UNIVERSITE BRETAGNE LOIRE / MATHSTIC

Université Bretagne Sud

THÈSE DE DOCTORAT DE

L'UNIVERSITÉ BRETAGNE SUD
COMUE UNIVERSITÉ BRETAGNE LOIRE

ÉCOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs Interactions*

Par **Ngoc Phu HA**

Théorie topologique des champs quantiques pour la superalgèbre de Lie sl(2|1)

Thèse présentée et soutenue à Vannes, le 7 décembre 2018
Unité de recherche : Laboratoire des Mathématiques de Bretagne Atlantique CNRS UMR 6205
Thèse N° :

Rapporteurs avant soutenance :

Anna Beliakova Professeur Université de Zürich

Emmanuel Wagner MCF HDR Université de Bourgogne

Composition du Jury :

Christian Blanchet Professeur Université Paris de Diderot
Président

Emmanuel Wagner MCF HDR Université de Bourgogne

Azat M. Gainutdinov CR Université François Rabelais

Gwénaël Massuyeau Professeur Université de Bourgogne

Gaël Meigniez Professeur Université de Bretagne Sud

Bertrand Patureau-Mirand MCF HDR Université de Bretagne Sud
Directeur de thèse

Théorie topologique des champs quantiques pour la superalgèbre de Lie $\mathfrak{sl}(2|1)$

Ngoc Phu HA

LMBA CNRS UMR 6205
Université de Bretagne Sud

Remerciement

L'apprentissage et la recherche à Vannes sont mes meilleures expériences. Dans cette très jolie ville, j'ai rencontré des professeurs, des amis qui sont très sympatiques. Je tiens tout d'abord à remercier au fond du cœur mon directeur de thèse Monsieur Bertrand Patureau-Mirand. Il a fait partager sa motivation et ses connaissances pour ce domaine de recherche. Je tiens à remercier encore sa famille pour son accueil très familier et sympatiques.

Je suis très reconnaissant envers les professeurs Anna Beliakova et Emmanuel Wagner d'avoir pris le temps d'évaluer mon travail et de leurs commentaires très utiles. Je remercie les professeurs Christian Blanchet, Azat M. Gainutdinov, Gaël Meigniez et Gwénaël Massuyeau d'avoir d'accepté de faire partie de mon jury.

Pendant plus de trois ans de thèse, j'ai eu la chance de participer à des activités de recherches. Je remercie Gaël, Guillaume, Michèle, Sylvain, ... et les membres de l'équipe de recherche Géométrie et Topologie du LMBA pour le Séminaire Quimpériodique. Je me souviens des «Journées TQFT, groupes quantiques et invariants non commutatifs», je remercie Christian et Nathan pour ces journées à Paris. Je n'oublie pas les jours où j'étais au «Meeting on non-semisimple TFT and logarithmic CFT» à l'Universität Hamburg, merci à Ingo Runkel pour ces jours-là. Je remercie encore Christian, Hoel et la Fondation Louis D. pour «TQFT and categorification» à Cargèse. Merci aux membres du LMBA pour un bon environnement mathématique et pour la gentillesse des gens ici.

Je me souviens de bon moments où j'en ai bien profité, c'est le foot le vendredi avec Sylvain, Ronan et Du, l'algèbre de Hopf avec Salim, du café, des gâteaux avec les autres ... Je vous remercie pour ces bons moments. Je tiens à remercier mes amis et les doctorants dans le labo pour leur aide dans la vie : Merci à Du, à Kévin (pour sa présentation de la culturel française : du vin, du café, ...), à Jamila, à Thuy, à Erwan, à Phuc, à la famille Tan

Tra, à la famille Le Tho – Jean, à la famille bac Hung et ba noi, à Ronan, à Éric, à Jonathan, à Tarik, à Hui, à Cuong, à Thong et Vy, à Jean François, à Lucien, ... Ils m'ont aidé à garder l'équilibre entre la vie quotidienne et le travail de recherche.

Merci à mes collègues à l'Université de Hung Vuong, Phu Tho, Vietnam m'avoir remplacé dans l'enseignement pendant le temps où je faisais ma thèse en France.

Enfin, je me souviens toujours de ma fille Phuong Quynh, ma femme Huong, de mes parents, de ma soeur et mon frère au Vietnam. Ils ont été à mes côtés tout le temps de ma thèse. Je vous remercie beaucoup.

Théorie topologique des champs quantiques pour la superalgèbre de Lie $\mathfrak{sl}(2|1)$

Résumé

Ce texte étudie le groupe quantique $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ associé à la superalgèbre de Lie $\mathfrak{sl}(2|1)$ et une catégorie de ses représentations de dimension finie. L'objectif est de construire des invariants topologiques de 3-variétés en utilisant la notion de *trace modifiée*. D'abord nous prouvons que la catégorie $\mathscr{C}^H$ des modules de poids nilpotents sur $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ est enrubannée et qu'il existe une trace modifiée sur son idéal des modules projectifs. De plus $\mathscr{C}^H$ possède une structure relativement G-prémodulaire ce qui est une condition suffisante pour construire un invariant de 3-variétés à la Costantino-Geer-Patureau. Cet invariant est le coeur d'une $1+1+1$-TQFT (Topological Quantum Field Theory). D'autre part Hennings a proposé à partir d'une algèbre de Hopf de dimension finie une construction d'invariants qui dispense de considérer la catégorie de ses représentations. Nous montrons que le groupe quantique déroulé $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$ possède une complétion qui est une algèbre de Hopf enrubannée topologique. Nous construisons un invariant de 3-variétés à la Hennings en utilisant cette structure algébrique, une transformation de Fourier discrète et la notion de G-intégrales. L'intégrale dans une algèbre de Hopf est centrale dans la construction de Hennings. La notion de trace modifiée dans une catégorie s'est récemment révélée être une généralisation des intégrales dans les algèbres de Hopf de dimension finie. Dans un contexte plus général d'algèbre de Hopf de dimension infinie nous prouvons la relation formulée entre la trace modifiée et la G-intégrale.

Mots clés : group quantique déroulé, algèbre topologique localement convexe, TQFT, super-symétries, invariant de 3-variétés, trace modifiée.

Topological quantum field theory for Lie superalgebra $\mathfrak{sl}(2|1)$

Abstract

This text studies the quantum group $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ associated with the Lie superalgebra $\mathfrak{sl}(2|1)$ and a category of finite dimensional representations. The aim is to construct the topological invariants of 3-manifolds using the notion of *modified trace*. We first prove that the category $\mathscr{C}^H$ of the nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is ribbon and that there exists a modified trace on its ideal of projective modules. Furthermore, $\mathscr{C}^H$ possesses a relative G-premodular structure which is a sufficient condition to construct an invariant of 3-manifolds of Costantino-Geer-Patureau type. This invariant is the heart of a $1+1+1$-TQFT (Topological Quantum Field Theory). Next Hennings proposed from a finite dimensional Hopf algebra, a construction of invariants which does not require to consider the category of its representations. We show that the unrolled quantum group $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$ has a completion which is a topological ribbon Hopf algebra. We construct an invariant of 3-manifolds of Hennings type using this algebraic structure, a discrete Fourier transform, and the notion of G-integrals. The integral in a Hopf algebra is central in the construction of Hennings. The notion of modified trace in a category has recently been revealed to be a generalization of the integrals in a finite dimensional Hopf algebra. In a more general context of infinite dimensional Hopf algebras we prove the relation formulated between the modified trace and the G-integral.

Keywords: unrolled quantum group, locally convex topological algebra, TQFT, super-symmetries, invariant of 3-manifolds, modified trace.

Table des matières

Chapitre 1

Introduction

1.1 Contexte

Depuis les années 80, la naissance du polynôme de Jones (voir [27]) a ouvert une nouvelle direction de recherche pour les invariants topologiques d'entrelacs et de 3-variétés. Quelques années après, beaucoup d'invariants d'entrelacs ont été découverts qui sont des généralisations du polynôme de Jones. D'abord le polynôme à deux variables appelé "HOMFLY" qui est une généralisation du polynôme de Jones (le nom HOMFLY provient des noms de six mathématiciens Hoste, Ocneanu, Millett, Freyd, Lickorish, et Yetter qui découvrent simultanément ce polynôme) (voir [10]). Puis Kauffman a défini une autre généralisation et a construit un invariant d'entrelacs en bande indépendant de l'orientation, ... ([37]). Ensuite, dans les deux articles [43] en 1990, et [44] en 1991 N. Reshetikhin, V.G. Turaev et E. Witten ont introduit une méthode de construction d'invariant d'entrelacs (nommé invariant RT) et de 3-variétés (nommé invariant WRT). Le premier article a présenté la construction d'un foncteur F d'une catégorie des graphes en rubans vers une catégorie enrubannée $\mathscr{C}$. Ces graphes en rubans sont composés par les parties élémentaires comme des bandes, des coupons, des anneaux, ... Ils sont coloriés par des objets et des morphismes de la catégorie $\mathscr{C}$. Le foncteur F ne dépend que la classe d'isotopie des graphes plongés et il détermine un invariant des entrelacs. En utilisant des représentations du groupe quantique $\mathcal{U}_q\mathfrak{sl}(2)$ on retrouve le polynôme de Jones. Dans leur deuxième article, inspiré par les idées de E. Witten (voir [50]) ils ont utilisé une catégorie modulaire enrubannée $\mathscr{C}$ pour construire un invariant de 3-variétés.

Dans certains contextes, l'invariant RT se révèle être trivial, par exemple pour les représentations projectives du groupe quantique $\mathcal{U}_q\mathfrak{sl}(2)$ où q est une racine de l'unité. La raison qui cause ce phénomène est la nullité de dimension quantique de la représentation (e. g. [15]). Pour trouver des informations cachées dans cette situation N. Geer, B. Patureau-Mirand et V. Turaev ont proposé une méthode dont l'idée principale est le remplacement de la dimension quantique par la dimension modifiée dans la construction de l'invariant RT ([17]). La dimension modifiée est déterminée par une famille des formes $\Bbbk$-linéaires nommée une trace modifiée. Ces notions leur permettent de trouver un invariant F' non trivial même lorsque l'invariant RT est trivial. La trace modifiée et ses techniques fournissent un autre point de vue sur la construction des invariants topologiques. Avec F. Costantino ([8]) et F. Costantino et C. Blanchet ([4]) ils généralisent avec ces nouveaux invariants la construction WRT pour produire des invariants de 3-variétés et des TQFTs (Topological Quantum Field Theories).

Les superalgèbres de Lie ([28]) sont des généralisations des algèbres de Lie utilisées par les physiciens pour décrire les super symétries. Elles admettent, comme les algèbres de Lie une déformation et leurs représentations sont en partie connues. Par exemple, les représentations irréductibles de $\mathcal{U}_q\mathfrak{sl}(2|1)$ aux racines de l'unité sont décrites dans [1]. La construction de Reshetikhin et Turaev repose sur l'existence d'une catégorie de représentations semi-simples des groupes quantiques. Cette propriété fait défaut dans le cas des groupes quantiques associés aux superalgèbres de Lie. Ceci suggère d'essayer d'utiliser des traces modifiées pour contourner cette difficulté et de tenter de développer une construction similaire à celle de [8].

Dans une autre direction, M. Hennings a présenté une méthode de construction d'invariants de 3-variétés en utilisant une intégrale sur une algèbre de Hopf enrubannée de dimension finie ([26]). De plus, dans [46] V. G. Turaev a présenté une structure de π-cogèbre de Hopf, i.e. un ensemble d'algèbres indexées par les éléments d'un groupe π avec des applications nommées le produit, le coproduit, l'unité, la counité et l'antipode qui satisfont des axiomes de compatiblité. Puis A. Virelizier a démontré l'existence d'une intégrale et d'une trace sur π-structure dans [49]. L'intégrale sur une π-cogèbre de Hopf nommée π-intégrale est une généralisation de la notion de l'intégrale sur une algèbre de Hopf utilisée dans la construction de Hennings. En utilisant une π-cogèbre de Hopf unimodulaire enrubannée de type finie et une π-intégrale, ils ont construit un invariant de 3-variétés dans [48]. Récemment dans [2] une relation a été trouvé entre l'intégrale sur l'algèbre de Hopf H et la trace modifiée dans la catégorie correspondante H-mod : A. Beliakova, C. Blanchet et A. M. Gainutdinov ont notamment établi une formule reliant la trace modifiée et l'intégrale.

1.2 Présentation des objectifs

Motivé par la notion de la trace modifiée nous voulons développer ses techniques dans le contexte des représentations du groupe quantique $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ décrites dans [1]. On suppose pouvoir construire un invariant de l'entrelacs coloriés par ses représentations. Cela nous fournit le premier objectif : c'est la construction des invariants quantiques associés à la super algèbre de Lie $\mathfrak{sl}(2|1)$. Pour faire cela : d'abord on démontre qu'il existe une structure enrubannée dans la catégorie $\mathscr{C}^H$ des représentations nilpotentes des modules de poids sur $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$, ensuite on indique l'existence de la trace modifiée sur l'idéal $\mathbf{Proj}(\mathscr{C}^H)$ des modules projectifs dans $\mathscr{C}^H$. Cette trace modifiée nous donne un invariant des graphes enrubannés. De plus la catégorie enrubannée $\mathscr{C}^H$ a aussi une structure relativement G-prémodulaire, ce qui permet de construire un invariant de 3-variétés à la Witten-Reshetikhin-Turaev.

À partir d'un invariant de 3-variétés on sait avoir une chance de construire une famille des TQFTs. Par exemple, en utilisant la construction universelle présentée par C. Blanchet, N. Habegger, G. Masbaum and P. Vogel dans [5], une famille de TQFTs est construite dans [4] à partir de l'invariant quantique trouvé par F. Costantino, N. Geer and B. Patureau-Mirand [8]. Les TQFTs dans [4] sont construites à partir de l'invariant CGP associé à $\mathfrak{sl}(2)$ ([8]) qui est similaire à celui que l'on définit ici avec $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. C'est la raison qui a motivé le deuxième objectif : la construction de $1+1+1$-TQFTs à partir ces invariants de 3-variétés. Pour appliquer la construction de De Renzi ([42]) on montre que la catégorie $\mathscr{C}^H$ est une catégorie relativement G-modulaire.

M. Hennings dans [26] a proposé une manière de construire un invariant de 3-variétés à partir d'une algèbre de Hopf enrubannée de dimension finie à l'aide de l'intégrale. Inspiré par sa méthode, nous désirions construire un invariant de 3-variétés pour le groupe quantique $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. Néanmoins, la dimension de la superalgèbre de Hopf $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ n'est pas finie, cela cause des difficultés. Les travaux ont été motivés par les réflexions suivantes : Puisqu'il existe une trace modifiée sur idéal des modules projectifs dans $\mathscr{C}^H$ produisant un invariant de 3-variétés, nous conjecturons qu'il existe quand même une chose analogue pour la superalgèbre $\mathcal{U}^H = \mathcal{U}_\xi^H\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$. En d'autres termes, nous pourrions construire un invariant de 3-variétés à la Hennings avec la superalgèbre $\mathcal{U}^H$.

Ceci est effectivement réalisé en remplaçant l'intégrale par une intégrale graduée. Donc, à partir de la superalgèbre $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$, on a deux approches pour construire cet invariant. La première manière utilise la structure de la catégorie enrubannée $\mathscr{C}^H$ et la trace modifiée en dedans. L'autre manière utilise une structure d'algèbre topologique $\mathcal{U}^H$ et l'intégrale graduée. Ceci suggère une relation entre les deux objets : la trace modifiée dans une catégorie et l'intégrale graduée d'une algèbre de Hopf. Dans un article récent [2] les auteurs ont montré que la trace modifiée dans la catégorie H-mod est l'intégrale symétrisée de l'algèbre de Hopf de dimension finie H. Inspiré par la suggestion ci-dessus nous nous fixons deux objectifs supplémentaires. Le troisième objectif est de trouver la relation entre la trace modifiée des catégories de représentations des groupes quantiques et les intégrales des G-cogèbres de Hopf pivotales correspondantes.

Finalement, le dernier objectif est la construction d'un invariant de 3-variétés de type Hennings associé au groupe quantique déroulé $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ malgré que ce soit une algèbre de Hopf de dimension infinie. Cet invariant est construit en trois étapes : 1) l'introduction d'une topologie sur l'algèbre déroulée, 2) une transformation de Fourier discrète et 3) la version G-graduée de l'invariant de Hennings dû à A. Virelizier ([48]).

1.3 Résultats principaux

Le texte est composé de quatre chapitres. Ses résultats principaux sont présentés dans les trois derniers chapitres. En particulier ils sont la reproduction des articles [22], [21] et [20]. Dans le deuxième chapitre, nous démontrons que la catégorie paire $\mathscr{C}^H$ des modules de poids nilpotents du groupe quantique $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ est enrubannée par le théorème 2.4.4, la proposition 2.4.5 et qu'il existe une trace modifiée sur idéal des modules projectifs de $\mathscr{C}^H$ par le théorème 2.5.4. On construit un invariant de graphes enrubannées dans S^3 par cette trace avec le théorème 2.5.5. De plus, cette catégorie possède une structure relativement G-prémodulaire avec $G = (\mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}, +)$, cela implique une construction d'invariants de 3-variétés similaire à celle dans [8] par le théorème 2.6.4. Ses résultats sont présentés dans l'article *Topological invariants from quantum groups* $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ *at roots of unity* ([22]). Dans ce chapitre nous rajoutons aussi une partie complémentaire où on montre que la catégorie paire $\mathscr{C}^H$ des modules de poids nilpotents du groupe quantique $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ est relativement G-modulaire d'après le sens de De Renzi ([42]) par la proposition 2.7.2. Cela nous permet de construire une famille de $1+1+1$-TQFTs étendues et graduées par $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Le troisième chapitre parle de la relation proche entre trace modifiée et intégrale. Soit H une algèbre de Hopf de dimension finie, il existe un élément de H^* appelé intégrale sur H qui est utilisé pour construire l'invariant de 3-variétés de Hennings (voir [26]). À partir de cette notion on peut définir la notion d'intégrale symétrisée et prouver une formule où chaque intégrale symétrisée sur H correspond à une trace modifiée sur l'idéal H-pmod des H-modules projectifs dans la catégorie H-mod (voir [2]). Pour généraliser ce résultat dans le contexte où la dimension de l'algèbre de Hopf H peut être infinie, on a défini la notion d'une G-cogèbre de Hopf pivotale. Une G-intégrale sur une G-cogèbre de Hopf pivotale nous permet de définir une G-intégrale symétrisée. Elle coïncide avec une trace modifiée sur l'idéal des H-modules projectifs. Autrement dit, si H est une algèbre de Hopf (sa dimension peut être infinie) et G est un groupe, on peut parfois former une G-cogèbre de Hopf pivotale $(H_g)_{g\in G}$ à partir de quotients de H. Les relations entre les G-intégrales sur $(H_g)_{g\in G}$ et les traces modifiées sur l'idéal des H-modules projectifs de dimension finie sont établies par le théorème 3.1.1. Ceci nous a permit une autre approche de construction de l'invariant de 3-variétés [22] à partir des intégrales symétrisées. Dans cette partie nous donnons aussi une application (voir Section 3.5) de la relation entre G-intégrale et trace modifiée par des calculs pour le groupe quantique associé à l'algèbre de Lie $\mathfrak{sl}(2)$ et la catégorie correspondante. Ces résultats sont prépubliés sur arXiv ([21]).

Le quatrième chapitre revient au groupe quantique déroulé $\mathcal{U}^H$ générant la catégorie $\mathscr{C}^H$. Soit W l'espace vectoriel de dimension finie sur $\mathbb{C}$ avec une base $\{e_1^p e_3^\rho e_2^\sigma f_1^{p'} f_3^{\rho'} f_2^{\sigma'}\ 0\le \rho,\sigma,\rho',\sigma'\le 1,\ 0\le p,p'\le \ell-1\}$. Le groupe quantique $\mathcal{U}^H$ est isomorphe à $W\otimes\mathbb{C}[k_1^{\pm1},k_2^{\pm1},h_1,h_2]$. Nous considérons l'injection de $\mathcal{U}^H$ dans $W\otimes\mathcal{H}(h_1,h_2)$ où $\mathcal{H}(h_1,h_2)$ est l'espace vectoriel des fonctions holomorphes sur $\mathbb{C}^2$. On peut voir chaque élément de $W\otimes\mathcal{H}(h_1,h_2)$ comme une fonction holomorphe à valeurs dans W. Puis on peut déterminer une topologie sur cet espace : c'est la topologie de la convergence uniforme sur les ensembles compacts. Nous démontrons que cette superalgèbre de Hopf possède une structure de superalgèbre de Hopf enrubannée au sens topologique. C'est à dire que cette topologie est compatible avec la structure d'algèbre de Hopf (cf. théorème 4.2.17). Sa bosonization est une algèbre de Hopf topologique enrubannée. Cette algèbre nous donne d'abord une construction d'invariant universel de l'entrelacs par le théorème 4.3.2 et puis une G-cogèbre de Hopf pivotale de type finie $\mathcal{U}^\sigma$ par la proposition 4.4.2 où chaque composante de $\mathcal{U}^\sigma$ est le quotient de l'algèbre par l'idéal engendré par $k_i^\ell - \xi^{\ell\alpha_i}$ pour $i=1,2$. Les G-intégrales sur $\mathcal{U}^\sigma$, l'invariant universel et une transformation de Fourier discrète nous permettent de construire un invariant de 3-variétés de type Hennings par le théorème 4.4.15. La méthode présentée dans ce chapitre pourrait se généraliser dans le contexte des groupes quantiques déroulés. Ces résultats sont prépubliés sur arXiv ([20]).

Au début de chaque chapitre, nous redéfinissons les notions nécessaires et rappelons les résultats préliminaires. En conséquence chaque chapitre pourrait être lu indépendamment des autres.

Chapter 2

Topological invariants from quantum group $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ at roots of unity

This chapter contains two parts, the first one with six sections is the content of the paper [22] in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, the second one is to prove the category $\mathscr{C}^H$ is relative G-modular.

Résumé. Dans ce chapitre, nous construisons des invariants d'entrelacs et des invariants de 3-variétés à partir du groupe quantique associé à la superalgèbre de Lie $\mathfrak{sl}(2|1)$. La construction est basée sur des représentations nilpotentes irréductibles finies du groupe quantique $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ où ξ est une racine de l'unité d'ordre impair. Ces constructions utilisent la notion de trace modifiée présentée par Geer, Kujawa et Patureau-Mirand [13] et la catégorie relativement G-modulaire présentée par Costantino, Geer et Patureau-Mirand [8].

Abstract. In this chapter we construct link invariants and 3-manifold invariants from the quantum group associated with the Lie superalgebra $\mathfrak{sl}(2|1)$. The construction is based on nilpotent irreducible finite dimensional representations of quantum group $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ where ξ is a root of unity of odd order. These constructions use the notion of modified trace presented by Geer, Kujawa and Patureau-Mirand [13] and relative G-modular category presented by Costantino, Geer and Patureau-Mirand [8].

MSC: 57M27, 17B37
Key words: Lie superalgebra, quantum group, link invariant, 3-manifold.

2.1 Introduction

The vanishing of the dimension of an object V in a ribbon category $\mathscr{C}$ is an obstruction when one studies the Reshetikhin-Turaev link invariant. If the dimension of a simple object V of $\mathscr{C}$ is zero, then the quantum invariants of all (framed oriented) links with components labelled by V are equal to zero, i.e. they are trivial. To overcome this difficulty, the authors N. Geer, B. Patureau-Mirand and V. Turaev introduced the notion of a modified dimension (see [17]). The modified dimension may be non-zero when $\dim_{\mathscr{C}}(V) = 0$. Using the modified dimension, for example on the class of projective simple objects, they defined an isotopy invariant $F^{'}(L)$ (the renormalized Reshetikhin-Turaev link invariant) for any link L whose components are labelled with objects of $\mathscr{C}$ under the only assumption that at least one of the labels belongs to the set of projective ambidextrous objects. Here $F^{'}(L)$ is a nontrivial link invariant (see [17]). This modified dimension is used to construct the quantum invariants in [8], [14].

The existence of the modified dimension generalizes the definition of modified traces (see [12]). In the article [13], the authors showed that a necessary and sufficient condition for the existence of a modified trace on an ideal generated by a simple object J is that J is an ambidextrous object. Recently the existence of an ambidextrous object has been shown in the context of factorizable finite tensor categories [11].

The Lie superalgebras (see [28]) are the generalizations of Lie algebras in the category of super vector spaces. They are used among others by physicists to describe supersymmetry. Deformations of these superalgebras and their representations are partially known. For the Lie superalgebra $\mathfrak{sl}(2|1)$ one can define a Hopf superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ which is a deformation of the universal enveloping algebra. Its irreducible representations at roots of unity are described in [1]. Using these representations and developing the idea of modified traces open up the method for constructing a quantum invariant of framed links with components labelled by irreducible representations.

The aim of this chapter is to construct a link invariant and a 3-manifold invariant from quantum group $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ at a root of unity of odd order. Note that the Lie superalgebra $\mathfrak{sl}(2|1)$ having superdimension zero, $\mathfrak{sl}(2|1)$-weight functions are trivial. Hence combining them with the Kontsevich integral or the LMO invariant also give trivial link and 3-manifold invariants. The chapter contains six sections. In Section 2.2, we recall the monoidal category, pivotal category, braided category, ribbon category and, Hopf superalgebra definitions. In Section 2.3 and 2.4, we describe the quantum superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ where ξ is a root of unity of odd order and by adding two elements h_1, h_2 to $\mathcal{U}_\xi\mathfrak{sl}(2|1)$, we have the Hopf superalgebra $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. Using this extension we can construct a non semi-simple ribbon category $\mathscr{C}^H$ of the nilpotent simple finite dimensional representations of $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. In Section 2.5 we prove that a typical module over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is an ambidextrous module and that a modified trace exists on the ideal of projective modules `Proj`. This modified trace will be used to construct a link invariant. In Section 2.6, we prove that the category $\mathscr{C}^H$ is relative G-premodular ([8]) and we construct a 3-manifold invariant using this property.

2.2 Preliminaries

2.2.1 Monoidal category

Definition 2.2.1 ([33, 29]). *A monoidal category $\mathscr{C}$ is a category enhanced with a bifunctor called tensor product $\cdot \otimes \cdot : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a unit object $\mathbb{I}$ such that there are natural isomorphisms*

$$\mathbb{I} \otimes \cdot \cong \cdot \otimes \mathbb{I} \cong \mathrm{Id}_{\mathscr{C}} \quad \text{and} \quad (\cdot \otimes \cdot) \otimes \cdot \cong \cdot \otimes (\cdot \otimes \cdot), \tag{2.2.1}$$

fulfilling the Pentagon Axiom and the Triangle Axiom.

We call *strict monoidal category* a monoidal category $\mathscr{C}$ whose the isomorphisms (2.2.1) are identities. In our examples the morphisms in (2.2.1) are simply the morphisms of the underlying vector spaces and are in the following regarded as equality. We write $V \in \mathscr{C}$ to denote an object V in the category $\mathscr{C}$ and call $\mathrm{Hom}_{\mathscr{C}}(V, W)$ the morphisms in $\mathscr{C}$ from $V \in \mathscr{C}$ to $W \in \mathscr{C}$ and $\mathrm{End}_{\mathscr{C}}(V) = \mathrm{Hom}_{\mathscr{C}}(V, V)$.

We say that $\mathscr{C}$ is a monoidal $\mathbb{C}$-linear category if for all $V, W \in \mathscr{C}$, the morphisms $\mathrm{Hom}_{\mathscr{C}}(V, W)$ form a $\mathbb{C}$-vector space and the composition and the tensor product are bilinear and $\mathrm{End}_{\mathscr{C}}(\mathbb{I}) \cong \mathbb{C}$. An object $V \in \mathscr{C}$ is *simple* if and only if $\mathrm{End}_{\mathscr{C}}(V) \cong \mathbb{C}$ as a unitary $\mathbb{C}$-algebra. An object $W \in \mathscr{C}$ is a direct sum of $V_1, ..., V_n \in \mathscr{C}$ if there is for $i = 1, ..., n$, $f_i \in \mathrm{Hom}_{\mathscr{C}}(V_i, W)$, $g_i \in \mathrm{Hom}_{\mathscr{C}}(W, V_i)$ such that $g_i \circ f_i = \mathrm{Id}_{V_i}$, $g_i \circ f_j = 0$ for $i \neq j$ and $\sum_{i=1}^n f_i \circ g_i = \mathrm{Id}_W$. An object $W \in \mathscr{C}$ is *semi-simple* if it is a direct sum of simple objects. The category $\mathscr{C}$ is *semi-simple* if all objects are semi-simple and $\mathrm{Hom}_{\mathscr{C}}(V, W) = \{0\}$ for any pair of non-isomorphic simple objects in $\mathscr{C}$.

2.2.2 Pivotal category

Definition 2.2.2. *Let $\mathscr{C}$ be a monoidal category and $A, B \in \mathscr{C}$. A duality between A and B is given by a pair of morphisms* $(\alpha \in \mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, B \otimes A), \beta \in \mathrm{Hom}_{\mathscr{C}}(A \otimes B, \mathbb{I}))$ *such that*

$$(\beta \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \alpha) = \mathrm{Id}_A \quad \text{and} \quad (\mathrm{Id}_B \otimes \beta) \circ (\alpha \otimes \mathrm{Id}_B) = \mathrm{Id}_B\,. \tag{2.2.2}$$

A *pivotal* category (or *sovereign*) is a strict monoidal category $\mathscr{C}$, with a unit object $\mathbb{I}$, equipped with the data for each object $V \in \mathscr{C}$ of its *dual object* $V^* \in \mathscr{C}$ and of four morphisms

$$\overrightarrow{\text{ev}}_V \colon V^* \otimes V \to \mathbb{I}, \qquad \overrightarrow{\text{coev}}_V \colon \mathbb{I} \to V \otimes V^*,$$
$$\overleftarrow{\text{ev}}_V \colon V \otimes V^* \to \mathbb{I}, \qquad \overleftarrow{\text{coev}}_V \colon \mathbb{I} \to V^* \otimes V$$

such that $(\overrightarrow{\text{ev}}_V, \overrightarrow{\text{coev}}_V)$ and $(\overleftarrow{\text{ev}}_V, \overleftarrow{\text{coev}}_V)$ are dualities which induce the same functor duality and the same natural isomorphism $(V \otimes W)^* \cong W^* \otimes V^*$. Thus, the right and left dual coincide in $\mathscr{C}$: for every morphism $h : V \to W$, we have

$$\begin{aligned} h^* &= (\overrightarrow{\text{ev}}_W \otimes \text{Id}_{V^*}) \circ (\text{Id}_{W^*} \otimes h \otimes \text{Id}_{V^*}) \circ (\text{Id}_{W^*} \otimes \overrightarrow{\text{coev}}_V) \\ &= (\text{Id}_{V^*} \otimes \overleftarrow{\text{ev}}_W) \circ (\text{Id}_{V^*} \otimes h \otimes \text{Id}_{W^*}) \circ (\overleftarrow{\text{coev}}_V \otimes \text{Id}_{W^*}) : W^* \to V^* \end{aligned}$$

and for $V, W \in \mathscr{C}$, the isomorphisms $\gamma_{V,W} : W^* \otimes V^* \to (V \otimes W)^*$ are given by

$$\begin{aligned} \gamma_{V,W} &= (\overrightarrow{\text{ev}}_W \otimes \text{Id}_{(V\otimes W)^*}) \circ (\text{Id}_{W^*} \otimes \overrightarrow{\text{ev}}_V \otimes \text{Id}_{W\otimes(V\otimes W)^*}) \circ (\text{Id}_{W^*\otimes V^*} \otimes \overrightarrow{\text{coev}}_{V\otimes W}) \\ &= (\text{Id}_{(V\otimes W)^*} \otimes \overleftarrow{\text{ev}}_V) \circ (\text{Id}_{(V\otimes W)^*\otimes V} \otimes \overleftarrow{\text{ev}}_W \otimes \text{Id}_{V^*}) \circ (\overleftarrow{\text{coev}}_{V\otimes W} \otimes \text{Id}_{W^*\otimes V^*}). \end{aligned}$$

The family of isomorphisms

$$\Phi = \{\Phi_V = (\overleftarrow{\text{ev}}_V \otimes \text{Id}_{V^{**}}) \circ (\text{Id}_V \otimes \overrightarrow{\text{coev}}_{V^*}) : V \to V^{**}\}_{V\in\mathscr{C}}$$

is a monoidal natural isomorphism called the pivotal structure.

Definition 2.2.3. *Given a multiplicative group G, we call the category $\mathscr{C}$ pivotal G-graded $\mathbb{k}$-linear if there exists a family of full subcategories $(\mathscr{C}_\alpha)_{\alpha\in G}$ of $\mathscr{C}$ such that*

1. $\mathbb{I} \in \mathscr{C}_1$.
2. $\forall(\alpha, \beta) \in G^2,\ \forall(V, W) \in \mathscr{C}_\alpha \times \mathscr{C}_\beta,\ \text{Hom}_{\mathscr{C}}(V, W) \neq \{0\} \Rightarrow \alpha = \beta$.
3. $\forall V \in \mathscr{C},\ \exists n \in \mathbb{N},\ \exists(\alpha_1, ..., \alpha_n) \in G^n,\ \exists V_i \in \mathscr{C}_{\alpha_i}$ *for* $i = 1, ..., n$ *such that* $V \simeq V_1 \oplus ... \oplus V_n$.
4. $\forall(V, W) \in \mathscr{C}_\alpha \times \mathscr{C}_\beta,\ V \otimes W \in \mathscr{C}_{\alpha\beta}$.
5. $\forall \alpha \in G,\ \mathscr{C}_\alpha$ *does not reduce to null object.*

2.2.3 Ribbon category

A *braided* category is a tensor category $\mathscr{C}$ provided with a braiding c : for all objects V and W of $\mathscr{C}$, we have an isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V.$$

These isomorphisms are natural and for all objects U, V and W of $\mathscr{C}$, we have

$$c_{U,V\otimes W} = (\mathrm{Id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{Id}_W) \text{ and } c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_U \otimes c_{V,W}).$$

If the category $\mathscr{C}$ is pivotal and braided, we can define a family of natural isomorphisms

$$\theta_V = \mathrm{ptr}_R(c_{V,V}) = (\mathrm{Id}_V \otimes \overleftarrow{\mathrm{ev}}_V) \circ (c_{V,V} \otimes \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_V \otimes \overrightarrow{\mathrm{coev}}_V) : V \to V.$$

We say that θ is a *twist* if it is compatible with the dual in the following sense

$$\forall V \in \mathscr{C}, \theta_{V^*} = (\theta_V)^*$$

which is equivalent to

$$\theta_V = \mathrm{ptr}_L(c_{V,V}) = (\overrightarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_{V^*} \otimes c_{V,V}) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_V) : V \to V.$$

A *ribbon category* is a braided pivotal category in which the family of isomorphisms θ is a twist.

2.2.4 Hopf superalgebras

We recall some notions (see also [15], [39]). A super space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ over $\mathbb{C}$. An element $x \in V$ is called even (resp. odd) if $x \in V_{\overline{0}}$ (resp. $x \in V_{\overline{1}}$). For the super spaces U, V the set of the morphisms between them denoted by $\mathrm{Hom}_{\mathbb{C}}(U, V)$ is the super space of linear maps given by

$$\mathrm{Hom}_{\mathbb{C}}(U,V)_{\overline{0}} = \mathrm{Hom}_{\mathbb{C}}(U_{\overline{0}}, V_{\overline{0}}) \oplus \mathrm{Hom}_{\mathbb{C}}(U_{\overline{1}}, V_{\overline{1}}) \text{ and}$$
$$\mathrm{Hom}_{\mathbb{C}}(U,V)_{\overline{1}} = \mathrm{Hom}_{\mathbb{C}}(U_{\overline{0}}, V_{\overline{1}}) \oplus \mathrm{Hom}_{\mathbb{C}}(U_{\overline{1}}, V_{\overline{0}}).$$

Denote $\underline{\otimes}$ the usual tensor product in the category $\mathrm{Vect}_{\mathbb{C}}$. We call even category $\mathrm{SVect}_{\overline{0}}$ the category whose the objects are the super spaces and the morphisms are the even morphisms. Category $\mathrm{SVect}_{\overline{0}}$ is monoidal with the operator $\otimes$: For $U, V \in \mathrm{SVect}_{\overline{0}}$ their tensor product is the vector space $U \underline{\otimes} V$ with the parity given by

$$(U \otimes V)_{\overline{0}} = U_{\overline{0}} \underline{\otimes} V_{\overline{0}} \oplus U_{\overline{1}} \underline{\otimes} V_{\overline{1}} \text{ and}$$
$$(U \otimes V)_{\overline{1}} = U_{\overline{0}} \underline{\otimes} V_{\overline{1}} \oplus U_{\overline{1}} \underline{\otimes} V_{\overline{0}},$$

for $f \in \mathrm{Hom}_{\mathbb{C}}(U, U'),\ g \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ the tensor product $f \otimes g$ is given by

$$f \otimes g = \begin{cases} f \underline{\otimes} g \text{ on } U_{\overline{0}} \underline{\otimes} V \\ (-1)^{\overline{g}} f \underline{\otimes} g \text{ on } U_{\overline{1}} \underline{\otimes} V \end{cases} .$$

This means that $f \otimes g(x \otimes y) = (-1)^{\overline{g}.\overline{x}} f(x) \otimes g(y)$.
Further, $\mathrm{SVect}_{\overline{0}}$ is also a symmetric monoidal category with symmetry isomorphisms $\tau_{U,V} :\ U \otimes V \simeq V \otimes U$ given by $\tau_{U,V}(u \otimes v) = (-1)^{\overline{u}.\overline{v}} v \otimes u$. Note that the category SVect of the super spaces with all morphisms is not a symmetric monoidal category because in general $(\mathrm{Id} \otimes g) \circ (f \otimes \mathrm{Id}) \neq (f \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes g)$.

We call *Hopf superalgebra* a Hopf algebra object in $\mathrm{SVect}_{\overline{0}}$. That is a super $\mathbb{C}$-vector space H endowed with five even $\mathbb{C}$-linear maps called product, unit, coproduct, counit and antipode

$$m : H \otimes H \to H,\ \eta : \mathbb{C} \to H,\ \Delta : H \to H \otimes H,\ \varepsilon : H \to \mathbb{C} \text{ and } S : H \to H$$

satisfying the axioms:

1. the product m is associative on H admitting $1_H = \eta(1)$ as unity.
2. the coproduct Δ is coassociative, i.e. $(\Delta \otimes \mathrm{Id}_H) \circ \Delta = (\mathrm{Id}_H \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \mathrm{Id}_H) \circ \Delta = (\mathrm{Id}_H \otimes \varepsilon) \circ \Delta = \mathrm{Id}_H$.
3. Δ and ε are algebra morphisms where the associative product in $H \otimes H$ is determined by $(m \otimes m) \circ (\mathrm{Id}_H \otimes \tau_{H,H} \otimes \mathrm{Id}_H)$.
4. $m \circ (S \otimes \mathrm{Id}_H) \circ \Delta = m \circ (\mathrm{Id}_H \otimes S) \circ \Delta = \eta \circ \varepsilon$.

Let H be a Hopf superalgebra. An even grouplike element $\phi \in H$ is said a pivotal element if $\Delta(\phi) = \phi \otimes \phi,\ \varepsilon(\phi) = 1$ and for all $h \in H,\ S^2(h) = \phi h \phi^{-1}$. The pair (H, ϕ) of a Hopf superalgebra and a pivot ϕ is called a pivotal Hopf superalgebra (see [39]).
Let (H, ϕ) be a Hopf superalgebra, let $H\text{-mod}_{\overline{0}}$ be the even category of finite dimensional modules over H. If V is an object of $H\text{-mod}_{\overline{0}}$ we denote by $\rho_V :\ H \to \mathrm{End}_{\mathbb{C}}(V)$ the representation of H in the module V.

Proposition 2.2.4 ([39]). *The category H-mod$_{\bar{0}}$ has the structure of a pivotal category with dual morphisms given by*

$$\overrightarrow{\mathrm{ev}}_V\colon e_i^*\otimes e_j\mapsto e_i^*(e_j)=\delta_i^j,\qquad \overrightarrow{\mathrm{coev}}_V\colon 1\mapsto\sum_i e_i\otimes e_i^*,$$
$$\overleftarrow{\mathrm{ev}}_V\colon e_j\otimes e_i^*\mapsto(-1)^{\deg e_j}e_i^*(\phi_0.e_j),\quad \overleftarrow{\mathrm{coev}}_V\colon 1\mapsto\sum_i(-1)^{\deg e_i}e_i^*\otimes(\phi_0^{-1}.e_i)$$

where $(e_i)_i$ is a basis of V and $(e_i^*)_i$ is its basis dual.

