MULTIVARIABLE LINK INVARIANTS ARISING FROM LIE SUPERALGEBRAS OF TYPE I

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Abstract. In this paper we construct new links invariants from a type I basic Lie superalgebra $\mathfrak{g}$. The construction uses the existence of an unexpected replacement of the vanishing quantum dimension of typical module, by non-trivial "fake quantum dimensions." Using this, we get a multivariable link invariant associated to any one parameter family of irreducible $\mathfrak{g}$-modules.

Introduction

Let $\mathfrak{g}$ be a Lie superalgebra of type I, i.e. $\mathfrak{g}$ is equal to $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2|2n)$. Here we assume that $m \neq n$. Let r be equal to $m+n-1$ if $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $n+1$ if $\mathfrak{g} = \mathfrak{osp}(2|2n)$.

The quantum dimension associated to a deformed typical $\mathfrak{g}$-module is zero. This implies that the usual Reshetikhin-Turaev quantum group link invariants arising from such a module is trivial. In this paper we show that the usual Reshetikhin-Turaev construction can be renormalized by non-zero "fake quantum dimensions". In Section 3, we will use this modified construction to define non-trivial multivariable link invariants. We will discuss these multivariable invariants in more detail later in the introduction, now let us consider how this modified construction fits into the general theory of quantum invariants and the representation theory of Lie superalgebras.

The work of this paper has lead to a variety of new mathematical ideas and relationships. The authors are working on three subsequent papers which we will now discuss.

- The first of these papers is joint work with V. Turaev. The paper [9] will contain a renormalization of the Reshetikhin-Turaev functor of a ribbon Ab-category, by "fake quantum dimensions". In the case of simple Lie algebras these "fake quantum dimensions" are proportional to the genuine quantum dimensions. More interestingly, this paper will contain two examples where the genuine quantum dimensions vanish but the "fake quantum

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dimensions" are non-zero and lead to non-trivial link invariants. The first of these examples recover the hierarchy of invariants defined by Akutsu, Deguchi and Ohtsuki [1], using a regularize of the Markov trace and nilpotent representations of quantized sl(2) at a root of unity. These invariants contain Kashaev's quantum dilogarithm invariants of knot (see [16]). The second example, is the invariants defined in this paper.

The definition of the "fake quantum dimensions" given in [9] is abstract where the analogous definition in this paper is given by explicit formulas. One can use general theory to show that these definitions are equivalent. The explicit formulas given in this paper are useful when one wants to compute the invariant or compare it to other invariants.

- In the second subsequent paper the authors will use the explicit formulas for the "fake quantum dimensions" to define "fake superdimensions" of typical representations of the Lie superalgebra $\mathfrak{g}$. These "fake superdimensions" are non-zero and lead to a kind of supertrace on the category representations of $\mathfrak{g}$ which is non-trivial and invariant. These statements are completely classical statements. However, the only proof we know of uses the quantum algebra and low-dimensional topology developed in this paper.
- We will now discuss the final subsequent paper in relation with the multivariable invariants defined in this paper. In Section 3 we will show that for $c \in \mathbb{N}^{r-1}$ the pair $(\mathfrak{g}, c)$ gives rise to a multivariable link invariant $M_{\mathfrak{g}}^{c}$. These invariants associate a variable to each component of the link. There are only a handful of such invariants including the multivariable Alexander polynomial and the ones defined in [1]. All of these invariants are related to the invariants defined in this paper.

Let us now explain these relationships. First, in [8] we plan on showing that the invariant $M_{\mathfrak{sl}(m|1)}^{(0,\dots,0)}$ specializes to the multivariable Alexander polynomial. Second, in order to define their link invariants the authors of [1] regularize the Markov trace. Although using different methods, the invariants of this paper have a similar regularization. In both case, the standard method using ribbon categories or the Markov trace is trivial. Moreover, both families of invariants are generalization of the multivariable Alexander polynomial. In [8], we plan on conjecturing that the invariants $M_{\mathfrak{sl}(m|1)}^{(0,\dots,0)}$, for $m \in \mathbb{N}$, specialize to the hierarchy of invariants defined in [1]. (Note this specialization depends on m and is different than the specialization related to the multivariable Alexander polynomial.)

Also, in [8] we will show that the invariants $M^{(0,\dots,0)}_{\mathfrak{sl}(m|1)}$ are related to other invariants. Let us briefly discuss this now. In [22], H. Murakami and J. Murakami show that the invariants of [1] and the set of colored Jones polynomials have a non-trivial intersection. Moreover, they show that this intersection contains Kashaev's invariants. This result led to a reformulation of Kashaev's Volume Conjecture (see [22]). In [8], we show that a similar result holds for the invariants $M^{(0,\dots,0)}_{\mathfrak{sl}(m|1)}$; namely, that the intersection of the set of multivariable link invariants $\{M^{(0,\dots,0)}_{\mathfrak{sl}(m|1)}\}_{m\geq 2}$ and the set of colored HOMFLY-PT polynomials contains Kashaev's invariants. We also show that the invariants $M^{(0,\dots,0)}_{\mathfrak{sl}(m|1)}$ are multivariable generalization of the set of two variable invariants defined by Links and Gould [3, 10].

Lie superalgebras have previously been used to construct invariants with more than one variable. For example see [7, 10, 11, 19]. It should be noted that in [10, 11, 19] the use of cutting one strand of a link in order to avoid the vanishing of the quantum dimension has already been used. In these papers all the strands of a braid are colored with the same modules and the authors use the Markov trace to construct a non-trivial invariant. Here we use links whose components are colored with different modules and work with ribbon functors. As explained above, this approach gives the potential for new constructions in low-dimensional topology and applications in representation theory.

We now state the main results of this paper more precisely. First, we recall some results from the theory of Lie superalgebras (for more details see Section 1). Every irreducible finite-dimensional $\mathfrak{g}$-module has a highest weight $\lambda \in \mathfrak{h}^*$ (where $\mathfrak{h}$ is the Cartan sub-superalgebra). Moreover, the set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules are in one to one correspondence with the set of dominant weights. These modules are parametrized by $\mathbb{N}^{r-1} \times \mathbb{C}$ and are divided into two classes: typical and atypical. Each highest weight $\mathfrak{g}$-module V can be deformed to a highest weight topologically free $U_h(\mathfrak{g})$-module $\widetilde{V}$, where $U_h(\mathfrak{g})$ is the Drinfeld-Jimbo superalgebra associated to $\mathfrak{g}$. We say $\widetilde{V}$ is a typical $U_h(\mathfrak{g})$-module if V is a typical $\mathfrak{g}$-module.

Let F be the usual Reshetikhin-Turaev functor from the category of framed tangles colored by topologically free $U_h(\mathfrak{g})$-modules of finite rank to the category of $U_h(\mathfrak{g})$-modules (see [23]). In Section 2, we define a map d from the set of typical $U_h(\mathfrak{g})$-module to the ring $\mathbb{C}[[h]][h^{-1}]$.

If T_λ is a framed $(1,1)$-tangle colored by $U_h(\mathfrak{g})$-modules such that the open string is colored by the deformed typical module $\widetilde{V}(\lambda)$ of highest weight λ, then $F(T_\lambda) = x.\, \mathrm{Id}_{\widetilde{V}(\lambda)}$, for some x in $\mathbb{C}[[h]]$. We set $F'(T_\lambda) = x.\, \mathsf{d}(\lambda)$.

If L is the colored link given by the closure of T_λ, we will see at the end of Section 1 that $F(L) = x.\, \mathrm{qdim}(\widetilde{V}(\lambda)) = 0$. For this reason we think of d as a replacement for the quantum dimension qdim. In Section 2, we show that this regularization makes the map F' into a well defined framed colored link invariant. In particular, we prove the following theorem.

Theorem 1. *The map F' induces a well defined invariant of framed links colored by at least one typical $U_h(\mathfrak{g})$-module. In other words, if L is a framed link colored by $U_h(\mathfrak{g})$-modules and the closure of T_λ is equal to L then the map given by $L \mapsto F'(T_\lambda)$ is a well defined framed colored link invariant.*

In Section 3 we will use F' to show that there exists multivariable link invariants $M^c_{\mathfrak{sl}(m|n)}$ and $M^c_{\mathfrak{osp}(2|2n)}$ for each $m, n \in \mathbb{N}^*$ and $c \in \mathbb{N}^{r-1}$. Let us now make this statement precise.

Let $c \in \mathbb{N}^{r-1}$. Then for all but a finite number of $a \in \mathbb{C}$ the module corresponding to (c, a) is typical. Let $\mathbb{T}_c$ be this set of complex numbers. Let $\lambda^c_a \in \mathfrak{h}^*$ be the weight corresponding to (c, a). The proof of the following theorem can be found in Section 3.

Theorem 2. *Let L' be a framed link with k ordered components. Let L be the non-framed link which underlies L'. For each $c \in \mathbb{N}^{r-1}$, there exists an multivariable link invariant $M^c_{\mathfrak{g}}$ with the following properties.*

1. *If $k = 1$ then $M^c_{\mathfrak{g}}(L)$ takes values in $(M^c_1(q, q_1))^{-1}\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$,*
2. *If $k \geq 2$ then $M^c_{\mathfrak{g}}(L)$ takes values in $\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \ldots, q_k^{\pm 1}]$,*
3. *If $(\xi_1, \ldots, \xi_k) \in (\mathbb{T}_c)^k$ and the ith component of L' is colored by the typical module of weight $\lambda^c_{\xi_i}$ then*

$$F'(L') = \mathrm{e}^{\sum lk_{ij} < \lambda^c_{\xi_i}, \lambda^c_{\xi_j} + 2\rho > h/2}\, M^c_{\mathfrak{g}}(L)|_{q_i = \mathrm{e}^{\xi_i h/2}}$$

where (lk_{ij}) is the linking matrix of L'; the bilinear form $< .,. >$, and element $\rho \in \mathfrak{h}$ are defined in subsection 1.1; and M^c_1 is defined in Lemma 3.2.

