambda = \{a+b\}\{a+b+1\}\{a+b+2\}$ and p_2 is the projection on the factor $\widetilde{V}(1, a+b) \subset \widetilde{V}(0,a) \otimes \widetilde{V}(0,b)$. This is true because $\boldsymbol{I}^{ab}_{ab}(-+)$ (resp. $\boldsymbol{I}^{ab}_{ab}(+-)$) is $\{a+b\}\{a+b+1\}$ times (resp. $\{a+b+1\}\{a+b+2\}$ times) the projection p_1 (resp. p_3) on $\widetilde{V}(0, a+b)$ (resp $\widetilde{V}(0, a+b+1)) \subset \widetilde{V}(0,a) \otimes \widetilde{V}(0,b)$. We have $F(T) = \sum \alpha_i p_i$ where the α_i are "Laurent polynomial" (this is true because $p_j(v_i \otimes v_i) = \delta_i^j v_i \otimes v_i$, thus $F(T)(v_i \otimes v_i) = \alpha_i v_i \otimes v_i$ for $i = 1,2,3$ and remark 4.4). Thus $F'(\widehat{T \circ p}) = \alpha_2 F'(\widehat{p})$.

Now one can compute $F'(\widehat{p}) = -\{a+b+2\} - \{a+b\} = -(q+1/q)\{a+b+1\}$ and $F'(\widehat{T \circ p})_{|h=i\pi} = 0$. □

To prove Theorem 3 we introduce a modified version of Turaev's axioms for the Conway map ([17] section 4):

Lemma 4.8. *The Conway function is the map uniquely determined by*

(1) *∇ assigns to each ordered oriented link L in S^3 an element of the field $\mathbb{C}(t_1, \dots, t_n)$ where n is the number of components of L.*
(2) *$\nabla(L)$ is unchange under (ambient) isotopy of the link L.*
(3) *$\nabla(unknot) = (t_1 - t_1^{-1})^{-1}$.*
(4) *If $n \geq 2$ then $\nabla(L) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.*
(5) *The one variable function $\widetilde{\nabla}(L) = \nabla(L)(t, t, \dots, t) \in \mathbb{C}[t^{\pm 1}]$ is unchanged by a reordering of the components of L.*
(6) *(Conway identity)*

$$\widetilde{\nabla}\left(\right.$$  $$\left. \right) - \widetilde{\nabla}\left(\right.$$  $$\left. \right) = (t - t^{-1})\widetilde{\nabla}\left(\right.$$  $$\left. \right)$$

(7) *(Modified doubling axiom). If L^+ (resp. L^-) is obtained from the link $L = L_1 \cup \dots \cup L_n$ by replacing the component L_i by its $(2,1)$-cable (resp. by its $(2,-1)$-cable) then*

$$t_i \nabla(L^+)(t_1, \dots, t_n) - t_i^{-1} \nabla(L^-)(t_1, \dots, t_n) =$$
$$\left(\prod_{j \neq i} t_j^{lk_{ij}} \right) (t_i^2 - t_i^{-2}) \nabla(L)(t_1, \cdots, t_{i-1}, t_i^2, t_{i+1}, \cdots, t_n)$$

Proof. The difference with the axioms given by Turaev ([17]) are :

- He only considers map with values in $\mathbb{Q}(t_1, \dots, t_n)$ and the axiom 4 is replaced by $\nabla(L) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. This change still allows to consider $\widetilde{\nabla}$.
- The axiom 7 is replaced by the doubling axioms: If the link L^+ is obtained from the link $L = L_1 \cup \dots \cup L_n$ by replacing the component L_i by its $(2,1)$-cable then

$$\nabla(L^+)(t_1, \dots, t_n) = (T + T^{-1}) \nabla(L)(t_1, \cdots, t_{i-1}, t_i^2, t_{i+1}, \cdots, t_n)$$

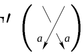
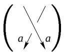
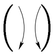
with $T = t_i \prod_{j \neq i} t_j^{lk_{ij}}$.

First the Conway function satisfies the modified doubling axiom because $\nabla(L^-)(t_1, \dots, t_n) = (T' + T'^{-1}) \nabla(L)(t_1, \cdots, t_{i-1}, t_i^2, t_{i+1}, \cdots, t_n)$ with $T' = t_i^{-1} \prod_{j \neq i} t_j^{lk_{ij}}$ (for example because L^- is the miror image of the $(2,1)$-cable of the miror image of L and $\nabla(\text{miror}(L)) = (-1)^{n+1} \nabla(L)$ (see [17])).