Proof. Let V be an object of H-mod$_{\bar{0}}$. Its dual is a $\mathbb{C}$-vector space $V^*=\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ provided with the action of $h\in H$ given by

$$(h,\varphi)\mapsto(-1)^{\deg h\deg\varphi}\varphi\circ\rho_V(S(h)).$$

First we show that four morphisms $\overrightarrow{\mathrm{ev}}_V$, $\overrightarrow{\mathrm{coev}}_V$, $\overleftarrow{\mathrm{ev}}_V$, $\overleftarrow{\mathrm{coev}}_V$ are invariant morphisms of H-mod$_{\bar{0}}$. It is clear for $\overrightarrow{\mathrm{ev}}_V$, $\overrightarrow{\mathrm{coev}}_V$, we prove $\overleftarrow{\mathrm{ev}}_V$ is invariant morphism. The invariant of the morphism $\overleftarrow{\mathrm{coev}}_V$ is proved similarly.
For $h\in H$, using the Sweedler notation $\Delta(h)=h_{(1)}\otimes h_{(2)}$ and denote $\deg x=|x|$, one computes

$$\begin{aligned}
\overleftarrow{\mathrm{ev}}_V\,(h.(e_j\otimes e_i^*)) &= (-1)^{|h_{(2)}||e_j|}\,\overleftarrow{\mathrm{ev}}_V\,(h_{(1)}e_j\otimes h_{(2)}e_i^*)\\
&= (-1)^{|h_{(2)}||e_j|+|e_i^*||h_{(2)}|}\,\overleftarrow{\mathrm{ev}}_V\,(h_{(1)}e_j\otimes e_i^*\circ S(h_{(2)}))\\
&= (-1)^{|h_{(2)}||e_j|+|e_i^*||h_{(2)}|+(|e_j|+|h_{(1)}|)(|h_{(2)}|+|e_i^*|)}e_i^*\left(S(h_{(2)})\phi h_{(1)}e_j\right)\\
&= (-1)^{|h_{(2)}||e_i^*|+|e_j||e_i^*|+|h_{(1)}||h_{(2)}|+|h_{(1)}||e_i^*|}e_i^*\left(\phi S^{-1}(h_{(2)})h_{(1)}e_j\right)\\
&= (-1)^{|h_{(2)}||e_i^*|+|e_j||e_i^*|+|h_{(1)}||e_i^*|}e_i^*\left(\phi S^{-1}(S(h_{(1)})h_{(2)})e_j\right)\\
&= (-1)^{|h_{(2)}||e_i^*|+|e_j||e_i^*|+|h_{(1)}||e_i^*|}\varepsilon(h)e_i^*(\phi e_j)\\
&= (-1)^{|h||e_i^*|+|e_j||e_i^*|}\varepsilon(h)e_i^*(\phi e_j)\\
&= (-1)^{|h||e_i^*|}\varepsilon(h)\,\overleftarrow{\mathrm{ev}}_V\,(e_j\otimes e_i^*).
\end{aligned}$$

If $|h|=1$ then $\varepsilon(h)=0$. This implies that

$$\overleftarrow{\mathrm{ev}}_V\,(h.(e_j\otimes e_i^*))=\varepsilon(h)\,\overleftarrow{\mathrm{ev}}_V\,(e_j\otimes e_i^*).$$

The duality of the pair $(\overrightarrow{\mathrm{ev}}_V,\overrightarrow{\mathrm{coev}}_V)$ is clear by definition. For $(\overleftarrow{\mathrm{ev}}_V,\overleftarrow{\mathrm{coev}}_V)$, one checks

$$(\overleftarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_V)\circ(\mathrm{Id}_V\otimes\overleftarrow{\mathrm{coev}}_V)=\mathrm{Id}_V\,.$$

For each e_j we have

$$\begin{aligned}
(\overleftarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_V)\circ(\mathrm{Id}_V\otimes\overleftarrow{\mathrm{coev}}_V)(e_j\otimes 1) &= \sum_i(-1)^{2|e_i|}e_i^*(\rho_V(\phi)e_j)\otimes\rho_V(\phi^{-1})e_i\\
&= \sum_i e_i^*(\rho_V(\phi)e_j)\otimes\rho_V(\phi^{-1})e_i.
\end{aligned}$$

Suppose $A = (a_{st})_{s,t}$ is the matrix of $\rho_V(\phi)$ then the matrix of $\rho_V(\phi^{-1})$ is $A^{-1} = (b_{st})_{s,t}$ in the basis $(e_i)_i$, one gets

$$\begin{aligned}
\sum_i e_i^*(\rho_V(\phi)e_j) \otimes \rho_V(\phi^{-1})e_i &= \sum_i e_i^* \left(\sum_s a_{sj} e_s\right) \otimes \sum_t b_{ti} e_t \\
&= \sum_i \left(\sum_s a_{sj} e_i^*(e_s)\right) \otimes \sum_t b_{ti} e_t \\
&= \sum_i a_{ij} \otimes \sum_t b_{ti} e_t \\
&= \sum_t \left(\sum_i b_{ti} a_{ij}\right) e_t \\
&= \sum_t \delta_j^t e_t \\
&= e_j.
\end{aligned}$$

By similar calculations one gets the equality

$$(\mathrm{Id}_{V^*} \otimes \overleftarrow{\mathrm{ev}}_V) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_{V^*}) = \mathrm{Id}_{V^*}.$$

Thus the pair of morphisms $(\overleftarrow{\mathrm{ev}}_V, \overleftarrow{\mathrm{coev}}_V)$ are dualities. □

2.3 Quantum superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$

In this section we define the superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ and we prove that it is a pivotal Hopf superalgebra. We also show that the Borel part of $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is a Nichols algebra.

2.3.1 Hopf superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$

Definition 2.3.1. *Let $\ell \geq 3$ be an odd integer and $\xi = \exp(\frac{2\pi i}{\ell})$. The superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is an associative superalgebra on $\mathbb{C}$ generated by the elements $k_1, k_2, k_1^{-1}, k_2^{-1}, e_1, e_2, f_1, f_2$ and the relations*

$$k_1k_2 = k_2k_1, \tag{2.3.1}$$

$$k_ik_i^{-1} = 1,\ i = 1, 2, \tag{2.3.2}$$

$$k_ie_jk_i^{-1} = \xi^{a_{ij}}e_j, k_if_jk_i^{-1} = \xi^{-a_{ij}}f_j\ i, j = 1, 2, \tag{2.3.3}$$

$$e_1f_1 - f_1e_1 = \frac{k_1 - k_1^{-1}}{\xi - \xi^{-1}}, e_2f_2 + f_2e_2 = \frac{k_2 - k_2^{-1}}{\xi - \xi^{-1}}, \tag{2.3.4}$$

$$[e_1, f_2] = 0, [e_2, f_1] = 0, \tag{2.3.5}$$

$$e_2^2 = f_2^2 = 0, \tag{2.3.6}$$
$$e_1^2 e_2 - (\xi + \xi^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0, \tag{2.3.7}$$
$$f_1^2 f_2 - (\xi + \xi^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0. \tag{2.3.8}$$

The last two relations are called the Serre relations. The matrix (a_{ij}) is given by $a_{11} = 2, a_{12} = a_{21} = -1, a_{22} = 0$. The odd generators are e_2, f_2.

We define $\xi^x := \exp(\frac{2\pi i x}{\ell})$, afterwards we will use the concepts

$$\{x\} = \xi^x - \xi^{-x},\ [x] = \frac{\xi^x - \xi^{-x}}{\xi - \xi^{-1}}.$$

Let define the odd elements $e_3 = e_1 e_2 - \xi^{-1} e_2 e_1,\ f_3 = f_2 f_1 - \xi f_1 f_2$. The Serre relations become

$$e_1 e_3 = \xi e_3 e_1,\ f_3 f_1 = \xi^{-1} f_1 f_3. \tag{2.3.9}$$

Furthermore

$$e_2 e_3 = -\xi e_3 e_2,\ f_3 f_2 = -\xi^{-1} f_2 f_3, \tag{2.3.10}$$
$$e_3 f_3 + f_3 e_3 = \frac{k_1 k_2 - k_1^{-1} k_2^{-1}}{\xi - \xi^{-1}}, \tag{2.3.11}$$
$$e_3^2 = f_3^2 = 0. \tag{2.3.12}$$

According to [31], $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is a Hopf superalgebra with the coproduct, counit and antipode as below

$$\Delta(e_i) = e_i \otimes 1 + k_i^{-1} \otimes e_i \ i = 1, 2,$$
$$\Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i \ i = 1, 2,$$
$$\Delta(k_i) = k_i \otimes k_i \ i = 1, 2,$$
$$S(e_i) = -k_i e_i, S(f_i) = -f_i k_i^{-1}, S(k_i) = k_i^{-1} \ i = 1, 2,$$
$$\varepsilon(k_i) = 1, \varepsilon(e_i) = \varepsilon(f_i) = 0 \ i = 1, 2.$$

The center and representations of $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ were studied by B. Abdesselam, D. Arnaudon and M. Bauer [1]. We focus on the case of nilpotent representations of type $\mathfrak{B}$ with the condition ℓ odd.

Remark 2.3.2. *1. Because* $(e_1 \otimes 1)(k_1^{-1} \otimes e_1) = \xi^2 (k_1^{-1} \otimes e_1)(e_1 \otimes 1)$ *and* $(\ell)_\xi := \frac{1-\xi^\ell}{1-\xi} = 0$ *then*

$$\Delta(e_1^\ell) = \sum_{m=0}^{\ell} \binom{\ell}{m}_\xi (e_1 \otimes 1)^m (k_1^{-1} \otimes e_1)^{\ell - m} = e_1^\ell \otimes 1 + k_1^{-\ell} \otimes e_1^\ell. \tag{2.3.13}$$

We have $\Delta^{op}(e_1^\ell) = 1 \otimes e_1^\ell + e_1^\ell \otimes k_1^{-\ell}$ *at the same time. It is known that* $e_1^\ell, f_1^\ell, k_1^\ell \in \mathcal{Z}$ *where* $\mathcal{Z}$ *is the center of* $\mathcal{U}_\xi\mathfrak{sl}(2|1)$*, so* $\Delta(e_1^\ell) \in \mathcal{Z} \otimes \mathcal{Z}$*. It follows that there exists no element* $R \in \mathcal{U}_\xi\mathfrak{sl}(2|1) \otimes \mathcal{U}_\xi\mathfrak{sl}(2|1)$ *such that* $\Delta^{op}(x) = R\Delta(x)R^{-1}\ \forall x \in \mathcal{U}_\xi\mathfrak{sl}(2|1)$*, i.e. the superalgebra* $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ *is not quasitriangular.*

2. *We think that the quotient superalgebra* $\mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$ *is not quasitriangular but a quotient like* $\mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell, k_1^\ell - 1, k_2^\ell - 1)$ *could be, a proof of this might be found along the lines of [35]. This is not the quotient that interests us in this chapter.*

3. *The unrolled version* $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ *(defined in Section 2.4) seems to be quasitriangular only in a topological sense (see [20]). However, we will show in Theorem 2.4.4 and Proposition 2.4.5 that some representations (the weight modules) form a ribbon category.*

It is commonly admitted that the superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$ has a Poincaré-Birkhoff-Witt basis $\{e_2^\rho e_3^\sigma e_1^p k_1^s k_2^t f_2^{\rho'} f_3^{\sigma'} f_1^{p'}, \rho, \sigma, \rho', \sigma' \in \{0,1\}, p, p' \in \{0, 1, ..., \ell-1\}, s, t \in \mathbb{Z}\}$ (see [1]). Nevertheless, we give in Appendix A.1 an elementary proof of this fact stated in Lemma 2.3.3. Its Borel part is a superalgebra $\mathcal{U}_\xi(\mathfrak{n}_+)$ which has a vector space basis $\{e_2^\rho e_3^\sigma e_1^p\ \rho, \sigma \in \{0,1\}, p \in \{0, 1, ..., \ell-1\}\}$. It is well known that $\mathcal{U}_\xi(\mathfrak{n}_+)$ is a Nichols algebra of diagonal type associated with the generalized Dynkin diagram $\overset{\xi^2}{\circ}\!\!\overset{\xi^{-2}}{\text{——}}\!\!\overset{-1}{\circ}$ (see [25]). We now explain this point of view. We consider the group algebra $B = \mathbb{C}G$ in which G is an abelian group generated by k_1, k_2, a vector space V on $\mathbb{C}$ generated by e_1, e_2. Here B is a Hopf algebra and $(V, \cdot, \delta)$ is a Yetter-Drinfeld module on B [25], where the action $\cdot : B \otimes V \to V$ of B on V is determined by

$$k_1 \cdot e_1 = \xi^2 e_1,\ \ k_1 \cdot e_2 = \xi^{-1} e_2,$$
$$k_2 \cdot e_1 = \xi^{-1} e_1,\ \ k_2 \cdot e_2 = -e_2,$$

the matrix determining the bicharacter is $(q_{ij})_{2\times 2}, q_{ij} = (-1)^{|i||j|}\xi^{a_{ij}}$ where $|1| = 0,\ |2| = 1$ and the coaction $\delta : V \to B \otimes V$ of B on V is given by

$$\delta(e_i) = k_i \otimes e_i\ \ i = 1, 2.$$

It is clear that $\delta(b \cdot v) = b_{(1)} v_{(-1)} S(b_{(3)}) \otimes b_{(2)} \cdot v_{(0)} = v_{(-1)} \otimes b \cdot v_{(0)}$ for all $b \in B, v \in V$. Here we use the Sweedler notation and write $(\Delta \otimes \mathrm{Id})\Delta(b) = b_{(1)} \otimes b_{(2)} \otimes b_{(3)},\ \ \delta(v) = v_{(-1)} \otimes v_{(0)}$ for $b \in B, v \in V$.
Using Hopf algebra B and Yetter-Drinfeld module V we can determine the Nichols algebra $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ where $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ is the tensor algebra of V with the braided copoduct $\widetilde{\Delta}(v) = 1 \otimes v + v \otimes 1$ and counit

$\varepsilon(v) = 0$ for $v \in V$, $\mathcal{J}(V)$ is the maximal coideal in degree ≥ 2 of $T(V)$. We now check that e_2^2 and the Serre relation $w = e_1e_3 - \xi e_3e_1$ are in $\mathcal{J}(V)$. We have $\widetilde{\Delta}(e_2^2) = \widetilde{\Delta}(e_2)\widetilde{\Delta}(e_2) = (1 \otimes e_2 + e_2 \otimes 1)(1 \otimes e_2 + e_2 \otimes 1) = 1 \otimes e_2^2 + (k_2 \cdot e_2) \otimes e_2 + e_2 \otimes e_2 + e_2^2 \otimes 1 = 1 \otimes e_2^2 + e_2^2 \otimes 1$, so $e_2^2 \in \mathcal{J}(V)$.

We calculate

$$\begin{aligned}\widetilde{\Delta}(e_3) &= \widetilde{\Delta}(e_1)\widetilde{\Delta}(e_2) - \xi^{-1}\widetilde{\Delta}(e_2)\widetilde{\Delta}(e_1)\\ &= (1 \otimes e_1 + e_1 \otimes 1)(1 \otimes e_2 + e_2 \otimes 1) - \xi^{-1}(1 \otimes e_2 + e_2 \otimes 1)(1 \otimes e_1 + e_1 \otimes 1)\\ &= 1 \otimes e_1e_2 + (k_1 \cdot e_2) \otimes e_1 + e_1 \otimes e_2 + e_1e_2 \otimes 1\\ &\qquad - \xi^{-1}\left(1 \otimes e_2e_1 + (k_2 \cdot e_1) \otimes e_2 + e_2 \otimes e_1 + e_2e_1 \otimes 1\right)\\ &= 1 \otimes e_3 + e_3 \otimes 1 + (1 - \xi^{-2})e_1 \otimes e_2.\end{aligned}$$

And a similar calculation gives us

$$\begin{aligned}\widetilde{\Delta}(e_1)\widetilde{\Delta}(e_3) = 1 \otimes e_1e_3 + \xi e_3 \otimes e_1 + (1 - \xi^{-2})\xi^2 e_1 \otimes e_1e_2\\ + e_1 \otimes e_3 + e_1e_3 \otimes 1 + (1 - \xi^{-2})e_1^2 \otimes e_2,\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Delta}(e_3)\widetilde{\Delta}(e_1) = 1 \otimes e_3e_1 + \xi e_1 \otimes e_3 + e_3 \otimes e_1 + e_3e_1 \otimes 1\\ + (1 - \xi^{-2})e_1 \otimes e_2e_1 + (1 - \xi^{-2})\xi^{-1}e_1^2 \otimes e_2.\end{aligned}$$

Thus we have

$$\begin{aligned}\widetilde{\Delta}(w) &= \widetilde{\Delta}(e_1)\widetilde{\Delta}(e_3) - \xi\widetilde{\Delta}(e_3)\widetilde{\Delta}(e_1)\\ &= 1 \otimes w + w \otimes 1 + (\xi^2 - 1)e_1 \otimes e_1e_2 + e_1 \otimes e_3\\ &\qquad - \xi^2 e_1 \otimes e_3 - (\xi - \xi^{-1})e_1 \otimes e_2e_1\\ &= 1 \otimes w + w \otimes 1.\end{aligned}$$

By maximality of $\mathcal{J}(V)$, this implies that $w \in \mathcal{J}(V)$. The bosonization of $\mathcal{B}(V)$ is then isomorphic to a Hopf subalgebra of the bosonization of the Hopf superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$.

Lemma 2.3.3. *The set of vectors* $\{e_2^{\rho}e_3^{\sigma}e_1^{p}k_1^{s}k_2^{t}f_2^{\rho'}f_3^{\sigma'}f_1^{p'}\ \rho, \sigma, \rho', \sigma' \in \{0,1\},\ p, p' \in \{0, 1, ..., \ell - 1\},\ s, t \in \mathbb{Z}\}$ *is a basis of* $\mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^{\ell}, f_1^{\ell})$.

Proof. See in Appendix A.1. □

2.3.2 Pivotal Hopf superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$

Recall that the even category of representations of a superalgebra is the category of representations in which one restricts to the morphisms of even degree.

Proposition 2.3.4. *Given* $\phi_0 = k_1^{-\ell} k_2^{-2}$*, so* $\forall u \in \mathcal{U}_\xi\mathfrak{sl}(2|1), S^2(u) = \phi_0 u \phi_0^{-1}$.

Proof. This can be verified for generator elements k_i, e_i, f_i, $i = 1, 2$. □

It follows that the Hopf superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ provided with the pivotal element $\phi_0 = k_1^{-\ell} k_2^{-2}$ is pivotal superalgebra (see [39]).

Let $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$ be the category of finite dimensional modules over $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ with even morphisms then $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$ is a pivotal category thanks to Proposition 2.2.4. If V is an object of $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$, its dual is a $\mathbb{C}$-vector space $V^* = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ provided with the action of u given by $(u, \varphi) \mapsto (-1)^{\deg u \deg \varphi} \varphi \circ \rho_V(S(u))$ where $\rho_V : \mathcal{U}_\xi\mathfrak{sl}(2|1) \to \mathrm{End}_\mathbb{C}(V)$ is the representation of $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. The unit element of category $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$ is the module $\mathbb{C}$ provided with the representation $\varepsilon : \mathcal{U}_\xi\mathfrak{sl}(2|1) \to \mathbb{C} \cong \mathrm{End}_\mathbb{C}(\mathbb{C})$.

If one has a basis $(e_i)_i$ of V with dual basis $(e_i^*)_i$, recall that the dual morphisms given by

$$\overrightarrow{\mathrm{ev}}_V \colon e_i^* \otimes e_j \mapsto e_i^*(e_j) = \delta_i^j, \qquad \overrightarrow{\mathrm{coev}}_V \colon 1 \mapsto \sum_i e_i \otimes e_i^*,$$

$$\overleftarrow{\mathrm{ev}}_V \colon e_j \otimes e_i^* \mapsto (-1)^{\deg e_j} e_i^*(\phi_0 . e_j), \quad \overleftarrow{\mathrm{coev}}_V \colon 1 \mapsto \sum_i (-1)^{\deg e_i} e_i^* \otimes (\phi_0^{-1} . e_i).$$

2.4 Category of nilpotent weight modules

This section allows to define the superalgebra $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ from $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. Then we define the even category $\mathscr{C}^H$ of nilpotent finite dimensional weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ and prove that this category is G-graded and ribbon. The category $\mathscr{C}^H$ is used to construct the topological invariants in next sections.

2.4.1 Typical module

We call nilpotent weight $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-module an object of $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$ on which $e_1^\ell = f_1^\ell = 0$ and k_1, k_2 are diagonalizable. Let $\mathscr{C}$ be the full subcategory of $\mathcal{U}_\xi\mathfrak{sl}(2|1)\text{-mod}_{\bar{0}}$ formed by all nilpotent weight modules over $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. One can check, for example see Equation (2.3.13) that the tensor product and the dual of nilpotent weight modules are nilpotent weight modules.

Each nilpotent simple weight module (called "of type $\mathfrak{B}$" in Section 5.2 [1]) is determined by the highest weight $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$ and is denoted by V_{μ_1,μ_2} or V_μ. Its highest weight vector $w_{0,0,0}$ satisfies

$$e_1 w_{0,0,0} = 0, \qquad e_2 w_{0,0,0} = 0,$$
$$k_1 w_{0,0,0} = \lambda_1 w_{0,0,0}, \qquad k_2 w_{0,0,0} = \lambda_2 w_{0,0,0}$$

where $\lambda_i = \xi^{\mu_i}$ with $i = 1, 2$.
For $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$ we say that $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-module V_μ is typical if it is a simple module of dimension 4ℓ. Other simple modules are said to be atypical. The basis of a typical module is formed by vectors $w_{\rho,\sigma,p} = f_2^\rho f_3^\sigma f_1^p w_{0,0,0}$ where $\rho, \sigma \in \{0, 1\}, 0 \le p < \ell$. The odd elements are $w_{0,1,p}$ and $w_{1,0,p}$, others are even. The representation of typical $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-module V_{μ_1,μ_2} is determined by

$$k_1 w_{\rho,\sigma,p} = \lambda_1 \xi^{\rho-\sigma-2p} w_{\rho,\sigma,p}, \tag{2.4.1}$$
$$k_2 w_{\rho,\sigma,p} = \lambda_2 \xi^{\sigma+p} w_{\rho,\sigma,p}, \tag{2.4.2}$$
$$f_1 w_{\rho,\sigma,p} = \xi^{\sigma-p} w_{\rho,\sigma,p+1} - \rho(1-\sigma)\xi^{-\sigma} w_{\rho-1,\sigma+1,p}, \tag{2.4.3}$$
$$f_2 w_{\rho,\sigma,p} = (1-\rho) w_{\rho+1,\sigma,p}, \tag{2.4.4}$$
$$e_1 w_{\rho,\sigma,p} = -\sigma(1-\rho)\lambda_1 \xi^{-2p+1} w_{\rho+1,\sigma-1,p} + [p][\mu_1 - p + 1] w_{\rho,\sigma,p-1}, \tag{2.4.5}$$
$$e_2 w_{\rho,\sigma,p} = \rho[\mu_2 + p + \sigma] w_{\rho-1,\sigma,p} + \sigma(-1)^\rho \lambda_2^{-1} \xi^{-p} w_{\rho,\sigma-1,p+1}. \tag{2.4.6}$$

where $\rho, \sigma \in \{0, 1\}$ and $p \in \{0, 1, ..., \ell - 1\}$.

We also have $V_\mu \simeq V_{\mu+\vartheta} \Leftrightarrow \vartheta \in (\ell\mathbb{Z})^2$.

Remark 2.4.1. *The module V_μ is typical if $[\mu_1 - p + 1] \neq 0$ $\forall p \in \{1, ..., \ell - 1\}$ ($\mu_1 \neq p - 1 + \frac{\ell}{2}\mathbb{Z}$ $\forall p \in \{1, ..., \ell - 1\}$) and $[\mu_2][\mu_1 + \mu_2 + 1] \neq 0$ ($\mu_2 \neq \frac{\ell}{2}\mathbb{Z}, \mu_1 + \mu_2 \neq -1 + \frac{\ell}{2}\mathbb{Z}$)* (see [1]).

We call $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ the $\mathbb{C}$-superalgebra generated by $e_i,\ f_i,\ k_i,\ k_i^{-1}$ and h_i for $i = 1, 2$ with Relations (2.3.1) - (2.3.8) plus the relations

$$[h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [h_i, h_j] = 0,\ [h_i, k_j] = 0\ i, j = 1, 2.$$

The superalgebra $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is a Hopf superalgebra where Δ, S and ε are determined as in $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ and by

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,\ S(h_i) = -h_i,\ \varepsilon(h_i) = 0\ i = 1, 2.$$

Note that $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ can be seen as a semidirect product of $\mathbb{C}[h_1, h_2]$ acting on $\mathcal{U}_\xi\mathfrak{sl}(2|1)$.

Let $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-mod$_{\bar{0}}$ be the category of finite dimensional modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ with even morphisms then $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-mod$_{\bar{0}}$ is a pivotal category thanks also to Proposition 2.2.4. We call nilpotent weight $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-module an object of $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-mod$_{\bar{0}}$ on which $e_1^\ell = f_1^\ell = 0$ and $\xi^{h_i} = k_i$ for $i = 1, 2$ are diagonalizable. Let $\mathscr{C}^H$ be the full subcategory of $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-mod$_{\bar{0}}$ formed by all nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. The category $\mathscr{C}^H$ is pivotal similar to $\mathscr{C}$ (see Section 2.3.2).

We define the actions of $h_i, i = 1, 2$ on the basis of V_{μ_1,μ_2} by

$$h_1 w_{\rho,\sigma,p} = (\mu_1 + \rho - \sigma - 2p)w_{\rho,\sigma,p},\ \ h_2 w_{\rho,\sigma,p} = (\mu_2 + \sigma + p)w_{\rho,\sigma,p}.$$

Thus V_{μ_1,μ_2} is a weight module of $\mathscr{C}^H$. A module in $\mathscr{C}^H$ is said to be typical if, seen as a $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-module, it is typical. For each module V we denote $\overline{V}$ the same module with the opposite parity. We set $G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}$ and for each $\overline{\mu} \in G$ we define $\mathscr{C}^H_{\overline{\mu}}$ as the subcategory of weight modules which have their weights in the coset $\overline{\mu}$ (modulo $\mathbb{Z} \times \mathbb{Z}$). So $\{\mathscr{C}^H_{\overline{\mu}}\}_{\overline{\mu}\in G}$ is a G-graduation (where G is an additive group): let $V \in \mathscr{C}^H_{\overline{\mu}}, V' \in \mathscr{C}^H_{\overline{\mu}'}$, then the weights of $V \otimes V'$ are congruent to $\overline{\mu} + \overline{\mu}'$ (modulo $\mathbb{Z} \times \mathbb{Z}$). Furthermore, if $\overline{\mu} \neq \overline{\mu}'$ then $\mathrm{Hom}_{\mathscr{C}^H}(V, V') = 0$ because a morphism preserves weights.

We also define

$$G_s = \{\overline{g} \in G \textit{ such that } \exists\ V \in \mathscr{C}^H_{\overline{g}} \textit{ simple and atypical}\}.$$

It follows from [1] that

$$G_s = \left\{\overline{0}, \overline{\frac{1}{2}}\right\} \times \mathbb{C}/\mathbb{Z} \cup \mathbb{C}/\mathbb{Z} \times \left\{\overline{0}, \overline{\frac{1}{2}}\right\} \cup \left\{(\overline{\mu_1}, \overline{\mu_2}) : \overline{\mu_1} + \overline{\mu_2} \in \left\{\overline{0}, \overline{\frac{1}{2}}\right\}\right\}.$$

2.4.2 Character of representations of $\mathcal{U}^H_\xi\mathfrak{sl}(2|1)$

Definition 2.4.2. *The character of a weight module V is*

$$\chi_V = \sum_\mu \dim(E_\mu(V)) X_1^{\mu_1} X_2^{\mu_2}$$

where $E_\mu(V)$ is the proper subspace of the proper value $\mu = (\mu_1, \mu_2)$ of (h_1, h_2).

Note that we do not use the concept of a super-character defined as above by replacing the dimension by the super-dimension.

A finite dimensional representation of $\mathcal{U}_\xi\mathfrak{gl}(2)$, subalgebra generated by e_1, f_1, k_i is defined by $V = Span_{\mathbb{C}}\{v_0, ..., v_{\ell-1}\}$ [1]

$$\begin{aligned}
&k_1 v_p = \lambda_1 \xi^{-2p} v_p \text{ with } p \in \{0, 1, ..., \ell - 1\},\\
&f_1 v_p = v_{p+1} \text{ with } p \in \{0, 1, ..., \ell - 2\} \text{ and } f_1 v_{\ell-1} = 0,\\
&e_1 v_p = [p][\mu_1 - p + 1]v_{p-1},\ \xi^{\mu_1} = \lambda_1,\\
&k_2 v_p = \lambda_2 \xi^p v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}.
\end{aligned}$$

It extends to the generators h_1, h_2 by

$$\begin{aligned}
&h_1 v_p = (\mu_1 - 2p)v_p \text{ with } p \in \{0, 1, ..., \ell - 1\},\\
&h_2 v_p = (\mu_2 + p)v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}
\end{aligned}$$

so that $\xi^{h_i} = k_i, i = 1, 2$ on V. We have the character of representation of $\mathcal{U}_\xi\mathfrak{gl}(2)$

$$\chi_{V^{\mathfrak{gl}(2)}_{\mu_1,\mu_2}} = X_1^{\mu_1} X_2^{\mu_2} \frac{1-x^\ell}{1-x} \text{ where } x = X_1^{-2}X_2.$$

In the case of a typical representation, the nilpotent representation V_{μ_1,μ_2} of $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ with highest weight (μ_1, μ_2) is determined by

$$k_1 w_{\rho,\sigma,p} = \lambda_1 \xi^{\rho-\sigma-2p} w_{\rho,\sigma,p},$$
$$k_2 w_{\rho,\sigma,p} = \lambda_2 \xi^{\sigma+p} w_{\rho,\sigma,p}$$

with $h_1 w_{\rho,\sigma,p} = (\mu_1 + \rho - \sigma - 2p) w_{\rho,\sigma,p}$ and $h_2 w_{\rho,\sigma,p} = (\mu_2 + \sigma + p) w_{\rho,\sigma,p}$. So the nilpotent representation V_{μ_1,μ_2} has the following character

$$\chi_{V^{\mathfrak{sl}(2|1)}_{\mu_1,\mu_2}} = \chi_{V^{\mathfrak{gl}(2)}_{\mu_1,\mu_2,\rho=\sigma=0}} + \chi_{V^{\mathfrak{gl}(2)}_{\mu_1,\mu_2,\rho=1,\sigma=0}} + \chi_{V^{\mathfrak{gl}(2)}_{\mu_1,\mu_2,\rho=0,\sigma=1}} + \chi_{V^{\mathfrak{gl}(2)}_{\mu_1,\mu_2,\rho=\sigma=1}}$$
$$= X_1^{\mu_1} X_2^{\mu_2} \frac{1-x^\ell}{1-x}(1+X_1)(1+X_1 x). \quad (2.4.7)$$

2.4.3 Braided category $\mathscr{C}^H$

Let $\mathcal{U}_q\mathfrak{sl}(2|1)$ be the $\mathbb{C}(q)$-subsuperalgebra of the h-adic quantized enveloping superalgebra of $\mathfrak{sl}(2|1)$ generated by the elements e_i, f_i, k_i, k_i^{-1} for $1 \le i \le 2$ where $q = e^h \in \mathbb{C}[[h]][h^{-1}]$. Let $\mathcal{A} = \mathbb{C}[q, q^{-1}, (\ell-1)_q!^{-1}]$. Let $\mathcal{U}_\mathcal{A}\mathfrak{sl}(2|1)$ be the $\mathcal{A}$-subsuperalgebra of $\mathcal{U}_q\mathfrak{sl}(2|1)$ generated by the elements e_i, f_i, k_i, k_i^{-1} for $1 \le i \le 2$ and the relations (2.3.1) - (2.3.12) in which ξ is replaced by q.

The $\mathbb{C}$-superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ can be seen as the specialisation at $q = \xi$ of $\mathcal{U}_\mathcal{A}\mathfrak{sl}(2|1)$, i.e. $\mathcal{U}_\xi\mathfrak{sl}(2|1) = \mathcal{U}_\mathcal{A}\mathfrak{sl}(2|1)/\,(q-\xi)\mathcal{U}_\mathcal{A}\mathfrak{sl}(2|1)$ (see also [6]). Then $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is a superalgebra over $\mathbb{C}$ with generators e_i, f_i, k_i, k_i^{-1} for $1 \le i \le 2$ and relations (2.3.1) - (2.3.12).

In articles [31, 51] the authors showed that $\mathcal{R}^q = \check{\mathcal{R}}^q \mathcal{K}_q$ where

$$\check{\mathcal{R}}^q = \sum_{i=0}^{\infty} \frac{\{1\}^i e_1^i \otimes f_1^i}{(i)_q!} \sum_{j=0}^{1} \frac{(-\{1\})^j e_3^j \otimes f_3^j}{(j)_q!} \sum_{k=0}^{1} \frac{(-\{1\})^k e_2^k \otimes f_2^k}{(k)_q!},$$

$(0)_q! = 1, (n)_q! := (1)_q(2)_q \dots (n)_q, (k)_q = \frac{1-q^k}{1-q}$ and $\mathcal{K}_q = q^{-h_1\otimes h_2 - h_2 \otimes h_1 - 2h_2\otimes h_2}$ is a universal R-matrix element of superalgebra $\mathcal{U}_q\mathfrak{sl}(2|1)$. That is, we have the following relations in the h-adic completion of these algebras

$$(\Delta \otimes \mathrm{Id})(\mathcal{R}^q) = \mathcal{R}^q_{13}\mathcal{R}^q_{23}, \ (\mathrm{Id} \otimes \Delta)(\mathcal{R}^q) = \mathcal{R}^q_{13}\mathcal{R}^q_{12}, \ \Delta^{op}(x)\mathcal{R}^q = \mathcal{R}^q\Delta(x)$$

for all $x \in \mathcal{U}_q\mathfrak{sl}(2|1)$. The superalgebra $\mathcal{U}_q\mathfrak{sl}(2|1)$ has a Poincaré-Birkhoff-Witt basis $\{e_1^{p'} e_3^{\sigma'} e_2^{\rho'} h_1^{s_1} h_2^{s_2} f_2^\rho f_3^\sigma f_1^p, p, p' \in \mathbb{N}, \rho, \sigma, \rho', \sigma' \in \{0,1\}, s_1, s_2 \in \mathbb{N}\}$.

Using this basis we can write $\mathcal{U}_q\mathfrak{sl}(2|1)$ as a direct sum $\mathcal{U}_q\mathfrak{sl}(2|1) = \mathcal{U}^< \oplus I$ where $\mathcal{U}^<$ is a $\mathbb{C}(q)$-module generated by the elements $e_1^{p'}e_3^{\sigma'}e_2^{\rho'}h_1^{s_1}h_2^{s_2}f_3^{\sigma}f_2^{\rho}f_1^{p}$ for $0 \le p, p' < \ell; \rho, \sigma, \rho', \sigma' \in \{0,1\}, s_1, s_2 \in \mathbb{N}$ and I is generated by the other monomials. Set $p : \mathcal{U}_q\mathfrak{sl}(2|1) \to \mathcal{U}^<$ the projection with kernel I. We define

$$\mathcal{R}^< = p \otimes p(\mathcal{R}^q) = p \otimes \mathrm{Id}(\mathcal{R}^q) = \mathrm{Id} \otimes p(\mathcal{R}^q).$$

The proposition below shows that the "truncated R-matrix" $\mathcal{R}^<$ satisfies the properties of an R-matrix "modulo truncation".

Proposition 2.4.3. $\mathcal{R}^<$ *satisfies:*

1. $(p \otimes p \otimes p)(\Delta \otimes \mathrm{Id}(\mathcal{R}^<)) = (p \otimes p \otimes p)\mathcal{R}^<_{13}\mathcal{R}^<_{23}$,
2. $(p \otimes p \otimes p)(\mathrm{Id} \otimes \Delta(\mathcal{R}^<)) = (p \otimes p \otimes p)\mathcal{R}^<_{13}\mathcal{R}^<_{12}$,
3. $(p \otimes p)(\mathcal{R}^<\Delta^{op}(x)) = (p \otimes p)(\Delta(x)\mathcal{R}^<)$ *for all* $x \in \mathcal{U}_q\mathfrak{sl}(2|1)$.

Proof. The above relations and $p \circ p = p$ give us $(p \otimes p \otimes p)(\Delta \otimes \mathrm{Id}(\mathcal{R}^q)) = (p \otimes p \otimes p)(\Delta \otimes \mathrm{Id})(\mathrm{Id} \otimes p(\mathcal{R}^q)) = (p \otimes p \otimes p)(\Delta \otimes \mathrm{Id})(\mathcal{R}^<)$. At the same time $(p \otimes p \otimes p)(\mathcal{R}^q_{13}\mathcal{R}^q_{23}) = (p \otimes p \otimes p)((p \otimes \mathrm{Id} \otimes \mathrm{Id})(\mathcal{R}^q_{13})(\mathrm{Id} \otimes p \otimes \mathrm{Id})(\mathcal{R}^q_{23})) = (p \otimes p \otimes p)(\mathcal{R}^<_{13}\mathcal{R}^<_{23})$. So

$$(p \otimes p \otimes p)(\Delta \otimes \mathrm{Id}(\mathcal{R}^<)) = (p \otimes p \otimes p)\mathcal{R}^<_{13}\mathcal{R}^<_{23}. \tag{2.4.8}$$

Similarly we also have

$$(p \otimes p \otimes p)(\mathrm{Id} \otimes \Delta)(\mathcal{R}^<) = (p \otimes p \otimes p)(\mathcal{R}^<_{13}\mathcal{R}^<_{12}). \tag{2.4.9}$$

For the third equality, it is enough to check on the generator elements.

It is true when $x = h_i$ because $\Delta(h_i)$ is symmetric and $\Delta(h_i)(e_j \otimes f_j) = e_j \otimes h_i f_j + h_i e_j \otimes f_j = e_j \otimes f_j(h_i - a_{ij}) + e_j(h_i + a_{ij}) \otimes f_j = e_j \otimes f_j(1 \otimes (h_i - a_{ij}) + (h_i + a_{ij}) \otimes 1) = (e_j \otimes f_j)\Delta(h_i)$.

For $x = e_i$ we have $(p \otimes p)(\Delta^{op}(e_i)\mathcal{R}^q) = (p \otimes p)(1 \otimes e_i + e_i \otimes k_i^{-1})\mathcal{R}^q = (p \otimes p)((1 \otimes e_i)\mathcal{R}^q) + (p \otimes p)((e_i \otimes k_i^{-1})\mathcal{R}^q) = (p \otimes p)((1 \otimes e_i)\mathcal{R}^<) + (p \otimes p)((e_i \otimes k_i^{-1})\mathcal{R}^<) = (p \otimes p)(\Delta^{op}(e_i)\mathcal{R}^<)$. On the other side $(p \otimes p)(\mathcal{R}^q\Delta(e_i)) = (p \otimes p)(\mathcal{R}^<\Delta(e_i))$. So we have $(p \otimes p)(\Delta^{op}(e_i)\mathcal{R}^<) = (p \otimes p)(\mathcal{R}^<\Delta(e_i))$.

For $x = f_i$ we proceed analogously. So we deduce that

$$(p \otimes p)(\Delta^{op}(x)\mathcal{R}^<) = (p \otimes p)(\mathcal{R}^<\Delta(x)) \ \forall x \in \mathcal{U}_q\mathfrak{sl}(2|1).$$

□

Let $\mathcal{K}$ be the operator in $\mathscr{C}^H \otimes \mathscr{C}^H$ defined by

$$\mathcal{K} = \xi^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}$$

that is $\forall V, W \in \mathscr{C}^H, \mathcal{K}_{V\otimes W} = \exp\left(\rho_{V\otimes W}(\frac{2i\pi}{\ell}(-h_1\otimes h_2 - h_2\otimes h_1 - 2h_2\otimes h_2))\right)$ is a linear map on the finite dimensional vector space $V\otimes W$. For example, if $w_{\rho,\sigma,p}\otimes w_{\rho',\sigma',p'} \in V_\mu\otimes V_{\mu'}$, one has

$$\begin{aligned}&\mathcal{K}_{V\otimes W}(w_{\rho,\sigma,p}\otimes w_{\rho',\sigma',p'})\\ &= \xi^{-(\mu_1+\rho-\sigma-2p)(\mu_2'+\sigma'+p')-(\mu_2+\sigma+p)(\mu_1'+\rho'-\sigma'-2p')-2(\mu_2+\sigma+p)(\mu_2'+\sigma'+p')} w_{\rho,\sigma,p}\otimes w_{\rho',\sigma',p'}.\end{aligned}$$

We have

$$\Delta\otimes\mathrm{Id}(\mathcal{K}) = \mathcal{K}_{13}\mathcal{K}_{23},\ \mathrm{Id}\otimes\Delta(\mathcal{K}) = \mathcal{K}_{13}\mathcal{K}_{12}. \tag{2.4.10}$$

Let $\check{\mathcal{R}}^<$ be the universal truncated quasi R-matrix of $\mathcal{U}_q\mathfrak{sl}(2|1), q = e^h \in \mathbb{C}[[h]]$ given by $\check{\mathcal{R}}^< = p\otimes p(\check{\mathcal{R}}^q) = \mathrm{Id}\otimes p(\check{\mathcal{R}}^q) = p\otimes\mathrm{Id}(\check{\mathcal{R}}^q)$, i.e.

$$\check{\mathcal{R}}^< = \sum_{i=0}^{\ell-1}\frac{\{1\}^i e_1^i\otimes f_1^i}{(i)_q!}\sum_{j=0}^{1}\frac{(-\{1\})^j e_3^j\otimes f_3^j}{(j)_q!}\sum_{k=0}^{1}\frac{(-\{1\})^k e_2^k\otimes f_2^k}{(k)_q!}.$$

Set $\check{\mathcal{R}} = \check{\mathcal{R}}^<|_{q=\xi}$, i.e.

$$\check{\mathcal{R}} = \sum_{i=0}^{\ell-1}\frac{\{1\}^i e_1^i\otimes f_1^i}{(i)_\xi!}\sum_{j=0}^{1}\frac{(-\{1\})^j e_3^j\otimes f_3^j}{(j)_\xi!}\sum_{k=0}^{1}\frac{(-\{1\})^k e_2^k\otimes f_2^k}{(k)_\xi!} \in \mathcal{U}_\xi^H\mathfrak{sl}(2|1)\otimes\mathcal{U}_\xi^H\mathfrak{sl}(2|1).$$

Theorem 2.4.4. *The operator $\mathcal{R} = \check{\mathcal{R}}\mathcal{K}$ led to a braiding $\{c_{V,W}\}$ in the category $\mathscr{C}^H$ where $c_{V,W} : V\otimes W \to W\otimes V$ is determined by $v\otimes w \mapsto \tau(\mathcal{R}(v\otimes w))$. Here $\tau : V\otimes W \to W\otimes V,\ v\otimes w \mapsto (-1)^{\deg v \deg w} w\otimes v$.*

Proof. It is sufficient to prove that the operator $\mathcal{R}$ satisfies

$$\Delta\otimes\mathrm{Id}(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \mathrm{Id}\otimes\Delta(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \mathcal{R}\Delta^{op}(x) = \Delta(x)\mathcal{R} \tag{2.4.11}$$

for all $x \in \mathcal{U}_\xi^H\mathfrak{sl}(2|1)$.