More generally we define in Theorem 3 (page 19) a $(k+1)$-multivariable invariant of ordered links whose k components are colored by different elements of $\mathbb{N}^{r-1}$.

In [7] the authors give proves of Theorem 1 and Theorem 2 in the case of $\mathfrak{sl}(2|1)$ and $c = (0, \ldots, 0)$. Many of the results, in [7], are proved by calculations made by hand. In this paper our proofs are based on more general techniques rooted in the representation theory of $\mathfrak{g}$.

Still these techniques require us to assume that $\mathfrak{g}$ is a Lie superalgebras of type I. We make this assumption because the character formulas for typical modules of other basic classical Lie superalgebras (i.e. not of type I), are more complicated. Also, basic classical Lie superalgebras of type II, do not have one-parameter families of modules and thus no natural way to construct multivariable invariants. However, for these Lie superalgebras one should still be able to define a map d and thus a framed colored link invariant.

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1. Preliminaries

In the section we review background material that will be used in the following sections.

A *super-space* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{C}$. We denote the parity of a homogeneous element $x \in V$ by $\bar{x} \in \mathbb{Z}_2$. We say x is even (odd) if $x \in V_{\bar{0}}$ (resp. $x \in V_{\bar{1}}$). A *Lie superalgebra* is a super-space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a super-bracket $[\,,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ that preserves the $\mathbb{Z}_2$-grading, is super-antisymmetric ($[x,y] = -(-1)^{\bar{x}\bar{y}}[y,x]$), and satisfies the super-Jacobi identity (see [14]). Throughout, all modules will be $\mathbb{Z}_2$-graded modules (module structures which preserve the $\mathbb{Z}_2$-grading, see [14]).

1.1. Lie superalgebras of type I.

In this subsection we recall notation and properties related to Lie superalgebras of type I and modules over such Lie superalgebras. Modules over Lie superalgebras of type I are different in nature than modules over semi-simple Lie algebras. For example, each Lie superalgebra of type I has one parameter families of modules.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra of type I, i.e. $\mathfrak{g}$ is equal to $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2|2n)$. We will assume that $m \neq n$. Let $\mathfrak{b}$ be the distinguished Borel sub-superalgebra of $\mathfrak{g}$. Then $\mathfrak{b}$ can be written as the direct sum of a Cartan sub-superalgebra $\mathfrak{h}$ and a positive nilpotent sub-superalgebra $\mathfrak{n}_+$. Moreover, $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let W be the Weyl group of the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$.

Let $\Delta^+_{\bar{0}}$ (resp. $\Delta^+_{\bar{1}}$) be the even (resp. odd) positive roots. Let $\rho_{\bar{0}}$ (resp. $\rho_{\bar{1}}$) denote the half sum of all the even (resp. odd) positive roots. Set $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$. A positive root is called *simple* if it cannot be decomposed into a sum of two positive roots.

A Cartan matrix associated to a Lie superalgebra is a pair consisting of a $r \times r$ matrix $A = (a_{ij})$ and a set $\tau \subset \{1, \dots, r\}$ determining the parity of the generators. Let (A, τ) be the Cartan arising from $\mathfrak{g}$ and the distinguished Borel sub-superalgebra $\mathfrak{b}$. Here the set $\tau = \{s\}$ consists of only one element because of our choice of Borel sub-algebra $\mathfrak{b}$. (See the appendix.)

There are $d_1, \dots, d_r$ in $\{\pm 1, \pm 2\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Here we will assume $d_1 = 1$. Let $< .,. >$ be the symmetric non-degenerate form on $\mathfrak{h}$ determined by $< h_i, h_j >= d_j^{-1} a_{ij}$. This form gives an identification of $\mathfrak{h}$ and $\mathfrak{h}^*$. Moreover, the form $< .,. >$ induces a W-invariant bilinear form on $\mathfrak{h}^*$, which we will also denote by $< .,. >$.

By Proposition 1.5 of [15] there exists $e_i \in \mathfrak{n}_+$, $f_i \in \mathfrak{n}_-$ and $h_i \in \mathfrak{h}$ for $i = 1, \dots, r$ such that the Lie superalgebra $\mathfrak{g}$ is generated by e_i, f_i, h_i where

$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = - a_{ij} f_j.$$

Let $\lambda \in \mathfrak{h}^*$ be a linear functional on $\mathfrak{h}$. Kac [14] defined a $\mathfrak{g}$ irreducible highest weight module $V(\lambda)$ of weight λ with a highest weight vector v_0 having the property that $h.v_0 = \lambda(h) v_0$ for all $h \in \mathfrak{h}$ and $\mathfrak{n}_+ v_0 = 0$. Let $a_i = \lambda(h_i)$. In [14] Kac showed that $V(\lambda)$ is finite-dimensional if and only if $a_i \in \mathbb{N}$ for $i \neq s$. Therefore, a_s can be an arbitrary complex number. Irreducible finite-dimensional $\mathfrak{g}$-modules are divided into two classes typical and atypical.

There are many equivalent definitions for a weight module to be typical (see [15]). Here we say that $V(\lambda)$ is typical if it splits in any finite-dimensional $\mathfrak{g}$-module. By Theorem 1 of [15] this is equivalent to requiring that

$$< \lambda + \rho, \alpha > \neq 0 \tag{1}$$

for all $\alpha \in \Delta_{\bar{1}}^+$. If $V(\lambda)$ is typical we will say the weight λ is typical. Also note that typical weights are dense in the space of weights corresponding to finite-dimensional modules. In particular, if $a_i \in \mathbb{N}$ for $1 \leq i \leq r$ and $i \neq s$ then there are only finitely many atypical weights with $a_i = \lambda(h_i)$. Furthermore, if λ is atypical then $a_s = \lambda(h_s) \in \mathbb{Z}$.

Let $\Lambda \simeq \mathbb{Z}^{r-1} \times \mathbb{C}$ be the group (or weight "lattice") of weights taking integer values on h_i for $i \neq s$. For $\alpha \in \Lambda$ we denote its image in $\mathbb{Z}[\Lambda]$ by e^{α}.

Let L'_0, L'_1 and L_1 be the following elements of the ring $\mathbb{Z}[\Lambda]$ of characters:

$$L'_0 = \prod_{\alpha \in \Delta_{\bar{0}}^+} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2}) \quad , \quad L_1 = \prod_{\alpha \in \Delta_{\bar{1}}^+} (\mathrm{e}^{\alpha/2} + \mathrm{e}^{-\alpha/2})$$

$$\text{and } L_1' = \prod_{\alpha\in\Delta_{\bar 1}^+} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2}).$$

If $V(\lambda)$ is typical then from (2.2′) of [15] we have the following formula for the (super-) character of $V(\lambda)$:

$$\begin{aligned}\mathrm{sch}(V(\lambda)) &= (L_1'/L_0')\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w(\lambda+\rho)}\\ \mathrm{ch}(V(\lambda)) &= (L_1/L_0')\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w(\lambda+\rho)}\end{aligned}\tag{2}$$

with $\epsilon : W \to \{\pm 1\}$ the function given by $\epsilon(w) = (-1)^c$ where c is the number of reflections in the expression of w.

Proposition 1.1. *The super-character (resp. the character) of a typical $\mathfrak{g}$-module $V(\lambda)$ has the form $\mathrm{sch}(V(\lambda)) = \chi_1'\chi_0(\lambda)$ (resp. $\mathrm{ch}(V(\lambda)) = \chi_1\chi_0(\lambda)$) where*

$$\chi_1' = \prod_{\alpha\in\Delta_{\bar 1}^+}(1-\mathrm{e}^{-\alpha}) \quad , \quad \chi_1 = \prod_{\alpha\in\Delta_{\bar 1}^+}(1+\mathrm{e}^{-\alpha})$$

and $\chi_0(\lambda)$ is the character of the even finite-dimensional irreducible $\mathfrak{g}_{\bar 0}$-module with highest weight λ.

Proof. Let $V(\lambda)$ be a typical $\mathfrak{g}$-module. From [15] Proposition 1.7(c) we have that $w(\rho_1) = \rho_1$ for all $w \in W$ and so equation (2) can be rewritten as

$$\begin{aligned}\mathrm{sch}(V(\lambda)) &= (L_1'/L_0')\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w(\lambda+\rho_0)-\rho_1}\\ &= \prod_{\alpha\in\Delta_{\bar 1}^+}(1-\mathrm{e}^{-\alpha})\frac{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w(\lambda+\rho_0)}}{\prod_{\alpha\in\Delta_{\bar 0}^+}(\mathrm{e}^{\alpha/2}-\mathrm{e}^{-\alpha/2})}\end{aligned}\tag{3}$$

where the last equality follows from the fact that ρ_1 is the half sum of the positive odd roots. Now the fraction in equation (3) is the character of the even finite-dimensional irreducible $\mathfrak{g}_{\bar 0}$-module with highest weight λ. A similar argument shows that $\mathrm{ch}(V(\lambda)) = \chi_1\chi_0(\lambda)$. □

Remark 1.2. *In formulas (1)-(3) we use the fact that $\mathfrak{g}$ is a Lie superalgebras of type I. In particular, in the notation of [15] we have $\Delta_{\bar 0}^+ = \overline{\Delta_{\bar 0}^+}$ and $\epsilon' = \epsilon$.*

Remark 1.3. *Here we only consider "even" irreducible modules: $V(\lambda)$, i.e. modules with an even highest weight vector. Every such irreducible module has an "odd" analog $V(\lambda)^-$ obtained by just taking the opposite* *parity. $V(\lambda)^-$ is isomorphic with $V(\lambda)$ with an odd isomorphism. Remark that the tensor product of two "even" modules may contain "odd" modules. The character and super-character of an "odd" module is*

$$\mathrm{ch}(V(\lambda)^-) = \mathrm{ch}(V(\lambda)) \quad \textit{and} \quad \mathrm{sch}(V(\lambda)^-) = -\,\mathrm{sch}(V(\lambda)).$$