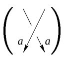
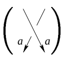
For the unicity, consider two maps ∇_1 and ∇_2 satisfying axioms 1–7. The proof of unicity given by Turaev has two steps: The first step uses axioms 1–6 to show that the two one variable specializations are the same: $\widetilde{\nabla}_1 = \widetilde{\nabla}_2$. This part applies in our context without any change.

Now, using the modified doubling axiom, one can show by induction on $\mathbb{N}^n$ that for any $(a_1, \dots, a_n) \in \mathbb{N}^n$, one has $\nabla_1(t^{2^{a_1}}, \dots, t^{2^{a_n}}) = \nabla_2(t^{2^{a_1}}, \dots, t^{2^{a_n}})$. But this implies that for any fixed ordered link L (with $n \geq 2$ components), the Laurent polynomial $\nabla_1(L) - \nabla_2(L)$ is zero. □

Proof of Theorem 3. If $k \in \mathbb{N}$, set $t_k = q_k^2$ and $M'(q_1, \ldots, q_n) = i.M(i, q_1, \ldots, q_n)$. It is clear that M' satisfies the axioms 1–5 of lemma 4.8 (we neglect the fact that M' leave a priori in the extension $\mathbb{C}[q_k^{\pm 1}]$ of $\mathbb{C}[t_k^{\pm 1}]$).
From equations (21) and (22) one can easily check that

$t^{a+a^2} F' \Big($  $\Big) - t^{-a-a^2} F' \Big($  $\Big) = (t^a - t^{-a}) F' \Big($  $\Big)$

and this implies that M' satisfies the axiom of the Conway identity.
Now for the modified doubling axiom, by equation (27) and its miror analog, one has

$t^{2a+a^2} F' \Big($  $\Big) - t^{-2a-a^2} F' \Big($  $\Big) = -i\boldsymbol{I}_{a,a}^{a,a}(-+)$

We choose a framing on L so that $lk_{ii} = 0$. Hence $L^{\pm}$ can be obtain from L by replacing L_i by two parallel copies modified in a small ball as in the left hand side of the previous equality. This with the second equation of (22) gives

$$t^{2a_i + a_i^2} F'(L^+) - t^{-2a_i - a_i^2} F'(L^-) = i(t^{2a_i} - t^{-2a_i}) F'(L)$$

where the k-th component of L^+, L^- and L is colored by a_k except the i-th component of L which is colored by $2a_i$. Now as $lk_{ik}(L^{\pm}) = lk_{ki}(L^{\pm}) = 2lk_{ik}(L)$ for $k \neq i$ and $lk_{ii}(L^{\pm}) = \pm 1$, the framing correction gives

$$\begin{aligned} &t^{a_i} M'(L^+)(t^{a_1}, \ldots, t^{a_n}) - t^{-a_i} M'(L^-)(t^{a_1}, \ldots, t^{a_n}) = \\ &\qquad \left(\prod_{j \neq i} t^{lk_{ij} a_j} \right) (t^{2a_i} - t^{-2a_i}) M'(L)(t^{a_1}, \cdots, t^{a_{i-1}}, t^{2a_i}, t^{a_{i+1}}, \cdots, t^{a_n}) \end{aligned}$$

Hence M' is the Conway function. □

Remark that Proposition 4.7 is a complete set of skein relations for a generalization of the Conway potential function ∇ to colored trivalent graphs.

5. Some examples

One can use the skein relations developed in Section 4 to compute the invariant M. In this section we give the results of such computations.

For a knot K, the invariant M is not a Laurent polynomial but it has the form $M_0 + P$ with $M_0 = \frac{1}{(q_1 - q_1^{-1})(qq_1 - (qq_1)^{-1})} = M(\text{unknot})$ and $P \in \mathbb{Z}[q^{\pm}, q_1^{\pm}]$. In fact $M(K)$ is just the Links-Gould invariant. More precisely we get

$$LG_K(q, p)_{|p = q_1\sqrt{q}} = (q_1 - q_1^{-1})(qq_1 - (qq_1)^{-1}) M(K)(q, q_1)$$

with the convention of [20]. In this last paper, the values of LG for the first prime knots (up to 10 crossings) are computed. The corresponding values of M can be deduced. As an example, in [20] the value for the trefoil 3_1 is presented by

$$LG_{3_1} = 1 + 2q^2 - (q + q^3)(p^2 + p^{-2}) + q^2(p^4 + p^{-4})$$

and our computation gives

$$M(3_1) = M_0 \; + \; q^2(qq_1^2 + (qq_1^2)^{-1})$$

this result and several other computations for knots are in agreement with the computations of [20].