Let $\chi_q : \mathcal{U}_q\mathfrak{sl}(2|1)\otimes\mathcal{U}_q\mathfrak{sl}(2|1) \to \mathcal{U}_q\mathfrak{sl}(2|1)\otimes\mathcal{U}_q\mathfrak{sl}(2|1)$ be the automorphism determined by $x\otimes y \mapsto \mathcal{K}_q(x\otimes y)\mathcal{K}_q^{-1}$, this one induces an automorphism $\chi_\xi : \mathcal{U}_\xi^H\mathfrak{sl}(2|1)\otimes\mathcal{U}_\xi^H\mathfrak{sl}(2|1) \to \mathcal{U}_\xi^H\mathfrak{sl}(2|1)\otimes\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$. We consider the element $\check{\mathcal{R}}^<$ of $\mathcal{U}_q\mathfrak{sl}(2|1)\otimes\mathcal{U}_q\mathfrak{sl}(2|1)$, Proposition 2.4.3 implies the relations

$$\Delta\otimes\mathrm{Id}(\check{\mathcal{R}}) = \check{\mathcal{R}}_{13}\,(\chi_\xi)_{13}\,(\check{\mathcal{R}}_{23}), \tag{2.4.12}$$

$$\mathrm{Id}\otimes\Delta(\check{\mathcal{R}}) = \check{\mathcal{R}}_{13}\,(\chi_\xi)_{13}\,(\check{\mathcal{R}}_{12}), \tag{2.4.13}$$

$$\check{\mathcal{R}}\,(\chi_\xi)\,(\Delta^{op}(x)) = \Delta(x)\check{\mathcal{R}} \text{ for all } x \in \mathcal{U}_\xi^H\mathfrak{sl}(2|1). \tag{2.4.14}$$

We will prove the equality (2.4.12), and that the other two are similar. From the first equality of the Proposition 2.4.3, we deduce that $(\Delta\otimes\mathrm{Id})(\check{\mathcal{R}}^<\mathcal{K}_q) =$

$\check{\mathcal{R}}^{<}_{13}(\mathcal{K}_q)_{13}\check{\mathcal{R}}^{<}_{23}(\mathcal{K}_q)_{23}$. The term in the left of this equality is equal to $(\Delta\otimes \mathrm{Id})(\check{\mathcal{R}}^{<})(\Delta\otimes\mathrm{Id})(\mathcal{K}_q) = \Delta\otimes\mathrm{Id}(\check{\mathcal{R}}^{<})(\mathcal{K}_q)_{13}(\mathcal{K}_q)_{23}$. The right one is equal to $\check{\mathcal{R}}^{<}_{13}(\mathcal{K}_q)_{13}\check{\mathcal{R}}^{<}_{23}(\mathcal{K}_q)_{23} = \check{\mathcal{R}}^{<}_{13}\,(\chi_q)_{13}\,(\check{\mathcal{R}}^{<}_{23})(\mathcal{K}_q)_{13}(\mathcal{K}_q)_{23}$. Now because $\mathcal{K}_q$ is invertible, the result is $\Delta\otimes\mathrm{Id}(\check{\mathcal{R}}^{<}) = \check{\mathcal{R}}^{<}_{13}\,(\chi_q)_{13}\,(\check{\mathcal{R}}^{<}_{23})$.

The element $\check{\mathcal{R}}^{<}$ has no pole when q is a root of unity of order ℓ. Hence we can specialize this relation at $q=\xi$ and $\Delta\otimes\mathrm{Id}(\check{\mathcal{R}}) = \check{\mathcal{R}}_{13}\,(\chi_\xi)_{13}\,(\check{\mathcal{R}}_{23})$. Finally, as operators on $V_1\otimes V_2\otimes V_3$ in which $V_1,V_2,V_3\in\mathscr{C}^H$, Equation (2.4.10) implies that

$$\begin{aligned}\Delta\otimes\mathrm{Id}(\mathcal{R}) &= (\Delta\otimes\mathrm{Id})(\check{\mathcal{R}})(\Delta\otimes\mathrm{Id})(\mathcal{K})\\ &= \check{\mathcal{R}}_{13}\,(\chi_\xi)_{13}\,(\check{\mathcal{R}}_{23})\mathcal{K}_{13}\mathcal{K}_{23}\\ &= \check{\mathcal{R}}_{13}\mathcal{K}_{13}\check{\mathcal{R}}_{23}\mathcal{K}_{13}^{-1}\mathcal{K}_{13}\mathcal{K}_{23}\\ &= \check{\mathcal{R}}_{13}\mathcal{K}_{13}\check{\mathcal{R}}_{23}\mathcal{K}_{23}\\ &= \mathcal{R}_{13}\mathcal{R}_{23}.\end{aligned}$$

Thus the relations of equation (2.4.11) hold. □

The category $\mathscr{C}^H$ is pivotal and braided with the braiding $c_{V,W}: V\otimes W\to W\otimes V,\ v\otimes w\mapsto \tau\circ\mathcal{R}(v\otimes w)$ where $V,W\in\mathscr{C}^H$.

2.4.4 Ribbon category $\mathscr{C}^H$

To prove the next proposition we will use the semi-simplicity of $\mathscr{C}_g$ ($g\in G\backslash G_s$) which is proven later in Theorem 2.4.14.

Proposition 2.4.5. *The family of isomorphisms $\theta_V : V\to V$ determined by $\theta_V = (\mathrm{Id}_V\otimes \overleftarrow{\mathrm{ev}}_V)(c_{V,V}\otimes\mathrm{Id}_{V^*})(\mathrm{Id}_V\otimes\overrightarrow{\mathrm{coev}}_V)$, $V\in\mathscr{C}^H$ is a twist. That is $\theta_V=\theta'_V$ $\forall V\in\mathscr{C}^H$ where $\theta'_V = (\overrightarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_V)(\mathrm{Id}_{V^*}\otimes c_{V,V})(\overleftarrow{\mathrm{coev}}_V\otimes\mathrm{Id}_V)$.*

Proof. Firstly, if V is a typical module of highest weight $\mu=(\mu_1,\mu_2), V\in\mathscr{C}^H_g, g\in G\backslash G_s$, we have $\theta'_V = (\overrightarrow{\mathrm{ev}}_V\otimes\mathrm{Id}_V)(\mathrm{Id}_{V^*}\otimes c_{V,V})(\overleftarrow{\mathrm{coev}}_V\otimes\mathrm{Id}_V) = X_1X_2X_3$.
We use the vector of lowest weight $(\mu_1-2\ell+2,\mu_2+\ell)$ of V, $w_{1,1,\ell-1}:=w_\infty$, to calculate.

$$\begin{aligned}X_3(w_\infty) &= \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}w^*_{\rho,\sigma,p}\otimes\phi_0^{-1}w_{\rho,\sigma,p}\otimes w_\infty\\ &= \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{\ell\mu_1+2\mu_2+2\sigma+2p}w^*_{\rho,\sigma,p}\otimes w_{\rho,\sigma,p}\otimes w_\infty.\end{aligned}$$

$X_2X_3(w_\infty) = \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{\ell\mu_1+2\mu_2+2\sigma+2p}w^*_{\rho,\sigma,p}\otimes(\tau\circ\mathcal{R})(w_{\rho,\sigma,p}\otimes w_\infty)$.

$$\begin{aligned}\mathcal{K}(w_{\rho,\sigma,p}\otimes w_\infty) &= \xi^{-h_1\otimes h_2-h_2\otimes h_1-2h_2\otimes h_2}w_{\rho,\sigma,p}\otimes w_\infty\\ &= \xi^{-\mu_1(\mu_2+\sigma+p+\ell)-\mu_2(\mu_1+2\mu_2+\sigma+\rho+2)-2(\sigma+p)}w_{\rho,\sigma,p}\otimes w_\infty.\end{aligned}$$

$$\begin{aligned}&X_2X_3(w_\infty)\\ &=\sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{\ell\mu_1+2\mu_2}\xi^{-\mu_1(\mu_2+\sigma+p+\ell)-\mu_2(\mu_1+2\mu_2+\sigma+\rho+2)}w^*_{\rho,\sigma,p}\otimes w_\infty\otimes w_{\rho,\sigma,p}\\ &=\sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+2\mu_2+\sigma+\rho)}w^*_{\rho,\sigma,p}\otimes w_\infty\otimes w_{\rho,\sigma,p}.\end{aligned}$$

So

$$\begin{aligned}X_1X_2X_3(w_\infty) &= \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+2\mu_2+\sigma+\rho)}w^*_{\rho,\sigma,p}(w_\infty)\otimes w_{\rho,\sigma,p}\\ &= \xi^{-\mu_1(\mu_2+\ell)-\mu_2(\mu_1+2\mu_2+2)}w_\infty.\end{aligned}$$

Secondly, we have

$$\theta_V = (\mathrm{Id}_V\otimes \overleftarrow{\mathrm{ev}}_V)(c_{V,V}\otimes\mathrm{Id}_{V^*})(\mathrm{Id}_V\otimes\overrightarrow{\mathrm{coev}}_V) = Y_1Y_2Y_3.$$

$Y_3(w_{0,0,0}) = \sum_{\rho,\sigma,p} w_{0,0,0}\otimes w_{\rho,\sigma,p}\otimes w^*_{\rho,\sigma,p}$,
$Y_2Y_3(w_{0,0,0}) = \sum_{\rho,\sigma,p}(\tau\circ\mathcal{R})(w_{0,0,0}\otimes w_{\rho,\sigma,p})\otimes w^*_{\rho,\sigma,p}$ where
$\mathcal{K}(w_{0,0,0}\otimes w_{\rho,\sigma,p}) = \xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+\rho-\sigma-2p)-2\mu_2(\mu_2+\sigma+p)}w_{0,0,0}\otimes w_{\rho,\sigma,p}$ and
$\mathcal{R}(w_{0,0,0}\otimes w_{\rho,\sigma,p}) = \xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+\rho-\sigma-2p)-2\mu_2(\mu_2+\sigma+p)}w_{0,0,0}\otimes w_{\rho,\sigma,p}$.
$Y_2Y_3(w_{0,0,0}) = \sum_{\rho,\sigma,p}\xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+\rho-\sigma-2p)-2\mu_2(\mu_2+\sigma+p)}w_{\rho,\sigma,p}\otimes w_{0,0,0}\otimes w^*_{\rho,\sigma,p}$.

$$\begin{aligned}&Y_1Y_2Y_3(w_{0,0,0})\\ &=\sum_{\rho,\sigma,p}\xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+\rho-\sigma-2p)-2\mu_2(\mu_2+\sigma+p)}w_{\rho,\sigma,p}\otimes w^*_{\rho,\sigma,p}((-1)^{\rho+\sigma}\phi_0 w_{0,0,0})\\ &=\sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{-\mu_1(\mu_2+\sigma+p)-\mu_2(\mu_1+\rho-\sigma-2p)-2\mu_2(\mu_2+\sigma+p)}w_{\rho,\sigma,p}\otimes w^*_{\rho,\sigma,p}(\xi^{-\ell\mu_1-2\mu_2}w_{0,0,0})\\ &=\xi^{-2\mu_1\mu_2-2\mu_2^2-2\mu_2-\ell\mu_1}w_{0,0,0}\\ &=\xi^{-\mu_1(\mu_2+\ell)-\mu_2(\mu_1+2\mu_2+2)}w_{0,0,0}.\end{aligned}$$

We can deduce that $\theta_V = \theta'_V$ for every typical module V with highest weight $\mu = (\mu_1,\mu_2), V\in\mathscr{C}^H_g, g\in G\backslash G_s$. Note that the calculation does not change if we reverse the parity of vectors. So we have the affirmation for a semi-simple module in degree $g\in G\backslash G_s$. Let a module $W\in\mathscr{C}^H_g, g\in G$. By Theorem 2.4.14 it exists $h\in G$ such that $\mathscr{C}^H_h,\mathscr{C}^H_{g+h}$ are semi-simple. For a module $V\in\mathscr{C}^H_h$ we have $W\otimes V\in\mathscr{C}^H_{g+h}$ is semi-simple.

Because $\theta_{W\otimes V} = (\theta_W\otimes\theta_V)c_{V,W}c_{W,V} = \theta'_{W\otimes V} = (\theta'_W\otimes\theta'_V)c_{V,W}c_{W,V}$ and $\theta_V = \theta'_V$, we deduce that $\theta_W = \theta'_W\,\forall W\in\mathscr{C}^H$, i.e. the family θ_V is a twist. □

Lemma 2.4.6. *Let $\mu = (\mu_1, \mu_2) \in \mathbb{C} \times \mathbb{C}$, then the value of the twist θ_{V_μ} on a simple module V_μ with highest weight μ is $\xi^{-\ell\mu_1 - 2\mu_2(1+\mu_1+\mu_2)} \operatorname{Id}_{V_\mu}$. That is,*

$$\theta_{V_\mu} = \xi^{-\ell\mu_1 - 2\mu_2(1+\mu_1+\mu_2)} \operatorname{Id}_{V_\mu} = -\xi^{-2(\alpha_2^2+\alpha_1\alpha_2)} \operatorname{Id}_{V_\mu}$$

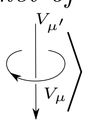
where $\alpha = (\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\ell}{2})$.

Proof. By the proof of Proposition 2.4.5, $\theta_{V_\mu} = \xi^{-\ell\mu_1 - 2\mu_2(1+\mu_1+\mu_2)} \operatorname{Id}_{V_\mu}$. $\square$

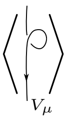
The category $\mathscr{C}^H$ is a braided pivotal category with a twist, i.e. $\mathscr{C}^H$ is a ribbon category.

Let $\mathcal{T}$ be the ribbon category of $\mathscr{C}^H$-colored oriented ribbon graphs in the sense of Turaev [47].
The set of morphisms $\mathcal{T}(((V_1, \pm), ..., (V_n, \pm)), ((W_1, \pm), ..., (W_n, \pm)))$ is a space of linear combinations of $\mathscr{C}$-colored ribbon graphs. The ribbon Reshetikhin-Turaev functor $F : \mathcal{T} \to \mathscr{C}^H$ is defined by the Penrose graphical calculus.

Definition 2.4.7. *If $T \in \mathcal{T}((V_\mu, +), (V_\mu, +))$ where V_μ is a simple weight module of $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$, then $F(T) = x. \operatorname{Id}_{V_\mu} \in \operatorname{End}_{\mathcal{U}_\xi^H\mathfrak{sl}(2|1)}(V_\mu)$ for $x \in \mathbb{C}$. We define the bracket of T by $\langle T \rangle = x$. For example, if $V_\mu, V_{\mu'} \in \mathscr{C}^H$, we define*

$S'(V_\mu, V_{\mu'}) = \Big\langle$  $\Big\rangle$.

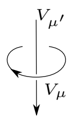
We write $S'(\mu, \mu')$ for $S'(V_\mu, V_{\mu'})$.

Another example is the bracket of the twist $\Big\langle$  $\Big\rangle = -\xi^{-2(\alpha_2^2+\alpha_1\alpha_2)}$, $(\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\ell}{2})$.

Proposition 2.4.8. *Let $V = V_\mu$ be a typical module, $V' = V_{\mu'}$ be a simple module, then*

$$S'(\mu, \mu') = \xi^{-4\alpha_2\alpha_2' - 2(\alpha_2\alpha_1' + \alpha_1\alpha_2')} \frac{\{\ell\alpha_1'\}\{\alpha_2'\}\{\alpha_2' + \alpha_1'\}}{\{\alpha_1'\}}$$

where $\alpha = (\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\ell}{2}), \alpha' = (\alpha_1', \alpha_2') = (\mu_1' - \ell + 1, \mu_2' + \frac{\ell}{2})$.

Proof. Let $S = S(\mu, \mu') \in \operatorname{End}_{\mathbb{C}}(V_{\mu_1', \mu_2'})$ be the endomorphism determined by the diagram . We have

$$\begin{aligned} S'(\mu, \mu') \operatorname{Id}_{V_{\mu'}} &= (\operatorname{Id}_{V'} \otimes \overleftarrow{\operatorname{ev}}_V)(c_{V,V'} \otimes \operatorname{Id}_{V^*})(c_{V',V} \otimes \operatorname{Id}_{V^*})(\operatorname{Id}_{V'} \otimes \overrightarrow{\operatorname{coev}}_V) \\ &= X_1 X_2 X_3 X_4. \end{aligned}$$

The definition gives us
$X_4(w'_{0,0,0}) = \sum_{\rho,\sigma,p} w'_{0,0,0} \otimes w_{\rho,\sigma,p} \otimes w^*_{\rho,\sigma,p}$ and
$X_3X_4(w'_{0,0,0}) = \sum_{\rho,\sigma,p}(\tau \circ \mathcal{R})(w'_{0,0,0} \otimes w_{\rho,\sigma,p}) \otimes w^*_{\rho,\sigma,p}$.
$\mathcal{K}(w'_{0,0,0} \otimes w_{\rho,\sigma,p}) = \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} w'_{0,0,0} \otimes w_{\rho,\sigma,p}$.
$\mathcal{R}(w'_{0,0,0} \otimes w_{\rho,\sigma,p}) = \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} w'_{0,0,0} \otimes w_{\rho,\sigma,p}$.
So

$$X_3X_4(w'_{0,0,0}) = \sum_{\rho,\sigma,p} \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} \cdot w_{\rho,\sigma,p} \otimes w'_{0,0,0} \otimes w^*_{\rho,\sigma,p}.$$

$$X_2X_3X_4(w'_{0,0,0}) = \sum_{\rho,\sigma,p} \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} \cdot (\tau \circ \mathcal{R})(w_{\rho,\sigma,p} \otimes w'_{0,0,0}) \otimes w^*_{\rho,\sigma,p}.$$

Furthermore, the element $(\check{\mathcal{R}} - 1)(w_{\rho,\sigma,p} \otimes w'_{0,0,0}) \in V_{\mu_1,\mu_2} \otimes V_{\mu'_1,\mu'_2}$ is a sum of vectors of the form $v' \otimes w'$ where w' is a weight vector of $V_{\mu'_1,\mu'_2}$ and v' is a weight vector of V_{μ_1,μ_2} which has a higher weight than $w_{\rho,\sigma,p}$.

$$X_2X_3X_4(w'_{0,0,0}) = \sum_{\rho,\sigma,p} (\xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} \cdot w'_{0,0,0} \otimes w_{\rho,\sigma,p} \otimes w^*_{\rho,\sigma,p} + \sum_k w'_k \otimes v'_k \otimes z_k).$$

$$\begin{aligned}
&X_1X_2X_3X_4(w'_{0,0,0}) \\
&= \sum_{\rho,\sigma,p} \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)} w'_{0,0,0} \otimes (-1)^{\rho+\sigma} w^*_{\rho,\sigma,p}(\phi_0 w_{\rho,\sigma,p}) \\
&= \sum_{\rho,\sigma,p} \xi^{-\mu'_1(\mu_2+\sigma+p)-\mu'_2(\mu_1+\rho-\sigma-2p)-2\mu'_2(\mu_2+\sigma+p)-\ell\mu_1-2(\mu_2+\sigma+p)} w'_{0,0,0} \\
&= \xi^{-(2\mu_2+\mu_1+1)(2\mu'_2+\mu'_1+1)+(\mu_1+1)(\mu'_1+1)-\ell(\mu'_1+\mu_1+1)} \frac{\{\ell(\mu'_1+1)\}\{\mu'_2\}\{\mu'_2+\mu'_1+1\}}{\{\mu'_1+1\}} w'_{0,0,0} \\
&= \xi^{-4\alpha_2\alpha'_2-2(\alpha_2\alpha'_1+\alpha_1\alpha'_2)} \frac{\{\ell\alpha'_1\}\{\alpha'_2\}\{\alpha'_2+\alpha'_1\}}{\{\alpha'_1\}} w'_{0,0,0}.
\end{aligned}$$

By the definition $S(\mu,\mu')(w'_{0,0,0}) = S'(\mu,\mu')w'_{0,0,0}$, we deduce the proposition. □

Definition 2.4.9. *If* $\mu = (\mu_1,\mu_2) \in \left(\mathbb{C}\backslash\frac{1}{2}\mathbb{Z} \cup (-1+\frac{\ell}{2}\mathbb{Z})\right) \times \mathbb{C}\backslash\frac{\ell}{2}\mathbb{Z}$ *and* $\mu_2 + \mu_1 + 1 \in \mathbb{C}\backslash\frac{\ell}{2}\mathbb{Z}$, *we define*

$$d(\mu) = \frac{\{\mu_1+1\}}{\ell\{\ell\mu_1\}\{\mu_2\}\{\mu_2+\mu_1+1\}} = \frac{\{\alpha_1\}}{\ell\{\ell\alpha_1\}\{\alpha_2\}\{\alpha_1+\alpha_2\}},$$

so there is a symmetry

$$d(\mu')S'(\mu,\mu') = d(\mu)S'(\mu',\mu).$$

2.4.5 Semi-simplicity of category $\mathscr{C}^H$

Remember that $G = \mathbb{C}/\mathbb{Z}\times\mathbb{C}/\mathbb{Z}$ and $G_s = \{\overline{g}\in G$ *such that* $\exists\ V\in\mathscr{C}^H_g$ *simple and atypical*$\}$. Recall that we denote with a bar a module with opposite parity. Then if $V\in\mathscr{C}^H$, $\overline{V}\simeq V\otimes\overline{\mathbb{I}}$.

Lemma 2.4.10. *If $\mathscr{C}^H_{\overline{\mu}}$ is semi-simple, then a module of $\mathscr{C}^H_{\overline{\mu}}$ is determined up to an isomorphism and parity by its character: let $V = V_1\oplus ...\oplus V_m$ be a decomposition of V into simple modules and let V' be a module with the same character then*

$$V'\simeq V_1\otimes\varepsilon_1\oplus ...\oplus V_m\otimes\varepsilon_m$$

where $\varepsilon_i\in\{\mathbb{I},\overline{\mathbb{I}}\}$ for $1\leq i\leq m$.

The above lemma and the character of representation $V_{\mu_1,\mu_2}\otimes V_{\mu'_1,\mu'_2}$ gives us the following theorem.

Theorem 2.4.11. *Let $V_\mu, V_{\mu'}$ be two typical modules. If $\overline{\mu+\mu'}\notin G_s$ then*

$$\begin{aligned}V_{\mu_1,\mu_2}\otimes V_{\mu'_1,\mu'_2} = \oplus_{k=0}^{\ell-1}(&V_{\mu_1+\mu'_1-2k,\mu_2+\mu'_2+k}\oplus\overline{V}_{\mu_1+\mu'_1-2k+1,\mu_2+\mu'_2+k}\\ &\oplus\overline{V}_{\mu_1+\mu'_1-2k,\mu_2+\mu'_2+k+1}\oplus V_{\mu_1+\mu'_1-2k-1,\mu_2+\mu'_2+k+1})\end{aligned}\tag{2.4.15}$$

where $\overline{V}$ is the module V with opposite parity.

Proof. According to the formula (2.4.7), we have

$$\begin{aligned}\chi_{V^{\mathfrak{sl}(2|1)}_{\mu_1,\mu_2}\otimes V^{\mathfrak{sl}(2|1)}_{\mu'_1,\mu'_2}} &= \chi_{V^{\mathfrak{sl}(2|1)}_{\mu_1,\mu_2}}\chi_{V^{\mathfrak{sl}(2|1)}_{\mu'_1,\mu'_2}}\\ &= X_1^{\mu_1+\mu'_1}X_2^{\mu_2+\mu'_2}\frac{1-x^\ell}{1-x}(1+X_1)(1+X_1x)\sum_{k=0}^{\ell-1}(X_1^{-2}X_2)^k(1+X_1+X_2+X_1^{-1}X_2)\\ &= \frac{1-x^\ell}{1-x}(1+X_1)(1+X_1x)\sum_{k=0}^{\ell-1}X_1^{\mu_1+\mu'_1-2k}X_2^{\mu_2+\mu'_2+k}+X_1^{\mu_1+\mu'_1-2k+1}X_2^{\mu_2+\mu'_2+k}\\ &\quad+X_1^{\mu_1+\mu'_1-2k}X_2^{\mu_2+\mu'_2+k+1}+X_1^{\mu_1+\mu'_1-2k-1}X_2^{\mu_2+\mu'_2+k+1}.\end{aligned}$$

The analysis of parity of highest weight vectors allows to conclude. ☐

Remark 2.4.12. *Not all terms in the decomposition of the above theorem are distinct.*

We defined a graduation $\mathscr{C}^H = \bigoplus_{\overline{\mu}\in G} \mathscr{C}^H_{\overline{\mu}}$. Let `Proj` be the subcategory of $\mathscr{C}^H$ containing projective modules, `Proj` is an ideal (see [12]), i.e. `Proj` is closed under retracts and absorbent for the tensor product. We have the following proposition.

Proposition 2.4.13. *For $\overline{\mu} \in G$, the three conditions below are equivalent*

1. *All the simple $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-modules of $\mathscr{C}_{\overline{\mu}}$ are projective.*
2. *The category $\mathscr{C}_{\overline{\mu}}$ is semi-simple.*
3. *The $\mathbb{C}$-superalgebra of finite dimension $\mathcal{U}/(k_1^\ell - \xi^{\ell\overline{\mu_1}}, k_2^\ell - \xi^{\ell\overline{\mu_2}})$ is semi-simple where $\mathcal{U} = \mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$.*

Proof. The equivalence is classic knowing that $\mathscr{C}_{\overline{\mu}}$ is also a category of the $\mathcal{U}/(k_1^\ell - \xi^{\ell\overline{\mu_1}}, k_2^\ell - \xi^{\ell\overline{\mu_2}})$-modules. □

Theorem 2.4.14.
1. *If $\overline{\mu} \in G\backslash G_s$ then $\mathscr{C}^H_{\overline{\mu}}$ is semi-simple.*
2. *A typical $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-module is projective.*

We select and fix a $\overline{\mu} \in G\backslash G_s$, denote $\mu_i = (\mu_1 + i_1, \mu_2 + i_2) \in \overline{\mu}$, $i_1, i_2 = 0, 1, ..., \ell - 1$, that is $\mu_i \in \{(\mu_1 + i_1, \mu_2 + i_2) : \ i_1, i_2 = 0, 1, ..., \ell - 1\}$. We have the two following lemmas.

Lemma 2.4.15. *For all $\mu_i, \mu_j \in \overline{\mu} : \mu_i \neq \mu_j$ there exists $z_{ij} \in \mathcal{Z}$ such that $\chi_{\mu_i}(z_{ij}) \neq \chi_{\mu_j}(z_{ij})$ where $\chi_{\mu_i}(z_{ij}) \in \mathbb{C}$ is defined by $\rho_{\mu_i}(z_{ij}) = \chi_{\mu_i}(z_{ij})\,\mathrm{Id}_{V_{\mu_i}}$.*

Proof. We consider $\mu = (\mu_1, \mu_2), \mu' = (\mu_1 + k, \mu_2 + m)$ $k, m = 0, 1, ..., \ell - 1$. We suppose that $\forall\ z \in \mathcal{Z} : \chi_\mu(z) = \chi_{\mu'}(z)$. Consider the central elements C_p where $p \in \mathbb{Z}$ (see [1]). We have

$$\chi_\mu(C_p) = (\xi - \xi^{-1})^2\xi^{(2p-1)(\mu_1+2\mu_2)}[\mu_2][\mu_2 + \mu_1 + 1],$$
$$\chi_{\mu'}(C_p) = (\xi - \xi^{-1})^2\xi^{(2p-1)(\mu_1+2\mu_2+k+2m)}[\mu_2 + m][\mu_2 + \mu_1 + k + m + 1].$$

Because $\chi_\mu(C_p) = \chi_{\mu'}(C_p)$ and $[\mu_2][\mu_2 + \mu_1 + 1] \neq 0$, we deduce that

$$\begin{cases} \frac{\chi_\mu(C_{p+1})}{\chi_\mu(C_p)} = \frac{\chi_{\mu'}(C_{p+1})}{\chi_{\mu'}(C_p)} \\ \chi_\mu(C_p) = \chi_{\mu'}(C_p). \end{cases}$$

This is equivalent to

$$\begin{cases} \xi^{2(\mu_1+2\mu_2)} = \xi^{2(\mu_1+2\mu_2+k+2m)} \\ \xi^{(2p-1)(\mu_1+2\mu_2)}[\mu_2][\mu_2 + \mu_1 + 1] = \xi^{(2p-1)(\mu_1+2\mu_2+k+2m)}[\mu_2 + m][\mu_2 + \mu_1 + k + m + 1], \end{cases}$$

which implies

$$2(k + 2m) = 0 \text{ (modulo } \ell\mathbb{Z}) \tag{2.4.16}$$

and

$$[\mu_2][\mu_2+\mu_1+1] = \xi^{k+2m}[\mu_2+m][\mu_2+\mu_1+k+m+1]. \tag{2.4.17}$$

Because ℓ odd, Equation (2.4.16) implies $k+2m=0$ (modulo $\ell\mathbb{Z}$) $\Leftrightarrow$ $k+m=-m$ (modulo $\ell\mathbb{Z}$). On the other hand, Equation (2.4.17) is equivalent to $[a][b]=[a+m][b-m] \Leftrightarrow -[a-b+m][m]=0 \Leftrightarrow [-\mu_1-1+m][m]=0 \Rightarrow m=0$ where $a=\mu_2, b=\mu_1+\mu_2+1$. Because $m=0$, we have $k=0$ (modulo $\ell\mathbb{Z}$) $\Rightarrow k=0$. □

Lemma 2.4.16. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, I be a finite set and consider a family of $\mathbb{C}$-linear functions $a_i:\mathcal{V}\to\mathbb{C}$, $i\in I$. If for all $i\neq j$ $\exists\, u_{ij}\in\mathcal{V}$ such that $a_i(u_{ij})\neq a_j(u_{ij})$, then it exists $u_0\in\mathcal{V}$ such that $\forall\, i\neq j$ $a_i(u_0)\neq a_j(u_0)$.*

Proof. We set $u=\sum_{i\neq j}x_{ij}u_{ij}\in\mathcal{V}$ with $x_{ij}\in\mathbb{C}$, $i,j\in I$. We denote $x=(x_{ij})\in\mathbb{C}^N$. We consider the set $X=\{x\in\mathbb{C}^N\ \exists i\neq j\ a_i(u)=a_j(u)\}=\{x\in\mathbb{C}^N:\ \sum_{i\neq j}(a_i(u_{ij})-a_j(u_{ij}))x_{ij}=0\}$, this is a finite reunion of hyperplanes of $\mathbb{C}^N$. This proves that $\exists x\notin X$ and this x does not have the above property. That is, it exists $u_0\in\mathcal{V}$ such that $a_i(u_0)\neq a_j(u_0)$ for all $i\neq j$. □

Now we introduce a new basis of module V_μ. This basis diagonalize the operator Ω in the proof of Theorem 2.4.14. We set

$$w'_{\rho,\sigma,p}=\begin{cases} w_{\rho,\sigma,p} & \text{if } \rho=\sigma=0,1\\ f_1^p w_{1,0,0} & \text{if } \rho=1,\sigma=0\\ e_1^{(\ell-1)-p}w_{0,1,r-1} & \text{if } \rho=0,\sigma=1\end{cases}$$

where $p=0,...,\ell-1$. For the basis $\{w'_{\rho,\sigma,p}\}$ we have the actions

$$\begin{aligned}
k_1w'_{\rho,\sigma,p}&=\xi^{\mu_1+\rho-\sigma-2p}w'_{\rho,\sigma,p},\\
k_2w'_{\rho,\sigma,p}&=\xi^{\mu_2+\sigma+p}w'_{\rho,\sigma,p},\\
e_1w'_{1,0,p}&=[p][\mu_1+2-p]w'_{1,0,p-1},\\
f_1w'_{1,0,p}&=w'_{1,0,p+1},\\
e_1w'_{0,1,p}&=w'_{0,1,p-1},\\
f_1w'_{0,1,p}&=[p+1][\mu_1-(p+1)]w'_{0,1,p+1}.
\end{aligned}$$

Proof of Theorem 2.4.14. We begin to show that $\mathscr{C}_{\overline{\mu}}$ is semi-simple. We set $\mathcal{A}=\mathcal{U}/(k_1^\ell-\xi^{\ell\overline{\mu_1}},k_2^\ell-\xi^{\ell\overline{\mu_2}})$. The density theorem implies that the application $\rho:\mathcal{A}\to\prod_{\mu_i}\mathrm{End}(V_{\mu_i})\cong\prod_{i=1}^{\ell^2}\mathcal{M}_{4\ell}(\mathbb{C})$ is surjective. We give here an elementary proof.

By Lemma 2.4.15 and 2.4.16, it exists an element $z \in \mathcal{Z}$ such that $\forall\ \mu_i \neq \mu_j\ \chi_{\mu_i}(z) \neq \chi_{\mu_j}(z)$ and we set $z_i = \chi_{\mu_i}(z)\ i = 1, ..., \ell^2$ and we introduce the ideal $J = \prod_{i=1}^{\ell^2}(z - z_i)\mathcal{A}$.

Firstly, we consider the representation $\rho : \mathcal{A}/(z - z_i) \to \mathrm{End}_{\mathbb{C}}(V_{\mu_i})$. We will prove that ρ is a surjection. We have $\mathrm{End}_{\mathbb{C}}(V_{\mu_i}) \cong \mathcal{M}_{4\ell}(\mathbb{C})$. We consider the elements $\Omega = \frac{k_1\xi + k_1^{-1}\xi^{-1}}{\{1\}^2} + f_1e_1 = \frac{k_1\xi^{-1} + k_1^{-1}\xi}{\{1\}^2} + e_1f_1, c = k_1k_2^2, k_1$ in $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. The actions of these elements on the basis $w'_{\rho,\sigma,p}$ are defined by

$$\begin{aligned}
&\Omega w'_{0,0,p} = (\xi^{\mu_1+1} + \xi^{-\mu_1-1})w'_{0,0,p}, \Omega w'_{1,1,p} = (\xi^{\mu_1+1} + \xi^{-\mu_1-1})w'_{1,1,p},\\
&\Omega w'_{0,1,p} = \frac{\xi^{\mu_1} + \xi^{-\mu_1}}{\{1\}^2}w'_{0,1,p}, \Omega w'_{1,0,p} = \frac{\xi^{\mu_1+2} + \xi^{-\mu_1-2}}{\{1\}^2}w'_{1,0,p},\\
&cw'_{\rho,\sigma,p} = \xi^{\mu_1+2\mu_2+\rho+\sigma}w'_{\rho,\sigma,p},\\
&k_1w'_{\rho,\sigma,p} = \xi^{\mu_1+\rho-\sigma-2p}w'_{\rho,\sigma,p}.
\end{aligned}$$

We now check that for all $w'_{\rho,\sigma,m} \neq w'_{\rho',\sigma',j}\ \exists\ u \in \{\Omega, c, k_1\}$ such that $\chi_{\mu_i}^{\rho,\sigma,m}(u) \neq \chi_{\mu_i}^{\rho',\sigma',j}(u)$ where $\rho(u)w'_{\rho,\sigma,m} = \chi_{\mu_i}^{\rho,\sigma,m}(u)w'_{\rho,\sigma,m}$ for $\rho, \sigma, \rho', \sigma' \in \{0, 1\},\ m, j \in \{0, ..., \ell - 1\}$. Indeed, if $\rho + \sigma \neq \rho' + \sigma'$ then we select $u = c$ and we have $cw'_{\rho,\sigma,m} \neq cw'_{\rho',\sigma',j}$. If $\rho+\sigma = \rho'+\sigma'$ then we consider two cases: if $(\rho, \sigma) \neq (\rho', \sigma')$ we select $u = \Omega$ and $\Omega w'_{\rho,\sigma,m} \neq \Omega w'_{\rho',\sigma',j}$; if $(\rho, \sigma) = (\rho', \sigma')$ we select $u = k_1$ and we have $k_1w'_{\rho,\sigma,m} \neq k_1w'_{\rho',\sigma',j}$ because $m \neq j$.

By Lemma 2.4.16 it exists a vector $u_0 \in \mathbb{C}\langle\Omega, c, k_1\rangle$-space generated by the elements Ω, c, k_1 such that $\chi_{\mu_i}^{\rho,\sigma,m}(u_0) \neq \chi_{\mu_i}^{\rho',\sigma',j}(u_0)$ for all $w'_{\rho,\sigma,m} \neq w'_{\rho',\sigma',j}$. The matrix B determined by the application $\rho(u_0)$ is a diagonal matrix which has 4ℓ different eigenvalues. The image of the projection on the i-th eigenspace of B is the matrix $E_{ii},\ i = 1, ..., 4\ell$. Hence the matrix E_{ii} is in the image of ρ.

For $i \in \{1, ..., \ell^2\}, j \in \{1, ..., 4\ell\}$ we have $\rho(\mathcal{A}/(z - z_i))(v_j) \subset V_{\mu_i}$ (here we denote v_j the j-th vector of the basis) and V_{μ_i} is simple. Thus we deduce $\rho(\mathcal{A}/(z - z_i))(v_j) = V_{\mu_i}$. This proves that it exists $a_0 \in \mathcal{A}/(z - z_i)$ such that $\rho(a_0)(v_j) = v_n\ \forall n \in \{1, ..., 4\ell\}$.

The endomorphism $\rho(a_0)$ determines the matrix $(\rho(a_0))$ where $\rho(a_0)_{jn} = 1$. The matrix E_{jn} is equal to $E_{jj}\rho(a_0)_{jn}E_{nn}$, i.e. the matrix E_{jn} is the image of an element in $\mathcal{A}/(z - z_i)$. So the application ρ is a surjection. This implies that the application $\prod_{i=1}^{\ell^2}\mathcal{A}/(z - z_i) \to \prod_{i=1}^{\ell^2}\mathcal{M}_{4\ell}(\mathbb{C})$ is surjective.

Secondly, the composition $\prod_{i=1}^{\ell^2}\mathcal{A}/(z - z_i) \to \mathcal{A}/J \to \prod_{i=1}^{\ell^2}\mathcal{A}/(z - z_i)$ is the identity. Thus, the application $\mathcal{A}/J \to \prod_{i=1}^{\ell^2}\mathcal{A}/(z - z_i)$ is surjective. We deduce a series of surjections $\mathcal{A} \twoheadrightarrow \mathcal{A}/J \twoheadrightarrow \prod_{i=1}^{\ell^2}\mathcal{A}/(z - z_i) \twoheadrightarrow \prod_{i=1}^{\ell^2}\mathcal{M}_{4\ell}(\mathbb{C})$, this sequence determines the surjection $\mathcal{A} \twoheadrightarrow \prod_{i=1}^{\ell^2}\mathcal{M}_{4\ell}(\mathbb{C})$.

Furthermore, the two algebras $\mathcal{A}$ and $\prod_{i=1}^{\ell^2}\mathcal{M}_{4\ell}(\mathbb{C})$ have the same dimension $16\ell^4$. This implies that this surjection is an isomorphism. This demonstrates that $\mathcal{A}$ is semi-simple. The category $\mathscr{C}_{\overline{\mu}}$ is also semi-simple.