1.2. **The quantization $U_h(\mathfrak{g})$.** Let h be an indeterminate. Set $q = \mathrm{e}^{h/2}$. We adopt the following notations:

$$q^z = \mathrm{e}^{zh/2} \quad \text{and} \quad \{z\} = q^z - q^{-z}.$$

Definition 1.4 ([24])**.** *Let $\mathfrak{g}$ be a Lie superalgebra of type I. Let $(A, \{s\})$ be the Cartan matrix arising from the distinguished Borel sub-super-algebra (see section 1.1). Let $U_h(\mathfrak{g})$ be the $\mathbb{C}[[h]]$-Hopf superalgebra generated by the elements h_i, E_i and F_i, $i = 1 \cdots r$, satisfying the relations:*

$$[h_i, h_j] = 0, \qquad [h_i, E_j] = a_{ij}E_j, \qquad [h_i, F_j] = -\,a_{ij}F_j,$$

$$[E_i, F_j] = \delta_{i,j}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \qquad E_s^2 = F_s^2 = 0,$$

plus the quantum Serre-type relations (see Definition 4.2.1 of [24]*). Here $[,]$ is the super-commutator given by $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$. All generators are even except for E_s and F_s which are odd. The coproduct, counit and antipode are given by*

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + q^{-h_i} \otimes E_i, & \epsilon(E_i) &= 0 & S(E_i) &= -\,q^{h_i}E_i\\
\Delta(F_i) &= F_i \otimes q^{h_i} + 1 \otimes F_i, & \epsilon(F_i) &= 0 & S(F_i) &= -\,F_i q^{-h_i}\\
\Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i, & \epsilon(h_i) &= 0 & S(h_i) &= -\,h_i.
\end{aligned}$$

Khoroshkin, Tolstoy [17] and Yamane [24] showed that $U_h(\mathfrak{g})$ has an explicit R-Matrix R. We will now recall their results. Let $\exp_q(x) := \sum_{n=0}^{\infty} x^n/(n)_q!$ be the "q-exponential," where $(n)_q! := (1)_q(2)_q \cdots (n)_q$ and $(k)_q := (1 - q^k)/(1 - q)$. Fix a normal ordering on the set of positive roots Δ^+. Let E_α and F_α for $\alpha \in \Delta^+$ be the q-analogs of the Cartan-Weyl generators where $E_{\alpha_i} = E_i$ and $F_{\alpha_i} = F_i$ for any simple root α_i (see Section 3 of [17]). Let $\{e_\alpha, f_\alpha, h_i : \alpha \in \Delta^+, i \in \{1, ..., r\}\}$ be the Cartan-Weyl basis of $\mathfrak{g}$. For $\alpha \in \Delta^+$ let a_α be the function of q defined by

$$[E_\alpha, F_\alpha] = a_\alpha(q^{h_\alpha} - q^{-h_\alpha})/(q - q^{-1})$$

where $h_\alpha := [e_\alpha, f_\alpha]$. Let

$$\check{R}_\alpha := \exp_q\big((-1)^{\bar{\alpha}}a_\alpha^{-1}(q - q^{-1})(E_\alpha \otimes F_\alpha)\big).$$

Also, let

$$\check{R} = \prod_{\alpha\in\Delta^+} \check{R}_\alpha, \tag{4}$$

where the order in the product is given by the chosen fixed normal ordering. Let (d_{ij}) be the inverse of the matrix (a_{ij}/d_j). Set

$$K = q^{\sum_{i,j}^r d_{ij} h_i \otimes h_j}. \tag{5}$$

Then the R-matrix is of the form $R = \check{R}K$. Remark for $\alpha \in \Delta_{\bar{1}}^+$ it follows from Zhang [25, 26] that $a_\alpha = 1$. One can also see from Yamane [24] that the R-matrix has the above form.

We say a $U_h(\mathfrak{g})$-module W is topologically free of finite rank if it is isomorphic as a $\mathbb{C}[[h]]$-module to $V[[h]]$, where V is a finite-dimensional $\mathfrak{g}$-module. Let $\mathcal{M}$ be the category of topologically free of finite rank $U_h(\mathfrak{g})$-modules. A standard argument shows that $\mathcal{M}$ is a ribbon category (for details see [5]). Let V, W be objects of $\mathcal{M}$. We denote the braiding and twist morphisms of $\mathcal{M}$ as

$$c_{V,W} : V \otimes W \to W \otimes V, \qquad \theta_V : V \to V$$

respectively. We also denote the duality morphisms of $\mathcal{M}$ as

$$b_V : \mathbb{C}[[h]] \to V \otimes V^*, \qquad d'_V : V \otimes V^* \to \mathbb{C}[[h]]$$

Let $\mathcal{T} = Rib_{\mathcal{M}}$ be the ribbon category of ribbon graphs over $\mathcal{M}$ in the sense of Turaev (see [23] chapter I) where coupons are colored by even morphisms. The set of morphisms $\mathcal{T}((V_1, \dots, V_n), (W_1, \dots, W_m))$ is a space of formal linear combinations of ribbon graphs colored by objects of $\mathcal{M}$. Let F be the usual ribbon functor from $\mathcal{T}$ to $\mathcal{M}$ (see [23]).

In [6], Geer shows that a weight $\mathfrak{g}$-module $V(\lambda)$ can be deformed to a weight $U_h(\mathfrak{g})$-module $\widetilde{V}(\lambda)$. In particular, Geer shows that the characters of $V(\lambda)$ and $\widetilde{V}(\lambda)$ are equal and that as a super-space $\widetilde{V}(\lambda)$ is equal to $V(\lambda)[[h]]$.

We say that $V \in \mathcal{M}$ is *irreducible* if $\mathrm{End}_{U_h(\mathfrak{g})}(V) = \mathbb{C}[[h]].\,\mathrm{Id}_V$. Then the deformation $\widetilde{V}(\lambda)$ is irreducible for every finite-dimensional irreducible weight $\mathfrak{g}$-module $V(\lambda)$.

It is well known that the super-dimension of any typical $\mathfrak{g}$-module is zero. Then an argument using the Kontsevich integral shows that the quantum dimension of any deformed typical $U_h(\mathfrak{g})$-module factors as $x.\,\mathrm{sdim}(V)$ (for some $x \in \mathbb{C}[[h]]$) and thus is zero. It follows that the functor F is zero on all closed ribbon graph over $\mathcal{M}$ with at least one color which is a deformed typical module. For this reason it can be difficult to construct nontrivial link invariants from typical $\mathfrak{g}$-modules.

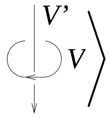
2. Proof of Theorem 1

In this section we define the map d and prove a series of lemmas which lead to the proof of Theorem 1.

Definition 2.1. *If $T \in \mathcal{T}(V,V)$ where $V \in \mathcal{M}$ is irreducible then $F(T) = x.\,\mathrm{Id}_V \in \mathrm{End}_{U_h(\mathfrak{g})}(V)$ for some $x \in \mathbb{C}[[h]]$. We define the bracket of T to be*

$$< T >= x.$$

For example, if V, V' are modules of $\mathcal{M}$ such that V' is irreducible, we define

$$S'(V,V') = \Big\langle$$ 

$$\Big\rangle.$$

When $V = \widetilde{V}(\lambda)$ and $V' = \widetilde{V}(\mu)$ are irreducible highest weight modules with highest weights λ and μ we write $S'(\lambda,\mu)$ for $S'(V,V')$.

For any weight β, let $\varphi_\beta : \mathbb{Z}[\Lambda] \to \mathbb{C}[[h]]$ be given by

$$\mathrm{e}^\alpha \mapsto q^{2<\alpha,\beta>}. \tag{6}$$

Proposition 2.2.

$$S'(\lambda,\mu) = \varphi_{\mu+\rho}(\mathrm{sch}(V(\lambda))).$$

Proof. Let (v_i) be a basis of $V(\lambda)$ such that v_i is a weight vector of weight $\alpha_i \in \mathfrak{h}^*$.

Let w_μ be a highest weight vector of $\widetilde{V}(\mu)$. Recall the R-matrix is of the form $R = \check{R}K$. We will use equation (5) to show

$$K(w_\mu \otimes v_i) = q^{<\mu,\alpha_i>} w_\mu \otimes v_i \qquad K(v_i \otimes w_\mu) = q^{<\mu,\alpha_i>} v_i \otimes w_\mu. \tag{7}$$

Indeed, let v and v' be two weight vectors of respective weight ν and ν'. As (a_{ij}/d_j) is the matrix of $< .,. >_{|\mathfrak{h}\otimes\mathfrak{h}}$ in the basis (h_i) and so its inverse (d_{ij}) is the matrix of $< .,. >_{|\mathfrak{h}^*\otimes\mathfrak{h}^*}$ in the dual basis. Now the scalars $x_i^{(\prime)} = \nu^{(\prime)}(h_i)$ by which h_i acts on $v^{(\prime)}$ are precisely the coordinates of $\nu^{(\prime)}$ in the basis dual to (h_i). Hence $\sum d_{ij} h_i \otimes h_j (v \otimes v') = (\sum x_i d_{ij} x'_j) v \otimes v' =< \nu, \nu' > v \otimes v'$.

We now give two facts. Let v be any weight vector of $\widetilde{V}(\lambda)$ of weight η.

Fact 1: $R(w_\mu \otimes v) = q^{<\mu,\eta>}(w_\mu \otimes v)$.
This fact follows from equations (4) and (7) and the property that $E_\alpha w_\mu = 0$ for $\alpha \in \Delta^+$.

