Appendix to Quantum Link Invariant from the Lie Superalgebra $\mathfrak{D}_{2\,1,\alpha}$

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Appendix

A The universal R-matrix of $U_h\mathfrak{D}_{2\,1}$

We follow a renormalisation of H. Thys [2], to make appear at least one of the symmetry : H. Thys defines $U_h\mathfrak{D}_{2\,1}$ has the superalgebra topologically (for the h-adic topology) generated by E_i, F_i, H_i, $i = 1,2,3$ with the relations (for all $i,j = 1,2,3$)

$$\begin{gathered}
H_iH_j = H_jH_i \\
[H_i, E_j] = a_{ij}E_j, \\
[H_i, F_j] = -a_{ij}F_j, \\
[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{Q_i - Q_i^{-1}}, \\
E_1^2 = F_1^2 = 0, \quad [E_2, E_3] = [F_2, F_3] = 0, \\
E_i^2E_1 - (Q_i + Q_i^{-1})E_iE_1E_i + E_1E_i^2 = 0 \text{ if } i = 2,3, \\
F_i^2F_1 - (Q_i + Q_i^{-1})F_iF_1F_i + F_1F_i^2 = 0 \text{ if } i = 2,3,
\end{gathered}$$

where $(a_{ij}) = \begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ is the Cartan matrix , and all the generators are even but E_1 and F_1,

$$Q_2 = \mathrm{e}^{\frac{h}{2}}, \quad Q_1 = \frac{1}{Q_2}, \quad Q_3 = Q_2^{\alpha}, \quad K_i = Q_i^{H_i},$$

Take $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 = 0$ and $a_3 = \alpha a_2$. I will adopt the notations :

$$e_i = E_i, \quad f_i = F_i, \quad h_i = H_i \text{ if } i = 2,3$$

$$f_1 = F_1, \quad e_1 = \frac{Q_1 - Q_1^{-1}}{\beta} E_1, \quad h_1 = \frac{1}{1+\alpha}(2H_1 - H_2 - \alpha H_3)$$

$$q_i = \mathrm{e}^{\frac{a_i}{2}h}, \quad k_i = q_i^{h_i} \text{ if } i = 1,2,3$$

where β is h times a unit of $R[[h]]$. Replacing h with $a_2 h$, the presentation of $U_h\mathfrak{D}_{2\,1}$ is given by :

$$\begin{gathered}
h_i h_j = h_j h_i \\
[h_i, e_j] = b_{ij} e_j, \\
[h_i, f_j] = -b_{ij} f_j, \\
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \text{ if } (i,j) \neq (1,1) \\
[e_1, f_1] = \frac{1}{\beta}\left(\left(\frac{k_1}{k_2 k_3}\right)^{\frac{1}{2}} - \left(\frac{k_2 k_3}{k_1}\right)^{\frac{1}{2}}\right) \\
e_1^2 = f_1^2 = 0, \quad [e_2, e_3] = [f_2, f_3] = 0, \\
e_i^2 e_1 - (q_i + q_i^{-1}) e_i e_1 e_i + e_1 e_i^2 = 0 \text{ if } i = 2,3, \\
f_i^2 f_1 - (q_i + q_i^{-1}) f_i f_1 f_i + f_1 f_i^2 = 0 \text{ if } i = 2,3,
\end{gathered}$$

where (b_{ij}) is the matrix, $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ all the generators are even but e_1 and f_1.

So that the universal R-matrix computed in [2] can be written

$$R = \exp_1\left(\frac{E_{\beta_1} \otimes F_{\beta_1}}{\varphi_1}\right) \cdots \exp_7\left(\frac{E_{\beta_7} \otimes F_{\beta_7}}{\varphi_7}\right) \exp\left(\frac{h}{2}\sum_i a_i h_i \otimes h_i\right) \in U_h\mathfrak{D}_{2\,1}\hat{\otimes} U_h\mathfrak{D}_{2\,1}$$