Now we prove that $\mathscr{C}_{\overline{\mu}}^H$ is semi-simple. Let V^H be a module in $\mathscr{C}_{\overline{\mu}}^H$. Set $W = \operatorname{Ker} e_1 \cap \operatorname{Ker} e_2 \cap \operatorname{Ker} e_3$, it is a vector space of the highest weight vectors (the weights for (h_1, h_2)). We call $\{v_j\}_{j=1}^n$ a basis of weight vectors of W, we have $h_i v_j = \mu_j^i v_j, i = 1, 2$. So each v_j generates a $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-module V_j,

$$V_j = \mathcal{U}_\xi^H\mathfrak{sl}(2|1).v_j = \mathcal{U}_\xi\mathfrak{sl}(2|1).v_j = \mathcal{U}_{-}.v_j$$

where $\mathcal{U}_{-} = \operatorname{Alg}\langle f_1, f_2, f_3\rangle \subset \mathcal{U}_\xi\mathfrak{sl}(2|1)$ and $\dim(\mathcal{U}_{-}) = 4\ell$. Thus $\dim(V_j) \leq 4\ell$ and V_j is simple (because there is no module in $\mathscr{C}_{\overline{\mu}}^H$ of dimension strictly between 0 and 4ℓ).

Set $V' = \sum_{i=1}^n V_i \subset V^H$. We can write $V^H = V' \oplus V''$ as a $\mathcal{U}_\xi\mathfrak{sl}(2|1)$-module. However $W \subset V'$ which implies $V'' = 0$ (because there is no highest weight vector in V'') and $V^H = V' = \sum_{i=1}^n V_i$. Because the V_i are simple, so $V^H = \bigoplus_{i\in I} V_i$ where $I \subset \{1, ..., n\}$. Thus V^H is semi-simple.

For the second assertion (2), if $V \in \mathscr{C}_{\overline{\mu}}^H$ and $\mathscr{C}_{\overline{\mu}}^H$ is semi-simple, then V is projective. If not, (2) follows from $S'(V_\mu, V) \neq 0$ where V_μ is any projective typical module which implies that V is a direct factor of $V_\mu \otimes V \otimes V_\mu^* \in \mathtt{Proj}$. This implies that V is a projective module. □

2.5 Modified traces on projective modules

In this section we recall the definition of an ambidextrous module presented by N. Geer, B. Patureau-Mirand and V. Turaev in [17] and of a modified trace on an ideal in a category introduced by N. Geer, J. Kujawa and B. Patureau-Mirand in [12]. We prove there exists a modified trace on the ideal of projective modules of the category $\mathscr{C}^H$. The modified trace allows us to construct an invariant of embedded graphs in Theorem 2.5.5.

2.5.1 Ambidextrous modules

For each object V of the category $\mathscr{C}$ and any endomorphism f of $V \otimes V$ set

$$\operatorname{ptr}_R(f) = (\operatorname{Id}_V \otimes \overleftarrow{\operatorname{ev}}_V) \circ (f \otimes \operatorname{Id}_{V^*}) \circ (\operatorname{Id}_V \otimes \overrightarrow{\operatorname{coev}}_V) \in \operatorname{End}(V),$$
$$\operatorname{ptr}_L(f) = (\overrightarrow{\operatorname{ev}}_V \otimes \operatorname{Id}_V) \circ (\operatorname{Id}_{V^*} \otimes f) \circ (\overleftarrow{\operatorname{coev}}_V \otimes \operatorname{Id}_V) \in \operatorname{End}(V).$$

In the ribbon category $\mathscr{C}^H$ of nilpotent weight $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-modules, we say that a module V is *ambidextrous* if V simple and $\operatorname{ptr}_L(f) = \operatorname{ptr}_R(f)$ for all $f \in \operatorname{End}(V \otimes V)$ (see [17]).

Theorem 2.5.1. *Each typical module V_μ of category $\mathscr{C}^H$ is an ambidextrous module.*

Proof. We will prove this theorem in two steps:
Step 1. Proving the existence of two nonzero $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-invariant vectors x_-w_+ and x_+w_-.
Step 2. Applying Theorem 3.1.3 [13] gives us the affirmation that V_μ is ambidextrous.

Call v_+, v'_+ the highest weight vectors of V_μ, V_μ^* and v_-, v'_- the lowest weight vectors of V_μ, V_μ^*. Set $x_- = f_2f_3f_1^{\ell-1}, x_+ = e_2e_3e_1^{\ell-1}, w_+ = v_+\otimes v'_+, w_- = v_-\otimes v'_-$. We will prove that the two vectors x_-w_+ and x_+w_- are $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-invariant.

We consider the actions of generator elements e_i, h_i, f_i on x_-w_+. The highest weight vector (resp. lowest) of V_μ is $v_+ = w_{0,0,0}$ (resp. $v_- = w_{1,1,\ell-1}$). The highest weight vector (resp. lowest) of V_μ^* is $v'_+ = w^*_{1,1,\ell-1}$ (resp. $v'_- = w^*_{0,0,0}$).

The weight of vector $w_+ = v_+\otimes v'_+$ is equal to the sum of the weights of v_+ and v'_+. That is weight$(w_+) = (\mu_1,\mu_2) + (-\mu_1+2\ell-2, -\mu_2-\ell) = (2\ell-2,-\ell)$. Furthermore, weight$(x_-w_+) =$ weight$(f_2f_3f_1^{\ell-1}w_+) =$ weight$(f_2f_1f_2f_1^{\ell-1}w_+) = -\ell$weight$(e_1)-2$weight$(e_2)+$weight$(w_+) = -\ell(2,-1)-2(-1,0)+(2\ell-2,-\ell) = (0,0)$. It implies that $h_ix_-w_+ = 0$.

We also have the relations below between the generator elements in $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ (see (B1) [1]):

$$f_1f_2^\rho f_3^\sigma f_1^p = \xi^{\rho-\sigma}f_2^\rho f_3^\sigma f_1^{p+1} - \rho(1-\sigma)\xi^{-\rho}f_2^{\rho-1}f_3^{\sigma+1}f_1^p,$$
$$f_2f_2^\rho f_3^\sigma f_1^p = (1-\rho)f_2^{\rho+1}f_3^\sigma f_1^{p+1},$$
$$[e_1, f_2^\rho f_3^\sigma f_1^p] = \sigma(1-\rho)(-1)^\sigma f_2^{\rho+1}f_3^{\sigma-1}f_1^p\xi^{h_1-2p+1} + [p]f_2^\rho f_3^\sigma f_1^{p-1}[h_1-p+1],$$
$$e_2f_2^\rho f_3^\sigma f_1^p - (-1)^{\rho+\sigma}f_2^\rho f_3^\sigma f_1^p e_2 = \rho f_2^{\rho-1}f_3^\sigma f_1^p[h_2+p+\sigma] + \sigma(-1)^\rho f_2^\rho f_3^{\sigma-1}f_1^{p+1}\xi^{-h_2-p}$$

where $(p,\rho,\sigma)\in\mathbb{N}\times\{0,1\}\times\{0,1\}$. With the above relations, it is easy to check $f_ix_-w_+ = 0$.

The fourth relation above gives us $e_2f_2f_3f_1^{\ell-1} - f_2f_3f_1^{\ell-1}e_2 = f_3f_1^{\ell-1}[h_2+\ell]$. Because $e_2(v_+\otimes v'_+) = 0$ and $[h_2+\ell](v_+\otimes v'_+) = 0$, we deduce $e_2x_-w_+ = 0$.

The third relation gives $[e_1, f_2f_3f_1^{\ell-1}] = [\ell-1]f_2f_3f_1^{\ell-2}[h_1-\ell+2]$. Because $e_1(v_+\otimes v'_+) = 0$ and $[h_1-\ell+2](v_+\otimes v'_+) = 0$, we deduce $e_1x_-w_+ = 0$.

Consequently, we conclude that x_-w_+ is an $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-invariant vector. The demonstration that the vector x_+w_- is $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$-invariant is analogous using the relations obtained by applying the automorphism ω of superalgebra $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ where $\omega(e_i) = (-1)^{\deg e_i}f_i, \omega(f_i) = (-1)^{\deg f_i}e_i, \omega(k_i) = k_i^{-1}, \omega(h_i) = -h_i, i = 1,2$.

Furthermore $\Delta x_- = x_- \otimes 1+$ a sum of tensor products of two elements of $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ with negative weight. Thus $\Delta x_- w_+$ contains the nonzero vector $x_- v_+ \otimes v'_+ = f_2 f_3 f_1^{\ell-1} v_+ \otimes v'_+ = w_{1,1,\ell-1} \otimes v'_+$. We conclude that the vector $x_- w_+$ is nonzero. Similarly, the vector $x_+ w_-$ is nonzero.

For step 2, we use the following results:

The decomposition of the tensor product $V \otimes V^*$ is a direct sum of indecomposable modules

$$V \otimes V^* = P_1 \oplus ... \oplus P_m.$$

The set of invariant vectors $w \in V \otimes V^*$ is in bijection with $\overrightarrow{\text{coev}}_V\ (\mathbb{C})$ because $\text{Hom}_{\mathscr{C}}(\mathbb{C}, V \otimes V^*) \cong \text{Hom}_{\mathscr{C}}(V, V) \cong \mathbb{C}$.

The vector w_+ (resp. w_-) is the highest weight vector (resp. lowest weight vector) of $V \otimes V^*$. Then there exists a unique integer k (resp. l) such that $w_+ \in P_k$ (resp. $w_- \in P_l$). The weight of w_+ (resp. w_-) is $\lambda_+ = (2\ell - 2, -\ell)$ (resp. $\lambda_- = (-2\ell + 2, \ell)$). Because $\lambda_- = -\lambda_+$ and $(V \otimes V^*)^* \simeq (V \otimes V^*)$, this implies $P_k^* \simeq P_l$.

In addition, $\overrightarrow{\text{coev}}_V\ (1) \in P_l, \overrightarrow{\text{coev}}_V\ (1) \in P_k$ because $x_+ P_l \subset P_l, x_- P_k \subset P_k$, then $P_k = P_l$. That is $P_k = P_k^*$. By Theorem 3.1.3 [13], it gives us the affirmation that V_μ ambidextrous. □

Remark 2.5.2. *All typical modules are projective and ambidextrous.*

2.5.2 Modified traces on the projective modules

Definition 2.5.3. *Let $\mathcal{I}$ be an ideal of $\mathscr{C}$* (see [12]). *The family of linear applications $t = (t_V : \text{End}_{\mathscr{C}}(V) \to \Bbbk)_{V \in \mathcal{I}}$ is a* trace (modified trace) *on $\mathcal{I}$ if it satisfies:*
$\forall U, V \in \mathcal{I}, \forall W \in \mathscr{C}$,

$$\forall f \in \text{Hom}_{\mathscr{C}}(U, V), \forall g \in \text{Hom}_{\mathscr{C}}(V, U), t_V(f \circ g) = t_U(g \circ f)$$

$$\forall f \in \text{End}_{\mathscr{C}}(V \otimes W),\ t_{V \otimes W}(f) = t_V(\text{ptr}_R(f)).$$

We also have

$$\forall f \in \text{End}_{\mathscr{C}}(W \otimes V),\ t_{W \otimes V}(f) = t_V(\text{ptr}_L(f)).$$

Given V as a typical module. The module V is ambidextrous and projective. This implies that the ideal generated by this module is $\mathcal{I}_V = \texttt{Proj}$ (see [12]). Hence the modified trace is also defined on non simple projective modules:

Theorem 2.5.4. *There exists a unique modified trace $t = \{t_P\}_{P \in \mathit{Proj}}$ on the ideal Proj of projective modules of $\mathscr{C}^H$,*

$$t_P : \mathrm{End}(P) \to \mathbb{C}, P \in \mathit{Proj}.$$

If $P = V_\mu$ is a typical module, then $t_{V_\mu}(f) = \langle f \rangle d(\mu), f \in \mathrm{End}(V_\mu)$, $d(\mu) = t_{V_\mu}(\mathrm{Id}_{V_\mu})$ is determined by Definition 2.4.9.

2.5.3 Invariants of embedded graphs

Recall that $\mathscr{C}^H$ is the $\mathbb{C}$-linear ribbon category of nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$, `Proj` is the ideal of projective modules and t is a trace on `Proj`.

We call $\mathscr{G}$ the set of $\mathscr{C}^H$-colored closed ribbon graphs, that are the $\mathscr{C}^H$-colored ribbon graphs in S^3. We have $\mathscr{G} \cong \mathrm{End}_{\mathcal{T}}(\emptyset)$.

We use the concept of a cutting presentation of $\mathscr{C}^H$-colored closed ribbon graph: If a diagram T represents a $\mathscr{C}^H$-colored ribbon graph which is an endomorphism of $\mathcal{T}$, its lower and upper parts are formed by the same sequences of k vertical colored strands. It is then possible, as for a braid of k strands, to consider the closure $\widehat{T}$ obtained by joining its k top vertices to its k bottom vertices by k parallel strands. This construction is actually the categorical trace in $\mathcal{T}$: we have $\widehat{T} = \mathrm{tr}_{\mathcal{T}}(T) \in \mathrm{End}_{\mathcal{T}}(\emptyset)$. We say that T is a cutting presentation with k strands of the closed graph $\widehat{T}$ and that $\widehat{T}$ is the closure of T (see [39]).

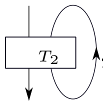
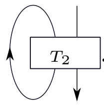
A closed graph T of $\mathcal{T}$ is said to be $\mathscr{C}^H$-colored *admissible* if there is at least one strand of T colored by $P \in$ `Proj`. Let $\mathscr{G}_a$ be the set of isotopy classes of $\mathscr{C}^H$-colored admissible ribbon graphs.

From the trace t on `Proj` we have the theorem below.

Theorem 2.5.5. *The application*

$$\begin{aligned} F' : \mathscr{G}_a &\to \mathbb{C} \\ \widehat{T} &\mapsto t_P(F(T)) \end{aligned}$$

is well defined. Here, $P \in \mathit{Proj}, T \in \mathrm{End}_{\mathcal{T}}((P,+))$ is a cutting presentation with one strand of $\widehat{T}$. That is to say the complex number $t_P(F(T))$ does not depend on the choice of T but only of the isotopy class of the $\mathscr{C}^H$-colored graph $\widehat{T}$.

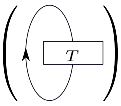
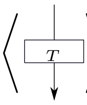
Proof. First, we select an edge of $\widehat{T}$ and cut, we have the graph T. Then, we select and cut a second edge of $\widehat{T}$, we have the graph T'. By cutting $\widehat{T}$ in both these places, one obtains a graph $T_2 \in \mathrm{End}_{\mathscr{G}_a}((P,+),(P',+))$ which is a presentation with two strands of $\widehat{T}$ and such that $T =$ , $T' =$ .

Finally we use the properties of the compatibility of trace t:

$$\begin{aligned} t_P(F(T)) &= t_P(\mathrm{ptr}_R(F(T_2))) = t_{P\otimes P'}(F(T_2)) \\ &= t_{P'}(\mathrm{ptr}_L(F(T_2))) = t_{P'}(F(T')). \end{aligned}$$

□

Remark 2.5.6. *In the case $P = V_\mu$ typical, we have*

$F'\left(\right.$  $\left.\right) = d(\mu)\left\langle\right.$  $\left.\right\rangle$.

The affirmation of the above theorem gives us a link invariant in the following corollary.

Corollary 2.5.7. *Let L be an oriented link with n ordered components then the application $F' : \{$admissible $\mathbb{C}^2$-coloring of $L\} \to \mathbb{C}$ determines a meromorphic function $f_L : \mathbb{C}^{2n} \to \mathbb{C}$ of the $2n$ complex parameters defining the coloring.*

2.6 Invariant of 3-manifolds

In the article [8] the authors constructed $\mathscr{C}$-decorated 3-manifold invariants where $\mathscr{C}$ is a ribbon category. In the previous section, it was proven that $\mathscr{C}^H$ is a ribbon category, this suggests we construct an invariant of $\mathscr{C}^H$-decorated 3-manifolds. We recall some concepts, definitions and results from [8].

2.6.1 Relative G-(pre)modular categories

Let $\mathscr{C}$ be a $\Bbbk$-linear ribbon category where $\Bbbk$ is a field. A set of objects of $\mathscr{C}$ is said to be commutative if for any pair $\{V, W\}$ of these objects, we have $c_{V,W} \circ c_{W,V} = \mathrm{Id}_{W\otimes V}$ and $\theta_V = \mathrm{Id}_V$. Let $(Z, +)$ be a commutative group. A *realization* of Z in $\mathscr{C}$ is a commutative set of objects $\{\varepsilon^t\}_{t\in Z}$ such that $\varepsilon^0 = \mathbb{I}, \mathrm{qdim}(\varepsilon^t) = 1$ and $\varepsilon^t \otimes \varepsilon^{t'} = \varepsilon^{t+t'}$ for all $t, t' \in Z$.

A realization of Z in $\mathscr{C}$ induces an action of Z on isomorphism classes of objects of $\mathscr{C}$ by $(t, V) \mapsto \varepsilon^t \otimes V$. We say that $\{\varepsilon^t\}_{t\in Z}$ is a *free realization* of Z in $\mathscr{C}$ if this action is free. This means that $\forall t \in Z\backslash\{0\}$ and for any simple object $V \in \mathscr{C}, V \otimes \varepsilon^t \not\simeq V$. We call *simple Z-orbit* the reunion of isomorphism classes of an orbit for this action.

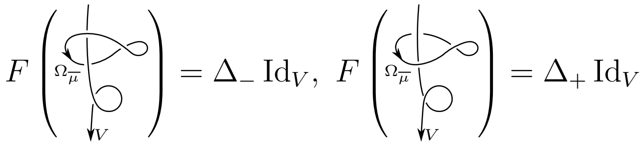
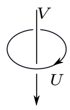


Figure 2.1 – $V \in \mathscr{C}_g$ and $\Omega_{\overline{\mu}}$ is a Kirby color of degree μ.

Definition 2.6.1 ([8]). *Let $(G, \times)$ and $(Z, +)$ be two commutative groups. A $\Bbbk$-linear ribbon category $\mathscr{C}$ is G-modular relative to $\mathcal{X}$ with modified dimension* d *and periodicity group Z if*

1. *the category $\mathscr{C}$ has a G-grading $\{\mathscr{C}_g\}_{g \in G}$,*
2. *the group Z has a free realization $\{\varepsilon^t\}_{t \in Z}$ in $\mathscr{C}_1$ (where $1 \in G$ is the unit),*
3. *there is a $\mathbb{Z}$-bilinear application $G \times Z \to \Bbbk^\times, (g,t) \mapsto g^{\bullet t}$ such that $\forall V \in \mathscr{C}_g, \forall t \in Z, c_{V,\varepsilon^t} \circ c_{\varepsilon^t,V} = g^{\bullet t} \operatorname{Id}_{\varepsilon^t \otimes V}$,*
4. *there exists $\mathcal{X} \subset G$ such that $\mathcal{X}^{-1} = \mathcal{X}$ and G cannot be covered by a finite number of translated copies of $\mathcal{X}$, in other words $\forall g_1, ..., g_n \in G, \cup_{i=1}^n (g_i \mathcal{X}) \neq G$,*
5. *for all $g \in G \setminus \mathcal{X}$, the category $\mathscr{C}_g$ is semi-simple and its simple objects are in the reunion of a finite number of simple Z-orbits,*
6. *there exists a nonzero trace* t *on ideal* `Proj` *of projective objects of $\mathscr{C}$ and* d *is the associated modified dimension,*
7. *there exists an element $g \in G \setminus \mathcal{X}$ and an object $V \in \mathscr{C}_g$ such that the scalar Δ_+ defined in Figure 2.1 is nonzero; similarly, there exists an element $g \in G \setminus \mathcal{X}$ and an object $V \in \mathscr{C}_g$ such that the scalar Δ_- defined in Figure 2.1 is nonzero,*
8. *the morphism $S(U,V) = F(H(U,V)) \neq 0 \in \operatorname{End}_{\mathscr{C}}(V)$, for all simple objects $U, V \in$* `Proj`*, where*

$$H(U,V) =$$


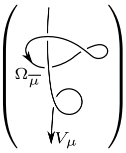
$$\in \operatorname{End}_{\mathscr{C}}((V,+)).$$

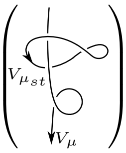
The category $\mathscr{C}^H$ of $\mathcal{U}_\xi^H \mathfrak{sl}(2|1)$-modules is G-modular relative to $\mathcal{X}$. Indeed, we have $\mathscr{C}^H$ being G-graded by $G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}$. We set $Z = \mathbb{Z} \times \mathbb{Z}$ and $\{\varepsilon^n\}_{n \in Z}$ the set of simple highest weight modules $n = (n_1\ell, n_2\ell)$, i.e. ε^n is a $\mathcal{U}_\xi^H \mathfrak{sl}(2|1)$-module of dimension 1 (with the basis $\{w\}$) determined by

$h_1 w = n_1 \ell w, h_2 w = n_2 \ell w, e_i w = f_i w = 0$. Because $c_{\varepsilon^m, \varepsilon^n} = \tau$ and $\theta_{\varepsilon^n} = \mathrm{Id}$, the two conditions (1) and (2) of Definition 2.6.1 are satisfied.

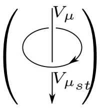
We consider a typical module V_μ. We have $c_{\varepsilon^n, V_\mu}(w \otimes w_{\rho,\sigma,p}) = \tau \circ \mathcal{R}(w \otimes w_{\rho,\sigma,p}) = \xi^{-n_1\ell\mu_2 - n_2\ell\mu_1 - 2n_2\ell\mu_2} w_{\rho,\sigma,p} \otimes w$. Next $c_{V_\mu,\varepsilon^n} \circ c_{\varepsilon^n, V_\mu}(w \otimes w_{\rho,\sigma,p}) = c_{V_\mu,\varepsilon^n}(\xi^{-n_1\ell\mu_2 - n_2\ell\mu_1 - 2n_2\ell\mu_2} w_{\rho,\sigma,p} \otimes w) = \xi^{-2n_1\ell\mu_2 - 2n_2\ell\mu_1 - 4n_2\ell\mu_2} w \otimes w_{\rho,\sigma,p} = \xi^{-2\ell(\mu_2 n_1 + (\mu_1 + 2\mu_2)n_2)} w \otimes w_{\rho,\sigma,p}$. So we can determine the $\mathbb{Z}$-bilinear application $G \times Z \to \mathbb{C}^\times, (\overline{\mu}, n) \mapsto \xi^{-2\ell(\mu_2 n_1 + (\mu_1 + 2\mu_2)n_2)}$ which satisfies $c_{V_\mu,\varepsilon^n} \circ c_{\varepsilon^n, V_\mu}(w \otimes w_{\rho,\sigma,p}) = \xi^{-2\ell(\mu_2 n_1 + (\mu_1+2\mu_2)n_2)} \mathrm{Id}_{\varepsilon^n \otimes V_\mu}(w \otimes w_{\rho,\sigma,p})$. This means that we have condition (3) of the definition. Condition (4) is also satisfied with $\mathcal{X} = G_s = \left\{\overline{0}, \overline{\tfrac{1}{2}}\right\} \times \mathbb{C}/\mathbb{Z} \cup \mathbb{C}/\mathbb{Z} \times \left\{\overline{0}, \overline{\tfrac{1}{2}}\right\} \cup \left\{(\overline{\mu_1}, \overline{\mu_2}) : \overline{\mu_1} + \overline{\mu_2} \in \left\{\overline{0}, \overline{\tfrac{1}{2}}\right\}\right\}$. It was proven that $\mathscr{C}_g^H$ is semi-simple for $g \in G \setminus G_s$ (Theorem 2.4.14) and $V_\mu \otimes \varepsilon^n \simeq V_{\mu + \ell n}$, i.e. the condition (5) is satisfied. Theorem 2.5.4 implies that condition (6) is true.

To compute Δ_-, we first use the graphical calculus

$F\Big($ 

$\Big) = \sum_{s,t=0}^{\ell-1} d(\mu_{st}) F\Big($ 

$\Big)$

$= \sum_{s,t=0}^{\ell-1} d(\mu_{st}) \left\langle \theta_{V_\mu}^{-1} \right\rangle \left\langle \theta_{V_{\mu_{st}}^*}^{-1} \right\rangle F\Big($ 

$\Big)$

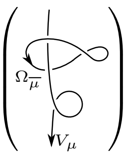
$$= \sum_{s,t=0}^{\ell-1} d(\mu_{st}) \left\langle \theta_{V_\mu}^{-1} \right\rangle \left\langle \theta_{V_{\mu_{st}}^*}^{-1} \right\rangle S'(\mu_{st}, \mu) \, \mathrm{Id}_{V_\mu}.$$

We have

$$\left\langle \theta_{V_\mu}^{-1} \right\rangle = -\xi^{2(\alpha_2^2 + \alpha_1\alpha_2)}, \left\langle \theta_{V_{\mu_{st}}^*}^{-1} \right\rangle = -\xi^{2((\alpha_2+t)^2 + (\alpha_1+s)(\alpha_2+t))}$$

and $S'(\mu_{st}, \mu) = \xi^{-4\alpha_2(\alpha_2+t) - 2(\alpha_2(\alpha_1+s) + \alpha_1(\alpha_2+t))} \dfrac{1}{\ell d(\mu)}$.

Thus

$F\Big($ 

$\Big) = \sum_{s,t=0}^{\ell-1} \dfrac{d(\mu_{st})}{\ell d(\mu)} \xi^{2(t^2+st)} \, \mathrm{Id}_{V_\mu}$

$$= \frac{1}{\ell d(\mu)\{\ell\alpha_1\}} \sum_{s,t=0}^{\ell-1} \frac{\{\alpha_1 + s\}}{\{\alpha_2 + t\}\{\alpha_1 + \alpha_2 + s + t\}} \xi^{2(t^2+st)} \, \mathrm{Id}_{V_\mu}$$

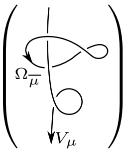
$$= \frac{1}{\ell d(\mu)\{\ell\alpha_1\}} \sum_{s,t=0}^{\ell-1} \left(\frac{\xi^{-(\alpha_2+t)}}{\{\alpha_2+t\}} - \frac{\xi^{-(\alpha_1+\alpha_2+s+t)}}{\{\alpha_1+\alpha_2+s+t\}} \right) \xi^{2(t^2+st)} \, \mathrm{Id}_{V_\mu}.$$

Because

$$\begin{aligned}\sum_{s,t=0}^{\ell-1}\frac{\xi^{-(\alpha_2+t)}\xi^{2(t^2+st)}}{\{\alpha_2+t\}} &= \sum_{t=0}^{\ell-1}\xi^{2t^2}\frac{\xi^{-(\alpha_2+t)}}{\{\alpha_2+t\}}\sum_{s=0}^{\ell-1}\xi^{2st}\\ &= \sum_{t=0}^{\ell-1}\xi^{2t^2}\frac{\xi^{-(\alpha_2+t)}}{\{\alpha_2+t\}}\ell\delta_t^0\\ &= \frac{\ell\xi^{-\alpha_2}}{\{\alpha_2\}},\end{aligned}$$

$$\begin{aligned}\sum_{s,t=0}^{\ell-1}\frac{\xi^{-(\alpha_1+\alpha_2+s+t)}\xi^{2(t^2+st)}}{\{\alpha_1+\alpha_2+s+t\}} &= -\sum_{s,t=0}^{\ell-1}\xi^{2(t^2+st)}\frac{1}{1-\xi^{2(\alpha_1+\alpha_2+s+t)}}\\ &= -\sum_{s,t=0}^{\ell-1}\xi^{2(t^2+st)}\sum_{k=0}^{\infty}\xi^{2k(\alpha_1+\alpha_2+s+t)}\\ &= -\sum_{k=0}^{\infty}\sum_{t=0}^{\ell-1}\xi^{2(t^2+k\alpha_1+k\alpha_2+kt)}\sum_{s=0}^{\ell-1}\xi^{2(k+t)s}\\ &= -\sum_{k=0}^{\infty}\sum_{t=0}^{\ell-1}\xi^{2(t^2+k\alpha_1+k\alpha_2+kt)}\ell\delta^0_{t+k \bmod \ell\mathbb{N}}\\ &= -\ell\left(1+\sum_{t=0}^{\ell-1}\xi^{2t^2}\sum_{j=1}^{\infty}\xi^{2(\ell j-t)(\alpha_1+\alpha_2+t)}\right)\\ &= -\ell\left(1+\sum_{t=0}^{\ell-1}\xi^{-2t(\alpha_1+\alpha_2)}\frac{\xi^{2\ell(\alpha_1+\alpha_2)}}{1-\xi^{2\ell(\alpha_1+\alpha_2)}}\right)\\ &= -\ell+\frac{\ell\xi^{\alpha_1+\alpha_2}}{\{\alpha_1+\alpha_2\}}\end{aligned}$$

then

F ()

$$\begin{aligned} &= \frac{1}{d(\mu)\{\ell\alpha_1\}}\left(\frac{1}{\xi^{\alpha_2}\{\alpha_2\}}-\frac{\xi^{\alpha_1+\alpha_2}}{\{\alpha_1+\alpha_2\}}+1\right)\mathrm{Id}_{V_\mu}\\ &= \frac{1}{\{\alpha_1\}}\Big(\{\alpha_1+\alpha_2\}\xi^{-\alpha_2}-\{\alpha_2\}\xi^{\alpha_1+\alpha_2}+\{\alpha_2\}\{\alpha_1+\alpha_2\}\Big)\,\mathrm{Id}_{V_\mu}\\ &= \frac{1}{\{\alpha_1\}}\{\alpha_1\}=\mathrm{Id}_{V_\mu}\,.\end{aligned}$$

This means that $\Delta_-=1$.

By using the automorphism ω of superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ where $\omega(e_i)=(-1)^{\deg e_i}f_i, \omega(f_i)=(-1)^{\deg f_i}e_i, \omega(k_i)=k_i^{-1}, \omega(h_i)=-h_i, i=1,2$ and computing we also have $\Delta_+=1$. Condition (8) is obviously true.

Hence category $\mathscr{C}^H$ is relatively G-modular.

2.6.2 Invariants of 3-manifolds

Definition 2.6.2. *Let (M, T, ω) be a triple where M is a compact connected oriented 3-manifold, $T \subset M$ is a $\mathscr{C}^H$-colored ribbon graph (possibly empty) and $\omega \in H^1(M \setminus T, G)$.*

1. *The triple (M, T, ω) is* compatible *if each edge e of T is colored by an element of $\mathscr{C}_{\omega(m_e)}$ where m_e is an oriented meridian of the edge e.*
2. *Let $L \cup T \subset S^3$ where L is an oriented link in $S^3 \setminus T$ which gives a presentation of (M, T) by surgery. The presentation $L \cup T$ is* computable *if for each component L_i of L whose meridian is denoted m_i, we have $\omega(m_i) \notin \mathcal{X}$.*

We suppose that (M, T, ω) is a compatible triple.

Definition 2.6.3. *The formal linear combination $\Omega_{\overline{\mu}} = \sum_{\mu_i \in \overline{\mu}} d(V_{\mu_i}) V_{\mu_i}$ is a Kirby color of degree $\overline{\mu} \in G \setminus G_s$ if $\{V_{\mu_i}\}$ is a set of representatives of simple Z-orbits of $\mathscr{C}_{\overline{\mu}}$.*

Theorem 2.6.4. *Let (M, T, ω) a compatible triple admitting a computable presentation $L \cup T \subset S^3$ then*

$$N(M, T, \omega) = F^{'}(L_\omega \cup T)$$

is a well defined topological invariant, i.e. depends only on the diffeomorphism class of the triple (M, T, ω) where L_ω is obtained as the link L in which we have colored the i-th component L_i by a Kirby color of degree $\omega(m_i)$ where m_i is a meridian of L_i.

2.6.3 Example

We consider an example in the case $\ell = 3$. Let M be the lens space $L(5, 2)$ which is given by surgery presentation on the Hopf link L (Figure 2.2). It has two oriented components $L_i, i = 1, 2$ with framings $3, 2$ and let m_i be an oriented meridian of L_i. The linking matrix of L with respect to the components L_i is

$$\mathrm{lk} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $\omega \in H^1(M \setminus T, G)$ and suppose that the triple $(M, \emptyset, \omega)$ is computable. We compute the values $\omega = (\omega^1, \omega^2)$ where $\overline{\mu} = \omega^1 = \omega(m_1), \overline{\mu}' = \omega^2 =$

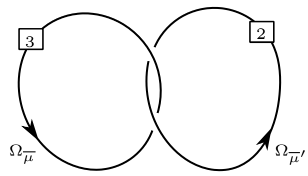


Figure 2.2 – Surgery presentation of L(5,2)

$\omega(m_2)$ from the equations $3\overline{\mu}+\overline{\mu}'=0$ and $\overline{\mu}+2\overline{\mu}'=0$ (in $\mathbb{C}/\mathbb{Z}\times\mathbb{C}/\mathbb{Z}$). Hence $\overline{\mu}=(\frac{k}{5},\frac{2k}{5}),\overline{\mu}'=(\frac{2k}{5},\frac{4k}{5}), k=1,...,4$. Here we set $\omega_k=(\omega_k^1,\omega_k^2),\ \omega_k^1=(\frac{k}{5},\frac{2k}{5}),\ \omega_k^2=(\frac{2k}{5},\frac{4k}{5}), k=1,...,4$. We have $\omega_4=-\omega_1,\ \omega_3=-\omega_2$. Using variables as in Lemma 2.4.6 we have $(\alpha_1,\alpha_2)=\overline{\mu}+(-\ell+1,\frac{\ell}{2})=(\frac{k}{5}-2,\frac{2k}{5}+\frac{3}{2}),(\alpha_1',\alpha_2')=\overline{\mu}'+(-\ell+1,\frac{\ell}{2})=(\frac{2k}{5}-2,\frac{4k}{5}+\frac{3}{2})$.

We color the i-th component L_i by a Kirby color of degree $\omega(m_i)$, i.e. $\Omega_{\omega(m_1)}=\Omega_{\overline{\mu}}=\sum_{s,t=0}^{2}d(\alpha_{st})V_{\alpha_{st}}$ and $\Omega_{\omega(m_2)}=\Omega_{\overline{\mu}'}=\sum_{i,j=0}^{2}d(\alpha_{ij}')V_{\alpha_{ij}'}$ where $\alpha_{st}=(\alpha_1+s,\alpha_2+t),\alpha_{ij}'=(\alpha_1'+i,\alpha_2'+j)$. By Lemma 2.4.6, Proposition 2.4.8 we have

$$N(M,\emptyset,\omega)=\sum_{s,t}\sum_{i,j}d(\alpha_{st})d(\alpha_{ij}')\left\langle\theta_{V_{\alpha_{st}}}\right\rangle^3\left\langle\theta_{V_{\alpha_{ij}'}}\right\rangle^2 d(\alpha_{st})S'(\alpha_{ij}',\alpha_{st})$$

in which

$$\begin{aligned}
d(\alpha_{st}) &= \frac{\{\alpha_1+s\}}{\ell\{\ell(\alpha_1+s)\}\{\alpha_2+t\}\{\alpha_1+\alpha_2+s+t\}},\\
\left\langle\theta_{V_{\alpha_{st}}}\right\rangle &= -\xi^{-2\left((\alpha_2+t)^2+(\alpha_1+s)(\alpha_2+t)\right)},\\
\left\langle\theta_{V_{\alpha_{ij}'}}\right\rangle &= -\xi^{-2\left((\alpha_2'+j)^2+(\alpha_1'+i)(\alpha_2'+j)\right)},\\
S'(\alpha_{ij}',\alpha_{st}) &= \frac{1}{\ell d(\alpha_{st})}\xi^{-4(\alpha_2'+j)(\alpha_2+t)-2\left((\alpha_2'+j)(\alpha_1+s)+(\alpha_1'+i)(\alpha_2+t)\right)}.
\end{aligned}$$

Using computer algebra software Sagemath, we have ($\xi^{\frac{1}{10}}$ has degree 8 over $\mathbb{Q}$)

$$\begin{aligned}
N(M,\emptyset,\pm\omega_1) &= \frac{1}{15}\left(-2\xi^{\frac{7}{10}}-2\xi^{\frac{3}{5}}-2\xi^{\frac{1}{2}}+2\xi^{\frac{2}{5}}+5\xi^{\frac{3}{10}}+2\xi^{\frac{1}{10}}\right),\\
N(M,\emptyset,\pm\omega_2) &= \frac{1}{15}\left(-7\xi^{\frac{7}{10}}-2\xi^{\frac{3}{5}}+4\xi^{\frac{1}{2}}+4\xi^{\frac{2}{5}}+2\xi^{\frac{3}{10}}+5\xi^{\frac{1}{10}}-4\right).
\end{aligned}$$

In this case, the result $N(M,\emptyset,\omega)=N(M,\emptyset,-\omega)$ is consistent with $(M,\emptyset,\omega)\simeq(M,\emptyset,-\omega)$.

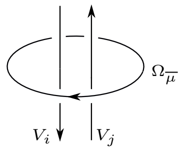


Figure 2.3 – Morphism of the relative modularity condition

2.7 Relative G-modular category $\mathscr{C}^H$

This section proves the category $\mathscr{C}^H$ has a relative G-modular structure. Following M. De Renzi [42] this implies the invariant N in Section 2.6 extends to a family of $1+1+1$-TQFTs.

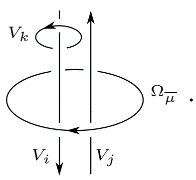
Definition 2.7.1 ([42])**.** *A pre-modular G-category $\mathscr{C}$ relative to $\mathcal{X}$ with modified dimension* d *and periodicity group Z is said a modular G-category relative to (G, Z) if it satisfies the modular condition, i.e. it exists a relative modularity parameter $\zeta \in \mathbb{C}^*$ such that*

$$d(V_i) f_{ij}^{\overline{\mu}} = \begin{cases} \zeta(\overrightarrow{\mathrm{coev}}_{V_i} \circ \overleftarrow{\mathrm{ev}}_{V_i}) & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for all $\overline{\mu}, \overline{\nu} \in G \setminus \mathcal{X}$ and for all $i, j \in \overline{\nu}$ which V_i, V_j are not in the same Z-orbit, where $f_{ij}^{\overline{\mu}}$ is the morphism determined by the $\mathscr{C}$-colored ribbon tangle depicted in Figure 2.3 under Reshetikhin-Turaev functor F.

Proposition 2.7.2. *The category $\mathscr{C}^H$ of nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is modular G-category relative to (G, Z) where $G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}$ and $Z = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. In Section 2.6 we proven that the category $\mathscr{C}^H$ of nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is G-premodular category relative to (G, Z). Now we show that this category is a relative G-modular category. It is necessary to verify the relative modularity condition. We consider the morphism f which is represented by the diagram



By the handle-slide the circle colored by V_k along the circle of $f_{ij}^{\overline{\mu}}$ and an isotopy we have two equalities given by the diagrams as in Figure 2.4. It

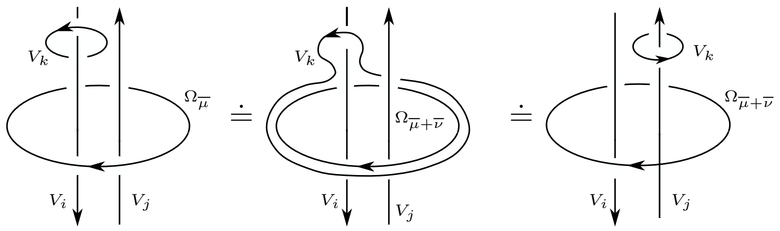


Figure 2.4 – Second Kirby's move on f

follows that

$$S'(V_k, V_i) f_{ij}^{\overline{\mu}} = S'(V_k, V_j) f_{ij}^{\overline{\mu}+\overline{\nu}} \text{ for all } V_k \in \mathscr{C}_{\overline{\nu}}.$$

It implies

$$f_{ij}^{\overline{\mu}+\overline{\nu}} = \frac{S'(V_{k_1}, V_i)}{S'(V_{k_1}, V_j)} f_{ij}^{\overline{\mu}} = \frac{S'(V_{k_2}, V_i)}{S'(V_{k_2}, V_j)} f_{ij}^{\overline{\mu}} \text{ for } V_{k_1}, V_{k_2} \in \mathscr{C}_{\overline{\nu}}.$$

We denote the highest weights of V_i, V_j, V_{k_1} and V_{k_2} by $(\nu_1 + i_1, \nu_2 + i_2), (\nu_1 + j_1, \nu_2 + j_2), (\nu_1 + s_1, \nu_2 + s_2)$ and $(\nu_1 + t_1, \nu_2 + t_2)$ for $0 \leq i_1, i_2, j_1, j_2, s_1, s_2, t_1, t_2 \leq \ell - 1$. By Proposition 2.4.8 we have

$$S'(V_{k_1}, V_i) = \xi^{-4(\nu_2+s_2)(\nu_2+i_2)-2((\nu_2+s_2)(\nu_1+i_1)+(\nu_1+s_1)(\nu_2+i_2))} \frac{1}{\ell d(V_i)},$$
$$S'(V_{k_1}, V_j) = \xi^{-4(\nu_2+s_2)(\nu_2+j_2)-2((\nu_2+s_2)(\nu_1+j_1)+(\nu_1+s_1)(\nu_2+j_2))} \frac{1}{\ell d(V_j)},$$
$$S'(V_{k_2}, V_i) = \xi^{-4(\nu_2+t_2)(\nu_2+i_2)-2((\nu_2+t_2)(\nu_1+i_1)+(\nu_1+t_1)(\nu_2+i_2))} \frac{1}{\ell d(V_i)},$$
$$S'(V_{k_2}, V_j) = \xi^{-4(\nu_2+t_2)(\nu_2+j_2)-2((\nu_2+t_2)(\nu_1+j_1)+(\nu_1+t_1)(\nu_2+j_2))} \frac{1}{\ell d(V_j)}.$$

Hence

$$\frac{S'(V_{k_1}, V_i)}{S'(V_{k_1}, V_j)} = \xi^{-4(\nu_2+s_2)(i_2-j_2)-2((\nu_2+s_2)(i_1-j_1)+(\nu_1+s_1)(i_2-j_2))} \frac{d(V_j)}{d(V_i)},$$
$$\frac{S'(V_{k_2}, V_i)}{S'(V_{k_2}, V_j)} = \xi^{-4(\nu_2+t_2)(i_2-j_2)-2((\nu_2+t_2)(i_1-j_1)+(\nu_1+t_1)(i_2-j_2))} \frac{d(V_j)}{d(V_i)}.$$

We see that

$$\frac{S'(V_{k_1}, V_i)}{S'(V_{k_1}, V_j)} : \frac{S'(V_{k_2}, V_i)}{S'(V_{k_2}, V_j)} = \xi^{-4(s_2-t_2)(i_2-j_2)-2((s_2-t_2)(i_1-j_1)+(s_1-t_1)(i_2-j_2))}$$

and the term $-4(s_2-t_2)(i_2-j_2)-2\left((s_2-t_2)(i_1-j_1)+(s_1-t_1)(i_2-j_2)\right)$ is determined by a symmetric bilinear non-degenerate B from $(\mathbb{Z}/\ell\mathbb{Z})^2 \times (\mathbb{Z}/\ell\mathbb{Z})^2$ to $\mathbb{Z}/\ell\mathbb{Z}$ which has the matrix $B=(b_{ij})_{2\times 2}$ where $b_{11}=0,\ b_{12}=b_{21}=-2$ and $b_{22}=-4$. It deduces that for all $i\neq j\in(\mathbb{Z}/\ell\mathbb{Z})^2$ it exists $k_1\neq k_2\in(\mathbb{Z}/\ell\mathbb{Z})^2$ such that $B(i-j,k_1-k_2)\neq 0$. Thus for all $i\neq j\in\overline{\nu}$ it exists $k_1\neq k_2\in\overline{\nu}$ such that $\dfrac{S'(V_{k_1},V_i)}{S'(V_{k_1},V_j)}\neq\dfrac{S'(V_{k_2},V_i)}{S'(V_{k_2},V_j)}$, it implies that $f_{ij}^{\overline{\mu}}=0$ if $i\neq j$.