Fact 2: All the pure tensors of the element $(\check{R}-1)(v\otimes v_\mu)\in \widetilde{V}(\lambda)\otimes\widetilde{V}(\mu)$ are of the form $v'\otimes w'$ where w' is a weight vector of $\widetilde{V}(\mu)$ and v' is a weight vector of $\widetilde{V}(\lambda)$ whose weight is of strictly higher order than that of the weight of v.
Fact 2 is true because $E_\alpha^n v$ (for $\alpha\in\Delta^+$ and $n\in\mathbb{N}^*$) is zero or a weight vector whose weight is of strictly higher order than the weight of v.

We will now compute $S'(\lambda,\mu)$ directly. Let V be an object of $\mathcal{M}$ and recall that the duality morphisms $b_V:\mathbb{C}[[h]]\to V\otimes V^*$ and $d'_V: V\otimes V^*\to\mathbb{C}[[h]]$ are defined as follows. The morphism b_V is the $\mathbb{C}[[h]]$-linear extension of the coevaluation map on the underlying $\mathfrak{g}$-module. In particular,

$$b_{\widetilde{V}(\lambda)}(1)=\sum v_i\otimes v_i^* \tag{8}$$

As in the case of semi-simple Lie algebras we have

$$d'_{\widetilde{V}(\lambda)}(v\otimes f)=(-1)^{\bar{v}\bar{f}}f(q^{2<\eta,\rho>}v) \tag{9}$$

where v is a weight vector of $\widetilde{V}(\lambda)$ of weight $\eta\in\mathfrak{h}^*$. Note in the above equation $q=\mathrm{e}^{h/2}$. Consider the element $S\in\mathrm{End}_{U_h(\mathfrak{g})}(\widetilde{V}(\mu))$ given by

$$(\mathrm{Id}_{\widetilde{V}(\mu)}\otimes d'_{\widetilde{V}(\lambda)})\circ(c_{\widetilde{V}(\lambda),\widetilde{V}(\mu)}\otimes\mathrm{Id}_{\widetilde{V}(\lambda)^*})\circ(c_{\widetilde{V}(\mu),\widetilde{V}(\lambda)}\otimes\mathrm{Id}_{\widetilde{V}(\lambda)^*})\circ(\mathrm{Id}_{\widetilde{V}(\mu)}\otimes b_{\widetilde{V}(\lambda)}).$$

To simplify notation set $S=(X_1)(X_2)(X_3)(X_4)$ where X_i is the corresponding morphism in the above formula. The morphism S is determined by its value on the highest weight w_μ. By definition $S(w_\mu)=S'(\lambda,\mu)w_\mu$, so it suffices to compute $S(w_\mu)$.

$$\begin{aligned}
S(w_\mu)&=(X_1)(X_2)(X_3)\Big(w_\mu\otimes\sum_i(v_i\otimes v_i^*)\Big)\\
&=(X_1)(X_2)\Big(\sum_i q^{<\mu,\alpha_i>}v_i\otimes w_\mu\otimes v_i^*\Big)\\
&=(X_1)\Big(\sum_i\big(q^{2<\mu,\alpha_i>}w_\mu\otimes v_i\otimes v_i^*\big)+\sum_k w'_k\otimes v'_k\otimes z_k\Big)\\
&=\sum_i(-1)^{\bar{v}_i}q^{2<\mu+\rho,\alpha_i>}w_\mu \qquad (10)
\end{aligned}$$

where $z_k=v_i^*$ (for some i), v'_k is a weight vector of $\widetilde{V}(\lambda)$ whose weight is of strictly higher weight than that of the weight of z_k^* and w'_k is a weight vector of $\widetilde{V}(\mu)$. Moreover, the first equality of the above equation follows from (8), the second from Fact 1, the third from (7) and Fact 2, and finally the fourth from (9) and the fact that $z_k(v'_k)=0$. The key observation in this proof is that Facts 1 and 2 imply that in the above computation the only contribution of the action of the R-matrix comes from K.

Since the super-character of $V(\lambda)$ is equal to $\sum(-1)^{\bar{\alpha}_i}\,\mathrm{e}^{\alpha_i}$ the proposition follows from equation (10). □

Corollary 2.3. *Let λ be a typical weight. Then $S'(\lambda,\mu)=0$ if and only if μ is atypical.*

Proof. From Proposition 1.1 we have

$$S'(\lambda,\mu)=\prod_{\alpha\in\Delta_{\bar{1}}^{+}}\varphi_{\mu+\rho}(1-\mathrm{e}^{-\alpha})\varphi_{\mu+\rho}(\chi_0(\lambda)).$$

Since $\chi_0(\lambda)$ is a character of an even finite-dimensional $\mathfrak{g}_{\bar{0}}$-module we have $\varphi_{\mu+\rho}(\chi_0(\lambda))\neq 0$ for any weight μ. Now by the definition of a typical module (see equation (1)) we have $\prod_{\alpha\in\Delta_{\bar{1}}^{+}}\varphi_{\mu+\rho}(1-\mathrm{e}^{-\alpha})$ is non-zero if and only if μ is typical. □

Recall the definitions of φ_λ and L'_i given in equation (6) and Subsection 1.1, respectively.

Lemma 2.4. *Let λ be a typical weight. Then $\frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_{\lambda+\rho}(L'_1)\varphi_\rho(L'_0)}$ is an element of $h^{-k_{\mathfrak{g}}}\mathbb{C}[[h]]$ where $k_{\mathfrak{g}}$ is the number of odd positive roots, i.e. $k_{\mathfrak{g}}=|\Delta_{\bar{1}}^{+}|$.*

Proof. We have $L'_1=\mathrm{e}^{\rho_1}\prod_{\alpha\in\Delta_{\bar{1}}^{+}}(1-\mathrm{e}^{-\alpha})$. Therefore, since λ is typical then by equation (1) we have $\varphi_{\lambda+\rho}(L'_1)$ is non-zero element of $\mathbb{C}[[h]]$. Moreover, since the product in L'_1 is taken over $\Delta_{\bar{1}}^{+}$ we have $h^{-|\Delta_{\bar{1}}^{+}|}\varphi_{\lambda+\rho}(L'_1)$ is invertible in $\mathbb{C}[[h]]$. The lemma follows from the fact that $\varphi_{\lambda+\rho}(L'_0)/\varphi_\rho(L'_0)\in\mathbb{C}[[h]]$. □

Definition 2.5. *Let λ be a typical weight. Define*

$$\mathsf{d}(\widetilde{V}(\lambda))=\mathsf{d}(\lambda)=\frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_{\lambda+\rho}(L'_1)\varphi_\rho(L'_0)}\in h^{-k_{\mathfrak{g}}}\mathbb{C}[[h]]$$

where we use the notation of Lemma 2.4.

Lemma 2.6. *Let λ and μ be typical weights. Then*

$$\mathsf{d}(\mu)S'(\lambda,\mu)=\mathsf{d}(\lambda)S'(\mu,\lambda)$$

Proof. We have

$$\begin{aligned}\mathsf{d}(\mu)\varphi_{\mu+\rho}\big(\operatorname{sch}(V(\lambda))\big) &= \varphi_\rho(L_0')^{-1}\varphi_{\mu+\rho}\left(\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w(\lambda+\rho)}\right)\\ &= \varphi_\rho(L_0')^{-1}\sum_{w\in W}\epsilon(w)q^{2<w(\lambda+\rho),\mu+\rho>}\\ &= \varphi_\rho(L_0')^{-1}\sum_{w\in W}\epsilon(w)q^{2<w(\mu+\rho),\lambda+\rho>}\\ &= \mathsf{d}(\lambda)\varphi_{\lambda+\rho}\big(\operatorname{sch}(V(\mu))\big)\end{aligned}$$

where the third equality follows from the fact that the form $<.,.>$ is symmetric and W-invariant on $\mathfrak{h}^*$. □

Remark 2.7. *One can easily check that Proposition 2.2 is also true if one replaces $V(\lambda)$ or $V(\mu)$ with its odd analog $V(\lambda)^-$ or $V(\mu)^-$, respectively. As $\operatorname{sch}(V(\lambda)^-) = -\operatorname{sch}(V(\lambda))$, the appropriate extension of d for "odd" modules is $\mathsf{d}(V(\lambda)^-) = -\mathsf{d}(V(\lambda))$. Hence Lemma 2.6 is still valid for "odd" modules.*

The following lemma shows that there is a typical $\mathfrak{g}$-module whose tensor product with itself is multiplicity free. This implies that the endomorphism ring of this tensor product is commutative, which we use to prove Lemma 2.9.

Lemma 2.8. *There exists a typical weight module $V(\lambda_0)$ such that $V(\lambda_0)\otimes V(\lambda_0)$ splits as a direct sum of irreducible typical modules with no multiplicity.*

Proof. Let α be an irrational number. Let λ_0 be the weight determined by $\lambda_0(h_s) = \alpha$ and $\lambda_0(h_i) = 0$ for $i \neq s$. We will show that $V(\lambda_0)$ has the desired properties.

Let $\chi_1 = \prod_{\alpha\in\Delta_{\bar{1}}^+}(1+\mathrm{e}^{-\alpha})$. From Proposition 1.1 we have that the character of the typical $\mathfrak{g}$-module $V(\lambda)$ is of the form

$$\operatorname{ch}(V(\lambda)) = \chi_1\chi_0(\lambda) \tag{11}$$

where $\chi_0(\lambda)$ is the character of an even finite-dimensional $\mathfrak{g}_{\bar{0}}$-module. For $V(\lambda_0)$ we have $\chi_0(\lambda_0) = \mathrm{e}^{\lambda_0}$. Therefore, $\operatorname{ch}(V(\lambda_0)\otimes V(\lambda_0)) = \chi_1(\chi_1\,\mathrm{e}^{2\lambda_0})$. From equation (11) it is enough to show that $\chi_1\,\mathrm{e}^{2\lambda_0}$ is a sum of characters of distinct irreducible $\mathfrak{g}_{\bar{0}}$-modules. (The corresponding $\mathfrak{g}$-module will be typical because the weights that appear in $\chi_1(\chi_1\,\mathrm{e}^{2\lambda_0})$ take irrational values on h_s.)