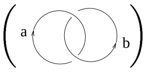
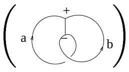
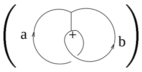
Remark 5.1. *In* [4] *the Links-Gould invariant is computed using an R-matrix of $U_q\mathfrak{gl}(2|1) \simeq U_q\mathbb{T}_1 \otimes U_q\mathfrak{sl}(2|1)$ (isomorphism of Hopf algebra) where $U_q\mathbb{T}_1$ is the (co-)commutative Hopf algebra of polynoms in one primitive variable c. They only consider one representation $\widetilde{V}(0,\alpha)$ (for a generic value of α) that is obtained from the one of lemma 3.1 by making c acting by the scalar α on it. Hence they compute the Reshetikhin-Turaev invariant of a $(1,1)$-tangle T colored by $\widetilde{V}(0,\alpha)$ with an R-matrix that differs form our by a scalar $q^{2\alpha(\alpha+1)}$. This scalar exactly correct the framing of the tangle.*
The Links-Gould invariant is given by $< T >$ as in defintion 2.1. Remark that the specialization $b = a$ of lemma 2.6 (which is not trivial for tangles with several components) gives a proof that the Links-Gould invariant is a well defined invariant of the link closure of T.
So again with the convention of [20] *where $p = q^{\alpha+\frac{1}{2}}$, we get that for any link L,*

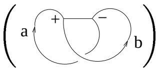
$$LG(L)(q, q^{\alpha+\frac{1}{2}}) = (q^{\alpha} - q^{-\alpha})(q^{\alpha+1} - q^{-\alpha-1})M(L)(q, q^{\alpha}, q^{\alpha}, \ldots, q^{\alpha})$$

It is more interesting to see the value for links with several components. For example, let H be the Hopf link (with negative crossings) then

$$M(H)(q, q_1, q_2) = q$$

and thus $M(H)$ does not depend of the two colors. This result can be deduced from proposition 2.2 but we use it to illustrate the skein relations:

$F'\Big($  $\Big) = \frac{q^{2ab+a+b}q^{-a}}{\{a+b+1\}\{a\}} F'\Big($  $\Big) + \frac{q^{2ab+a+b}q^{b+1}}{\{a+b+1\}\{b+1\}} F'\Big($  $\Big)$

$- \frac{q^{2ab+a+b}}{\{a\}\{b+1\}} F'\Big($  $\Big)$

$$= \frac{q^{2ab+a+b}q^{-a}}{\{a+b+1\}\{a\}} q^{2ab} + \frac{q^{2ab+a+b}q^{b+1}}{\{a+b+1\}\{b+1\}} q^{2(a+1)(b+1)} - \frac{q^{2ab+a+b}}{\{a\}\{b+1\}} q^{2a(b+1)}$$

$$= q^{4ab+2a+2b}q \quad \text{and } M(H) = q.$$

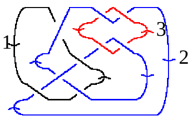
The first equality is obtained from the miror analog of (18). For the second we use equations (14). (Remark that the image by F' of any typical planar "Theta" (Θ) graph is equal to 1.)

The Borromean link B ($B = L6a4$ in the Thistlethwaite link table) has 3 symmetric components:

$$M(B)(q, q_1, q_2, q_3) = \delta(q) + \delta(q_1)\delta(q_2)\delta(q_3)$$
$$\text{where } \delta(x) = (x - 1/x)(qx - 1/(qx))$$

This link is noted 6^3_2 in [20] and we check $LG(6^3_2) = \delta(\frac{p}{\sqrt{q}})M(B)(q, \frac{p}{\sqrt{q}}, \frac{p}{\sqrt{q}}, \frac{p}{\sqrt{q}})$.

The first link with trivial multivariable Alexander polynomial in the Thistlethwaite link table is $L9n27$

$L9n27 =$ 

(see the "Knot Atlas" at the url http://katlas.math.toronto.edu). We have

$$M(L9n27)(q,q_1,q_2,q_3) = \{1\}\{2\}(q^2q_2^2 + q_2^{-2} - 2)$$

this is coherent with $\nabla(L9n27) = 0$ and this show in particular, that M is strictly stronger than the Conway potential function.

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