If $i=j$ we have $f_{ii}^{\overline{\mu}}=f_{ii}^{\overline{\mu}+\overline{\nu}}$ for $\overline{\mu},\overline{\nu}\in G\setminus G_s$. We see that $f_{ii}^{\overline{\mu}}\in\mathrm{End}_{\mathcal{U}^H}(V_i\otimes V_i^*)$ and $W=V_i\otimes V_i^*$ has a vector $\mathcal{U}^H$-invariant y. As $\mathrm{Hom}_{\mathcal{U}^H}(V_i\otimes V_i^*,\mathbb{C})\simeq\mathrm{Hom}_{\mathcal{U}^H}(\mathbb{C},V_i\otimes V_i^*)\simeq\mathrm{End}_{\mathcal{U}^H}(V_i)\simeq\mathbb{C}\,\mathrm{Id}_{V_i}$ then these imply that two morphisms $f_{ii}^{\overline{\mu}}$ and $\overrightarrow{\mathrm{coev}}_{V_i}\circ\overleftarrow{\mathrm{ev}}_{V_i}$ are proportional, i. e. there is a $\lambda\in\mathbb{C}^*$ such that $f_{ii}^{\overline{\mu}}=\lambda\,\overrightarrow{\mathrm{coev}}_{V_i}\circ\overleftarrow{\mathrm{ev}}_{V_i}$.

First, we show the existence of vector invariant y. Let $V_k\in\mathscr{C}_{\overline{\nu}}$, by Lemma 4.9 of [8] we can do a handle-slide move on the circle component of the graph representing $f_{ii}^{\overline{\mu}}\otimes\mathrm{Id}_{V_k}$ to obtain the equality

$$c_{W,V_k}\circ(f_{ii}^{\overline{\mu}}\otimes\mathrm{Id}_{V_k})=c_{V_k,W}^{-1}\circ(f_{ii}^{\overline{\mu}+\overline{\nu}}\otimes\mathrm{Id}_{V_k})=c_{V_k,W}^{-1}\circ(f_{ii}^{\overline{\mu}}\otimes\mathrm{Id}_{V_k}).$$

The braidings $c_{W,V_k},c_{V_k,W}^{-1}:W\otimes V_k\to V_k\otimes W$ are given by $c_{W,V_k}=\tau^s\circ\mathcal{R}$ and $c_{V_k,W}^{-1}=\mathcal{R}^{-1}\circ\tau^s$ where $\mathcal{R}=\check{\mathcal{R}}\mathcal{K}$ with

$$\check{\mathcal{R}}=\sum_{i=0}^{\ell-1}\frac{\{1\}^i e_1^i\otimes f_1^i}{(i)_\xi!}(1-e_3\otimes f_3)(1-e_2\otimes f_2),$$
$$(0)_\xi!=1,(i)_\xi!=(1)_\xi(2)_\xi\cdots(i)_\xi,(k)_\xi=\frac{1-\xi^k}{1-\xi}$$
$$\text{and }\mathcal{K}=\xi^{-h_1\otimes h_2-h_2\otimes h_1-2h_2\otimes h_2}.$$

Let $x\neq 0$ be a weight vector of weight 0 of W and $v\in V_k$ be an even weight vector of weight $\nu=(\nu_1,\nu_2)$, set $y=f_{ii}^{\overline{\mu}}(x)\in W$.
Let W_+' be the vector space generated by $\{e_1^{i_1}e_3^{i_3}e_2^{i_2}y\mid i_1+i_2+i_3>1\text{ for }0\leq i_1\leq\ell-1,\ 0\leq i_2,i_3\leq 1\}$, W_-' be the vector space generated by $\{f_1^{i_1}f_3^{i_3}f_2^{i_2}v\mid i_1+i_2+i_3>1\text{ for }0\leq i_1\leq\ell-1,\ 0\leq i_2,i_3\leq 1\}$, V_+' be the vector space generated by $\{e_1^{i_1}e_3^{i_3}e_2^{i_2}v\mid i_1+i_2+i_3>1\text{ for }0\leq i_1\leq\ell-1,\ 0\leq i_2,i_3\leq 1\}$ and V_-' be the vector space generated by $\{f_1^{i_1}f_3^{i_3}f_2^{i_2}y\mid i_1+i_2+i_3>1\text{ for }0\leq i_1\leq\ell-1,\ 0\leq i_2,i_3\leq 1\}$. Because the weight of x is 0 then $\mathcal{K}(y\otimes v)=y\otimes v$. Hence

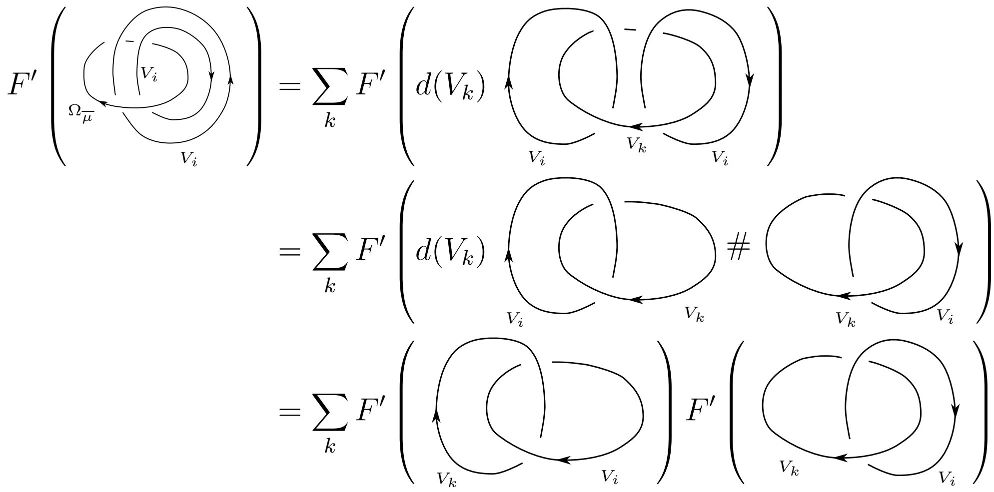
$$\begin{aligned}&c_{W,V_k}(y\otimes v)=\tau^s\circ\check{\mathcal{R}}\mathcal{K}(y\otimes v)\\&=v\otimes y+(\xi-\xi^{-1})f_1v\otimes e_1y+f_3v\otimes e_3y+f_2v\otimes e_2y+W_-'\otimes W_+'\end{aligned}$$

and

$$
\begin{aligned}
&c^{-1}_{V_k,W}(y\otimes v) = \mathcal{R}^{-1}\circ\tau^s(y\otimes v) = (S\otimes \mathrm{Id}_{\mathcal{U}^H})(\mathcal{R})(v\otimes y)\\
&= (S\otimes \mathrm{Id}_{\mathcal{U}^H})\left(v\otimes y + (\xi-\xi^{-1})e_1v\otimes f_1y - e_3v\otimes f_3y - e_2v\otimes f_2y + V_+^{'}\otimes V_-^{'}\right)\\
&= v\otimes y - (\xi-\xi^{-1})k_1e_1v\otimes f_1y + k_1k_2e_3v\otimes f_3y + k_2e_2v\otimes f_2y + S\left(V_+^{'}\right)\otimes V_-^{'}.
\end{aligned}
$$

Setting the above equations equal we have $e_1y = f_1y = 0$ and $e_2y = f_2y = 0$. By the relations $e_1f_1 - f_1e_1 = \frac{k_1-k_1^{-1}}{\xi-\xi^{-1}}, e_2f_2 + f_2e_2 = \frac{k_2-k_2^{-1}}{\xi-\xi^{-1}}$, it implies that $k_i^2y = y$ for $i = 1, 2$ and also since k_i act as ξ^{h_i} and the weights of W are in $\mathbb{Z}\times\mathbb{Z}$, we have that the eigenvalues of k_i are in $\xi^{\mathbb{Z}}$ which does not contain -1 (note that ℓ is odd). Thus $k_iy = y$ for $i = 1, 2$ and y is an invariant vector of W.

Second, we compute λ in $f_{ii}^{\overline{\mu}} = \lambda\, \overrightarrow{\mathrm{coev}}_{V_i}\circ \overleftarrow{\mathrm{ev}}_{V_i}$. We consider the value F' of the braid closure of the graphs in this equality.



$$
\begin{aligned}
&= \sum_k d(V_i)S'\left(V_k, V_i\right) d(V_i)S'\left(V_k^*, V_i\right)\\
&= \sum_k d^2(V_i)S'\left(V_k, V_i\right) S'\left(V_k^*, V_i\right)
\end{aligned}
$$

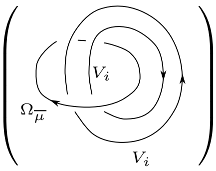
where $\Omega_{\overline{\mu}} = \sum_{k\in\overline{\mu}} d(V_k)V_k$ and the second equality by

$$F'(L_1\#_V L_2) = d^{-1}(V)F'(L_1)F'(L_2).$$

Furthermore

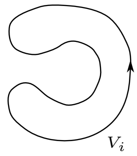
$$S'(V_{k_1}^*, V_i) = \xi^{4(\nu_2+s_2)(\nu_2+i_2)+2((\nu_2+s_2)(\nu_1+i_1)+(\nu_1+s_1)(\nu_2+i_2))}\frac{1}{\ell d(V_i)},$$

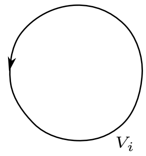
it implies that

$$F'\left(\right.$$


$$\left. \right) = \sum_{s_1,s_2=0}^{\ell-1} \frac{1}{\ell^2} = 1.$$

For the graph of $\overrightarrow{\text{coev}}_{V_i} \circ \overleftarrow{\text{ev}}_{V_i}$, the value F' of its closure is

$$F'\left(\right.$$


$$\left. \right) = F'\left(\right.$$


$$\left. \right) = d(V_i).$$

Hence $\lambda = d^{-1}(V_i)$ and it proves that $d(V_i) f_{ii}^{\overline{\mu}} = \overrightarrow{\text{coev}}_{V_i} \circ \overleftarrow{\text{ev}}_{V_i}$. □

We see that the relative modularity parameter $\zeta = \Delta_- \Delta_+ = 1$.

Chapter 3

Modified trace from pivotal Hopf G-coalgebra

This chapter is the content of the paper [21] available in https://arxiv.org/abs/1804.02416.

Résumé. Dans un article récent, les auteurs A. Beliakova, C. Blanchet et A. M. Gainutdinov ont montré que la trace modifiée sur la catégorie H-pmod des modules projectifs correspond à l'intégrale symétrisée sur l'algèbre de Hopf pivotale de dimension finie H. Nous généralisons ce théorème au contexte des catégories G-graduées et G-cogèbre de Hopf étudiée par Turaev-Virelizier. Nous montrons que la G-intégrale symétrisée sur une G-cogèbre de Hopf pivotale de type fini induit une trace modifiée dans la catégorie G-graduée associée.

Abstract. In a recent paper the authors A. Beliakova, C. Blanchet and A. M. Gainutdinov have shown that the modified trace on the category H-pmod of the projective modules corresponds to the symmetrised integral on the finite dimensional pivotal Hopf algebra H. We generalize this fact to the context of G-graded categories and Hopf G-coalgebra studied by Turaev-Virelizier. We show that the symmetrised G-integral on a finite type pivotal Hopf G-coalgebra induces a modified trace in the associated G-graded category.

MSC: 57M27, 17B37

Key words: modified trace, G-integral, symmetrised G-integral, pivotal Hopf G-coalgebra.

3.1 Introduction

The notion of a modified trace was introduced by N. Geer, J. Kujawa and B. Patureau-Mirand in the article [13]. This is one of the topological tools which can be used first to renormalize the Reshetikhin-Turaev invariant of links. Later F. Costantino, N. Geer and B. Patureau-Mirand used the modified trace to construct a class of invariants of 3-manifolds (CGP invariant) via link surgery presentations (see [8]). The modified trace is also used to construct invariants of 3-manifolds of Reshetikhin-Turaev type from quantum group associated to the Lie superalgebra $\mathfrak{sl}(2|1)$ (see Chapter 2) and for constructing the logarithmic invariant of Hennings type (see [3]). In order to construct invariant of 3-manifolds, M. Hennings proposed a method based on the theory of integral for a finite dimensional Hopf algebra (see [26]). The notion of integral was introduced by R. G. Larson and M. E. Sweedler in [34] and is studied in the book [41] of Radford. It is known that under some assumption, both the space of modified trace and that of integral are one dimensional (see [11, 41]). A close relation between the modified trace and the integral has been established recently in [2]. The authors proved that a symmetrised integral for a finite dimensional pivotal Hopf algebra gives a modified trace $\mathtt{t}$ on H-pmod with an explicit formula. We would like to adapt these results to the unrestricted quantum groups at roots of unity. They are infinite dimensional Hopf algebra but can be understood as a Hopf G-coalgebra organized into a bundles of algebra over a Lie group. For a finite type Hopf G-coalgebra $H = (H_\alpha)_{\alpha\in G}$ there exists a family of linear forms on H_α, called G-integral (see [49]). The aim of this chapter is to establish a correspondence between the G-integral for the finite type unimodular pivotal Hopf G-coalgebra H and the modified trace in the associated G-graded category H-mod. We introduce now these two notions.

G-integral

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S)$ be a Hopf G-coalgebra over a field $\Bbbk$ (see in Section 3.2). A *right G-integral* for the Hopf G-coalgebra H is a family of $\Bbbk$-linear forms $\mu = (\mu_\alpha : \ H_\alpha \to \Bbbk)_{\alpha\in G}$ satisfying

$$(\mu_\alpha \otimes \mathrm{Id}_{H_\beta})\Delta_{\alpha,\beta}(x) = \mu_{\alpha\beta}(x)1_\beta \text{ for any } x \in H_{\alpha\beta}. \qquad (3.1.1)$$

Similarly, a *left G-integral* $\mu^l_\alpha \in \prod_{\alpha\in G} H^*_\alpha$ satisfies

$$(\mathrm{Id}_{H_\alpha} \otimes \mu^l_\beta)\Delta_{\alpha,\beta}(x) = \mu^l_{\alpha\beta}(x)1_\alpha \text{ for any } x \in H_{\alpha\beta}.$$

The linear form μ_1 is an usual right integral for the Hopf algebra H_1 (see e.g [41]). If H is a finite type Hopf G-coalgebra, i.e. a Hopf G-coalgebra in which

$\dim(H_\alpha) < +\infty$ for any $\alpha \in G$, the space of right (resp. left) G-integral is known to be 1-dimensional (see e.g [49]).
A *pivotal Hopf G-coalgebra* is a pair (H, g), where the pivot is the family $g = (g_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha$ satisfying $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes g_\beta$ for any $\alpha, \beta \in G$, $\varepsilon(g_1) = 1_{\Bbbk}$, and $S_{\alpha^{-1}}S_\alpha(x) = g_\alpha x g_\alpha^{-1}$ for any $x \in H_\alpha$. Note that $g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha\in G}$, i.e. $g_\alpha^{-1} = S_{\alpha^{-1}}(g_{\alpha^{-1}})$ (see e.g [49]). In particular, g_1 is a pivotal element for H_1 and g_1 is invertible with $g_1^{-1} = S_1(g_1)$, $\varepsilon(g_1) = 0$ (see e.g [29]).
The *symmetrised right G-integral* on (H, g) associated with μ is the family $\widetilde{\mu} = (\widetilde{\mu}_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha^*$ defined by

$$\widetilde{\mu}_\alpha(x) := \mu_\alpha(g_\alpha x) \text{ for any } x \in H_\alpha.$$

Similarly, a *symmetrised left G-integral* on (H, g) is

$$\widetilde{\mu}^l_\alpha(x) := \mu^l_\alpha(g_\alpha^{-1}x) \text{ for any } x \in H_\alpha. \tag{3.1.2}$$

A pivotal Hopf G-coalgebra is *G-unibalanced* if its symmetrised right G-integral is also symmetrised left G-integral, i.e. $\widetilde{\mu}_\alpha = \widetilde{\mu}^l_\alpha$ for any $\alpha \in G$.
In the case (H, g) is unimodular, i.e. H_1 is unimodular, we show that the symmetrised G-integrals are symmetric linear forms on H and they are non-degenerate (see Proposition 3.2.7).

Modified trace

Let $\mathcal{C}$ be a *pivotal* $\Bbbk$-linear category [39]. Let $\mathbf{Proj}(\mathcal{C})$ be the tensor ideal of projective objects of $\mathcal{C}$. A *modified trace* on ideal $\mathbf{Proj}(\mathcal{C})$ is a family of $\Bbbk$-linear forms $\mathsf{t} = \{\mathsf{t}_P : \operatorname{End}_{\mathcal{C}}(P) \to \Bbbk\}_{P\in\mathbf{Proj}(\mathcal{C})}$ satisfying the cyclicity property and the partial trace property (see in Section 3.3.2).

Main results

Let $(H, g) = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S, g)$ be a finite type unimodular pivotal Hopf G-coalgebra. If t is a right (resp. left) modified trace on H-pmod, it defines a family of linear forms $\lambda^{\mathsf{t}} = (\lambda^{\mathsf{t}}_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha^*$ by $\lambda^{\mathsf{t}}_\alpha(h) = \mathsf{t}_{H_\alpha}(R_h)$ for $h \in H_\alpha$, H_α is a projective object of H-mod and R_h is the right multiplication of H_α.

Theorem 3.1.1. *The application $\mathsf{t} \mapsto \lambda^{\mathsf{t}}$ defined above gives a bijection between the space of right (resp. left) modified traces and the space of symmetrised right (resp. left) G-integrals.*
Furthermore, (H, g) is G-unibalanced if and only if the right modified trace is also left.

The chapter contains five section. In Section 3.2 we recall some definitions and results for a Hopf G-coalgebra, we also define a pivotal Hopf G-coalgebra, a symmetrised G-integral for a pivotal Hopf G-coalgebra H and prove that the symmetrised G-integrals are symmetric non-degenerate forms on H. Section 3.3 recall some results about modified traces and the proof of Reduction Lemma in the context of G-graded categories. In Section 3.4 we present the decomposition of tensor product $H_\alpha \otimes H_\beta$ and the proof of the main theorem. In Section 3.5 we give an application of the main theorem in the case associated to a quantization of the Lie algebra $\mathfrak{sl}(2)$.

3.2 Pivotal Hopf G-coalgebra

In this section, we recall some facts about Hopf G-coalgebra. For details see [46, 49]. We then define a pivotal Hopf G-coalgebra, a symmetrised G-integral and give some of its properties.

3.2.1 Pivotal Hopf G-coalgebra

Hopf G-coalgebra

Definition 3.2.1. *Let G be a multiplicative group. A G-coalgebra over a field $\Bbbk$ is a family $C = \{C_\alpha\}_{\alpha\in G}$ of $\Bbbk$-spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in G}$ of $\Bbbk$-linear maps (the coproduct) and a $\Bbbk$-linear map $\varepsilon :\ C_1 \to \Bbbk$ (the counit) such that*

1. *Δ is coassociative, i.e. for any $\alpha, \beta, \gamma \in G$,*

$$(\Delta_{\alpha,\beta} \otimes \mathrm{Id}_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{Id}_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

2. *for all $\alpha \in G$, $(\mathrm{Id}_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{Id}_{C_\alpha} = (\varepsilon \otimes \mathrm{Id}_{C_\alpha})\Delta_{1,\alpha}$.*

A Hopf G-coalgebra is a G-coalgebra $H = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon)$ endowed with a family $S = \{S_\alpha :\ H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in G}$ of $\Bbbk$-linear maps (the antipode) such that

1. *each H_α is an algebra with product m_α and unit element $1_\alpha \in H_\alpha$,*
2. *$\varepsilon :\ H_1 \to \Bbbk$ and $\Delta_{\alpha,\beta} :\ H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra homomorphisms for all $\alpha, \beta \in G$,*
3. *for any $\alpha \in G$*

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{Id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(\mathrm{Id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}.$$

The antipode automatically satisfies additional property:

Lemma 3.2.2. *Given a Hopf G-coalgebra $H = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon, S)$, then*

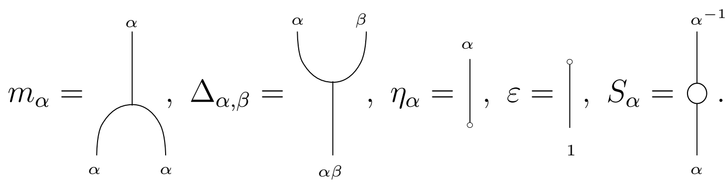


Figure 3.1 – The structural maps

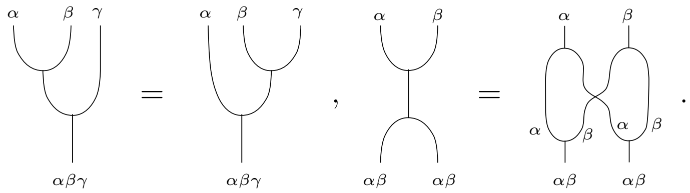


Figure 3.2 – The coassociativity and algebra homomorphism $\Delta_{\alpha,\beta}$

1. *$S_\alpha(xy) = S_\alpha(y)S_\alpha(x)$ for any $x, y \in H_\alpha$,*
2. *$S_\alpha(1_\alpha) = 1_{\alpha^{-1}}$,*
3. *$\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \tau(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ where $\tau : \ H_{\alpha^{-1}} \otimes H_{\beta^{-1}} \to H_{\beta^{-1}} \otimes H_{\alpha^{-1}}$ is the flip switching the two factors of $H_{\alpha^{-1}} \times H_{\beta^{-1}}$,*
4. *$\varepsilon S_1 = \varepsilon$.*

Graphical axioms for Hopf G-coalgebras

We will use the diagrams for the structural maps and the identities corresponding to the Hopf G-coalgebra $H = (H_\alpha)_{\alpha \in G}$. For simplicity we write α instead of H_α in the diagrams. Figure 3.1 presents the structural maps of the Hopf G-coalgebra which are the product, coproduct, unit, counit and the antipode, respectively. Note that these maps are in the category $\mathrm{Vect}_{\Bbbk}$ of finite dimensional vector spaces over a field $\Bbbk$.

The identity of the coassociativity and the algebra homomorphism $\Delta_{\alpha,\beta}$ are defined as in Figure 3.2. The antipode properties are shown in Figure 3.3. Finally, the compatibility between the antipode and the unit, counit are illustrated in Figure 3.4.

Example 3.2.3. *Let H be a possibly infinite dimensional pivotal Hopf algebra with the pivot ϕ. Suppose there is a commutative Hopf subalgebra C contained in the center of H (for example H can be the unrestricted quantum group in [7]; an other example will be detailed in Section 3.5). Let $G = \mathrm{Hom}_{Alg}(C, \Bbbk)$ be the group of characters on C with multiplication given by $gh = (g \otimes h) \circ \Delta$*

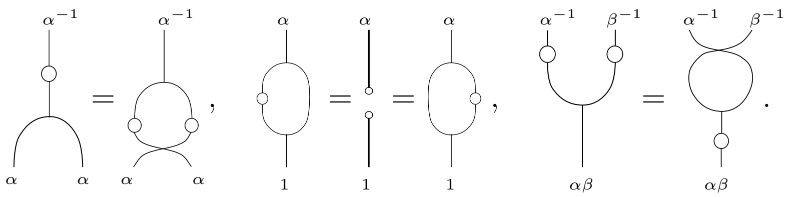


Figure 3.3 – The antipode properties

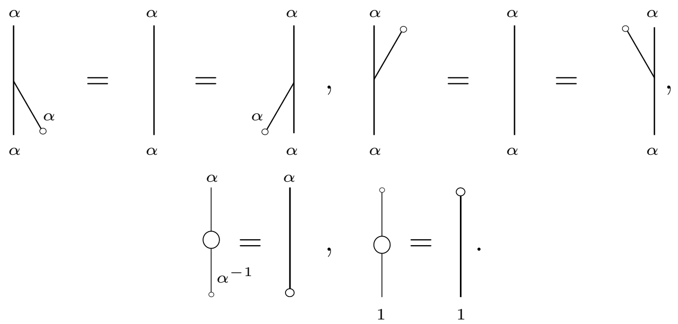


Figure 3.4 – The unit and counit

for $g, h \in G$ *and* $g^{-1} = g \circ S|_C$*. For* $g \in G$ *we define* $H_g = H \otimes_{g:C\to\Bbbk} \Bbbk = H/I_g$ *where* I_g *is the ideal generated by elements* $z - g(z)$ *for* $z \in C$*. Assume* $g = g_1 g_2$ *for* $g_1, g_2 \in G$*, then*

$$\begin{aligned}\Delta(z - g(z)) &= \Delta(z) - (g_1 \otimes g_2)(\Delta(z)) \\ &= z_{(1)} \otimes z_{(2)} - g_1(z_{(1)}) \otimes g_2(z_{(2)}) \\ &= \Big(z_{(1)} - g_1(z_{(1)})\Big) \otimes z_{(2)} + g_1(z_{(1)}) \otimes \Big(z_{(2)} - g_2(z_{(2)})\Big)\end{aligned}$$

where we used the Sweedler's notation $\Delta(z) = z_{(1)} \otimes z_{(2)}$*. This implies that* $\Delta(I_g) \subset I_{g_1} \otimes H + H \otimes I_{g_2}$*. We thus have that a well defined coproduct* Δ_{g_1,g_2} *given by the commutative diagram below*

$$\begin{array}{ccc} H & \xrightarrow{\ \Delta\ } & H \otimes H \\ \downarrow{\scriptstyle p_{g_1 g_2}} & & \downarrow{\scriptstyle p_{g_1} \otimes p_{g_2}} \\ H_{g_1 g_2} & \xrightarrow{\ \Delta_{g_1,g_2}\ } & H_{g_1} \otimes H_{g_2} \end{array}$$

where $p_g : H \to H_g$ *is the projective morphism. The family* $\{H_g\}_{g\in G}$ *with coproduct* $\Delta_{g,h}$ *is a* G*-coalgebra. It is also a Hopf* G*-coalgebra with the family* *of antipode given by the commutative diagram*

$$\begin{array}{ccc} H & \xrightarrow{S} & H \\ \downarrow{\scriptstyle p_g} & & \downarrow{\scriptstyle p_{g^{-1}}} \\ H_g & \xrightarrow{S_g} & H_{g^{-1}}. \end{array}$$

The family S_g for $g \in G$ is well defined since $S(z - g(z)) = S(z) - g(z) = S(z) - g^{-1}(S(z)) \in I_{g^{-1}}$.

We say a Hopf G-coalgebra H is of *finite type* if H_α is finite dimensional over $\Bbbk$ for all $\alpha \in G$.

Pivotal structure

We recall that a G-grouplike element of a Hopf G-coalgebra H is a family $g = (g_\alpha)_{\alpha\in G} \in \prod_\alpha H_\alpha$ such that $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes g_\beta$ for any $\alpha, \beta \in G$ and $\varepsilon(g_1) = 1_\Bbbk$. Note that g_1 is a grouplike element of the Hopf algebra H_1. It follows [49] that the set of the G-grouplike elements of H is a group and if $g = (g_\alpha)_{\alpha\in G}$, then $g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha\in G}$.

Definition 3.2.4. *A G-grouplike element $g \in H$ is called a pivot if $S_{\alpha^{-1}}S_\alpha(x) = g_\alpha x g_\alpha^{-1}$ for all $x \in H_\alpha$. The pair (H, g) of a Hopf G-coalgebra H and a pivot g is called a pivotal Hopf G-coalgebra.*

Remark that for a pivotal Hopf G-coalgebra $H = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon, S, g)$, H_1 is a pivotal Hopf algebra.

Example 3.2.5. *Let H be a Hopf G-coalgebra as in Example 3.2.3. Let ϕ_g be the image of ϕ in the quotient H_g. Then H is a pivotal Hopf G-coalgebra.*

3.2.2 Symmetrised right and left G-integrals

Let $H = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon, S)$ be a finite type pivotal Hopf G-coalgebra with right G-integral μ. The symmetrised right G-integral associated with μ is a family $\widetilde{\mu} = (\widetilde{\mu}_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha^*$ defined by $\widetilde{\mu}_\alpha(x) := \mu_\alpha(g_\alpha x)$ for any $x \in H_\alpha$.
Using the definition of the right G-integral, see Equation (3.1.1) and replacing $x \in H_{\alpha\beta}$ by $g_{\alpha\beta}x$ we get:

$$(\widetilde{\mu}_\alpha \otimes g_\beta)\Delta_{\alpha,\beta}(x) = \widetilde{\mu}_{\alpha\beta}(x)1_\beta. \tag{3.2.1}$$

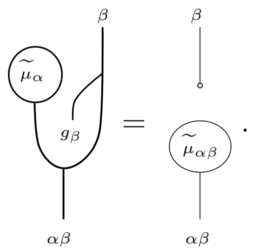


Figure 3.5 – The graphical representation of the relation for the right symmetrised G-integral

Similarly, the symmetrised left G-integral is defined by $\widetilde{\mu}^l_\alpha(x) := \mu^l_\alpha(g_\alpha^{-1}x)$ for any $x \in H_\alpha$. Applying (3.1.2) for $g_{\alpha\beta}^{-1}x,\ x \in H_{\alpha\beta}$ we get the defining relation for the symmetrised left G-integral:

$$(g_\alpha^{-1} \otimes \widetilde{\mu}^l_\beta)\Delta_{\alpha,\beta}(x) = \widetilde{\mu}^l_{\alpha\beta}(x)1_\alpha. \tag{3.2.2}$$

The graphical representation for Equality (3.2.1) is given in Figure 3.5. The graphical representation of the relation for the left symmetrised G-integral is similar.

Since the pivot is invertible Equation (3.2.1) for $\widetilde{\mu}$ is equivalent to Equation (3.1.1) for μ. As the space of right G-integrals is one-dimensional, relation (3.2.1) defines $\widetilde{\mu}$ uniquely (up to a scalar). Similarly the symmetrised left G-integral $\widetilde{\mu}^l$ defined by (3.2.2) is unique. Note also that the symmetrised G-integral for H_1 is the one in the sense of [2].

Recall that a left (resp. right) *cointegral* in H_1 is an element $\Lambda \in H_1$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in H_1$ ([2]).

Definition 3.2.6. *1. A Hopf G-coalgebra H is unimodular if the Hopf algebra H_1 is unimodular, this means that the spaces of left and right cointegrals in H_1 coincide.*

2. A family of linear forms $\varphi_\alpha \in H_\alpha^*$ for $\alpha \in G$ is symmetric non-degenerate if for any $\alpha \in G$ the associated bilinear forms $(x, y) \mapsto \varphi_\alpha(xy),\ x, y \in H_\alpha$ is.

Proposition 3.2.7. *Assume (H, g) is unimodular, then the symmetrised right (resp. left) G-integral for (H, g) is symmetric and non-degenerate.*

Proof. For any $\alpha \in G,\ x, y \in H_\alpha$, by [49, Lemma 7.1] we have

$$\widetilde{\mu}_\alpha(xy) = \mu_\alpha(g_\alpha xy) = \mu_\alpha(S_{\alpha^{-1}}S_\alpha(y)g_\alpha x) = \mu_\alpha(g_\alpha yx) = \widetilde{\mu}_\alpha(yx)$$

and by [49, Corollary 3.7] H^*_α is free left module rank one over H_α with basis $\{\mu_\alpha\}$ when the action is defined by

$$(h \rightharpoonup \mu_\alpha)(x) := \mu_\alpha(xh) \text{ for } h, x \in H_\alpha.$$

If $\widetilde{\mu}_\alpha(xy) = \mu_\alpha(g_\alpha xy) = \mu_\alpha(xyg_\alpha) = (yg_\alpha \rightharpoonup \mu_\alpha)(x) = 0$ for all $x \in H_\alpha$, then $yg_\alpha \rightharpoonup \mu_\alpha = 0$. It follows thus $y = 0$.
For the symmetrised left G-integral the proof is similar. □

Also note that the spaces of left and right G-integrals are not equal in general. We have a lemma.

Lemma 3.2.8. *The left G-integral for H can be chosen as $\mu^l_\alpha(x) = \mu_{\alpha^{-1}}(S_\alpha(x))$ for any $x \in H_\alpha$.*

Proof. By (3.1.1) we have

$$(\mu_{\alpha^{-1}} \otimes \mathrm{Id}_{H_{\beta^{-1}}})\Delta_{\alpha^{-1},\beta^{-1}}(S_{\beta\alpha}(x)) = \mu_{(\beta\alpha)^{-1}}(S_{\beta\alpha}(x))1_{\beta^{-1}} \text{ for any } x \in H_{\beta\alpha}.$$

Using Lemma 3.2.2 (3) $\Delta_{\alpha^{-1},\beta^{-1}}(S_{\beta\alpha}(x)) = (S_\alpha \otimes S_\beta)\Delta^{op}_{\beta,\alpha}(x)$ we get

$$(\mu_{\alpha^{-1}} \circ S_\alpha \otimes S_\beta)\Delta^{op}_{\beta,\alpha}(x) = (S_\beta \otimes \mu_{\alpha^{-1}} \circ S_\alpha)\Delta_{\beta,\alpha}(x) = \mu_{(\beta\alpha)^{-1}}(S_{\beta\alpha}(x))1_{\beta^{-1}}.$$

Applying S^{-1}_β to both sides of the last equality and $S^{-1}_\beta(1_{\beta^{-1}}) = 1_\beta$, we obtain that $(\mathrm{Id}_{H_\beta} \otimes \mu_{\alpha^{-1}} \circ S_\alpha)\Delta_{\beta,\alpha}(x) = (\mu_{(\beta\alpha)^{-1}} \circ S_{\beta\alpha})(x)1_\beta$, i.e. $\mu_{\alpha^{-1}} \circ S_\alpha$ satisfies the definition of the left G-integral. □

3.2.3 G-unibalanced Hopf algebras

Let $H = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon, S)$ be a finite type Hopf G-coalgebra with right G-integral μ. We call a *distinguished G-grouplike* of H (see e.g [49]) or *G-comodulus* of H a G-grouplike element $\mathbf{a} = (a_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha$ satisfying

$$(\mathrm{Id}_{H_\alpha} \otimes \mu_\beta)\Delta_{\alpha,\beta}(x) = \mu_{\alpha\beta}(x)a_\alpha \text{ for any } x \in H_{\alpha\beta}. \tag{3.2.3}$$

Note that a_1 is the comodulus element of the Hopf algebra H_1 (see [2]). By multiplying (3.2.3) with $\mathbf{a}^{-1}$ and replacing x by $a_{\alpha\beta}x$ we have

$$(\mathrm{Id}_{H_\alpha} \otimes \mu_\beta(a_\beta?))\Delta_{\alpha,\beta}(x) = \mu_{\alpha\beta}(a_{\alpha\beta}x)1_\alpha \tag{3.2.4}$$

where denote by $\mu_\beta(a_\beta?)$ the linear map $x \mapsto \mu_\beta(a_\beta x)$ for $x \in H_\beta$. This equality implies that $\mu_\beta(a_\beta?)$ is a left G-integral for H, i.e.

$$\mu^l_\beta(x) = \mu_\beta(a_\beta x). \tag{3.2.5}$$

This is another choice for left G-integral from right G-integral. This choice of the left G-integral is the same with the one in Lemma 3.2.8 by following proposition.

Proposition 3.2.9. *We have the relation* $\mu_{\alpha^{-1}}(S_\alpha(x)) = \mu_\alpha(a_\alpha x)$ *for any* $x \in H_\alpha$.

Proof. By (3.2.4) we get

$$(\mathrm{Id}_{H_\alpha} \otimes \mu_1(a_1?))\,\Delta_{\alpha,1}(x) = \mu_\alpha(a_\alpha x)1_\alpha \text{ for } x \in H_\alpha.$$

By Lemma 3.2.8 we get

$$(\mathrm{Id}_{H_\alpha} \otimes \mu_1 \circ S_1)\,\Delta_{\alpha,1}(x) = (\mu_{\alpha^{-1}} \circ S_\alpha)(x)1_\alpha \text{ for } x \in H_\alpha.$$

Furthermore, Proposition 4.7 [2] gives $\mu_1(S_1(x)) = \mu_1(a_1 x)$ for $x \in H_1$. This implies that $\mu_\alpha(a_\alpha x)1_\alpha = (\mu_{\alpha^{-1}} \circ S_\alpha)(x)1_\alpha$ for all $x \in H_\alpha$, i.e. $\mu_{\alpha^{-1}}(S_\alpha(x)) = \mu_\alpha(a_\alpha x)$ for any $x \in H_\alpha$. □

Recall that a finite type pivotal Hopf G-coalgebra (H, g) is G-unibalanced if its symmetrised right G-integral is also left.

Lemma 3.2.10. *Assume* (H, g) *is a unimodular pivotal Hopf* G*-coalgebra. Then* (H, g) *is* G*-unibalanced if and only if* $a_\alpha = g_\alpha^2$ *for any* $\alpha \in G$.

Proof. First, we assume that $a_\alpha = g_\alpha^2$. Applying (3.2.3) on $g_{\alpha\beta}x$ we have

$$(g_\alpha^{-1} \otimes \widetilde{\mu}_\beta)\Delta_{\alpha,\beta}(x) = \widetilde{\mu}_{\alpha\beta}(x)1_\alpha.$$

This equality states that $\widetilde{\mu}_\beta$ is a symmetrised left G-integral, i.e. $\widetilde{\mu}_\beta = \widetilde{\mu}^l_\beta$. Second, we assume that (H, g) is G-unibalanced. By applying the equality (3.2.5) on $g_\alpha^{-1}x$ and the G-unibalanced condition one gets

$$\mu^l_\alpha(g_\alpha^{-1}x) = \widetilde{\mu}^l_\alpha(x) = \widetilde{\mu}_\alpha(x) = \mu_\alpha(g_\alpha x) = \mu_\alpha(a_\alpha g_\alpha^{-1}x)$$

for any $x \in H_\alpha$. The last equality gives

$$\mu_\alpha\left((a_\alpha g_\alpha^{-1} - g_\alpha)x\right) = 0 \text{ for any } x \in H_\alpha.$$

By Proposition 3.2.7, μ_α is non-degenerate. Therefore, the above equality holds if and only if $a_\alpha = g_\alpha^2$. □

3.3 Traces on finite G-graded categories

In this section we recall some notions and results from [2]. Let (H, g) be a finite type unimodular pivotal Hopf G-coalgebra. We determine the pivotal structure in pivotal G-graded category H-mod. We also prove the Reduction Lemma in the context of G-graded categories and recall the close relation between a modified trace on H_1-pmod and a symmetrised integral for H_1 [2].