In the following two cases the roots are expressed in terms of the standard orthogonal basis (ϵ_i) of $\mathfrak{h}^*$ (see the Appendix).

Case 1: $\mathfrak{g} = \mathfrak{sl}(m|n)$. Set $\delta_i = \epsilon_{i+m}$. Recall that on $\mathfrak{h}$, $\sum \epsilon_i = \sum \delta_j$. The set of odd positive roots is given by $\Delta_{\bar{1}}^+ = \{\epsilon_i - \delta_j : 1 \le i \le m, 1 \le j \le n\}$. Set $x_i = \mathrm{e}^{\epsilon_i}$ and $y_j = \mathrm{e}^{\delta_j}$, then

$$\chi_1 = \prod_{\alpha \in \Delta_{\bar{1}}^+} (1 + \mathrm{e}^{-\alpha}) = \prod_{i,j} (1 + y_j/x_i) = (x_1 x_2 \cdots x_m)^{-n} \prod_{i,j} (x_i + y_j).$$

Let $s_\lambda(x) = s_\lambda(x_1, \ldots, x_m)$ be the Schur function associated to a partition λ. From the Cauchy identity we have

$$(x_1 x_2 \cdots x_m)^{-n} \prod_{i,j} (x_i + y_j) = (x_1 x_2 \cdots x_m)^{-n} \sum_{\lambda \subset (n^m)} s_\lambda(x) s_{\hat{\lambda}'}(y) \quad (12)$$

where the sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 \le n$, the complementary partition $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_m)$ is defined by $\hat{\lambda}_i = n - \lambda_{m+1-i}$, and $\hat{\lambda}'$ is the conjugate of $\hat{\lambda}$.

The Lie algebra $\mathfrak{g}_{\bar{0}}$ is isomorphic to $\mathfrak{sl}(m) \times \mathfrak{sl}(n) \times \mathbb{T}_1$. The central element acts by a complex number on any irreducible $\mathfrak{g}_{\bar{0}}$-module. This correspond to the $\mathbb{C}$-grading of $\mathbb{Z}[\Lambda]$. So the sum (12) splits as a sum of characters of $mn + 1$ distinct $\mathfrak{g}_{\bar{0}}$-modules following the spectral decomposition of the central element of $\mathfrak{g}_{\bar{0}}$:

$$\mathrm{e}^{2\lambda_0} \chi_1 = \sum_{i=0}^{mn} \left(\mathrm{e}^{2\lambda_0} (x_1 x_2 \cdots x_m)^{-n} \sum_{\substack{|\lambda| = i \\ \lambda \subset (n^m)}} s_\lambda(x) s_{\hat{\lambda}'}(y) \right)$$

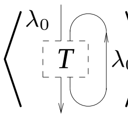
Now for fixed i, the functions $s_\lambda(x)$ with $|\lambda| = \lambda_1 + \cdots + \lambda_m = i$ are the characters of distinct irreducible $\mathfrak{sl}(m)$-modules (with highest weight $(\lambda_1 - \lambda_2, \ldots, \lambda_{m-1} - \lambda_m)$). Furthermore $s_{\hat{\lambda}'}(y)$ is also the character of an irreducible $\mathfrak{sl}(n)$-module so all the terms in this sum are characters of different irreducible $\mathfrak{g}_{\bar{0}}$-modules.

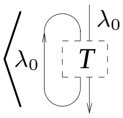
Case 2: $\mathfrak{g} = \mathfrak{osp}(2|2n)$. Set $\delta_i = \epsilon_{i+2}$ for $i = 1, \ldots, n$. The set of odd positive roots is given by $\Delta_{\bar{1}}^+ = \{\epsilon_1 \pm \delta_i : 1 \le i \le n\}$. Set $x_i = \mathrm{e}^{\delta_i}$ and $y = \mathrm{e}^{-\epsilon_1}$. Let $e_k(z_1, \ldots, z_p) = e_k(z)$ be the k^{th} elementary symmetric function in p variables $z_1, \ldots, z_p$. We have

$$\chi_1 = \prod_{i=1}^{n} (1 + y x_i)(1 + y/x_i) = \sum_{k=0}^{2n} e_k(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n) y^k.$$

The proof follows from the fact that $e_{2n-k}(x, x^{-1}) = e_k(x, x^{-1})$, $e_0 = 1$ and for $k = 1 \cdots n$, $e_k(x, x^{-1}) = \sum_{i=0}^{\lfloor n/2 \rfloor} \Gamma_{k-2i}$ where Γ_j is the character of the irreducible $\mathfrak{sp}(2n)$-module with highest weight the j^{th} fundamental weight. (see [4] section 24.2 page 407). $\square$

Lemma 2.9. *Let $V(\lambda_0)$ be the module of Lemma 2.8. Let $\widetilde{V}_{\lambda_0}$ be the $U_h(\mathfrak{g})$-module which is the deformation of $V(\lambda_0)$. Then we have*

$\Big\langle$ 

$\Big\rangle = \Big\langle$ 

$\Big\rangle$

for all $T \in \mathcal{T}\big((\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}), (\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0})\big)$.

Proof. Set $E = \mathrm{End}(\widetilde{V}_{\lambda_0} \otimes \widetilde{V}_{\lambda_0})$.
Consider the following linear forms on E:

$$\mathrm{tr}_L(f) = (d_{\widetilde{V}_{\lambda_0}} \otimes \mathrm{Id}_{\widetilde{V}_{\lambda_0}}) \circ (\mathrm{Id}_{\widetilde{V}^*_{\lambda_0}} \otimes f) \circ (b'_{\widetilde{V}_{\lambda_0}} \otimes \mathrm{Id}_{\widetilde{V}_{\lambda_0}}) \in \mathrm{End}(\widetilde{V}_{\lambda_0}) \cong \mathbb{C}[[h]],$$

$$\mathrm{tr}_R(f) = (\mathrm{Id}_{\widetilde{V}_{\lambda_0}} \otimes d'_{\widetilde{V}_{\lambda_0}}) \circ (f \otimes \mathrm{Id}_{\widetilde{V}^*_{\lambda_0}}) \circ (\mathrm{Id}_{\widetilde{V}_{\lambda_0}} \otimes b_{\widetilde{V}_{\lambda_0}}) \in \mathrm{End}(\widetilde{V}_{\lambda_0}) \cong \mathbb{C}[[h]].$$

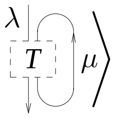
Let T be any element of $\mathcal{T}((\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}), (\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}))$. Let F is the functor described in subsection 1.2. In general,

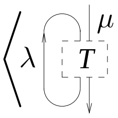
$$(\mathrm{tr}_R \circ F)(T) = (\mathrm{tr}_L \circ F)(c^{-1}_{\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}} T c_{\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}})$$

but from Lemma 2.8 we have that E is a commutative algebra and so $F(c^{-1}_{\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}} T c_{\widetilde{V}_{\lambda_0}, \widetilde{V}_{\lambda_0}}) = F(T)$. $\square$

The following lemma is a key ingredient in the proof of Theorem 1.

Lemma 2.10. *Let λ and μ be two typical weights. Then we have*

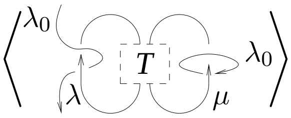
$\mathsf{d}(\lambda) \Big\langle$ 

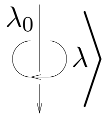
$\Big\rangle = \mathsf{d}(\mu) \Big\langle$ 

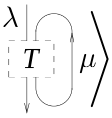
$\Big\rangle$

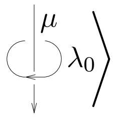
for all $T \in \mathcal{T}\Big(\big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big), \big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big)\Big)$.

Proof. Let $T \in \mathcal{T}\Big(\big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big), \big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big)\Big)$. Let $V(\lambda_0)$ be the module of Lemma 2.8. By definition we have

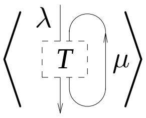
$\Big\langle$ 

$\Big\rangle = \Big\langle$ 

$\Big\rangle \Big\langle$ 

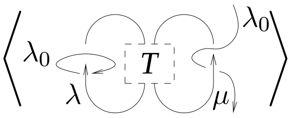
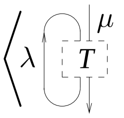
$\Big\rangle \Big\langle$ 

$\Big\rangle$

$= S'(\lambda, \lambda_0) S'(\lambda_0, \mu) \Big\langle$ 

$\Big\rangle$. (13)

Similarly,

$\Big\langle$  $\Big\rangle = S'(\mu, \lambda_0)S'(\lambda_0, \lambda) \Big\langle$  $\Big\rangle$. (14)

From Lemma 2.9 we have that the left sides of equations (13) and (14) are equal. Thus, the results follows from Lemma 2.6 and Corollary 2.3. □

Remark 2.11. *Let $\widetilde{V}_\lambda$ be the deformation of a typical $\mathfrak{g}$-module of highest weight λ. Compare the definition of d given in 2.5 with the quantum dimension of the $U_h(\mathfrak{g}_{\bar{0}})$-module V_0 with the same highest weight λ:*

$$\mathrm{qdim}(V_0) = \frac{\varphi_{\lambda+\rho_{\bar{0}}}(L'_0)}{\varphi_{\rho_{\bar{0}}}(L'_0)} = \frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_\rho(L'_0)} \quad \textit{so} \quad \mathsf{d}(\widetilde{V}_\lambda) = \frac{\mathrm{qdim}(V_0)}{\varphi_{\lambda+\rho}(L'_1)}.$$

Also, compare $\mathsf{d}(\widetilde{V}_\lambda)$ with the formula that gives the quantum dimension of $\widetilde{V}_\lambda$:

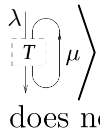
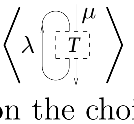
$$\mathrm{qdim}(\widetilde{V}_\lambda) = \varphi_\rho(\mathrm{sch}(\widetilde{V}_\lambda)) = \varphi_\rho(L'_1)\frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_\rho(L'_0)}$$

this last product vanishes because $\varphi_\rho(L'_1) = 0$

Lemma 2.10 suggests to consider d as a replacement for the quantum dimension. Also, remark that any map proportional to d would also be appropriate. For example $h^{k_\mathfrak{g}}\,\mathsf{d}$ which admit a classical limit ($h \to 0$). Hence one gets a classical analog of Lemma 2.10 ($\lim_{h\to 0} h^{k_\mathfrak{g}}\,\mathsf{d}$ is a "replacement" for the super-dimension) which seems unknown. In fact we can show, using the Kontsevich integral, that this classical version would be equivalent to Lemma 2.10. But surprisingly, we have not found a simpler "classical" proof of this result.