Where $E_{\beta_1} = e_3$, $E_{\beta_5} = e_1$, $E_{\beta_7} = e_2$, $F_{\beta_1} = f_3$, $F_{\beta_5} = f_1$, $F_{\beta_7} = f_2$ and

$$\begin{aligned}
E_{\beta_2} &= E_{\beta_5} E_{\beta_1} - q_3^{-1} E_{\beta_1} E_{\beta_5}, & F_{\beta_2} &= F_{\beta_5} F_{\beta_1} - q_3^{-1} F_{\beta_1} F_{\beta_5}, \\
E_{\beta_6} &= E_{\beta_7} E_{\beta_5} - q_2^{-1} E_{\beta_5} E_{\beta_7}, & F_{\beta_6} &= F_{\beta_7} F_{\beta_5} - q_2^{-1} F_{\beta_5} F_{\beta_7}, \\
E_{\beta_3} &= E_{\beta_7} E_{\beta_2} - q_2^{-1} E_{\beta_2} E_{\beta_7}, & F_{\beta_3} &= F_{\beta_7} F_{\beta_2} - q_2^{-1} F_{\beta_2} F_{\beta_7}, \\
E_{\beta_4} &= E_{\beta_5} E_{\beta_3} + q_1 E_{\beta_3} E_{\beta_5}, & F_{\beta_4} &= F_{\beta_5} F_{\beta_3} + q_1 F_{\beta_3} F_{\beta_5}.
\end{aligned}$$

where

$$\varphi_1 = -\frac{1}{q_3 - q_3^{-1}}, \quad \varphi_2 = \frac{1}{q_3\beta}, \quad \varphi_7 = -\frac{1}{q_2 - q_2^{-1}}, \quad \varphi_6 = \frac{1}{q_2\beta},$$
$$\varphi_4 = q_1^2 \frac{q_1 - q_1^{-1}}{\beta^2}, \quad \varphi_5 = -\frac{1}{\beta}, \quad \varphi_3 = -\frac{q_1}{\beta},$$
$$q^x = e^{xh/2}, \quad c_2 = c_3 = c_5 = c_6 = 0, \quad c_1 = 2a_3, \quad c_4 = 2a_1, \quad c_7 = 2a_2,$$
$$(n)_i = \begin{cases} \frac{q^{nc_i}-1}{q^{c_i}-1} \text{if } c_i \neq 0, \\ 1 \text{ else}, \end{cases}, \quad (n)_i! = (n)_i(n-1)_i \cdots (1)_i, \quad n \in \mathbb{N},\ 1 \leq i \leq 7,$$
$$\exp_i(a \otimes b) = \sum_{n \geq 0} \frac{a^n \otimes b^n}{(n)_i!}, \quad a, b \in U_h\mathfrak{D}_{2\,1}, \quad i = 1, \ldots, 7,$$
$$\exp(a \otimes b) = \sum_{n \geq 0} \frac{a^n \otimes b^n}{n!}, \quad a, b \in U_h\mathfrak{D}_{2\,1},$$

B The $U_h\mathfrak{D}_{2\,1}$-module L

The Hopf subalgebra of $U_h\mathfrak{D}_{2\,1}$ given by considering the first two simple root is isomorphic to a quantification $U_h\mathfrak{sl}(2,1)$ of the simple Lie superalgebra $\mathfrak{sl}(2,1) \subset \mathfrak{D}_{2\,1}$. As a $U_h\mathfrak{sl}(2,1)$-module, L split as the sum

$$L = h \oplus l \oplus v_+ \oplus v_-$$

where h is the trivial representation of dimension 1, l is the representation of $U_h\mathfrak{sl}(2,1)$ corresponding to the adjoint representation of $\mathfrak{sl}(2,1)$ and v_+, v_- are dual representations of $U_h\mathfrak{sl}(2,1)$ of dimension 4 (The weights of v_+ are β_1, β_2, β_3 and β_4).

Fix basis $\mathcal{B}_h$, $\mathcal{B}_l$, $\mathcal{B}_{v_+}$ and $\mathcal{B}_{v_-}$ of the $U_h\mathfrak{sl}(2,1)$ modules and let $\mathcal{B}$ be the union of these bases whose vectors have weight
$(0,0,0,\beta_5,\beta_6,\beta_7,-\beta_5,-\beta_6,-\beta_7,\beta_1,\beta_2,\beta_3,\beta_4,-\beta_1,-\beta_2,-\beta_3,-\beta_4)$. The matrix