3.3.1 Cyclic traces

Let $\mathcal{C}$ be a $\Bbbk$-linear category. We call *cyclic trace* on $\mathcal{C}$ a family of $\Bbbk$-linear maps

$$t = \{t_P : \operatorname{End}_{\mathcal{C}}(P) \to \Bbbk\}_{P \in \mathcal{C}} \tag{3.3.1}$$

satisfying cyclicity property, i.e. $t_V(gh) = t_U(hg)$ for $g \in \operatorname{Hom}_{\mathcal{C}}(U, V)$ and $h \in \operatorname{Hom}_{\mathcal{C}}(V, U)$ with $U, V \in \mathcal{C}$. We say that a cyclic trace t is *non-degenerate* if the pairings

$$\operatorname{Hom}_{\mathcal{C}}(M, P) \times \operatorname{Hom}_{\mathcal{C}}(P, M) \to \Bbbk, \ (f, g) \mapsto t_P(fg) \tag{3.3.2}$$

are non-degenerate for all $P, M \in \mathcal{C}$.

For a finite dimensional algebra A, let A-pmod be the category of projective A-modules. There is a bijection from the space of cyclic traces on A-pmod to the space of symmetric linear forms on A:

Lemma 3.3.1. *There is an isomorphism of algebras*

$$R : \ A^{op} \to \operatorname{End}_A(A)$$

given by

$$R(h) = R_h, \qquad R^{-1}(f) = f(1)$$

where R_h denotes the right multiplication with h, i.e. $R_h(x) = xh$ for any $x \in A$.

Lemma 3.3.1 implies that if t is a cyclic trace on A-pmod then

$$\lambda(h) = t_A(R_h) \tag{3.3.3}$$

defines a symmetric linear form on A.

Proposition 3.3.2. *[2, Proposition 2.4] A symmetric linear form λ on a finite dimensional algebra A extends uniquely to a family of cyclic traces $\{t_P : \operatorname{End}_A(P) \to \Bbbk\}_{P \in A\text{-pmod}}$ which satisfies Equality (3.3.3).*

If $f \in \operatorname{End}_A(P)$, one can find $a_i \in \operatorname{Hom}(A, P)$, $b_i \in \operatorname{Hom}(P, A)$ $i \in I$ for some finite set I such that $f = \sum_{i \in I} a_i b_i$ (see [2]). Then the cyclicity property of t implies that

$$t_P(f) = \sum_{i \in I} t_A(b_i a_i) = \sum_{i \in I} \lambda\left(b_i a_i(1)\right). \tag{3.3.4}$$

Furthermore, the non-degeneracy of the form linear λ is equivalent to the one of the pairings (3.3.2) determined by $(t_P)_{P \in A\text{-pmod}}$ in (3.3.4) (see [2], Theorem 2.6 where a stronger non-degeneracy condition for traces is considered).

3.3.2 Modified trace in pivotal category

Let $\mathcal{C}$ be a *pivotal* $\Bbbk$-linear category [39]. Then $\mathcal{C}$ is a strict monoidal $\Bbbk$-linear category, with a unit object $\mathbb{I}$, equipped with the data for each object $V \in \mathcal{C}$ of its dual object $V^* \in \mathcal{C}$ and of four morphisms

$$\overrightarrow{\mathrm{ev}}_V\colon\ V^* \otimes V \to \mathbb{I}, \qquad \overrightarrow{\mathrm{coev}}_V\colon\ \mathbb{I} \to V \otimes V^*,$$
$$\overleftarrow{\mathrm{ev}}_V\colon\ V \otimes V^* \to \mathbb{I}, \qquad \overleftarrow{\mathrm{coev}}_V\colon\ \mathbb{I} \to V^* \otimes V$$

such that $(\overrightarrow{\mathrm{ev}}_V, \overrightarrow{\mathrm{coev}}_V)$ and $(\overleftarrow{\mathrm{ev}}_V, \overleftarrow{\mathrm{coev}}_V)$ are dualities which induce the same functor duality which is monoidal. In the category $\mathcal{C}$ there is a family of isomorphisms

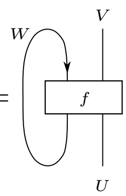
$$\Phi = \{\Phi_V = (\overleftarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_{V^{**}})(\mathrm{Id}_V \otimes \overrightarrow{\mathrm{coev}}_{V^*}) :\ V \to V^{**}\}_{V \in \mathcal{C}}$$

which is a monoidal natural isomorphism called the pivotal structure.
We recall the notion of a *modified trace* on ideal in a pivotal category $\mathcal{C}$ which be introduced in [13, 18]. Given $U, V, W \in \mathcal{C}$ and $f \in \mathrm{End}_{\mathcal{C}}(W \otimes V)$. The *left partial trace* (with respect to W) is the map

$$\mathrm{tr}^l_W :\ \mathrm{Hom}_{\mathcal{C}}(W \otimes U, W \otimes V) \to \mathrm{Hom}_{\mathcal{C}}(U, V)$$

defined for $f \in \mathrm{Hom}_{\mathcal{C}}(W \otimes U, W \otimes V)$ by

$$\mathrm{tr}^l_W(f) = (\overrightarrow{\mathrm{ev}}_W \otimes \mathrm{Id}_V)(\mathrm{Id}_{W^*} \otimes f)(\overleftarrow{\mathrm{coev}}_W \otimes \mathrm{Id}_U) =$$



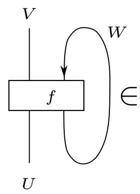
$\in \mathrm{Hom}_{\mathcal{C}}(U, V).$

The *right partial trace* (with respect to W) is the map

$$\mathrm{tr}^r_W :\ \mathrm{Hom}_{\mathcal{C}}(U \otimes W, V \otimes W) \to \mathrm{Hom}_{\mathcal{C}}(U, V)$$

defined for $f \in \mathrm{Hom}_{\mathcal{C}}(U \otimes W, V \otimes W)$ by

$$\mathrm{tr}^r_W(f) = (\mathrm{Id}_V \otimes \overleftarrow{\mathrm{ev}}_W)(f \otimes \mathrm{Id}_{W^*})(\mathrm{Id}_U \otimes \overrightarrow{\mathrm{coev}}_W) =$$



$\in \mathrm{Hom}_{\mathcal{C}}(U, V).$ (3.3.5)

Let $\mathbf{Proj}(\mathcal{C})$ be the tensor ideal of projective objects of $\mathcal{C}$. A *left modified trace* on $\mathbf{Proj}(\mathcal{C})$ is a cyclic trace t on $\mathbf{Proj}(\mathcal{C})$ satisfying

$$\mathsf{t}_{W \otimes P}(f) = \mathsf{t}_P(\mathrm{tr}^l_W(f))$$

for any $f \in \mathrm{End}_{\mathcal{C}}(W \otimes P)$ with $P \in \mathbf{Proj}(\mathcal{C})$ and $W \in \mathcal{C}$.

A *right modified trace* on $\mathbf{Proj}(\mathcal{C})$ is a cyclic trace t on $\mathbf{Proj}(\mathcal{C})$ satisfying

$$\mathsf{t}_{P\otimes W}(f) = \mathsf{t}_P(\mathrm{tr}^r_W(f))$$

for any $f \in \mathrm{End}_{\mathcal{C}}(P \otimes W)$ with $P \in \mathbf{Proj}(\mathcal{C})$ and $W \in \mathcal{C}$.

A *modified trace* on ideal $\mathbf{Proj}(\mathcal{C})$ is a cyclic trace t on $\mathbf{Proj}(\mathcal{C})$ which is both a left and right trace on $\mathbf{Proj}(\mathcal{C})$.

Next we define the category of H-mod which is a pivotal G-graded category.

3.3.3 Pivotal structure on H-mod

G-graded category

Given a multiplicative group G, we call the category $\mathcal{C}$ pivotal G-graded $\Bbbk$-linear if there exists a family of full subcategories $(\mathcal{C}_\alpha)_{\alpha\in G}$ of $\mathcal{C}$ such that

1. $\mathbb{I} \in \mathcal{C}_1$.
2. $\forall(\alpha,\beta) \in G^2,\ \forall(V,W) \in \mathcal{C}_\alpha \times \mathcal{C}_\beta,\ \mathrm{Hom}_{\mathcal{C}}(V,W) \neq \{0\} \Rightarrow \alpha = \beta$.
3. $\forall V \in \mathcal{C},\ \exists n \in \mathbb{N},\ \exists(\alpha_1,...,\alpha_n) \in G^n,\ \exists V_i \in \mathcal{C}_{\alpha_i}$ for $i = 1,...,n$ such that $V \simeq V_1 \oplus ... \oplus V_n$.
4. $\forall(V,W) \in \mathcal{C}_\alpha \times \mathcal{C}_\beta,\ V \otimes W \in \mathcal{C}_{\alpha\beta}$.
5. $\forall\alpha \in G,\ \mathcal{C}_\alpha$ does not reduce to null object.

Pivotal structure on H-mod

Let $(H,g) = (\{H_\alpha\}_{\alpha\in G}, \Delta, \varepsilon, S, g)$ be a finite type pivotal Hopf G-coalgebra, let $\mathcal{C}$ be the $\Bbbk$-linear category $\bigoplus_{\alpha\in G}\mathcal{C}_\alpha$ in which $\mathcal{C}_\alpha$ is H_α-mod the category of finite dimensional H_α-modules. An object V of $\mathcal{C}$ is a finite direct sum $V_{\alpha_1} \oplus ... \oplus V_{\alpha_n}$ where $V_{\alpha_i} \in \mathcal{C}_{\alpha_i}$. Each object V in H_α-mod has a dual $V^* = \mathrm{Hom}_{\Bbbk}(V,\Bbbk)$ in $H_{\alpha^{-1}}$-mod with the $H_{\alpha^{-1}}$ action defined by $(hf)(x) = f(S_{\alpha^{-1}}(h)x)$ for $h \in H_{\alpha^{-1}},\ f \in V^*$ and $x \in V$. The category $\mathcal{C}$ is a G-graded tensor category, i.e. for $V_\alpha \in \mathcal{C}_\alpha,\ V_\beta \in \mathcal{C}_\beta$ $V_\alpha \otimes V_\beta \in \mathcal{C}_{\alpha\beta}$ and for $\alpha \neq \beta$ $\mathrm{Hom}_{\mathcal{C}}(V_\alpha, V_\beta) = 0$.
Then $\mathcal{C}$ is a pivotal category with pivotal structure given by the left and right duality morphisms as follows. Assume that $\{v_j \mid j \in J\}$ is a basis of

$V \in H_\alpha$-mod and $\{v^j \mid j \in J\}$ is the dual basis of V^*, then

$$\overrightarrow{\mathrm{ev}}_V\colon\ V^*\otimes V \to \Bbbk, \qquad f\otimes v \mapsto f(v), \tag{3.3.6}$$
$$\overrightarrow{\mathrm{coev}}_V\colon\ \Bbbk \to V\otimes V^*, \qquad 1 \mapsto \sum_{j\in J} v_j \otimes v^j,$$
$$\overleftarrow{\mathrm{ev}}_V\colon\ V\otimes V^* \to \Bbbk, \qquad v\otimes f \mapsto f(g_\alpha v), \tag{3.3.7}$$
$$\overleftarrow{\mathrm{coev}}_V\colon\ \Bbbk \to V^*\otimes V, \qquad 1 \mapsto \sum_{i\in J} v^i \otimes g_\alpha^{-1} v_i.$$

We call H-pmod or $\mathbf{Proj}(\mathcal{C})$ the ideal of projective H-modules. As $\mathcal{C} = \bigoplus_{\alpha\in G}\mathcal{C}_\alpha$, the projective modules of $\mathcal{C}_\alpha$ are in H-pmod $\cap\, \mathcal{C}_\alpha = H_\alpha$-pmod.

Lemma 3.3.3. *Let (H, g) be a finite type pivotal Hopf G-coalgebra. Let t be a cyclic trace on H-pmod. Let $V \in H$-pmod and ${}_\varepsilon W \in H_1$-mod be endowed with the trivial action $\rho_{{}_\varepsilon W} = \varepsilon\,\mathrm{Id}_{{}_\varepsilon W}$. Then*

$$\forall f \in \mathrm{End}_{H\text{-}mod}(V\otimes {}_\varepsilon W),\ t_{V\otimes {}_\varepsilon W}(f) = t_V(\mathrm{tr}^r_{{}_\varepsilon W}(f)) \tag{3.3.8}$$

and

$$\forall f \in \mathrm{End}_{H\text{-}mod}({}_\varepsilon W\otimes V),\ t_{{}_\varepsilon W\otimes V}(f) = t_V(\mathrm{tr}^l_{{}_\varepsilon W}(f)). \tag{3.3.9}$$

Proof. Consider a decomposition of $\mathrm{Id}_{{}_\varepsilon W}$

$$\mathrm{Id}_{{}_\varepsilon W} = \sum_{i\in I} e_i\varphi_i \text{ where } \varphi_i :\ {}_\varepsilon W \to \Bbbk,\ e_i :\ \Bbbk \to {}_\varepsilon W,\ \varphi_i(e_j) = \delta_{ij}. \tag{3.3.10}$$

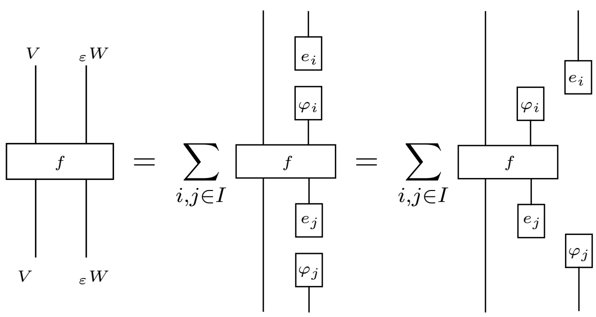
By setting $\widetilde{e}_i = \mathrm{Id}_V \otimes e_i :\ V \to V\otimes {}_\varepsilon W$ and $\widetilde{\varphi}_i = \mathrm{Id}_V \otimes \varphi_i :\ V\otimes {}_\varepsilon W \to V$ one gets

$$\mathrm{Id}_{V\otimes {}_\varepsilon W} = \sum_{i\in I} \widetilde{e}_i\widetilde{\varphi}_i. \tag{3.3.11}$$

For $f \in \mathrm{End}_{H\text{-mod}}(V\otimes {}_\varepsilon W)$, on the one hand we have

$$t_{V\otimes {}_\varepsilon W}(f) = \sum_{i\in I} t_{V\otimes {}_\varepsilon W}(f\widetilde{e}_i\widetilde{\varphi}_i) = \sum_{i\in I} t_V(\widetilde{\varphi}_i f\widetilde{e}_i) = \sum_{i\in I} t_V(f_{ii})$$

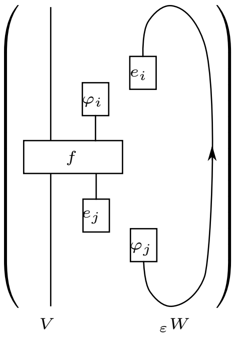
where $f_{ii} = \widetilde{\varphi}_i f\widetilde{e}_i \in \mathrm{End}_{H\text{-mod}}(V)$. In the above calculations, we use Equation (3.3.11) in the first equality and the cyclicity property in the second equality. On the other hand, each map $f \in \mathrm{End}_{H\text{-mod}}(V\otimes {}_\varepsilon W)$ is presented by graph below



$$= \sum_{i,j\in I} f_{ij}\otimes(e_i\varphi_j)\,\mathrm{Id}_{{}_\varepsilon W}$$

where $f_{ij} = \widetilde{\varphi}_i f \widetilde{e}_j \in \mathrm{End}_{H\text{-mod}}(V)$. From this graphical representation implies

$$t_V(\mathrm{tr}^r_{{}_\varepsilon W}(f)) = \sum_{i,j\in I} t_V \left(\right. \quad \left. \right) = \sum_{i\in I} t_V(f_{ii}).$$



Therefore Equality (3.3.8) holds.
Remark that the pivotal element acts trivially on ${}_\varepsilon W$ so the evaluation $\overleftarrow{\mathrm{ev}}_{{}_\varepsilon W}$ in H-mod is just the usual evaluation of $\mathrm{Vect}_{\Bbbk}$.

For Equality (3.3.9) the proof is similar. □

Reduction Lemma

We have a graded version of Reduction Lemma [2, Lemma 3.2]

Lemma 3.3.4. *Let (H, g) be a finite type unimodular pivotal Hopf G-coalgebra and $\lambda = (\lambda^\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H^*_\alpha$ be a family of symmetric linear forms and $\mathsf{t} = (\mathsf{t}^\alpha)_{\alpha\in G}$ be the associated cyclic traces. Then t is a right modified trace on H-pmod if and only if for all $\alpha, \beta \in G$ and for all $f \in \mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta)$*

$$\mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta}(f) = \mathsf{t}^\alpha_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(f)). \tag{3.3.12}$$

Similarly, t is a left modified trace on H-pmod if and only if for all $f \in \mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta)$

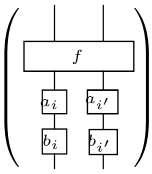
$$\mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta}(f) = \mathsf{t}^\beta_{H_\beta}(\mathrm{tr}^l_{H_\alpha}(f)).$$

Proof. The proof strictly follows the line of Reduction Lemma 3.2 [2]. The necessity is obvious. We now prove the sufficiency of the condition. By Proposition 3.3.2 for each $\alpha \in G$ the symmetric linear form λ^α induces the cyclic trace $\{\mathsf{t}^\alpha_P : \mathrm{End}_{H_\alpha}(P) \to \Bbbk\}_{P\in H_\alpha\text{-pmod}}$. We then prove that the cyclic trace t^α satisfies the right partial trace property.
First, let $P \in H_\alpha$-pmod, $P' \in H_\beta$-pmod and $f \in \mathrm{End}_{H_{\alpha\beta}}(P \otimes P')$. Suppose that Id_P and $\mathrm{Id}_{P'}$ have the decomposition

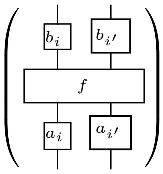
$$\mathrm{Id}_P = \sum a_i \circ b_i, \qquad \mathrm{Id}_{P'} = \sum a_{i'} \circ b_{i'} \tag{3.3.13}$$

where $a_i : H_\alpha \to P,\ b_i : P \to H_\alpha$ and $a_{i'} : H_\beta \to P',\ b_{i'} : P' \to H_\beta$. The modified trace of f is calculated as follows:

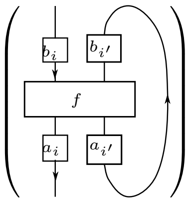
$$\mathsf{t}^{\alpha\beta}_{P\otimes P'}(f) = \mathsf{t}^{\alpha\beta}_{P\otimes P'}$$



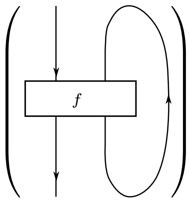
$$= \mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta} \tag{3.3.14}$$



$$= \mathsf{t}^{\alpha}_{H_\alpha}$$



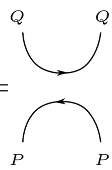
$$= \mathsf{t}^{\alpha}_{P}$$



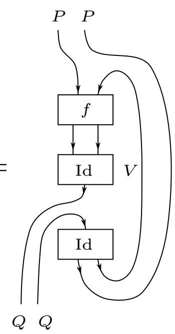
$$= \mathsf{t}^{\alpha}_{P}\left(\mathrm{tr}^r_{P'}(f)\right).$$

In this calculation, one uses (3.3.13) in the first equality, in the second equality one uses the cyclicity property of cyclic traces, the third equality thanks to (3.3.12) and finally one uses the duality morphisms to move $b_{i'}$ around the loop then applying again (3.3.13) and the cyclicity property.
Second, let $P \in H_\alpha$-pmod, $V \in H_\beta$-mod and $f \in \mathrm{End}_{H_{\alpha\beta}}(P\otimes V)$. Set $Q = P\otimes V$, note that $Q \in H_{\alpha\beta}$-pmod and $P\otimes P^*,\ Q\otimes Q^* \in H_1$-pmod. Consider two morphisms $A \in \mathrm{Hom}_{H_1\text{-mod}}(P\otimes P^*, Q\otimes Q^*)$ and $B \in \mathrm{Hom}_{H_1\text{-mod}}(Q\otimes Q^*, P\otimes P^*)$ are given by

$A =$



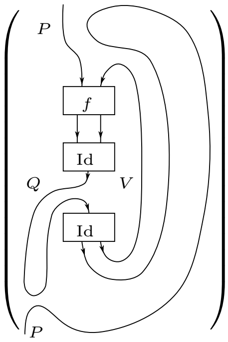
, $B =$

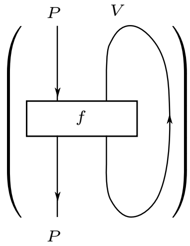


.

According to (3.3.14) one gets

$$\mathsf{t}^{1}_{P\otimes P^*}(B\circ A) = \mathsf{t}^{\alpha}_{P}\left(\mathrm{tr}^r_{P^*}(B\circ A)\right)$$

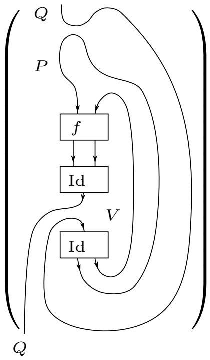
$= \mathsf{t}_P^{\alpha}$ 

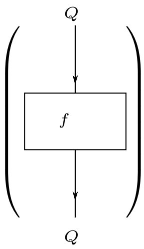
$= \mathsf{t}_P^{\alpha}$ 

$= \mathsf{t}_P^{\alpha}\left(\mathrm{tr}_V^r(f)\right).$

In above calculation, one applies the definition of the partial trace in second equality, in the third equality one uses the properties of the pivotal structure. Similarly we also have

$$\mathsf{t}^1_{Q\otimes Q^*}(A\circ B) = \mathsf{t}_Q^{\alpha\beta}\left(\mathrm{tr}^r_{Q^*}(A\circ B)\right)$$

$= \mathsf{t}_Q^{\alpha\beta}$ 

$= \mathsf{t}_Q^{\alpha\beta}$ 

$= \mathsf{t}^{\alpha\beta}_{P\otimes V}(f).$

Since the cyclicity property $\mathsf{t}^1_{P\otimes P^*}(B\circ A) = \mathsf{t}^1_{Q\otimes Q^*}(A\circ B)$, it follows that $\mathsf{t}^{\alpha\beta}_{P\otimes V}(f) = \mathsf{t}^{\alpha}_P\left(\mathrm{tr}^r_V(f)\right)$.
The proof in the case of the left modified trace is similar. $\square$

3.3.4 Applications of Theorem 3.1.1

Theorem 3.1.1 has two immediate consequences when $G = \{1\}$ or H is semi-simple. First, in degree 1 the symmetrised G-integral is also the symmetrised integral of H_1 and Theorem 3.1.1 recovers the main theorem of [2] that we recall here:

Theorem 3.3.5 ([2]). *Let (H, g) be a finite dimensional unimodular pivotal Hopf algebra over a field $\Bbbk$. Then the space of right (left) modified traces on H-pmod is equal to the space of symmetrised right (left) integrals, and hence is 1-dimensional. Moreover, the right modified trace on H-pmod is non-degenerate and determined by the cyclicity property and by*

$$\mathsf{t}_H(f) = \mu(gf(1)) \quad \textit{for any } f \in \mathrm{End}_H(H)\ .$$

Similarly, the left modified trace is non degenerate and determined by

$$\mathsf{t}_H(f) = \mu^l(g^{-1}f(1)) \quad \textit{for any } f \in \mathrm{End}_H(H)\ .$$

In particular, H is unibalanced if and only if the right modified trace is also left.

Second, for a finite type unimodular pivotal Hopf G-coalgebra (H, g), if H is semi-simple, i.e. H_α is semi-simple for all $\alpha \in G$ then H-pmod $= \mathcal{C}$. Then the categorical trace generates the space of modified traces on H-pmod: for any $f \in \mathrm{End}_\mathcal{C}(V)$, the right and left categorical trace are

$$\mathrm{tr}_V^\mathcal{C}(f) := \overleftarrow{\mathrm{ev}}_V\ (f \otimes \mathrm{Id}_V)\ \overrightarrow{\mathrm{coev}}_V \in \Bbbk,$$
$$^\mathcal{C}\,\mathrm{tr}_V(f) := \overrightarrow{\mathrm{ev}}_V\ (\mathrm{Id}_{V^*} \otimes f)\ \overleftarrow{\mathrm{coev}}_V \in \Bbbk.$$

As a corollary of Theorem 3.1.1 we then have the proposition.

Proposition 3.3.6. *Let (H, g) be a finite type unimodular pivotal Hopf G-coalgebra over a field $\Bbbk$. The right categorical trace $\mathrm{tr}^\mathcal{C}_{H_\alpha}$ and its left version $^\mathcal{C}\,\mathrm{tr}_{H_\alpha}$ are non-zero if and only if H_α-mod is semi-simple and in this case coincide up to a scalar with the trace maps*

$$f \mapsto \widetilde{\mu}_\alpha(f(1_\alpha)) \textit{ and } f \mapsto \widetilde{\mu}^l_\alpha(f(1_\alpha))$$

respectively, where $f \in \mathrm{End}_{H_\alpha}(H_\alpha)$.

3.4 Proof of the main theorem

3.4.1 Decomposition of tensor products of the regular representations

We denote by H_α the left H_α-module given by the left regular action. Let us denote by ${}_\varepsilon H_\beta$ the vector space underlying H_β equipped with the H_1-module structure given by

$$h.m = \varepsilon(h)m \text{ for } m \in {}_\varepsilon H_\beta,\ h \in H_1.$$

We will use Sweedler's notation: $\Delta_{\alpha,\beta}(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H_{\alpha\beta}$, $h_{(1)} \in H_\alpha, h_{(2)} \in H_\beta$.

Theorem 3.4.1. *Let $H = (H_\alpha)_{\alpha\in G}$ be a finite type Hopf G-coalgebra. Then*

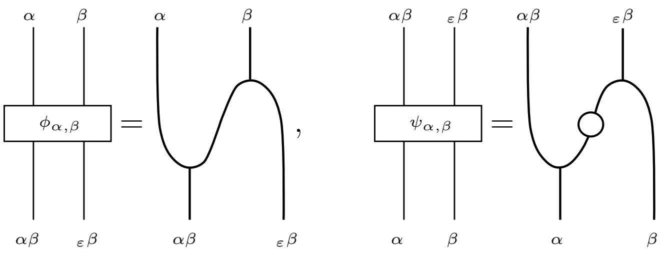


Figure 3.6 – The graphical representations of $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$

(1) the map

$$\begin{array}{rrcl}\phi_{\alpha,\beta}: & H_{\alpha\beta}\otimes {}_{\varepsilon}H_{\beta} & \to & H_{\alpha}\otimes H_{\beta}\\ & h\otimes m & \mapsto & h_{(1)}\otimes h_{(2)}m\end{array}$$

is an isomorphism of $H_{\alpha\beta}$-modules whose inverse is

$$\begin{array}{rrcl}\psi_{\alpha,\beta}: & H_{\alpha}\otimes H_{\beta} & \to & H_{\alpha\beta}\otimes {}_{\varepsilon}H_{\beta}\\ & x\otimes y & \mapsto & x_{(1)}\otimes S_{\beta^{-1}}(x_{(2)})y.\end{array}$$

(2) the map

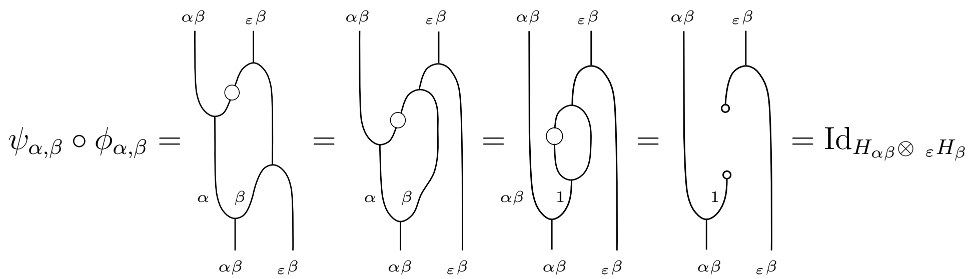
$$\begin{array}{rrcl}\phi^{l}_{\alpha,\beta}: & {}_{\varepsilon}H_{\alpha}\otimes H_{\alpha\beta} & \to & H_{\alpha}\otimes H_{\beta}\\ & m\otimes h & \mapsto & h_{(1)}m\otimes h_{(2)}\end{array}$$

is an isomorphism of $H_{\alpha\beta}$-modules whose inverse is

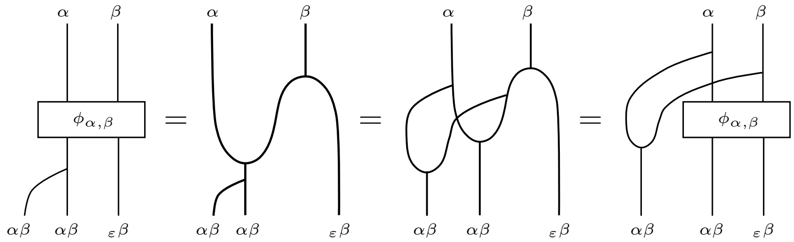
$$\begin{array}{rrcl}\psi^{l}_{\alpha,\beta}: & H_{\alpha}\otimes H_{\beta} & \to & {}_{\varepsilon}H_{\alpha}\otimes H_{\alpha\beta}\\ & x\otimes y & \mapsto & S^{-1}_{\alpha^{-1}}(y_{(1)})x\otimes y_{(2)}.\end{array}$$

We prove the theorem using graphical calculus with the graphical representations for Hopf G-coalgebras given in Section 3.2.1. The maps $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ are presented in Figure 3.6. The graphical representations for $\phi^{l}_{\alpha,\beta}$ and $\psi^{l}_{\alpha,\beta}$ are similar.

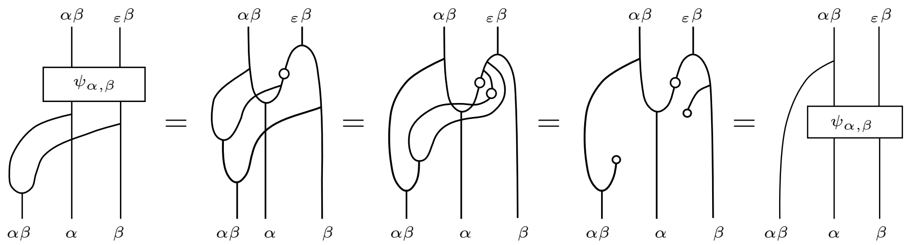
Proof. In order to prove part (1), we first check that $\phi_{\alpha,\beta}$ is left inverse to $\psi_{\alpha,\beta}$, by computing the composition one gets



where the associativity of the product m_β is used in the first equality, then we use the coassociativity of the coproduct in the second equality, and finally we use the antipode properties in the last equality. Similarly we have $\phi_{\alpha,\beta}\circ\psi_{\alpha,\beta} = \mathrm{Id}_{H_\alpha\otimes H_\beta}$.
Next we prove the map $\phi_{\alpha,\beta}$ is $H_{\alpha\beta}$-linear by diagrammic calculus:



where we used the property of the algebra homomorphism $\Delta_{\alpha,\beta}$ in the second equality and the associativity of multiplication in the third equality. The map $\psi_{\alpha,\beta}$ is also $H_{\alpha\beta}$-linear by:



where we used the property of the algebra homomorphism $\Delta_{\alpha,\beta}$ in the first equality, the coassociativity of coproduct and the antipode properties are used in the second equality, the associativity of multiplication and the antipode properties are used in the third equality, and we used the antipode properties in the last equality.

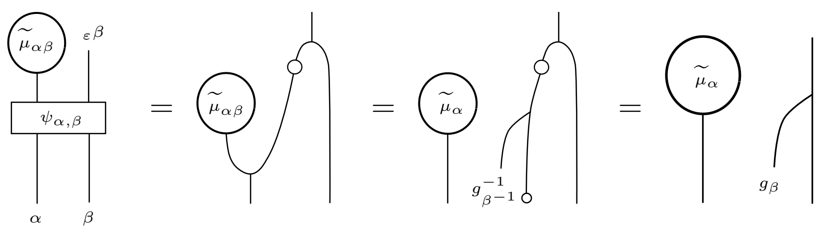
The proof of the part (2) is similar way. □

Proposition 3.4.2. *Let $H = (H_\alpha)_{\alpha\in G}$ be a finite type pivotal Hopf G-coalgebra. Then we have the equalities of linear maps:*

(1) $\phi_{\alpha,\beta}(1_{\alpha\beta}\otimes m) = 1_\alpha \otimes m$ *for* $m \in {}_\varepsilon H_\beta$,

(2) $(\widetilde{\mu}_{\alpha\beta}\otimes \mathrm{Id}'_\beta)\circ\psi_{\alpha,\beta} = \widetilde{\mu}_\alpha \otimes g_\beta \,\mathrm{Id}'_\beta$ *where* $\mathrm{Id}'_\beta : H_\beta \to {}_\varepsilon H_\beta$ *is the identity map in* $\mathrm{Vect}_\Bbbk$.

Proof. The equality (1) holds by the definition of the map $\phi_{\alpha,\beta}$. Part (2) follows from the diagrammic calculus in $\mathrm{Vect}_\Bbbk$:


(3.4.1)

where in the second equality of (3.4.1) we used the relation of the right symmetrised G-integral in Figure 3.5. □

3.4.2 Proof of Theorem 3.1.1

Let (H, g) be a finite type unimodular pivotal Hopf G-coalgebra, $\mathcal{C}$ be the pivotal G-graded category of H-modules. The existence of modified trace on $\mathbf{Proj}(\mathcal{C})$ follows from: 1) the existence of non-zero integral on H_1 2) the existence of modified trace in $\mathcal{C}_1$ by applying the results of [2] for H_1 and 3) the existence of the extension of ambidextrous trace in [18, Theorem 3.6]. Nevertheless we choose to give a direct proof of this fact following the lines of [2]. Furthermore, Theorem 3.1.1 also gives an explicit formula to compute the modified trace t from the integral and conversely.

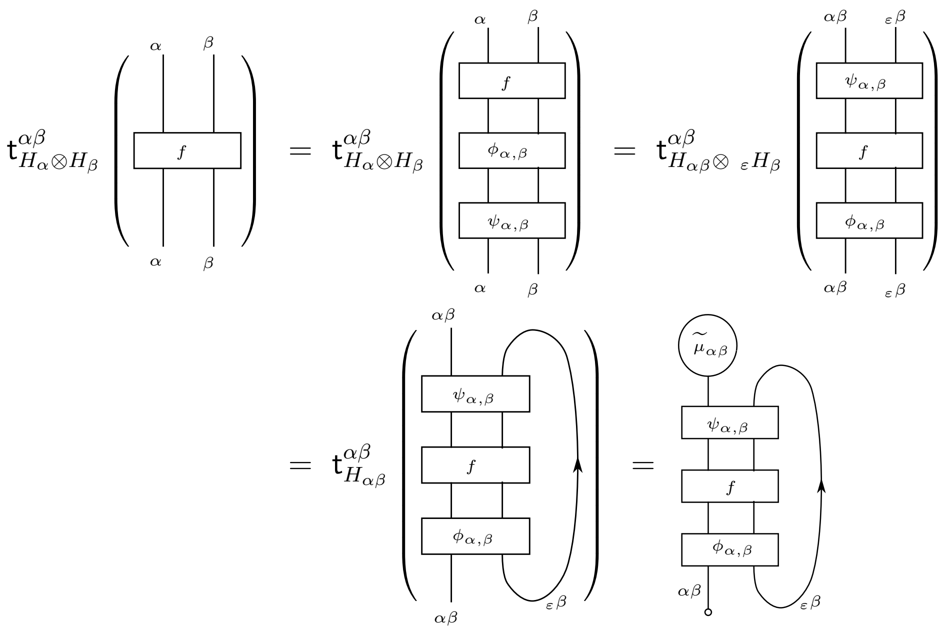
Proof of Theorem 3.1.1. First, we show that a right symmetrised G-integral provides a modified trace. Suppose that $\widetilde{\mu} = (\widetilde{\mu}_\alpha)_{\alpha\in G}$ is the right symmetrised G-integral for H. By Proposition 3.3.2 the family of the symmetric forms associated with $\widetilde{\mu}$ induces the family of cyclic traces $\mathsf{t} = (\mathsf{t}^\alpha)_{\alpha\in G}$ of H-pmod. Here $\mathsf{t}^\alpha = \{\mathsf{t}^\alpha_P : \operatorname{End}_{H_\alpha}(P) \to \Bbbk\}_{P\in H_\alpha\text{-pmod}}$ is determined by

$$\mathsf{t}^\alpha_{H_\alpha}(f) = \widetilde{\mu}_\alpha(f(1_\alpha)) \text{ for } f \in \operatorname{End}_{H_\alpha}(H_\alpha). \tag{3.4.2}$$

To show t is a modified trace, it is enough to check

$$\mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta}(f) = \mathsf{t}^\alpha_{H_\alpha}(\operatorname{tr}^r_{H_\beta}(f)) \text{ for any } f \in \operatorname{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta) \tag{3.4.3}$$

thanks to Reduction Lemma 3.3.4. The value of $\mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta}(f)$ in Equality (3.4.3) is calculated



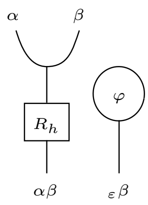
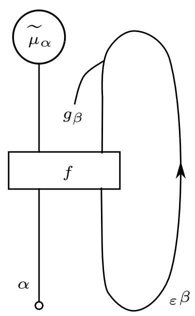


Figure 3.7 – The graphical representation of the map k



$$= \quad = \quad \mathsf{t}^{\alpha}_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(f)) \; .$$

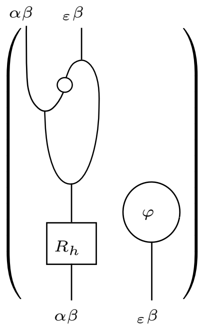
In the above calculation, we use Theorem 3.4.1 in the first equality; the cyclicity property of trace in the second equality; Lemma 3.3.3 in the third equality; Equation (3.4.2) in the fourth equality and in the fifth equality we use the two equalities in Proposition 3.4.2.