Proof of Theorem 1. Any closed ribbon graph $L \in \mathcal{T}(\emptyset, \emptyset)$ over $\mathcal{M}$ with at least one edge colored by a typical module $\widetilde{V}(\lambda)$ can be represented as the closure of $T_\lambda \in \mathcal{T}(\widetilde{V}(\lambda), \widetilde{V}(\lambda))$. We set

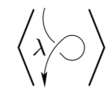
$$F'(L) = \mathsf{d}(\lambda) < T_\lambda >$$

. If L can also be represented as the closure of $T_\mu \in \mathcal{T}(\widetilde{V}(\mu), \widetilde{V}(\mu))$ for some typical weight μ then there exits $T \in \mathcal{T}\Big(\big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big), \big(\widetilde{V}(\lambda), \widetilde{V}(\mu)\big)\Big)$ such that $T_\lambda = \Big\langle$  $\Big\rangle$ and $T_\mu = \Big\langle$  $\Big\rangle$ so by Lemma 2.10 the definition of $F'(L)$ does not depend on the choice of T_λ. □

3. The multivariable invariant

In this section we construct families of multivariable invariants of links and prove Theorem 2.

Lemma 3.1. *Let λ be a dominant weight. Then the value of the twist $\theta_{\widetilde{V}(\lambda)}$ is $q^{<\lambda,\lambda+2\rho>}$. In other words,*



$$= q^{<\lambda,\lambda+2\rho>}.$$

Proof. The proof follow from Fact 1 in the proof of Proposition 2.2 and equation (9). $\square$

Let $c, d \in \mathbb{N}^{r-1}$. Let λ_a^c be the weight corresponding to an $(r-1)$-tuple $c = (c_1, \dots, c_{r-1}) \in \mathbb{N}^{r-1}$ and a complex parameter $a \in \mathbb{C}$. Let $\widetilde{V}_a^c$ be the $U_h(\mathfrak{g})$-module $\widetilde{V}(\lambda_a^c)$. Recall that $\mathbb{C} \setminus \mathbb{T}_c$ is finite where $\mathbb{T}_c$ is the set of complex numbers a such that λ_a^c is typical.

Lemma 3.2. *Recall the definition of d given in Definition 2.5. Let λ_a^c be typical. There exists integers n_α^c for $\alpha \in \Delta^+$ such that for*

$$M_0^c(q) := \prod_{\alpha \in \Delta_{\bar{0}}^+} \frac{q^{n_\alpha^c} - q^{-n_\alpha^c}}{q^{n_\alpha^0} - q^{-n_\alpha^0}} \in \mathbb{Z}[q^{\pm 1}]$$

$$\text{and} \quad M_1^c(q, q_1) := \prod_{\alpha \in \Delta_{\bar{1}}^+} (q_1 q^{n_\alpha^c} - q_1^{-1} q^{-n_\alpha^c}) \in \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$$

one has $\varphi_{\lambda_a^c+\rho}(L_0')/\varphi_\rho(L_0') = M_0^c(\mathrm{e}^{h/2})$ and $\varphi_{\lambda_a^c+\rho}(L_1') = M_1^c(\mathrm{e}^{h/2}, \mathrm{e}^{ha/2})$. In particular,

$$\mathsf{d}(\lambda_a^c) = \frac{M_0^c(\mathrm{e}^{h/2})}{M_1^c(\mathrm{e}^{h/2}, \mathrm{e}^{ah/2})}.$$

Proof. The existence of the integers n_α^c follows from an explicit computation (given in the Appendix) of the products $< \lambda_a^c + \rho, \alpha >$. As $M_0^c(q)$ (see Remark 2.11) is the quantum dimension of the $U_h(\mathfrak{g}_{\bar{0}})$-module with the highest weight λ_a^c, it is in $\mathbb{Z}[q^{\pm 1}]$. $\square$

In [18] Le proved that the quantum invariants arising from simple Lie algebras (with a suitable normalization) are Laurent polynomial in q. His proof uses Lusztig's canonical basis. We will now show that a similar statement holds for $\mathfrak{g}$. However in our case, the complex parameter of $\widetilde{V}_a^c$ will lead to a Laurent polynomial in more than one variable.

Lemma 3.3. *There exists bases B_a^c of $\widetilde{V}_a^c$ and elements $R^{c,d}(x,y,z) \in \mathrm{GL}(\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}])$ such that the action of the R-matrix on $\widetilde{V}_a^c \otimes \widetilde{V}_b^d$ in the basis $B_a^c \times B_b^d$ is given by $q^{<\lambda_a^c, \lambda_b^d>} R^{c,d}(q, q^a, q^b)$.*

Proof. We fix $a \in \mathbb{C} \setminus \mathbb{Q}$ and consider the ring of Laurent polynomials in two variables $A = \mathbb{Z}[q^{\pm 1}, q^{\pm a}] \subset \mathbb{C}[[h]]$. Let $\mathcal{U}_{\mathbb{Z}}^{\bar{0}}$ (resp. $\mathcal{U}_{\mathbb{Z}}$) be the A-sub-algebra of $U_h(\mathfrak{g}_{\bar{0}})$ (resp. of $U_h(\mathfrak{g})$) generated by $X := \{E_i^{(k)}, F_i^{(k)}, K_i, K_s : k \in \mathbb{N}, i = 1 \cdots r, i \neq s\}$ (resp. by $X \cup \{F_s, (q - q^{-1})E_s\}$) where $G^{(k)} := G^k/(k)_q!$ and $K_i := q^{h_i}$.

Recall that the R-matrix of $U_h(\mathfrak{g})$ is of the form $R = \check{R}K$ where $\check{R} = \prod_{\alpha \in \Delta^+} \check{R}_\alpha$. It follows from Lusztig ([20], see also [12]) that the quasi R-matrix of $U_h(\mathfrak{g}_{\bar{0}})$, $\check{R}_{\bar{0}} := \prod_{\alpha \in \Delta_{\bar{0}}^+} R_\alpha$ is an element of (a completion of) $\mathcal{U}_{\mathbb{Z}}^{\bar{0}} \otimes_A \mathcal{U}_{\mathbb{Z}}^{\bar{0}}$. By definition $R_\alpha = 1 + (q - q^{-1})E_\alpha \otimes F_\alpha$ for $\alpha \in \Delta_{\bar{1}}^+$. Combining the last two sentences we have that $\check{R}$ is an element of (a completion of) $\mathcal{U}_{\mathbb{Z}} \otimes_A \mathcal{U}_{\mathbb{Z}}$.

Let v be a highest weight vector of $\widetilde{V}_a^c$ and set $L = \mathcal{U}_{\mathbb{Z}} v$. We will show that there exists a basis $B_a^c \times B_b^d$ such that the entries of the endomorphism of $\widetilde{V}_a^c \otimes \widetilde{V}_b^d$ arising from $\check{R}$ are in $\mathbb{Z}[q^{\pm 1}, q^{\pm a}, q^{\pm b}]$. From the previous paragraph it suffices to show that there exists a basis for L such that the action of any element of $\mathcal{U}_{\mathbb{Z}}$ on L has a matrix (in this basis) whose entries are Laurent polynomials in q and q^a.

Let $B_{\bar{0}}$ be a Lusztig's canonical basis of $L_{\bar{0}} = \mathcal{U}_{\mathbb{Z}}^{\bar{0}} v = \bigoplus_{x \in B_{\bar{0}}} A.x$. Let $B_{\bar{1}}$ be the PBW basis of $U_h \mathfrak{n}_{\bar{1}}^-$ given by the $2^{|\Delta_{\bar{1}}^+|}$ ordered product of elements from any subset of $\{F_\alpha : \alpha \in \Delta_{\bar{1}}^+\}$ (see [25, 26, 3]). Let $B = \{f.x : f \in B_{\bar{1}}, x \in B_{\bar{0}}\}$. It follows from the PBW theorem that B generates $\widetilde{V}_a^c$ (because $U_h(\mathfrak{g}) = U_h \mathfrak{n}_{\bar{1}}^- U_h \mathfrak{n}_{\bar{0}}^- U_h \mathfrak{h} U_h \mathfrak{n}_{\bar{0}}^+ U_h \mathfrak{n}_{\bar{1}}^+$ and $\widetilde{V}_a^c = U_h(\mathfrak{g})v$). Then the character formula for typical modules gives that B is a $\mathbb{C}[[h]]$-basis for $\widetilde{V}_a^c$.