of ρ in $\mathcal{B}$ are given by

$$
\rho(e_3) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2{q_3}^{-1}\frac{[2]_2}{[2]_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2{q_3}^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\frac{[2]_1}{[2]_2} & -\frac{[4]_2}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\rho(f_3) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{[2]_3}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_3\frac{[2]_3[2]_1}{{[2]_2}^2} & -\frac{q_3}{2}\frac{[2]_3[4]_2}{{[2]_2}^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$\rho(e_1)$, $\rho(e_2)$, $\rho(f_1)$ and $\rho(f_2)$ leave stable the $U_h\mathfrak{sl}(2,1)$ modules. They are zero on h and their matrices $\rho(x)^m_m$ on the other modules are given by

$$\rho(e_1)_l^l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2\,\frac{[2]_2}{[4]_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,\frac{[2]_2}{[4]_2} & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q_2 & 0 \end{bmatrix} \quad \rho(f_1)_l^l = \begin{bmatrix} 0 & 0 & -\frac{[2]_2}{[4]_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{[2]_2}{[4]_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & {q_2}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(e_2)_l^l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\,{q_2}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{[4]_2}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(f_2)_l^l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{q_2}{2}\,\frac{[4]_2}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(e_1)_{v_+}^{v_+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{[2]_3}{[2]_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{[2]_1}{[2]_2} & 0 \end{bmatrix} \quad \rho(f_1)_{v_+}^{v_+} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(e_2)_{v_+}^{v_+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(f_2)_{v_+}^{v_+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(e_1)_{v_-}^{v_-} = \begin{bmatrix} 0 & -q_3\,\frac{[2]_3}{[2]_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1^{-1}\,\frac{[2]_1}{[2]_2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(f_1)_{v_-}^{v_-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ {q_3}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q_1 & 0 \end{bmatrix}$$

$$\rho(e_2)_{v_-}^{v_-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(f_2)_{v_-}^{v_-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

L possess the invariant bilinear form B whose matrices are given by

$$B_{h,h} = \left[\begin{array}{c} \frac{[2]_3[2]_1}{[2]_2{}^2} \end{array}\right] \quad B_{l,l} = \left[\begin{array}{cccccccc} \frac{1}{2}\frac{[4]_2}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\frac{[4]_2}{[2]_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2{}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_2{}^{-1} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 \end{array}\right]$$

$$B_{v_+,v_-} = \left[\begin{array}{cccc} q_3{}^{-1} & 0 & 0 & 0 \\ 0 & q_3{}^{-1} & 0 & 0 \\ 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_1 \end{array}\right] \quad B_{v_-,v_+} = \left[\begin{array}{cccc} q_3 & 0 & 0 & 0 \\ 0 & -q_3 & 0 & 0 \\ 0 & 0 & -q_1^{-1} & 0 \\ 0 & 0 & 0 & q_1^{-1} \end{array}\right]$$

If $\mathcal{B} = (v_1, v_2, \ldots, v_{17})$, then the map from L to $L \otimes L$ is determined by the image of the highest weight vector v_{13} which can be chosen to be

$$-\frac{1}{2}\frac{[4]_2}{[2]_2}v_1 \otimes v_{13} + \frac{1}{2}\frac{[4]_2}{[2]_2}v_{13} \otimes v_1 + \frac{[2]_1}{[2]_2}v_2 \otimes v_{13} - \frac{[2]_1}{[2]_2}v_{13} \otimes v_2$$
$$- v_4 \otimes v_{12} - q_1 v_{12} \otimes v_4 + q_2{}^{-1}v_5 \otimes v_{11} + q_3{}^{-1}v_{11} \otimes v_5$$

In this base, all the structure tensors (including the R-matrix) have coefficients in $\mathbb{Z}[\frac{1}{2}, q_1, q_2, q_3, [4]_1{}^{-1}, [4]_2{}^{-1}, [4]_3{}^{-1}]/(q_1 q_2 q_3 = 1)$.

References

[1] B. Patureau-Mirand - *Quantum Link Invariant from the Lie Superalgebras* $\mathfrak{D}_{2\,1,\alpha}$ (math.GT/0404548).

[2] H. Thys - *R-matrice universelle pour* $U_h(D(2,1,x))$ *et invariant d'entrelacs associé* Bull. Soc. Math. Fr. 130, No.2, 309-336 (2002).

[3] Y. M. Zou - *Deformation of the universal enveloping algebra of* $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, Canadian Math. Bull., Vol 39(4)(1996), 499-506.