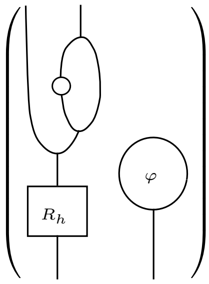
Second, assume that we have a right modified trace, and hence the symmetric form t^{α}_{P} on $\mathrm{End}_{H_\alpha}(P)$ for any projective module P and any $\alpha \in G$. In particular for any $\alpha, \beta \in G$ the symmetric forms $\mathsf{t}^{\alpha}_{H_\alpha}$ on $\mathrm{End}_{H_\alpha}(H_\alpha)$ and $\mathsf{t}^{\alpha\beta}_{H_\alpha \otimes H_\beta}$ on $\mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta)$ satisfy

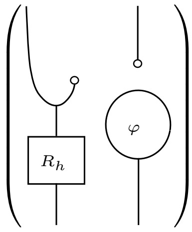
$$\mathsf{t}^{\alpha\beta}_{H_\alpha \otimes H_\beta}(f) = \mathsf{t}^{\alpha}_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(f)) \text{ for any } f \in \mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta). \tag{3.4.4}$$

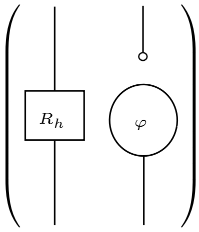
Let $\widetilde{\nu}_\alpha(h) = \mathsf{t}^{\alpha}_{H_\alpha}(R_h)$ for $R_h \in \mathrm{End}_{H_\alpha}(H_\alpha)$ with $h \in H_\alpha$. Then $\widetilde{\nu}_\alpha(f(1_\alpha)) = \mathsf{t}^{\alpha}_{H_\alpha}(f)$ for $f \in \mathrm{End}_{H_\alpha}(H_\alpha)$ (see Lemma 3.3.1). We prove that the family $\widetilde{\nu} = (\widetilde{\nu}_\alpha)_{\alpha \in G}$ satisfies the relation of the right symmetrised G-integral.
Consider the maps $k = \Delta_{\alpha,\beta} \circ (R_h \otimes \varphi) : H_{\alpha\beta} \otimes {}_{\varepsilon}H_\beta \to H_\alpha \otimes H_\beta$ for $h \in H_{\alpha\beta}$ and $\varphi \in {}_{\varepsilon}H^*_\beta$. Then k is a morphism of $H_{\alpha\beta}$-modules. The graphical representation of the map k is given in Figure 3.7. Let $\widetilde{f} = k \circ \psi_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_\alpha \otimes H_\beta$ then $\widetilde{f} \in \mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta)$. We now calculate the values of the modified trace for $\widetilde{f} \in \mathrm{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta)$ and $\mathrm{tr}^r_{H_\beta}(\widetilde{f}) \in \mathrm{End}_{H_\alpha}(H_\alpha)$.
On the one hand, we have

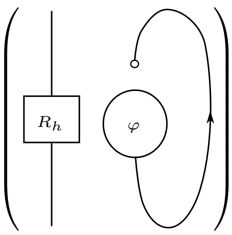
$$\mathsf{t}^{\alpha\beta}_{H_\alpha \otimes H_\beta}(\widetilde{f}) = \mathsf{t}^{\alpha\beta}_{H_\alpha \otimes H_\beta}(k \circ \psi_{\alpha,\beta}) = \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta} \otimes \, {}_{\varepsilon}H_\beta}(\psi_{\alpha,\beta} \circ k)$$

$= \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta}\otimes\, {}_{\varepsilon}H_{\beta}}$ 

$= \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta}\otimes\, {}_{\varepsilon}H_{\beta}}$ 

$= \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta}\otimes\, {}_{\varepsilon}H_{\beta}}$ 

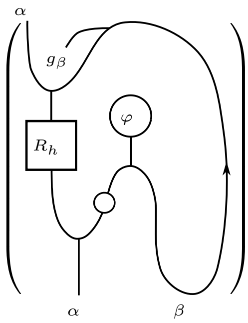
$= \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta}\otimes\, {}_{\varepsilon}H_{\beta}}$ 

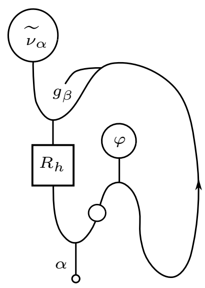
$= \mathsf{t}^{\alpha\beta}_{H_{\alpha\beta}}$ 

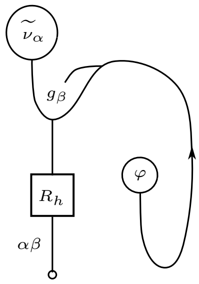
$= \widetilde{\nu}_{\alpha\beta}(h)\varphi(1_\beta).$

In the above calculations, we use the cyclicity property in the second equality; the coassociativity of the coproduct in the fourth equality; the antipode properties in the fifth equality and finally we use the partial trace property. On the other hand, we have

$$\mathsf{t}^{\alpha}_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(\widetilde{f})) = \mathsf{t}^{\alpha}_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(k \circ \psi_{\alpha,\beta}))$$

$= \mathsf{t}^{\alpha}_{H_\alpha}$ 

$=$ 

$=$  $= \widetilde{\nu}_\alpha(h_{(1)})\varphi(g_\beta h_{(2)})$

where we use the left evaluation $\overleftarrow{\mathrm{ev}}$ with the pivot g_β and the right coevaluation $\overrightarrow{\mathrm{coev}}$ in the second equality and $\Delta_{\alpha,\beta}(h) = h_{(1)} \otimes h_{(2)}$.
By Equality (3.4.4) one has $\mathsf{t}^{\alpha\beta}_{H_\alpha\otimes H_\beta}(\widetilde{f}) = \mathsf{t}^{\alpha}_{H_\alpha}(\mathrm{tr}^r_{H_\beta}(\widetilde{f}))$. This equality means that

$$\widetilde{\nu}_{\alpha\beta}(h)\varphi(1_\beta) = \widetilde{\nu}_\alpha(h_{(1)})\varphi(g_\beta h_{(2)}) \text{ for any } \varphi \in {}_\varepsilon H^*_\beta,\ h \in H_{\alpha\beta}.$$

This equality holds for any $\varphi \in {}_\varepsilon H^*_\beta$ implies that $\widetilde{\nu}_{\alpha\beta}(h)1_\beta = \widetilde{\nu}_\alpha(h_{(1)})g_\beta h_{(2)}$, i.e. $(\widetilde{\nu}_\alpha \otimes g_\beta)\Delta_{\alpha,\beta}(h) = \widetilde{\nu}_{\alpha\beta}(h)1_\beta$ for any $h \in H_{\alpha\beta}$. Therefore the family $\widetilde{\nu} = (\widetilde{\nu}_\alpha)_{\alpha\in G}$ is the right symmetrised G-integral for H.

For the case of the left modified trace the proof is similar. □

3.5 Modified trace for G-graded quantum $\mathfrak{sl}(2)$

In this section we present the symmetrised G-integral for the quantization of $\mathfrak{sl}(2)$ and the modified trace on ideal of projective modules of category of the weight modules over $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$. It explains clearly the relation between the symmetrised G-integral for a pivotal Hopf G-coalgebra and the modified trace in associated category $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$-mod.

3.5.1 Unrestricted quantum $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$

Let $\mathcal{U}_q\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra given by generators E, F, K, K^{-1} and relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

where $q = e^{\frac{i\pi}{r}}$ is a $2r^{th}$-root of unity. The algebra $\mathcal{U}_q\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned}
&\Delta(E) = 1\otimes E + E\otimes K, && \varepsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1}\otimes F + F\otimes 1, && \varepsilon(F) = 0, && S(F) = -KF,\\
&\Delta(K) = K\otimes K && \varepsilon(K) = 1, && S(K) = K^{-1}.
\end{aligned}$$

Let $\overline{\mathcal{U}} := \overline{\mathcal{U}}_q\mathfrak{sl}(2)$ be the algebra $\mathcal{U}_q\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$ and $C = \mathbb{C}[K^{\pm r}]$ be the commutative Hopf subalgebra in the center of $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$. The algebra $\overline{\mathcal{U}}$ is a pivotal Hopf algebra with the pivot $g = K^{1-r}$. Let $G = (\mathbb{C}/2\mathbb{Z}, +) \xrightarrow{\sim} \mathrm{Hom}_{Alg}(C, \mathbb{C})$, $\overline{\alpha} \mapsto (K^r \mapsto q^{r\alpha} := e^{i\pi\alpha})$ and let $\mathcal{U}_{\overline{\alpha}}$ be the algebra $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$ modulo the relations $K^r = q^{r\alpha}$ for $\overline{\alpha} \in G$. By applying Example 3.2.3 it follows that $\mathcal{U} = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is the Hopf G-coalgebra with the coproduct and the antipode are determined by the commutative diagrams:

$$\begin{array}{ccc} \overline{\mathcal{U}} & \xrightarrow{\Delta} & \overline{\mathcal{U}} \otimes \overline{\mathcal{U}} \\ \downarrow{\scriptstyle p_{\overline{\alpha}+\overline{\beta}}} & & \downarrow{\scriptstyle p_{\overline{\alpha}} \otimes p_{\overline{\beta}}} \\ \mathcal{U}_{\overline{\alpha}+\overline{\beta}} & \xrightarrow{\Delta_{\overline{\alpha},\overline{\beta}}} & \mathcal{U}_{\overline{\alpha}} \otimes \mathcal{U}_{\overline{\beta}} \end{array} \qquad \begin{array}{ccc} \overline{\mathcal{U}} & \xrightarrow{S} & \overline{\mathcal{U}} \\ \downarrow{\scriptstyle p_{\overline{\alpha}}} & & \downarrow{\scriptstyle p_{-\overline{\alpha}}} \\ \mathcal{U}_{\overline{\alpha}} & \xrightarrow{S_{\overline{\alpha}}} & \mathcal{U}_{-\overline{\alpha}} \end{array}$$

where $p_{\overline{\alpha}} : \overline{\mathcal{U}} \to \mathcal{U}_{\overline{\alpha}}$ is the projective morphism from $\overline{\mathcal{U}}$ to $\mathcal{U}_{\overline{\alpha}}$. The Hopf G-coalgebra $\mathcal{U} = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ has the pivotal structure given by $g_{\overline{\alpha}} = q^{-r\alpha}K$. For $\overline{\alpha} = \overline{0}$ the Hopf algebra $\mathcal{U}_{\overline{0}}$ is called the restricted quantum $\mathfrak{sl}(2)$, i.e. the algebra $\mathcal{U}_q\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$ and $K^r = 1$. The right $\overline{0}$-integral is the usual right integral given by

$$\mu_{\overline{0}}(E^m F^n K^l) = \eta\delta_{m,r-1}\delta_{n,r-1}\delta_{l,1}$$

where η is a constant. By definition of right G-integral (3.1.1) we get

$$\mu_{\overline{\alpha}}(E^m F^n K^l) = q^{r\alpha}\eta\delta_{m,r-1}\delta_{n,r-1}\delta_{l,1}.$$

One can show that the Hopf G-coalgebra $\{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is G-unibalanced. The symmetrised right G-integral for $\{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is determined by

$$\widetilde{\mu}_{\overline{\alpha}}(E^m F^n K^l) = \eta\delta_{m,r-1}\delta_{n,r-1}\delta_{l,0}. \tag{3.5.1}$$

3.5.2 Modified trace

Let $\mathcal{C}$ be the category of representations of the Hopf G-coalgebra $\mathcal{U}$ (see Section 3.3.3). Then $\mathcal{C}$ is equal to the G-graded category of finite dimensional weight modules over $\overline{\mathcal{U}}_q\mathfrak{sl}(2)$ (module in which K has a diagonalizable action). For $\alpha \in \mathbb{C}$ let V_α be a r-dimensional highest weight module of highest weight $\alpha + r - 1$ in $\mathcal{C}$ (see [9]). Recall the modified dimension $d(V_\alpha)$ of V_α for $\alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$ was computed:

$$d(V_\alpha) = \mathsf{t}_{V_\alpha}(\mathrm{Id}_{V_\alpha}) = d_0 \prod_{k=1}^{r-1} \frac{\{k\}}{\{\alpha + r - k\}} = d_0 \frac{r\{\alpha\}}{\{r\alpha\}} \tag{3.5.2}$$

where t is the modified trace on ideal $\mathbf{Proj}(\mathcal{C})$ of projective modules and d_0 is a non-zero complex number. In [9] for the analogous unrolled category, it is normalized by $d_0 = (-1)^{r-1}$. We now present the way to compute the modified dimension of V_α using the symmetrised G-integral.
By density theorem we have the isomorphism of algebras

$$\mathcal{U}_{\overline{\alpha}} \xrightarrow{\sim} \bigoplus_{k \in H_r} \mathrm{End}(V_{\alpha+2k})$$

where $H_r = \{-(r-1), -(r-3), ..., r-1\}$. Hence we have the isomorphism of left $\mathcal{U}_{\overline{\alpha}}$-modules:

$$\mathcal{U}_{\overline{\alpha}} \xrightarrow{\sim} \bigoplus_{k \in H_r} \mathrm{End}(V_{\alpha+2k}) \xrightarrow{\sim} \bigoplus_{k \in H_r} V_{\alpha+2k} \otimes {}_{\varepsilon}V^*_{\alpha+2k} \ .$$

Consider the quantum Casimir element of $\overline{\mathcal{U}}$ defined by

$$\Omega = FE + \frac{Kq + K^{-1}q^{-1}}{\{1\}^2} = EF + \frac{Kq^{-1} + K^{-1}q}{\{1\}^2}.$$

For $k \in \mathbb{N}$, by induction one gets

$$\prod_{i=0}^{k-1} \left(\Omega - \frac{q^{-2i-1}K + q^{2i+1}K^{-1}}{\{1\}^2} \right) = E^k F^k. \tag{3.5.3}$$

Lemma 3.5.1. *For $k \in \mathbb{N}$ then*

$$\Omega^k - E^k F^k \in Span_{\mathbb{C}}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}.$$

Proof. The proof is by induction on k. Indeed, by (3.5.3) $\Omega^k - E^k F^k \in \mathrm{Span}_{\mathbb{C}}\{\Omega^j K^i \mid j < k, \ i \in \mathbb{Z}\}$ which by the induction hypothesis is contained in $\mathrm{Span}_{\mathbb{C}}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}$. □

Following (3.5.1) we have the corollary.

Corollary 3.5.2. *For all $k \in \{0, ..., r-2\}$ we have $\widetilde{\mu}_{\overline{\alpha}}\left(\Omega^k\right) = 0$. For $k = r-1$ then $\widetilde{\mu}_{\overline{\alpha}}(\Omega^{r-1}) = \eta$.*

Proof. It follows from (3.5.1) that $\mathrm{Span}_{\mathbb{C}}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}$ is contained in the kernel of $\widetilde{\mu}_{\overline{\alpha}}$ for $k \in \{0, ..., r-2\}$. □

For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, Ω acts on V_α by the scalar w_α which is calculated as follows: Let v be a highest weight vector of V_α. The action of K on v defined by $Kv = q^{\alpha+r-1}v$. This implies that $\Omega v = \frac{q^{\alpha+r}+q^{-\alpha-r}}{\{1\}^2}v$, i.e. $w_\alpha = \frac{q^{\alpha+r}+q^{-\alpha-r}}{\{1\}^2}$. The elements $w_{\alpha+2k}$, $0 \leq k < r-1$ are distinct as $w_{\alpha+2i} - w_{\alpha+2j} =$

$\frac{\{i-j\}\{\alpha+r+i+j\}}{\{1\}^2} \neq 0$ for $i \neq j$.
We consider in $\mathcal{U}_{\overline{\alpha}}$ the element

$$L_\alpha(\Omega) = \frac{\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})}{\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k})}.$$

This element is the projector on $V_\alpha \otimes {}_\varepsilon V_\alpha^* \simeq \bigoplus_{k=1}^{r} V_\alpha$ as $L_\alpha(w_{\alpha+2k}) = \delta_{0,k}$.
The value of symmetrised right G-integral on $L_\alpha(\Omega)$ is

$$\widetilde{\mu}_{\overline{\alpha}}\left(L_\alpha(\Omega)\right) = \frac{1}{\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k})}\widetilde{\mu}_{\overline{\alpha}}\left(\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})\right).$$

Corollary 3.5.2 implies that

$$\widetilde{\mu}_{\overline{\alpha}}\left(\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})\right) = \widetilde{\mu}_{\overline{\alpha}}\left(\Omega^{r-1}\right) = \eta.$$

The equality $\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k}) = (-1)^{r-1}\prod_{k=1}^{r-1}\frac{\{k\}\{\alpha+k\}}{\{1\}^2}$ gives

$$\begin{aligned}\widetilde{\mu}_{\overline{\alpha}}\left(L_\alpha(\Omega)\right) &= (-1)^{r-1}\eta\prod_{k=1}^{r-1}\frac{\{1\}^2}{\{k\}\{\alpha+k\}}\\ &= \eta\prod_{k=1}^{r-1}\frac{\{1\}^2}{\{k\}^2}(-1)^{r-1}\prod_{k=1}^{r-1}\frac{\{k\}}{\{\alpha+r-k\}} = \frac{\{1\}^{2r-2}\eta}{r^3 d_0} r d(V_\alpha)\end{aligned}$$

where we used the identity $\prod_{k=1}^{r-1}\{k\}^2 = (-1)^{r-1}r^2$ in the last equality.
It is clear that the coefficient $\frac{\{1\}^{2r-2}\eta}{r^3 d_0}$ does not depend on α. This proves that $\widetilde{\mu}_{\overline{\alpha}}\left(L_\alpha(\Omega)\right) = r d(V_\alpha)$ with the choice $d_0 = \frac{\{1\}^{2r-2}\eta}{r^3}$ where $\eta = \widetilde{\mu}_{\overline{\alpha}}\left(E^{r-1}F^{r-1}\right)$.

Chapter 4

A Hennings type invariant of 3-manifolds from a topological Hopf superalgebra

This chapter is the content of the paper [20] available in https://arxiv.org/abs/1806.08277.

RÉSUMÉ. Nous prouvons que la superalgèbre quantique déroulée associée à la superalgèbre de Lie $\mathfrak{sl}(2|1)$ a une complétion qui est une superalgèbre enrubannée au sens topologique. En utilisant cette superalgèbre topologique enrubannée, nous construisons un invariant universel d'entrelacs. Nous l'utilisons pour construire un invariant de 3-variétés de type Hennings.

ABSTRACT. We prove the unrolled quantum superalgebra associated with the super Lie algebra $\mathfrak{sl}(2|1)$ has a completion which is a ribbon superalgebra in a topological sense. Using this topological ribbon superalgebra we construct its universal invariant of links. We use it to construct an invariant of 3-manifolds of Hennings type.

MSC: 57M27, 17B37

Key words: Lie superalgebra, unrolled quantum group, G-integral, invariant of 3-manifolds, Hennings invariant, topological Hopf superalgebra.

4.1 Introduction

The notion of an unrolled quantum group is introduced in [16] by N. Geer and B. Patureau-Mirand. Then an unrolled quantum group is a quantum group with some additional generators which should be thought of the logarithms of some other generators, for example in $\mathcal{U}_q^H\mathfrak{sl}(2)$ the additional generator is an element H with the relation $q^H = K$ (see [9, 16, 17]). This element H is a tool to construct a ribbon structure on representations of $\mathcal{U}_q^H\mathfrak{sl}(2)$. The category of weight modules of $\mathcal{U}_q^H\mathfrak{sl}(2)$ is ribbon and not semi-simple but the Hopf algebra is not ribbon. With this category $\mathcal{U}_q^H\mathfrak{sl}(2)$-mod one constructed the invariants of links and of 3-manifolds (see [9, 39]). For the Lie superalgebra $\mathfrak{sl}(2|1)$, the associated unrolled quantum group is denoted by $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ with two additional generators $h_1,\ h_2$ from the quantum group $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. Using this unrolled quantum group in Chapter 2 one has shown that the category $\mathscr{C}^H$ of nilpotent weight modules over $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is ribbon and relative G-(pre)modular and leads to an invariant of links and of 3-manifolds. The category $\mathscr{C}^H$ is ribbon thanks to the role of the additional elements h_1, h_2 which should be thought as the logarithms of $k_1,\ k_2$, i.e. $\xi^{h_i} = k_i$ for $i = 1, 2$. They help to construct quasitriangular ribbon structure in $\mathscr{C}^H$. The relations $\xi^{h_i} = k_i$ for $i = 1, 2$ also suggest that $k_1,\ k_2$ can be consider as holomorphic functions of h_1 and h_2 on $\mathbb{C}^2$. Following this idea we extend the superalgebra $\mathcal{U}^H = \mathcal{U}_\xi^H\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$ to a ribbon superalgebra $\widehat{\mathcal{U}^H}$ in a topological sense, the topology determined by the norm of uniform convergence on compact sets. Its bosonization $\widehat{\mathcal{U}^H}^\sigma$ is a ribbon algebra (see in Section 4.2.3).

It is known that for each ribbon Hopf algebra one can construct a universal link invariant (all links are framed and oriented) (see [23], [38]). In fact, one can show (see [3]) that a double braiding in a Hopf algebra is enough to construct a universal invariant for string links or bottom tangles. From some universal link invariants one could construct a 3-manifold invariant. There are many ways to do this. In [26], M. Hennings introduced a method of building an invariant of 3-manifolds by using a universal link invariant and a right integral. He worked with a finite dimensional ribbon algebra and this condition guarantees the existence of a right integral. In other way, A. Virelizier and V. Turaev constructed the invariants which called invariants of π-links and invariants of π-manifolds, the invariants of equivalence class of π-bundles or equivalently of manifolds equipped with a map from the fundamental group to π (see [48]). They began with a ribbon Hopf π-coalgebra of finite type to construct the invariants of π-links, after that they renormalized the invariant to invariant of π-manifolds by using π-integrals (see Chapter 3). Note that the π-integrals exist if and only if the group-coalgebra is of finite type (see [49]). From a Hopf algebra one can construct a Hopf group-coalgebra (see Chapter 3). In our case $\pi = G = (\mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}, +)$ is commutative, therefor a G-structure on a manifold M is a cohomology class

$\omega \in H^1(M,G)$ and a G-link is a G-structure on the complement of a link. The Hopf G-coalgebra associated with the ribbon algebra $\widehat{\mathcal{U}^H}^\sigma$ consists of quotients of $\mathcal{U}^H_{\overline{\alpha}}$ by the ideal $(k_i^\ell - \xi^{\ell\alpha_i},\ i=1,2)$ is ribbon but is not of finite type, i.e. the method of construction the invariant of 3-manifolds in [48] does not work here. We show that for $\widehat{\mathcal{U}^H}^\sigma$ there is an associated Hopf G-coalgebra $\mathcal{U}^\sigma$(see in Section 4.4) which is of finite type but is not ribbon. We will present an another approach to construct an invariant of 3-manifolds from $\widehat{\mathcal{U}^H}^\sigma$. We will use first the topological ribbon structure of $\widehat{\mathcal{U}^H}^\sigma$ to construct a universal invariant of links. The value of this invariant is represented by a product of a part which is a holomorphic function of variables $h_1,\ h_2$ and a part of elements in copies of $\mathcal{U}^\sigma$. Assume the link is a surgery link in S^3 that produces a closed 3-manifold M. Next we use a cohomology class $\omega \in H^1(M,G)$ and a discrete Fourier transform to reduce this element. This universal invariant of links allows to construct an invariant of 3-manifolds (M,ω) of Hennings type.

The chapter contains four sections. In Section 4.2 we construct the topological ribbon structure of $\mathcal{U}^H$ whose bosonization is a topological ribbon algebra. Section 4.3 builds the universal invariant of links from the topological ribbon superalgebra $\widehat{\mathcal{U}^H}$ and a factorization of the invariant. Finally, in Section 4.4 we define discrete Fourier transforms from the topological ribbon superalgebra to a finite type Hopf G-coalgebra. This leads to definition in Theorem 4.4.15 of an invariant of pair (M,ω) as above.

4.2 Topological ribbon Hopf superalgebra $\widehat{\mathcal{U}^H}$

In this section we recall the definition of Hopf superalgebra $\mathcal{U}^H_\xi\mathfrak{sl}(2|1)$ and we construct a topological ribbon Hopf superalgebra $\widehat{\mathcal{U}^H}$ which is a completion of $\mathcal{U}^H$. The topology used in the present chapter is the one of uniform convergence on compact sets for the vector space of holomorphic functions on $\mathbb{C}^2$. This topology defined for q a root of unity is very different from the widely studied h-adic topology used with $q = e^h \in \mathbb{C}[[h]]$. For example, a topological completion of the $\mathcal{U}_q\mathfrak{sl}(2)$ over $\mathbb{C}(q)$ can be seen in [24, 38].

4.2.1 Hopf superalgebra $\widehat{\mathcal{U}^H}$

Hopf superalgebra $\mathcal{U}^H_\xi\mathfrak{sl}(2|1)$

We recall here Definition 2.3.1 of the Hopf superalgebra $\mathcal{U}^H_\xi\mathfrak{sl}(2|1)$.

Definition 4.2.1. *Let $\ell \geq 3$ be an odd integer and $\xi = \exp(\frac{2\pi i}{\ell})$. The superalgebra $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is an associative superalgebra on $\mathbb{C}$ generated by the elements $k_1, k_2, k_1^{-1}, k_2^{-1}, e_1, e_2, f_1, f_2$ and the relations*

$$k_1k_2 = k_2k_1, \tag{4.2.1}$$
$$k_ik_i^{-1} = 1, \ \ i = 1, 2, \tag{4.2.2}$$
$$k_ie_jk_i^{-1} = \xi^{a_{ij}}e_j, \ \ k_if_jk_i^{-1} = \xi^{-a_{ij}}f_j \ i, j = 1, 2, \tag{4.2.3}$$
$$e_1f_1 - f_1e_1 = \frac{k_1 - k_1^{-1}}{\xi - \xi^{-1}}, \ \ e_2f_2 + f_2e_2 = \frac{k_2 - k_2^{-1}}{\xi - \xi^{-1}}, \tag{4.2.4}$$
$$[e_1, f_2] = 0, \ \ [e_2, f_1] = 0, \tag{4.2.5}$$
$$e_2^2 = f_2^2 = 0, \tag{4.2.6}$$
$$e_1^2e_2 - (\xi + \xi^{-1})e_1e_2e_1 + e_2e_1^2 = 0, \tag{4.2.7}$$
$$f_1^2f_2 - (\xi + \xi^{-1})f_1f_2f_1 + f_2f_1^2 = 0. \tag{4.2.8}$$

The last two relations are called the Serre relations. The matrix (a_{ij}) is given by $a_{11} = 2, \ a_{12} = a_{21} = -1, \ a_{22} = 0$. The odd generators are $e_2, \ f_2$.

We define $\xi^x := \exp(\frac{2\pi i x}{\ell})$, afterwards we will use the notation

$$\{x\} = \xi^x - \xi^{-x}.$$

According to [31], $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ is a Hopf superalgebra with the coproduct, counit and the antipode as below

$$\Delta(e_i) = e_i \otimes 1 + k_i^{-1} \otimes e_i \quad i = 1, 2,$$
$$\Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i \quad i = 1, 2,$$
$$\Delta(k_i) = k_i \otimes k_i \quad i = 1, 2,$$
$$S(e_i) = -k_ie_i, \ \ S(f_i) = -f_ik_i^{-1}, \ \ S(k_i) = k_i^{-1} \quad i = 1, 2,$$
$$\varepsilon(k_i) = 1, \ \ \varepsilon(e_i) = \varepsilon(f_i) = 0 \quad i = 1, 2.$$

We call $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ the $\mathbb{C}$-superalgebra generated by $e_i, \ f_i, \ k_i, \ k_i^{-1}$ and h_i for $i = 1, 2$ with Relations (4.2.1) - (4.2.8) plus the relations

$$[h_i, e_j] = a_{ij}e_j, \ \ [h_i, f_j] = -a_{ij}f_j, \ \ [h_i, h_j] = 0, \ \ [h_i, k_j] = 0 \ i, j = 1, 2.$$

The superalgebra $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ is a Hopf superalgebra where $\Delta, \ S$ and ε are determined as in $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ and by

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \ \ S(h_i) = -h_i, \ \ \varepsilon(h_i) = 0 \quad i = 1, 2.$$

Note that $\mathcal{U}_\xi^H\mathfrak{sl}(2|1)$ can be seen as a semidirect product of $\mathbb{C}[h_1, h_2]$ acting on $\mathcal{U}_\xi\mathfrak{sl}(2|1)$.

Define the odd elements $e_3 = e_1e_2 - \xi^{-1}e_2e_1,\ f_3 = f_2f_1 - \xi f_1f_2$. Denote by

$$\begin{aligned}\mathfrak{B}_+ &= \{e_1^p e_3^\rho e_2^\sigma,\ p \in \{0,1,...,\ell-1\},\ \rho,\sigma \in \{0,1\}\},\\ \mathfrak{B}_- &= \{f_1^{p'} f_3^{\rho'} f_2^{\sigma'},\ p' \in \{0,1,...,\ell-1\},\ \rho',\sigma' \in \{0,1\}\},\\ \mathfrak{B}_0 &= \{k_1^{s_1}k_2^{s_2},\ s_1,s_2 \in \mathbb{Z}\} \text{ and } \mathfrak{B}_h = \{h_1^{t_1}h_2^{t_2},\ t_1,t_2 \in \mathbb{N}\}.\end{aligned}$$

Let $\mathcal{U}^H = \mathcal{U}_\xi^H\mathfrak{sl}(2|1)/(e_1^\ell, f_1^\ell)$, this is a Hopf superalgebra.

Lemma 4.2.2. *The set of vectors $\mathfrak{B}_+\mathfrak{B}_0\mathfrak{B}_h\mathfrak{B}_-$ is a Poincaré-Birkhoff-Witt basis of $\mathcal{U}^H$.*

Proof. A proof of this lemma similar to that of Lemma 2.3.3 can be obtained by replacing $\mathbb{C}[k_1^{\pm1}, k_2^{\pm1}]$ with $\mathbb{C}[k_1^{\pm1}, k_2^{\pm1}, h_1, h_2]$. □

Topological Hopf superalgebra $\widehat{\mathcal{U}^H}$

We recall some notions of topological tensor product and nuclear spaces in [40, 19]. A locally convex space E is called nuclear, if all the compatible topologies on $E \otimes F$ agree for all locally convex spaces F, i.e. the topology on $E \otimes F$ compatible with $\otimes$ is unique. A topology is compatible with $\otimes$ if: 1) $\otimes :\ E \times F \to E \otimes F$ is continuous and 2) for all $(e, f) \in E' \times F'$ the linear form $e \otimes f :\ E \otimes F \to \mathbb{C},\ x \otimes y \mapsto e(x)f(y)$ is continuous [40]. For two nuclear spaces E and F the completion of the tensor product $E \otimes F$ endowed with its compatible topology is denoted $E\widehat{\otimes}F$. A finite dimensional space is nuclear, the tensor product of two nuclear spaces is nuclear space and a space is nuclear if only if its completion is nuclear [19]. The complete nuclear spaces form a symmetric monoidal category **Nuc** with the product $\widehat{\otimes}$ (see [40]).

A super nuclear space E is a $\mathbb{Z}/2\mathbb{Z}$-graded nuclear $E = E_{\overline{0}} \oplus E_{\overline{1}}$ where both $E_{\overline{0}}$ and $E_{\overline{1}}$ are closed in E. As for $\mathrm{SVect}_{\overline{0}}$ one can form the monoidal category $\mathbf{SNuc}_{\overline{0}}$ of super complete nuclear spaces with even morphisms. We call *topological Hopf superalgebra* a Hopf algebra object in the monoidal category $\mathbf{SNuc}_{\overline{0}}$. That is a super complete nuclear $\mathbb{C}$-space H endowed with the even $\mathbb{C}$-linear continuous maps called the product, unit, coproduct, counit and antipode

$$m : H\widehat{\otimes}H \to H,\ \eta : \mathbb{C} \to H,\ \Delta : H \to H\widehat{\otimes}H,\ \varepsilon : H \to \mathbb{C} \text{ and } S : H \to H$$

satisfy the axioms:

1. the product m is associative on H admitting $1_H = \eta(1)$ as unity.

2. the coproduct Δ is coassociative, i.e. $(\Delta\widehat{\otimes}\,\mathrm{Id}_H)\circ\Delta = (\mathrm{Id}_H\,\widehat{\otimes}\Delta)\circ\Delta$ and $(\varepsilon\widehat{\otimes}\,\mathrm{Id}_H)\circ\Delta = (\mathrm{Id}_H\,\widehat{\otimes}\varepsilon)\circ\Delta = \mathrm{Id}_H$.
3. Δ and ε are algebra morphisms where the associative product in $H\widehat{\otimes}H$ determined by $(m\widehat{\otimes}m)\circ(\mathrm{Id}_H\,\widehat{\otimes}\tau\widehat{\otimes}\,\mathrm{Id}_H)$.
4. $m\circ(S\widehat{\otimes}\,\mathrm{Id}_H)\circ\Delta = m\circ(\mathrm{Id}_H\,\widehat{\otimes}S)\circ\Delta = \eta\circ\varepsilon$.

The notion of a topological Hopf algebra is defined similarly.

If V is a finite dimensional $\mathbb{C}$-vector space we denote by $\mathcal{H}(V)$ the space of holomorphic functions on V endowed with the topology of uniform convergence on compact sets, it is nuclear space. We will also use the notation $\mathcal{H}(h_1,\ \ldots,\ h_n) := \mathcal{H}(V)$ if $h_1,\ \ldots,\ h_n$ are coordinate functions on V. Remark that we have $\mathcal{H}(V_1)\widehat{\otimes}\,\mathcal{H}(V_2)\simeq\mathcal{H}(V_1\times V_2)$ (Theorem 51.6 [45]) where $V_1,\ V_2$ are finite dimensional $\mathbb{C}$-vector spaces. For a quantum group, if $\mathfrak{H}$ is generated by Cartan generators and W is a finite dimensional vector space generated by other generators then elements of $W\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}^*)$ can be seen as W-valued holomorphic functions. We have the proposition.

Proposition 4.2.3. *Let $\mathfrak{H}_i$ be $\mathbb{C}$-vector spaces of dimension n_i and let W_i be finite dimensional vector spaces on $\mathbb{C}$ for $i=1,2$. Then*

$$(W_1\otimes\mathcal{H}(\mathfrak{H}_1^*))\widehat{\otimes}(W_2\otimes\mathcal{H}(\mathfrak{H}_2^*))\simeq(W_1\otimes W_2)\otimes\mathcal{H}(\mathfrak{H}_1^*\times\mathfrak{H}_2^*).$$

Proof. By the symmetric and associative properties of $\widehat{\otimes}$ we have

$$(W_1\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_1^*))\widehat{\otimes}(W_2\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_2^*))\simeq(W_1\widehat{\otimes}W_2)\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_1^*)\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_2^*).$$

Furthermore, by Theorem 51.6 [45] $\mathcal{H}(\mathfrak{H}_1^*)\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_2^*)\simeq\mathcal{H}(\mathfrak{H}_1^*\times\mathfrak{H}_2^*)$. It implies

$$(W_1\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_1^*))\widehat{\otimes}(W_2\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_2^*))\simeq(W_1\widehat{\otimes}W_2)\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_1^*\times\mathfrak{H}_2^*).$$

Since the spaces $W_i,\mathcal{H}(\mathfrak{H}_i^*)$ for $i=1,2$ are complete then $W_i\widehat{\otimes}\,\mathcal{H}(\mathfrak{H}_i^*)\simeq W_i\otimes\mathcal{H}(\mathfrak{H}_i^*)$. Thus we get

$$(W_1\otimes\mathcal{H}(\mathfrak{H}_1^*))\widehat{\otimes}(W_2\otimes\mathcal{H}(\mathfrak{H}_2^*))\simeq(W_1\otimes W_2)\otimes\mathcal{H}(\mathfrak{H}_1^*\times\mathfrak{H}_2^*).$$

□

The space of entire functions is a nuclear space obtained as the completion of polynomial functions for the topology of uniform convergence on compact sets. We use a similar completion to define a topological ribbon Hopf superalgebra from $\mathcal{U}^H$. That is a topological ribbon Hopf superalgebra $\widehat{\mathcal{U}^H}$ where the topology is constructed as follow. We consider $\mathcal{U}^H\simeq W\otimes_{\mathbb{C}}\mathbb{C}[h_1,h_2,k_1^{\pm1},k_2^{\pm1}]$ as a vector space on $\mathbb{C}$ where W is a finite dimensional vector space on $\mathbb{C}$ with the basis

$$\mathfrak{B}=\mathfrak{B}_+\mathfrak{B}_-.$$

Let $\mathfrak{H}$ be $\mathbb{C}$-vector space with basis $\{h_1, h_2\}$ and $\mathfrak{H}^*$ be its dual, let $\mathcal{H}(h_1, h_2)$ be the vector space of holomorphic functions on $\mathbb{C}^2 \simeq \mathfrak{H}^*$. Now $\mathbb{C}[h_1, h_2, k_1^{\pm 1}, k_2^{\pm 1}]$ embeds in $\mathcal{H}(h_1, h_2)$ by sending k_i to $\xi^{h_i} = \exp\left(\frac{2i\pi}{\ell} h_i\right)$.
Furthermore $\mathbb{C}[h_1, h_2, k_1^{\pm 1}, k_2^{\pm 1}]$ is dense in $\mathcal{H}(h_1, h_2)$ equipped with the topology of uniform convergence on compact sets. Thus $\mathcal{U}^H$ is embedded in $W \widehat{\otimes}_{\mathbb{C}} \mathcal{H}(h_1, h_2) \simeq W \otimes_{\mathbb{C}} \mathcal{H}(h_1, h_2)$, in particular $k_i = 1 \otimes \xi^{h_i} \in W \otimes_{\mathbb{C}} \mathcal{H}(h_1, h_2)$ for $i = 1, 2$. This space is nuclear. As $W \otimes \mathcal{H}(h_1, h_2)$ is complete and $\mathcal{U}^H$ is dense in it then the completion $\widehat{\mathcal{U}^H}$ of $\mathcal{U}^H$ is isomorphic to $W \otimes_{\mathbb{C}} \mathcal{H}(h_1, h_2)$, i.e. $\widehat{\mathcal{U}^H} \simeq W \otimes_{\mathbb{C}} \mathcal{H}(h_1, h_2)$.

In the following, we show that the completion $\widehat{\mathcal{U}^H}$ has the topological Hopf algebraic structure continuously extended from $\mathcal{U}^H$ with the coproduct $\Delta : \mathcal{U}^H \to \mathcal{U}^H \widehat{\otimes} \mathcal{U}^H$.

Remark 4.2.4. *For each $w_i \in \mathfrak{B}$ there exists $|w_i| = (|w_i|_1, |w_i|_2) \in \mathbb{Z}^2$ such that*

$$h_k w_i = w_i(h_k + |w_i|_k) \text{ and}$$
$$\forall w_i, w_j \in \mathfrak{B} \quad w_i w_j = \sum_m w_m c_{ij}^m (h_1, h_2),$$

here $|w_i|_k \in \mathbb{Z}$ is the weight of w_i for h_k with $k = 1, 2$.

Remark 4.2.5. *As $\widehat{\mathcal{U}^H} \simeq W \otimes_{\mathbb{C}} \mathcal{H}(h_1, h_2)$ then each $u \in \widehat{\mathcal{U}^H}$ can be written uniquely*

$$u = \sum_{1 \le i,j \le 4\ell} u_i Q_{ij}(h_1, h_2) v_j$$

where $u_i \in \mathfrak{B}_-,\ v_j \in \mathfrak{B}_+,\ Q_{ij}(h_1, h_2) \in \mathcal{H}(h_1, h_2)$ for $1 \le i, j \le 4\ell$.
Furthermore, by Remark 4.2.4 each $u \in \widehat{\mathcal{U}^H}$ can be also written

$$u = \sum_{1 \le i,j \le 4\ell} u_i v_j P_{ij}(h_1, h_2) = \sum_{i,j} Q'_{ij}(h_1, h_2) u_i v_j \tag{4.2.9}$$

where $u_i \in \mathfrak{B}_-,\ v_j \in \mathfrak{B}_+,\ P_{ij}(h_1, h_2),\ Q'_{ij}(h_1, h_2) \in \mathcal{H}(h_1, h_2)$ for $1 \le i, j \le 4\ell$.

Let K be a compact set in $\mathfrak{H}^*$. If $\phi \in K$ and $x(h_1, h_2) \in \mathcal{H}(h_1, h_2)$ then $\phi_* x(h_1, h_2)$ is the evaluation of x at ϕ, that is $\phi_* x(h_1, h_2) = x(\phi(h_1), \phi(h_2)) \in \mathbb{C}$. For $x = \sum_k w_k x_k(h_1, h_2) \in \widehat{\mathcal{U}^H}$, define a norm associated to K on $\widehat{\mathcal{U}^H}$ as follow

$$\begin{aligned} \|x\|_K = \|\sum_k w_k x_k(h_1, h_2)\|_K &= \sup_k \sup_{\phi \in K} |\phi_*(x_k(h_1, h_2))| \\ &= \sup_k \sup_{\phi \in K} |x_k(\phi(h_1), \phi(h_2))| = \sup_{\phi \in K} \|\phi_* x\|_\infty^{\mathfrak{B}} \end{aligned} \tag{4.2.10}$$

where $\phi_* x = \sum_k w_k \phi_* x_k \in W$.

Remark that all norms on W are equivalent so the choice above of the norm $\|.\|_\infty^{\mathfrak{B}}$ in the basis $\mathfrak{B}$ does not matter. In particular, $\|x\|_K = 1$ when $x \in \mathfrak{B}$. The set $\{\|.\|_K\}_{K \text{ compact}}$ induce the topology of uniform convergence on compact sets.
If E is a nuclear space, it is a locally convex space and its topology is generated by the open balls of the continuous semi-norms. A linear map $f : E \to F$ between nuclear spaces is continuous if and only if for any continuous semi-norm $\|.\|_F$ on F there exists a continuous semi-norm $\|.\|_E$ on E and a constant $\eta \in \mathbb{R}^+$ such that

$$\forall x \in E \ \|f(x)\|_F \leq \eta \|x\|_E.$$

The following three propositions show that the Hopf algebra maps on $\mathcal{U}^H$ are continuous. This implies that these maps induce a topological Hopf algebra structure on $\widehat{\mathcal{U}^H}$.