We claim that $L = \bigoplus_{y \in B} A.y$ and hence $B_a^c := B$ is a basis of L. This follows from the following two observations. First, since B is a $\mathbb{C}[[h]]$-basis for $\widetilde{V}_a^c$ we have that B_a^c is a linearly independent subset of L. Second, let $F = \bigoplus_{\alpha \in \Delta_{\bar{1}}^+} A.F_\alpha \subset \mathcal{U}_{\mathbb{Z}}$ and $E = \bigoplus_{\alpha \in \Delta_{\bar{1}}^+} A.E_\alpha \subset \mathcal{U}_{\mathbb{Z}}$. The commutation relations of $U_h(\mathfrak{g})$ (see [25, 26, 3]) imply that

$$\mathcal{U}_{\mathbb{Z}}^{\bar{0}}.F = F.\mathcal{U}_{\mathbb{Z}}^{\bar{0}} \qquad \qquad \mathcal{U}_{\mathbb{Z}}^{\bar{0}}.E = E.\mathcal{U}_{\mathbb{Z}}^{\bar{0}}.$$

From this equation and the commuting relation between E_α and F_β, for $\alpha, \beta \in \Delta_{\bar{1}}^+$, we have that any $g \in \mathcal{U}_{\mathbb{Z}}$ can be written as an element in $\mathcal{U}_{\mathbb{Z}}^{\bar{1},-} \mathcal{U}_{\mathbb{Z}}^{\bar{0}} \mathcal{U}_{\mathbb{Z}}^{\bar{1},+}$ where $\mathcal{U}_{\mathbb{Z}}^{\bar{1},\pm}$ is generated by $\{(q - q^{-1})E_\alpha : \alpha \in \Delta_{\bar{1}}^+\}$ or $\{F_\alpha : \alpha \in \Delta_{\bar{1}}^+\}$, respectively. In other words, for any $g \in \mathcal{U}_{\mathbb{Z}}$ the element gv can be written as an A-linear combination of elements in B_a^c any $g \in \mathcal{U}_{\mathbb{Z}}$. Hence, the free A-lattice generated by B is stable by any $u \in \mathcal{U}_{\mathbb{Z}}$ and $\check{R}$ acts as desired. Notice that all of the above could have be done over $\mathbb{Z}[q^{\pm 1}]$ independently of the complex parameter a except that some of the commuting relations involve Laurent polynomials in K_s. The element K_s acts (by multiplication) on the weight vectors of $\widetilde{V}_a^c$ by $q^a q^k$ (for some $k \in \mathbb{Z}$). So the action of any element of $\mathcal{U}_{\mathbb{Z}}$ has a matrix in the basis B_a^c whose coefficients are Laurent polynomials independant of a in the two variables q and q^a.

Finally we show that K acts appropriately. A vector in the basis B_a^c is a weight vector. Its weight differs from the highest weight by an element of the root lattice. But for a root α, one has $<\alpha, \lambda_a^c> \in \mathbb{Z}+a\mathbb{Z}$. This with equation (7) shows that $q^{-<\lambda_a^c,\lambda_b^d>}K$ acts by monomials of $\mathbb{Z}[q^{\pm 1}, q^{\pm a}, q^{\pm b}]$ on the elements of the bases $B_a^c \times B_b^d$. □

Remark 3.4. *One has* $<\lambda_a^c, \lambda_b^d> = \frac{n_1ab+n_2a+n_3b+n_4}{n_5}$ *where* $n_i \in \mathbb{N}$. *Thus, without the linking correction of Theorem 3, F' takes values in a more complicated ring than a ring of Laurent polynomials.*

Theorem 3. *Let L be a link with k ordered components colored by k $(r-1)$-tuples $\bar{c} = (\bar{c}_1, \dots, \bar{c}_k) \in (\mathbb{N}^{r-1})^k$. Then there exists a multivariable Laurent polynomial $M(L;\bar{c})$ with values in $\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \dots, q_k^{\pm 1}]$ if $k \geq 2$ and in $(M_1^{\bar{c}_1}(q,q_1))^{-1}\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}]$ if $k = 1$ such that if*

- *L' is any framed representative of L,*
- *$(\xi_1, \dots, \xi_k) \in \mathbb{T}_{\bar{c}_1} \times \cdots \times \mathbb{T}_{\bar{c}_k}$,*
- *the i^{th} component of L' is colored by the typical module $\widetilde{V}_{\xi_i}^{\bar{c}_i}$ with highest weight $\lambda_{\xi_i}^{\bar{c}_i}$,*

then one has

$$F'(L') = \mathrm{e}^{\sum lk_{ij} <\lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} + 2\rho > h/2} \, M(L;\bar{c})\big|_{q_i = \mathrm{e}^{\xi_i h/2}}$$

where lk_{ij} are the linking numbers of L' (when $i = j$ the lk_{ii} is the framing of L'_i).

Proof. Choose k complex numbers $\xi_1, \dots, \xi_k$ such that $(1, \xi_1, \dots, \xi_k)$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{C}$. Let $\phi : \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \dots, q_k^{\pm 1}] \to \mathbb{C}[[h]]$ be the ring map defined by

$$\phi(q) = \mathrm{e}^{\frac{h}{2}} \quad \text{and} \quad \phi(q_i) = \mathrm{e}^{\frac{\xi_i h}{2}}$$

Then ϕ is injective since the family $\{\phi(q^{t_0} q_1^{t_1} \cdots q_n^{t_n}) : (t_0, \dots, t_n) \in \mathbb{Z}^{n+1}\}$ is free.

Consider a $(1,1)$-tangle T obtained by opening the t^{th} component of L' for $t \in \{1 \cdots k\}$. By definition $F'(L') = \mathsf{d}(\lambda_{\xi_t}^{\bar{c}_t}) < T >$. Combining this with Lemmas 3.3 and 3.2 we have

$$F'(L') = \mathrm{e}^{\sum lk_{ij} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} > h/2} \frac{\phi\big(M_0^{\bar{c}_t}(q)\big)}{\phi\big(M_1^{\bar{c}_t}(q, q_t)\big)} \mathrm{Im}(\phi). \tag{15}$$

As $< \lambda_{\xi_i}^{\bar{c}_i}, 2\rho > \in \mathbb{Z} + \xi_i\mathbb{Z}$, (see Appendix) we can define

$$M(L; \bar{c}) := \frac{\phi^{-1}\left(\mathrm{e}^{-\sum lk_{ij} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} + 2\rho > h/2} M_1^{\bar{c}_1}(\mathrm{e}^{h/2}, \mathrm{e}^{\xi_1 h/2}) F'(L')\right)}{M_1^{\bar{c}_1}(q, q_1)}. \tag{16}$$

Note that Theorem 1 implies that $M(L; \bar{c})$ is well defined. In other words, we are always able to cut the first component of L'. We added the $< \lambda_{\xi_i}^{\bar{c}_i}, 2\rho >$ in equation (16) to make M a link invariant, i.e. framing independent (see Lemma 3.1).

Next we will show that if $k \geq 2$ then

$$F'(L') \in \mathrm{e}^{\sum lk_{ij} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} > h/2} \mathrm{Im}(\phi). \tag{17}$$

For $i = 1, 2$, let T_i be a $(1,1)$-tangles whose closure is L' and whose open strand is the ith component of L'. From Theorem 1 we have $F'(T_1) = F'(T_2)$. Then Lemma 3.3 implies the existence of Laurent polynomials P_1 and P_2 such that

$$\mathrm{e}^{-\sum lk_{ij} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} > h/2} F'(L) = \mathsf{d}(\lambda_{\xi_1}^{\bar{c}_1}) \phi(P_1) = \mathsf{d}(\lambda_{\xi_2}^{\bar{c}_2}) \phi(P_2). \tag{18}$$

By definition

$$\mathsf{d}(\lambda_{\xi_i}^{\bar{c}_i}) = \frac{M_0^{\bar{c}_i}(\mathrm{e}^{h/2})}{M_1^{\bar{c}_i}(\mathrm{e}^{h/2}, \mathrm{e}^{h\lambda_{\xi_i}^{\bar{c}_i}/2})}, \quad \text{for } i = 1, 2. \tag{19}$$

Thus

$$M_0^{\bar{c}_1}(q) M_1^{\bar{c}_2}(q, q_2) P_1 = M_0^{\bar{c}_2}(q) M_1^{\bar{c}_1}(q, q_1) P_2$$

where $M_1^{\bar{c}_1}(q, q_1)$ and $M_0^{\bar{c}_1}(q) M_1^{\bar{c}_2}(q, q_2)$ are relatively prime. Since $\mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$ is an unique factorization domain we have that $M_1^{c_1}(q, q_1)$ divides P_1 and so equation (17) holds. Combining this with equation (16) we have $M(L; \bar{c}) \in \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$.

Because of Lemma 3.3, $M(L; \bar{c})$ is independent of the choice $\xi = (\xi_1, \ldots, \xi_k)$ lying in the dense subset of $\mathbb{C}^n$ defined by the condition: $(1, \xi_1, \ldots, \xi_k)$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{C}$. Now the two maps F' and $\phi \circ M$ depend continuously of ξ so the relation between F' and M in Theorem 3 is valid for any $(\xi_1, \ldots, \xi_k) \in \mathbb{T}_{\bar{c}_1} \times \cdots \times \mathbb{T}_{\bar{c}_k}$. □

Proof of Theorem 2. Just apply Theorem 3 to the link L with all components colored by c. □

Appendix

The $\mathfrak{sl}(m|n)$ case. $\mathfrak{sl}(m|n)$ is the Lie superalgebra with Cartan matrix $A = (a_{ij})$ whose non zeros entries given by

$$\begin{aligned} a_{i,i} &= 2 & \text{except } a_{m,m} &= 0 \\ a_{i,i+1} &= -1 & \text{except } a_{m,m+1} &= 1 \\ a_{i+1,i} &= -1. \end{aligned}$$

Set $d_i = 1$ for $i = 1 \cdots m$ and $d_i = -1$ for $i > m$ then $(d_i a_{ij})$ is a symmetric matrix.

We can identify $\mathfrak{sl}(m|n)$ with the Lie superalgebra of super-trace zero $(m|n) \times (m|n)$ matrices. This standard representation is obtained by sending e_i on the elementary matrix $E_{i,i+1}$, f_i on $E_{i+1,i}$, h_i on $E_{i,i} - E_{i+1,i+1}$ if $i \neq m$ and $h_m = E_{m,m} + E_{m+1,m+1}$. The Cartan subalgebra $\mathfrak{h}$ with basis (h_i) is contained in the space of diagonal matrices X. The space X^* has a canonical basis $(\epsilon_1, \dots \epsilon_{m+n})$ which is dual to the basis formed by the matrices $E_{i,i}$. Set $\delta_i = \epsilon_{i+m}$, then $\mathfrak{h}$ is the kernel of the super-trace $\mathrm{str} = \sum \epsilon_i - \sum \delta_j$. Therefore, $\mathfrak{h}^*$ is the quotient of X^* by the super-trace.

The bilinear form on $\mathfrak{h}$ given by $< h_i, h_j >= d_j^{-1} a_{ij}$ is equal to the restriction on $\mathfrak{h}$ of $< H, H' >= \mathrm{str}(H.H')$. So it extends to the whole set of diagonal matrices and induces on its dual the bilinear form defined by $< \epsilon_i, \epsilon_j >= \delta_j^i$ for $i, j = 1 \cdots m$ and $< \delta_i, \delta_j >= -\delta_j^i$ for $i, j = 1 \cdots n$. Hence $\mathfrak{h}^*$ can also be identified as an euclidien space with $\mathrm{str}^\perp$.

The set of positive roots is $\Delta^+ = \Delta_{\bar{0}}^+ \cup \Delta_{\bar{1}}^+$ with

$$\Delta_{\bar{0}}^+ = \{\epsilon_i - \epsilon_j,\ 1 \le i < j \le m\} \cup \{\delta_i - \delta_j,\ 1 \le i < j \le n\}$$

$$\text{and} \quad \Delta_{\bar{1}}^+ = \{\epsilon_i - \delta_j\}$$

The half sums of positive roots are given by

$$2\rho_0 = \sum_i (m+1-2i)\epsilon_i + \sum_j (n+1-2j)\delta_j \quad , \quad 2\rho_1 = n\sum_i \epsilon_i - m\sum_j \delta_j$$

$$\text{and } \rho = \rho_0 - \rho_1 = \frac{1}{2}\left(\sum_i (m-n+1-2i)\epsilon_i + \sum_j (m+n+1-2j)\delta_j\right).$$

Up to the super-trace, representatives of the fundamental weights are given by

$$\begin{aligned} w_k &= \sum_{i=1}^{k} \epsilon_i \text{ for } k = 1 \cdots m-1 \\ w_m &= \sum_i \epsilon_i = \sum_j \delta_j \\ w_{m+k} &= -\sum_{j=k+1}^{n} \delta_j \text{ for } k = 1 \cdots n-1. \end{aligned}$$

Thus for the weight λ_a^c (with $c \in \mathbb{N}^{r-1}$ and $a \in \mathbb{T}_c$) of section 3:

$$\lambda_a^c = c_1 w_1 + \cdots + c_{m-1} w_{m-1} + a w_m + c_m w_{m+1} + \cdots + c_{m+n-2} w_{m+n-1}. \quad (20)$$

One has

$$\begin{aligned} < \lambda_a^c, \rho_0 > &= \frac{1}{2}\left(\sum_{i=1}^{m-1} i(m-i)c_i - \sum_{i=1}^{n-1} i(n-i)c_{m+n-1-i} \right) \\ < \lambda_a^c, \rho_1 > &= \frac{1}{2}\left(\sum_{i=1}^{m-1} nic_i + mna - \sum_{i=1}^{n-1} mic_{m+n-1-i} \right) \\ < \lambda_a^c, 2\rho > &= \sum_{i=1}^{m-1} i(m-n-i)c_i - mna + \sum_{i=1}^{n-1} i(m-n+i)c_{m+n-1-i} \end{aligned}$$

$$\begin{aligned} < \epsilon_i - \delta_j, \lambda_a^c + \rho > &= < \epsilon_i - \delta_j, \lambda_a^c > + < \epsilon_i - \delta_j, \rho > \\ &= \sum_{k=i}^{m-1} c_k + a - \sum_{k=m}^{m+n-1-j} c_k + (m+1-i-j) \end{aligned}$$

and for $i < j$,

$$\begin{aligned} < \epsilon_i - \epsilon_j, \lambda_a^c + \rho > &= j - i + \sum_{k=i}^{j-1} c_k \\ < \delta_i - \delta_j, \lambda_a^c + \rho > &= i - j - \sum_{k=m+n-i}^{m+n-j-1} c_k. \end{aligned}$$

(Remark that in all these scalar products, at least one of the two weights is orthogonal to str.) This gives the form of $\mathsf{d}(\lambda_a^c)$:

$$\begin{aligned}\mathsf{d}(\lambda_a^c) &= \frac{\varphi_{\lambda_a^c+\rho}(L'_0)}{\varphi_{\lambda_a^c+\rho}(L'_1)\varphi_\rho(L'_0)}\\ &= \prod_{\alpha\in\Delta_{\bar 0}^+}\frac{q^{<\lambda_a^c+\rho,\alpha>}-q^{-<\lambda_a^c+\rho,\alpha>}}{q^{<\rho,\alpha>}-q^{-<\rho,\alpha>}}\Big/\prod_{\alpha\in\Delta_{\bar 1}^+}(q^{<\lambda_a^c+\rho,\alpha>}-q^{-<\lambda_a^c+\rho,\alpha>}).\end{aligned}$$

The $\mathfrak{osp}(2|2n)$ case. $\mathfrak{osp}(2|2n)$ is the Lie superalgebra with Cartan matrix $A=(a_{ij})$ with non zeros entries given by

$$\begin{aligned} a_{i,i} &= 2 & &\text{except } a_{1,1}=0\\ a_{i,i+1} &= -1 & &\text{except } a_{1,2}=1 \text{ and } a_{n,n+1}=-2\\ a_{i+1,i} &= -1. \end{aligned}$$

Set $d_1=1$, $d_i=-1$ for $i=2\cdots n$ and $d_{n+1}=-2$. Then (d_ia_{ij}) is a symmetric matrix.

Consider the super-symmetric form on $\mathbb{C}^{2|2n}$ with matrix

$$B=\begin{pmatrix} I_2 & 0 & 0\\ 0 & 0 & I_n\\ 0 & -I_n & 0\end{pmatrix}.$$

Then $\mathfrak{osp}(2|2n)$ is the set of matrices of the form

$$X=\begin{pmatrix} A & B & C\\ -{}^tC & D & E\\ {}^tB & F & -{}^tD\end{pmatrix} \text{ with } {}^tA=-A,\ {}^tE=E,\ {}^tF=F.$$

We consider the Cartan sub-algebra with basis $e_1=\sqrt{-1}(E_{1,2}-E_{2,1})$ and $e_{i+1}=E_{2+i,2+i}-E_{2+n+i,2+n+i}$ for $i=1\cdots n$. The dual basis is denoted $(\epsilon,\delta_1,\dots,\delta_n)$. The Cartan elements are $h_i=e_i-e_{i+1}$ for $i=1\cdots n$ and $h_{n+1}=e_{n+1}$.

The bilinear form on $\mathfrak{h}$ given by $<H,H'>=\frac{1}{2}\operatorname{str}(H.H')$ induces a bilinear form on $\mathfrak{h}^*$ defined by $<\epsilon,\epsilon>=1$, $<\epsilon,\delta_i>=0$ and $<\delta_i,\delta_j>=-\delta_j^i$ for $i,j=1\cdots n$.

The set of positive roots is $\Delta^+=\Delta_{\bar 0}^+\cup\Delta_{\bar 1}^+$ with

$$\Delta_{\bar 0}^+=\{\delta_i\pm\delta_j,\ 1\le i<j\le n\}\cup\{2\delta_i\} \quad\text{and}\quad \Delta_{\bar 1}^+=\{\epsilon\pm\delta_i\}$$

The half sums of positive roots are given by

$$\rho_0=\sum_i(n+1-i)\delta_i \quad,\quad \rho_1=n\epsilon \quad\text{and}\quad \rho=\rho_0-\rho_1.$$

The fundamental weights are given by

$$w_1 = \epsilon$$

$$w_{k+1} = \epsilon + \sum_{i=1}^{k} \delta_i \text{ for } k = 1 \cdots n.$$

Thus for the weight λ_a^c (with $c \in \mathbb{N}^{r-1}$ and $a \in \mathbb{T}_c$) of section 3 :

$$\lambda_a^c = aw_1 + c_1 w_2 + \cdots + c_n w_{n+1}. \tag{21}$$

One has

$$< \lambda_a^c, \rho_1 >= n(a + \sum_i c_i) \text{ and } < \lambda_a^c, \rho_0 >= -\sum_{i=1}^{n} \frac{i(2n-i+1)}{2} c_i$$

$$\begin{aligned} < \epsilon \pm \delta_i, \lambda_a^c + \rho > &=< \epsilon \pm \delta_i, \lambda_a^c > &+& \quad < \epsilon \pm \delta_i, \rho > \\ &= a + \textstyle\sum_{k=1}^{n} c_k \mp \sum_{k=i}^{n} c_k &+& \quad (-n \mp (n+1-i)) \end{aligned}$$

and for $i < j$

$$< \delta_i - \delta_j, \lambda_a^c + \rho > = -\sum_{k=m+n-i}^{m+n-j-1} c_k + i - j.$$

This gives the form of $\mathsf{d}(\lambda_a^c)$:

$$\begin{aligned} \mathsf{d}(\lambda_a^c) &= \frac{\varphi_{\lambda_a^c+\rho}(L_0')}{\varphi_{\lambda_a^c+\rho}(L_1')\varphi_\rho(L_0')} \\ &= \prod_{\alpha\in\Delta_0^+} \frac{q^{<\lambda_a^c+\rho,\alpha>} - q^{-<\lambda_a^c+\rho,\alpha>}}{q^{<\rho,\alpha>} - q^{-<\rho,\alpha>}} \Big/ \prod_{\alpha\in\Delta_1^+} (q^{<\lambda_a^c+\rho,\alpha>} - q^{-<\lambda_a^c+\rho,\alpha>}). \end{aligned}$$

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