Proposition 4.2.6. *For each compact set $K \subset \mathfrak{H}^*$, there exists a compact set K' and a $\lambda_K \in \mathbb{R}$ such that $\forall\ x, y \in \mathcal{U}^H$, we have*

$$\|xy\|_K \leq \lambda_K \|x\|_{K'} \|y\|_K.$$

Proof. Given $x = \sum_i w_i x_i(h_1, h_2),\ y = \sum_j w_j y_j(h_1, h_2)$ then

$$\begin{aligned} xy &= \sum_i w_i x_i(h_1, h_2) \sum_j w_j y_j(h_1, h_2) \\ &= \sum_{i,j} w_i w_j x_i(h_1 + |w_j|_1, h_2 + |w_j|_2) y_j(h_1, h_2) \\ &= \sum_{i,j,k} w_k c_{i,j}^k(h_1, h_2) x_i(h_1 + |w_j|_1, h_2 + |w_j|_2) y_j(h_1, h_2). \end{aligned}$$

$$\begin{aligned} &\|xy\|_K \\ &= \sup_k \sup_{\phi \in K} |\sum_{i,j} c_{i,j}^k(\phi(h_1), \phi(h_2)) x_i(\phi(h_1) + |w_j|_1, \phi(h_2) + |w_j|_2) y_j(\phi(h_1), \phi(h_2))| \\ &\leq \sup_k \sup_{\phi \in K} |\sum_{i,j} c_{i,j}^k(\phi(h_1), \phi(h_2))| \sup_i \sup_{\phi \in K} |x_i(\phi(h_1) + |w_j|_1, \phi(h_2) + |w_j|_2)| \\ &\quad \sup_j \sup_{\phi \in K} |y_j(\phi(h_1), \phi(h_2))| \\ &= \lambda_K \|x\|_{K+C} \|y\|_K \end{aligned}$$

where $\lambda_K = \sup_k \sup_{\phi \in K} |\sum_{i,j} c_{i,j}^k(\phi(h_1), \phi(h_2))|$ and $C \subset \mathfrak{H}^*$ is the convex hull of weights of elements of $\mathfrak{B}$. □

Proposition 4.2.6 implies that the product on $\mathcal{U}^H$ is continuous but there does not seem to exist multiplicative seminorms on $\mathcal{U}^H$.
By Proposition 4.2.3 we have $\mathcal{U}^H \widehat{\otimes} \mathcal{U}^H \simeq W^{\otimes 2} \otimes \mathcal{H}(h_{i,j})$ where $h_{i,1} = h_i \otimes 1,\ h_{i,2} = 1 \otimes h_i$ for $i = 1, 2$ and the $h_{i,j}$ are seen as coordinates functions on $\mathfrak{H}^* \times \mathfrak{H}^*$. Thus we can write each $x \in \mathcal{U}^H \widehat{\otimes} \mathcal{U}^H$ form $x = \sum_k w_k x_k(h_{i,j})$ where $w_k \in \mathfrak{B} \otimes \mathfrak{B}$ and $x_k(h_{i,j}) \in \mathcal{H}(h_{i,j})$. We can define a norm of $x \in \mathcal{U}^H \widehat{\otimes} \mathcal{U}^H$ associated to a compact set $K_2 \subset \mathfrak{H}^* \times \mathfrak{H}^*$ by

$$\|x\|_{K_2} = \sup_k \sup_{\phi \in K_2} |\phi_* x_k(h_{i,j})| = \sup_k \sup_{\phi \in K_2} |x_k(\phi(h_{i,j}))|. \qquad (4.2.11)$$

Proposition 4.2.7. *For each compact set $K_2 \subset \mathbb{C}^4$, there exists a compact set $K \subset \mathbb{C}^2$ and a $\lambda_{K_2} \in \mathbb{R}$ such that $\forall\ x \in \mathcal{U}^H$, we have*

$$\|\Delta x\|_{K_2} \leq \lambda_{K_2} \|x\|_K.$$

Proof. Let U be a compact set, $U = U_1 \times U_2 \subset \mathfrak{H}^* \times \mathfrak{H}^* \simeq \mathbb{C}^4$. First there exists $\lambda_U \in \mathbb{R}$ such that for any $a, a', b, b' \in \mathcal{U}^H$ we have

$$\begin{aligned} \|(a \otimes b)(a' \otimes b')\|_{U_1 \times U_2} &= \|aa' \otimes bb'\|_{U_1 \times U_2} = \|aa'\|_{U_1} \|bb'\|_{U_2} \qquad (4.2.12) \\ &\leq \lambda_{U_1} \|a\|_{U_1 + C_1} \|a'\|_{U_1} \lambda_{U_2} \|b\|_{U_2 + C_2} \|b'\|_{U_2} \\ &= \lambda_{U_1} \lambda_{U_2} \|a\|_{U_1 + C_1} \|b\|_{U_2 + C_2} \|a'\|_{U_1} \|b'\|_{U_2} \\ &= \lambda_U \|a \otimes b\|_{U + C_1 \times C_2} \|a' \otimes b'\|_U \\ &= \lambda_U \|a \otimes b\|_{U'} \|a' \otimes b'\|_U \end{aligned}$$

where $\lambda_U = \lambda_{U_1} \lambda_{U_2}$ and $U' = U + C_1 \times C_2$. Second let a compact set $K_2 \subset \mathfrak{H}^* \times \mathfrak{H}^*$ and let $K \subset \mathfrak{H}^*$ be the compact set $\{\varphi + \psi |\ (\varphi, \psi) \in K_2\}$. For $x \in \mathcal{U}^H,\ x = \sum_j w_j x_j(h_1, h_2)$, we have

$$\begin{aligned} \|\Delta x\|_{K_2} &= \| \sum_j \Delta w_j \Delta x_j(h_1, h_2)\|_{K_2} \leq \sum_j \|\Delta w_j \Delta x_j(h_1, h_2)\|_{K_2} \\ &= \sum_j \| \sum_s w_j^{1,s} \otimes w_j^{2,s} x_j(h_{1,1} + h_{1,2}, h_{2,1} + h_{2,2})\|_{K_2} \\ &\leq \sum_j \sum_s \|w_j^{1,s} \otimes w_j^{2,s} x_j(h_{1,1} + h_{1,2}, h_{2,1} + h_{2,2})\|_{K_2} \\ &\leq \sum_j \sum_s \lambda_{K_2,j,s} \|w_j^{1,s} \otimes w_j^{2,s}\|_{K_2 + (C_1, C_2)} \|x_j(h_{1,1} + h_{1,2}, h_{2,1} + h_{2,2})\|_{K_2} \\ &\leq \sum_j \lambda_{K_2,j} \|x_j(h_{1,1} + h_{1,2}, h_{2,1} + h_{2,2})\|_{K_2} \end{aligned}$$

where the sums are finite and $\lambda_{K_2,j,s},\ \lambda_{K_2,j}$ are constants and in the fifth inequality one used Inequality (4.2.12). Furthermore, let $\mathfrak{H}$ be vector space on $\mathbb{C}$ with basis $\{h_1, h_2\}$. The symmetric algebra $S(\mathfrak{H} \times \mathfrak{H}) \simeq S(\mathfrak{H} \oplus \mathfrak{H}) \simeq S\mathfrak{H} \otimes S\mathfrak{H}$ (see [29]), it is a commutative algebra on $\mathbb{C}$ generated by $h_1 \otimes 1,\ h_2 \otimes 1,\ 1 \otimes h_1,\ 1 \otimes h_2$ and $\mathrm{Hom}_{Alg}(S\mathfrak{H} \otimes S\mathfrak{H}, \mathbb{C}) \simeq \mathrm{Hom}_{Alg}(S(\mathfrak{H} \times \mathfrak{H}), \mathbb{C}) \simeq \mathrm{Hom}_{Vect}(\mathfrak{H} \times \mathfrak{H}, \mathbb{C}) \simeq (\mathfrak{H} \times \mathfrak{H})^* \simeq \mathfrak{H}^* \times \mathfrak{H}^*$. This isomorphism allows that for $(\varphi, \psi) \in \mathfrak{H}^* \times \mathfrak{H}^*$ one has $(\varphi, \psi)(h_i \otimes 1) = \varphi(h_i)$ and $(\varphi, \psi)(1 \otimes h_i) = \psi(h_i)$ for $i = 1, 2$. It implies that

$$\begin{aligned} \|x_j(h_{1,1}+&h_{1,2},\ h_{2,1} + h_{2,2})\|_{K_2} \\ &= \sup_{(\varphi,\psi)\in K_2} |(\varphi, \psi)_* x_j\,(h_{1,1} + h_{1,2}, h_{2,1} + h_{2,2})\,| \\ &= \sup_{(\varphi,\psi)\in K_2} |(\varphi + \psi)_* x_j\,(h_1, h_2)\,| = \|x_j(h_1, h_2)\|_K. \end{aligned}$$

Hence

$$\|\Delta x\|_{K_2} \leq \sum_j \lambda_{K_2,j} \|x_j(h_1, h_2)\|_K \leq \lambda_{K_2} \|x\|_K$$

where λ_{K_2} is a constant. □

This proposition implies that the coproduct is continuous. The antipode S is also continuous by proposition below.

Proposition 4.2.8. *For each compact set $K \subset \mathfrak{H}^*$ there exists a compact set $K'' \subset \mathfrak{H}^*$ and a constant λ_K such that*

$$\|S(x)\|_K \leq \lambda_K \|x\|_{K''} \text{ for } x \in \mathcal{U}^H.$$

Proof. For $x = \sum_j w_j x_j(h_1, h_2) \in \mathcal{U}^H$ we have

$$\begin{aligned} \|S(x)\|_K &= \|\sum_j S(x_j(h_1, h_2))S(w_j)\|_K = \|\sum_j x_j(-h_1, -h_2)S(w_j)\|_K \\ &\leq \sum_j \|x_j(-h_1, -h_2)S(w_j)\|_K \\ &\leq \sum_j \lambda_{K,j} \|x_j(-h_1, -h_2)\|_{K'} \|S(w_j)\|_K \\ &\leq \sum_j \lambda'_{K,j} \|x_j(h_1, h_2)\|_{-K'} \leq \lambda_K \|x\|_{-K'} \end{aligned}$$

where $\lambda_{K,j},\ \lambda'_{K,j}$ and λ_K are constants. □

It is clear that the unit and counit are continuous. Hence the maps product, coproduct, unit, counit and the antipode of $\mathcal{U}^H$ are continuous (with the topology of uniform convergence on compact sets). Thus the topology of uniform convergence on compact sets of $\mathcal{U}^H$ is compatible with its algebraic structure. The maps product, coproduct, unit, counit and the antipode of

$\mathcal{U}^H$ continuously extend to the completion $\widehat{\mathcal{U}^H}$. Note that the coproduct $\mathcal{U}^H \to \mathcal{U}^H \otimes \mathcal{U}^H$ extends to $\widehat{\mathcal{U}^H} \to \mathcal{U}^H \widehat{\otimes} \mathcal{U}^H$. The space $\widehat{\mathcal{U}^H}$ endows with these continuous maps is a topological Hopf superalgebra.

Similarly, for $n \geq 2$ denote

$$h_{i,j} = 1 \otimes ... \otimes h_i \otimes ... \otimes 1 \tag{4.2.13}$$

where h_i is in j-th position for $1 \leq i \leq 2$ and $1 \leq j \leq n$. Then the completion of $\mathcal{U}^{H\otimes n}$ is topological vector space $\mathcal{U}^{H\widehat{\otimes}n} \simeq W^{\otimes n} \otimes \mathcal{H}(h_{i,j})$ with the topology of uniform convergence on compact sets. Here $W^{\otimes n}$ is the tensor product of n copies of W and $\mathcal{H}(h_{i,j})$ is the vector space of holomorphic functions of $2n$ variables $\{h_{i,j}\}_{i=1,2}^{j=1,\dots,n}$ in $\mathbb{C}^{2n}$. Note also that the maps $\Delta_i^{[n]} : \mathcal{U}^{H\otimes n} \to \mathcal{U}^{H\otimes(n+1)}$ and $\varepsilon_i^{[n]} : \mathcal{U}^{H\otimes n} \to \mathcal{U}^{H\otimes(n-1)}$ continuously extend to $\mathcal{U}^{H\widehat{\otimes}n}$, here $\Delta_i^{[n]}$ and $\varepsilon_i^{[n]}$ determined by

$$\Delta_i^{[n]} = \underbrace{\mathrm{Id} \otimes ... \otimes \mathrm{Id}}_{i-1} \otimes \Delta \otimes \underbrace{\mathrm{Id} \otimes ... \otimes \mathrm{Id}}_{n-i}$$

and

$$\varepsilon_i^{[n]} = \underbrace{\mathrm{Id} \otimes ... \otimes \mathrm{Id}}_{i-1} \otimes \varepsilon \otimes \underbrace{\mathrm{Id} \otimes ... \otimes \mathrm{Id}}_{n-i}$$

where Δ, ε are in i-th position. It follows that

$$\begin{aligned}
&\mathrm{Id} \otimes \Delta_i^{[n]} = \Delta_{i+1}^{[n+1]}, \Delta_i^{[n]} \otimes \mathrm{Id} = \Delta_i^{[n+1]},\\
&\Delta_i^{[n+1]} \circ \Delta_i^{[n]} = \Delta_{i+1}^{[n+1]} \circ \Delta_i^{[n]},\\
&\Delta_j^{[n+1]} \circ \Delta_i^{[n]} = \Delta_i^{[n+1]} \circ \Delta_j^{[n]} \quad i \neq j
\end{aligned}$$

and we denote $\Delta^{[n]}(x) = \sum x_{(1)} \otimes ... \otimes x_{(n)}$ for $x \in \mathcal{U}^H$. Hence, each element x of $\mathcal{U}^{H\widehat{\otimes}n}$ can be written $x = \sum_k w_k x_k(h_{i,j})$ where $w_k \in \mathfrak{B}^{\otimes n}$, $x_k(h_{i,j}) \in \mathcal{H}(h_{i,j}) := \mathcal{H}(\mathfrak{H}^{*n})$. In particular, the element $k_{i,j} := 1 \otimes ... \otimes k_i \otimes ... \otimes 1$ where k_i is in j-th position is equal to $\xi^{h_{i,j}} = 1 \otimes ... \otimes \xi^{h_i} \otimes ... \otimes 1$ for $i = 1, 2$ $j = 1, ..., n$. Let K be a compact set in $\mathbb{C}^{2n} \simeq Span_{\mathbb{C}}\{h_{i,j}\}^*$. As in Definition (4.2.10) we define

$$\|x\|_K = \sup_k \sup_{\phi \in K} |\phi_*(x_k(h_{i,j}))| = \sup_k \sup_{\phi \in K} |x_k(\phi(h_{i,j}))|.$$

Recall that $\mathscr{C}^H$ is the even category of finite dimensional nilpotent modules over $\mathcal{U}^H$ (see in Chapter 2).

Proposition 4.2.9. *For any $V_1, ..., V_n \in \mathscr{C}^H$ the representation $\rho_{V_1 \otimes ... \otimes V_n}$: $\mathcal{U}^{H\otimes n} \to \mathrm{End}_{\mathbb{C}}(V_1 \otimes ... \otimes V_n)$ continuously extends to a representation $\mathcal{U}^{H\widehat{\otimes}n} \to \mathrm{End}_{\mathbb{C}}(V_1 \otimes ... \otimes V_n)$.*

Proof. Let K be the compact set containing the weights of $V = V_1 \otimes \ldots \otimes V_n$. We have $\rho_V : \mathcal{U}^{H \otimes n} \to \mathrm{End}(V)$ be continuous on compact set K. Indeed, let $x \in \mathcal{U}^{H\otimes n}$ and write $x = \sum_k w_k x_k(h_{i,j})$. On the subspace of weights $\phi \in K$, $\rho_V\left(\sum_k w_k x_k(h_{i,j})\right)$ acts as $\left\|\rho_V\left(\sum_k w_k x_k(\phi(h_{i,j}))\right)\right\| \leq \sum_k \|w_k\|_K \|x_k\|_K \leq \lambda_K \|x\|_K$ with λ_K is a constant. It implies that ρ_V is continuous. This prove that it exists a continuous representation $\widehat{\rho_V} : \mathcal{U}^{H \widehat{\otimes} n} \to \mathrm{End}_{\mathbb{C}}(V)$. □

4.2.2 Topological ribbon superalgebra $\widehat{\mathcal{U}^H}$

It is known in Chapter 2 that the operator $\mathcal{R} = \check{\mathcal{R}}\mathcal{K}$ on $\mathscr{C}^H$ where

$$\check{\mathcal{R}} = \sum_{i=0}^{\ell-1} \frac{\{1\}^i e_1^i \otimes f_1^i}{(i)_\xi!} \sum_{\rho=0}^{1} \frac{(-\{1\})^\rho e_3^\rho \otimes f_3^\rho}{(\rho)_\xi!} \sum_{\delta=0}^{1} \frac{(-\{1\})^\delta e_2^\delta \otimes f_2^\delta}{(\delta)_\xi!} \in \mathcal{U}^H \otimes \mathcal{U}^H,$$

$$(0)_\xi! = 1, \ (i)_\xi! = (1)_\xi (2)_\xi \cdots (i)_\xi, \ (k)_\xi = \frac{1-\xi^k}{1-\xi} \quad \text{and}$$

$$\mathcal{K} = \xi^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2} \in \mathcal{U}^{H \widehat{\otimes} 2} \tag{4.2.14}$$

satisfies these conditions below

$$\begin{aligned} \Delta \otimes \mathrm{Id}(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ \mathrm{Id} \otimes \Delta(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}, \\ \mathcal{R}\Delta^{op}(x) &= \Delta(x)\mathcal{R} \text{ for all } x \in \mathcal{U}^H. \end{aligned}$$

This operator is given by action of an element $\mathcal{R}$ is in the completion $\mathcal{U}^{H\widehat{\otimes}2}$, so the proof of the lemma below follows the line of Theorem VIII.2.4 [29].

Lemma 4.2.10. *The element $\mathcal{R} = \check{\mathcal{R}}\mathcal{K}$ is a topological universal R-matrix of the topological Hopf superalgebra $\widehat{\mathcal{U}^H}$.*

The element $\mathcal{R}$ satisfies the properties

$$\begin{aligned} &\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \\ &(\varepsilon \otimes \mathrm{Id}_{\widehat{\mathcal{U}^H}})(\mathcal{R}) = 1 = (\mathrm{Id}_{\widehat{\mathcal{U}^H}} \otimes \varepsilon)(\mathcal{R}), \\ &(S \otimes \mathrm{Id}_{\widehat{\mathcal{U}^H}})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{Id}_{\widehat{\mathcal{U}^H}} \otimes S^{-1})(\mathcal{R}), \\ &(S \otimes S)(\mathcal{R}) = \mathcal{R}. \end{aligned}$$

The completion $\widehat{\mathcal{U}^H}$ of $\mathcal{U}^H$ is a Hopf $\mathbb{C}$-superalgebra which has a pivotal element $\phi_0 = k_1^{-\ell} k_2^{-2}$ (see Proposition 2.3.4). We define an even element θ, invertible and in the center of $\widehat{\mathcal{U}^H}$ by

$$\theta = \phi_0.(m \circ \tau^s \circ (\mathrm{Id} \otimes S)(\mathcal{R}))^{-1} \tag{4.2.15}$$

where $\tau^s : \mathcal{U}^H \hat{\otimes} \mathcal{U}^H \to \mathcal{U}^H \hat{\otimes} \mathcal{U}^H,\ x \otimes y \mapsto (-1)^{\deg x \deg y} y \otimes x$ is super-flip of $\mathcal{U}^H \hat{\otimes} \mathcal{U}^H$.
We now show that the completion $\widehat{\mathcal{U}^H}$ with the element θ will be a ribbon Hopf superalgebra.

Proposition 4.2.11. *The θ is a twist, i.e. the element θ satisfies*

1. $\varepsilon(\theta) = 1$,
2. $\Delta(\theta) = \tau^s(\mathcal{R}).\mathcal{R}.(\theta \otimes \theta)$,
3. $S(\theta) = \theta$.

Equalities (1) and (2) follow from the definition of θ. To prove (3), we need the following lemmas.
Let $\mathcal{U}^h$ be the sub-superalgebra of $\widehat{\mathcal{U}^H}$ of all elements commuting with $h_1,\ h_2$ we have the lemma.

Lemma 4.2.12. *For $u \in \widehat{\mathcal{U}^H},\ u \in \mathcal{U}^h$ if and only if u has the form*

$$u = \sum_{0 \le \rho,\sigma \le 1,\ 0 \le p \le \ell-1} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) e_2^{\rho} e_3^{\sigma} e_1^{p} \tag{4.2.16}$$

where $weight(y_{\rho,\sigma,p}) + weight(e_2^{\rho} e_3^{\sigma} e_1^{p}) = 0$ and $Q_{\rho,\sigma,p}(h_1, h_2) \in \mathcal{H}(h_1, h_2)$.

Proof. By Remark 4.2.5 each $u \in \widehat{\mathcal{U}^H}$ can be written uniquely

$$u = \sum_{1 \le i,j \le 4\ell} u_i Q_{ij}(h_1, h_2) v_j$$

where $u_i \in \mathfrak{B}_-,\ v_j \in \mathfrak{B}_+,\ Q_{ij}(h_1, h_2) \in \mathcal{H}(h_1, h_2)$ for $1 \le i, j \le 4\ell$.
For each h_k for $k = 1, 2$ we have

$$\begin{aligned} u h_k &= \sum_{1 \le i,j \le 4\ell} u_i Q_{ij}(h_1, h_2) v_j h_k \\ &= \sum_{1 \le i,j \le 4\ell} u_i Q_{ij}(h_1, h_2)(h_k + |v_j|_k) v_j \\ &= \sum_{1 \le i,j \le 4\ell} u_i (h_k + |v_j|_k) Q_{ij}(h_1, h_2) v_j \\ &= \sum_{1 \le i,j \le 4\ell} (h_k + |u_i|_k + |v_j|_k) u_i Q_{ij}(h_1, h_2) v_j \\ &= h_k u + \sum_{1 \le i,j \le 4\ell} (|u_i|_k + |v_j|_k) u_i Q_{ij}(h_1, h_2) v_j. \end{aligned}$$

It implies that u commutes with h_i if and only if the sum of the weights of u_i and v_j is zero. $\square$

Let $\mathcal{I}^+$ be a left ideal of $\widehat{\mathcal{U}^H}$ generated by $e_1,\ e_2$ and e_3, set $\mathcal{I} = \mathcal{I}^+ \cap \mathcal{U}^h$.

Lemma 4.2.13. *We have $\mathcal{I} = \mathcal{I}^+ \cap \mathcal{U}^h = \mathcal{I}^- \cap \mathcal{U}^h$ and $\mathcal{U}^h = \mathcal{H}(h_1, h_2) \bigoplus \mathcal{I}$ where $\mathcal{I}^-$ is right ideal generated by $f_1,\ f_2$ and f_3.*

Hence, $\mathcal{I}$ is a two-sided ideal and the projection $\varphi:\ \mathcal{U}^h \to \mathcal{H}(h_1, h_2)$ is a homomorphism of algebras called the Harish-Chandra homomorphism.

Proposition 4.2.14. *Let V_μ be a simple highest weight $\mathcal{U}^H$-module with highest weight $\mu = (\mu_1, \mu_2)$. Then for any $z \in Z(\widehat{\mathcal{U}^H})$ and any $v \in V_\mu$*

$$zv = \varphi(z)(\mu)v$$

where $\varphi(z)$ is in $\mathcal{H}(\mathfrak{H}^*)$ and $\varphi(z)(\mu)$ is its value at $\mu = (\mu_1, \mu_2)$.

Proof. Let $w_{0,0,0}$ be a highest weight vector generating V_μ and z a central element of $\widehat{\mathcal{U}^H}$. Following the lemmas above, z can be written

$$z = \varphi(z) + \sum_{(\rho,\sigma,p)\neq(0,0,0)} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) e_2^\rho e_3^\sigma e_1^p.$$

Since $e_2^\rho e_3^\sigma e_1^p w_{0,0,0} = 0$ for $(\rho, \sigma, p) \neq (0, 0, 0)$ and $h_i w_{0,0,0} = \mu_i w_{0,0,0}$ $i = 1, 2$, we get $zw_{0,0,0} = \varphi(z)(\mu_1, \mu_2)w_{0,0,0}$. If v is an arbitrary vector of V_μ, we have $v = xw_{0,0,0}$ for some x in $\widehat{\mathcal{U}^H}$. It implies that $zv = zxw_{0,0,0} = xzw_{0,0,0} = \varphi(z)(\mu_1, \mu_2)xw_{0,0,0} = \varphi(z)(\mu_1, \mu_2)v$. □

By using this proposition, we have

Proposition 4.2.15. *Let u be a central element of $\widehat{\mathcal{U}^H}$. If $\varphi(u) = 0$ then $u = 0$ where φ is Harish-Chandra homomorphism.*

Proof. Let u be a central element of $\widehat{\mathcal{U}^H}$ such that $\varphi(u) = 0$. Assume u is non-zero can be written as

$$u = \sum_{(\rho,\sigma,p)\neq(0,0,0)} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) e_2^\rho e_3^\sigma e_1^p$$

where $Q_{\rho,\sigma,p}(h_1, h_2)$ are non-zero functions in $\mathcal{H}(h_1, h_2)$, $0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1$ and $(\rho, \sigma, p) \neq (0, 0, 0)$.
Consider a typical highest weight $\mathcal{U}^H$-module V_μ generated by highest weight vector $w_{0,0,0}$. It is known that the set of $4r$ vectors $B^* = \{S^{-1}(e_2^\rho e_3^\sigma e_1^p)w^*_{0,0,0}\}$ forms a basis of V_μ^* where $0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1,\ \{w^*_{\rho,\sigma,p}\}$ is the dual basis of $\{w_{\rho,\sigma,p}\}$ of V_μ. In fact, the elements $S^{-1}(e_2^\rho e_3^\sigma e_1^p)$ form up to multiplication by $k_1^a k_2^b$ $a, b \in \mathbb{Z}$ a basis of the subalgebra $\mathcal{U}^+$ of $\mathcal{U}^H$ generated by $e_2^\rho e_3^\sigma e_1^p$ $0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1$. Since $\mathcal{U}^- w^*_{0,0,0} = \mathbb{C}w^*_{0,0,0}$

where $\mathcal{U}^-$ is subalgebra of $\widehat{\mathcal{U}^H}$ generated by $f_2^\rho f_3^\sigma f_1^p$ $0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1$, we have $Span_{\mathbb{C}}(B^*) \simeq \mathcal{U}^+ w^*_{0,0,0} \simeq \mathcal{U}^+\mathcal{U}^0\mathcal{U}^- w^*_{0,0,0} \simeq \mathcal{U}^H w^*_{0,0,0} \simeq V^*_\mu$ where $\mathcal{U}^0$ is subalgebra of $\widehat{\mathcal{U}^H}$ topologically generated by h_1, h_2. Furthermore $\operatorname{card}(B^*) = \dim V^*_\mu$, hence B^* is a basis of V^*_μ. It exists in V_μ a dual basis $B = \{\widetilde{w}_{\rho,\sigma,p}\ 0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1\}$ of B^* in V^*_μ, i.e. given (ρ, σ, p), for any $e_2^{\rho'} e_3^{\sigma'} e_1^{p'}$, $w^*_{0,0,0}(e_2^{\rho'} e_3^{\sigma'} e_1^{p'} \widetilde{w}_{\rho,\sigma,p}) = \delta_\rho^{\rho'} \delta_\sigma^{\sigma'} \delta_p^{p'}$.
On the one hand, Proposition 4.2.14 implies that $u\widetilde{w}_{\rho,\sigma,p} = 0$ for all $0 \le \rho, \sigma \le 1,\ 0 \le p \le \ell - 1$. On the other hand, we have that $e_2^{\rho_0} e_3^{\sigma_0} e_1^{p_0}$ is an element having minimal weight of ones in the items of sum

$$\sum_{(\rho,\sigma,p)\neq(0,0,0)} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) e_2^\rho e_3^\sigma e_1^p$$

such that $Q_{\rho_0,\sigma_0,p_0}(h_1, h_2) \neq 0$. It is clear that $e_2^\rho e_3^\sigma e_1^p \widetilde{w}_{\rho_0,\sigma_0,p_0} = 0$ for $e_2^\rho e_3^\sigma e_1^p$ having the weight higher than one of $e_2^{\rho_0} e_3^{\sigma_0} e_1^{p_0}$ and $e_2^\rho e_3^\sigma e_1^p \widetilde{w}_{\rho_0,\sigma_0,p_0} = \delta_\rho^{\rho_0} \delta_\sigma^{\sigma_0} \delta_p^{p_0} w_{0,0,0}$ for $e_2^\rho e_3^\sigma e_1^p$ having the weight equal one of $e_2^{\rho_0} e_3^{\sigma_0} e_1^{p_0}$. Hence we have

$$\begin{aligned}
&\sum_{(\rho,\sigma,p)\neq(0,0,0)} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) e_2^\rho e_3^\sigma e_1^p \widetilde{w}_{\rho_0,\sigma_0,p_0} \\
&= \sum_{\text{weight}(e_2^\rho e_3^\sigma e_1^p)=\text{weight}(e_2^{\rho_0} e_3^{\sigma_0} e_1^{p_0})} y_{\rho,\sigma,p} Q_{\rho,\sigma,p}(h_1, h_2) \delta_\rho^{\rho_0} \delta_\sigma^{\sigma_0} \delta_p^{p_0} w_{0,0,0} \\
&= y_{\rho_0,\sigma_0,p_0} Q_{\rho_0,\sigma_0,p_0}(h_1, h_2) w_{0,0,0} \\
&= Q_{\rho_0,\sigma_0,p_0}(\mu) w_{\rho_0,\sigma_0,p_0} = 0.
\end{aligned}$$

This result prove that $Q_{\rho_0,\sigma_0,p_0}(h_1, h_2) = 0$. Thus $u = 0$. □

Lemma 4.2.16. *Let $\rho_V : \mathcal{U}^H \to \operatorname{End}(V)$ be a nilpotent finite dimensional representation of $\mathcal{U}^H$. We have*

$$\rho_V(S(\theta)) = \rho_V(\theta).$$

Proof. Recall that the category $\mathscr{C}^H$ of nilpotent representations of $\mathcal{U}^H_\xi \mathfrak{sl}(2|1)$ is a ribbon category having the twist is the family of isomorphisms $\theta_V : V \to V,\ \forall V \in \mathscr{C}^H$, $\theta_V = \rho_V(\theta)$ where $\rho_V : \mathcal{U}^H \to \operatorname{End}(V)$ is a representation of $\mathcal{U}^H$ (see Chapter 2). It follows that $(\theta_V)^* = \theta_{V^*}\ \forall V \in \mathscr{C}^H$. In fact $(\theta_V)^* = (\rho_V(\theta))^* = (\overrightarrow{\operatorname{ev}}_V \otimes \operatorname{Id}_{V^*})(\operatorname{Id}_{V^*} \otimes \theta_V \otimes \operatorname{Id}_{V^*})(\operatorname{Id}_{V^*} \otimes \overrightarrow{\operatorname{coev}}_V) : V^* \to V^*$ has matrix $(\rho_V(\theta))^t$ where $(\rho_V(\theta))$ is the matrix of the endomorphism $\rho_V(\theta)$. Furthermore $\theta_{V^*} = \rho_{V^*}(\theta)$ has matrix $(\rho_V(S(\theta)))^t$, so we have

$$\rho_V(\theta) = \rho_V(S(\theta)). \tag{4.2.17}$$

□

Proof of Proposition 4.2.11. Set $z = S(\theta) - \theta$, z is in the center of $\widehat{\mathcal{U}^H}$. Let a weight module V_μ in $\mathscr{C}^H$ of weight μ and v is a weight vector of V_μ. By Proposition 4.2.14 and Equality (4.2.17) we have $\varphi(z)(\mu)v = zv = 0$. It implies that $\varphi(z)(\mu) = 0$. Furthermore $\varphi(z) \in \mathcal{H}(\mathfrak{H}^*)$, this deduces that $\varphi(z) = 0$, so $z = 0$ by Proposition 4.2.15, i.e. $S(\theta) = \theta$. □

Hence the results above give us the theorem.

Theorem 4.2.17. *The completion $\widehat{\mathcal{U}^H}$ of $\mathcal{U}^H$ is a topological ribbon superalgebra.*

4.2.3 Bosonization of $\widehat{\mathcal{U}^H}$

It is known that each ribbon superalgebra has an associated ribbon algebra, namely its bosonization (see [36]). For the ribbon superalgebra $\widehat{\mathcal{U}^H}$, its bosonization denoted by $\widehat{\mathcal{U}^H}^\sigma$, is a topological ribbon algebra by adding an element σ from $\widehat{\mathcal{U}^H}$, i.e. as an algebra, $\widehat{\mathcal{U}^H}^\sigma$ is the semi-direct product of $\widehat{\mathcal{U}^H}$ with $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ where the action of σ is given by

$$\sigma x = (-1)^{\deg x} x\sigma \quad \text{for } x \in \widehat{\mathcal{U}^H}. \tag{4.2.18}$$

The coproduct Δ^σ, the counit ε^σ and the antipode S^σ on $\widehat{\mathcal{U}^H}^\sigma$ given by

1. $\Delta^\sigma \sigma = \sigma \otimes \sigma$, $\Delta^\sigma(x) = \sum_i x_i \sigma^{\deg x_i'} \otimes x_i'$ where $\Delta(x) = \sum_i x_i \otimes x_i'$ for $x \in \widehat{\mathcal{U}^H}$,
2. $\varepsilon^\sigma(\sigma) = 1$, $\varepsilon^\sigma(x) = \varepsilon(x)$ for $x \in \widehat{\mathcal{U}^H}$ and
3. $S^\sigma(\sigma) = \sigma$, $S^\sigma(x) = \sigma^{\deg x} S(x)$ for $x \in \widehat{\mathcal{U}^H}$.

The universal $\mathcal{R}$-matrix $\mathcal{R}^\sigma$ in $\widehat{\mathcal{U}^H}^\sigma$ determined by

$$\mathcal{R}^\sigma = R_1 \sum_i R_i^1 \sigma^{\deg R_i^2} \otimes R_i^2$$

where $R_1 = \dfrac{1}{2}(1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma)$ and $\mathcal{R} = \sum_i R_i^1 \otimes R_i^2$ is the universal $\mathcal{R}$-matrix in $\widehat{\mathcal{U}^H}$. Note that the universal $\mathcal{R}$-matrix $\mathcal{R}^\sigma$ can be written by

$$\mathcal{R}^\sigma = \sum_i a_i \otimes b_i \sum_j \mathcal{K}_j^1 \otimes \mathcal{K}_j^2 \tag{4.2.19}$$

where the terms a_i, b_i do not contain h_1, h_2 for all i and $\mathcal{K} = \sum_j \mathcal{K}_j^1 \otimes \mathcal{K}_j^2$ is the Cartan part which contains only h_1, h_2 (see Equation (4.2.14)). Its inverse denotes

$$(\mathcal{R}^\sigma)^{-1} = \sum_j \overline{\mathcal{K}}_j^1 \otimes \overline{\mathcal{K}}_j^2 \sum_i \overline{a}_i \otimes \overline{b}_i. \tag{4.2.20}$$

The pivotal element of the ribbon algebra $\widehat{\mathcal{U}^H}^\sigma$ is $\phi_0^\sigma = \sigma\phi_0$. We denote $\mathcal{U}^\sigma$ the Hopf subalgebra of $\widehat{\mathcal{U}^H}^\sigma$ generated by elements e_i, f_i, k_i, k_i^{-1} for $i = 1, 2$ and σ. It is a pivotal Hopf algebra with a pivotal element ϕ_0^σ.

4.3 Universal invariant of link diagrams

It is well known that from a ribbon algebra one can construct a universal invariant of oriented framed links, for example one can see these constructions presented by K. Habiro (see [23]), M. Hennings (see [26]), L. Kauffman and D. E. Radford (see [30]), T. Ohtsuki (see [38]), ... In previous section we proved that $\widehat{\mathcal{U}^H}$ is a ribbon superalgebra in the topological sense so its bosonization is a topological ribbon algebra. This topological ribbon algebra allows to construct a universal invariant of oriented framed links. In this section we apply the methods above to reconstruct a universal invariant of oriented framed links associated with the unrolled quantum group $\mathcal{U}^H$. Then we will use this invariant to construct an invariant of 3-manifolds in the next section.

4.3.1 Category of tangles

We recall the category $\mathcal{T}$ of framed, oriented tangles (see [23], [29]). The objets are the tensor words of symbols $\downarrow$ and $\uparrow$, i.e. each word forms $x_1 \otimes ... \otimes x_n$ with $x_1, ..., x_n \in \{\downarrow, \uparrow\}, n \geq 0$. The tensor word of length 0 is denoted by $1 = 1_{\mathcal{T}}$. The morphisms $T: \ w \to w'$ between $w, w' \in Ob(\mathcal{T})$ are the isotopy classes of framed, oriented tangles in a cube $[0,1]^3$ such that the endpoints at the bottom are descriped by w and those at the top by w'.
The composition gf of a composable pair (f, g) of morphisms in $\mathcal{T}$ is obtained by placing g above f, and the tensor product $f \otimes g$ of two morphisms f and g is obtained by placing g on the right of f.
The braiding $c_{w,w'}: \ w \otimes w' \to w' \otimes w$ for $w, w' \in Ob(\mathcal{T})$ is the positive braiding of parallel of strings. The dual $w^* \in Ob(\mathcal{T})$ of $w \in Ob(\mathcal{T})$ is defined by $1^* = 1$, $\downarrow^* = \uparrow$, $\uparrow^* = \downarrow$ and

$$(x_1 \otimes ... \otimes x_n)^* = x_n^* \otimes ... \otimes x_1^* \quad \text{for } x_1, ..., x_n \in \{\downarrow, \uparrow\},\ n \geq 2.$$

For $w \in Ob(\mathcal{T})$, let

$$\overrightarrow{\mathrm{ev}}_w\colon\ w^* \otimes w \to 1,\ \overrightarrow{\mathrm{coev}}_w\colon\ 1 \to w \otimes w^*$$

denote the duality morphisms. For each object w in $\mathcal{T}$, let $t_w: \ w \to w$ denote the positive full twist defined by

$$t_w = (w \otimes \overrightarrow{\mathrm{ev}}_{w^*})(c_{w,w} \otimes w^*)(w \otimes \overrightarrow{\mathrm{coev}}_w).$$

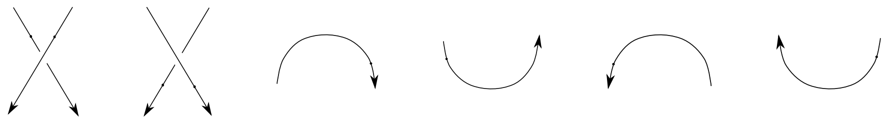


Figure 4.1 – The morphisms $c_{\downarrow,\downarrow}$, $c_{\downarrow,\downarrow}^{-1}$, $\overrightarrow{\mathrm{ev}}_{\downarrow}$, $\overrightarrow{\mathrm{coev}}_{\downarrow}$, $\overrightarrow{\mathrm{ev}}_{\uparrow}$, $\overrightarrow{\mathrm{coev}}_{\uparrow}$

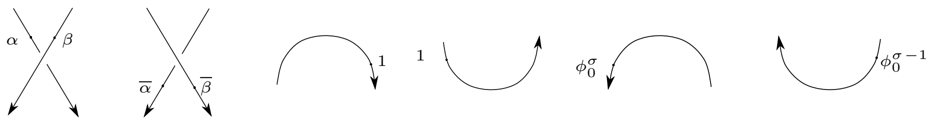


Figure 4.2 – Place elements on the strings

It is well known that $\mathcal{T}$ is generated as a monoidal category by the objects $\downarrow, \uparrow$ and the morphisms

$$c_{\downarrow,\downarrow},\ c_{\downarrow,\downarrow}^{-1},\ \overrightarrow{\mathrm{ev}}_{\downarrow},\ \overrightarrow{\mathrm{coev}}_{\downarrow},\ \overrightarrow{\mathrm{ev}}_{\uparrow},\ \overrightarrow{\mathrm{coev}}_{\uparrow}$$

which are represented in Figure 4.1.

A *string link* is a tangle without closed component whose arcs end at the same order as they start, with downwards orientation.

4.3.2 Universal invariant of link diagrams

We recall the notion of the 0^{th}-Hochschild homology for an algebra A, that is $\mathrm{HH}_0(A) := A/[A,A]$ where $[A,A] = \mathrm{Span}\{xy - yx :\ x, y \in A\}$. Let $L = L_1 \cup ... \cup L_n$ be a (framed, oriented) link diagram consisting of n ordered circle components $L_1, ..., L_n$ with $n \geq 0$. We use the method in Ohtsuki's book [38] to construct the universal invariant. It can be described by using the generators of $\mathcal{T}$ (see Figure 4.1).

We can put elements of $\widehat{\mathcal{U}^H}^\sigma$ on the strings of L according to the rule depicted in Figure 4.2 or in two Figures 4.2 and 4.3. For each $j = 1, ..., n$, we define $\widehat{\mathcal{J}}_{L_j}$ by first obtaining a word $\mathcal{J}_{L_j}^b$ to be the product of the elements put on the component L_j where these elements are read along the orientation of L_j starting from any point (base point) in L_j. Then set

$$\widehat{\mathcal{J}}_{L_j} = \mathrm{tr}_u(\mathcal{J}_{L_j}^b)$$

where $\mathrm{tr}_u :\ \widehat{\mathcal{U}^H}^\sigma \to \mathrm{HH}_0\left(\widehat{\mathcal{U}^H}^\sigma\right)$ is the universal trace and $\mathrm{HH}_0\left(\widehat{\mathcal{U}^H}^\sigma\right)$ is the

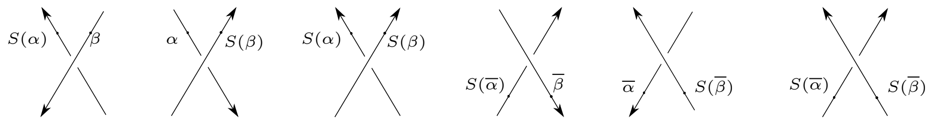


Figure 4.3 – The cases of crossings with upwards strings where $\mathcal{R}^\sigma = \sum \alpha \otimes \beta$ and $(\mathcal{R}^\sigma)^{-1} = \sum \overline{\alpha} \otimes \overline{\beta}$.

0^{th}-Hochschild homology for the algebra $\widehat{\mathcal{U}^H}^\sigma$. We define

$$\widehat{\mathcal{J}_L} = \sum \widehat{\mathcal{J}_{L_1}} \otimes \ldots \otimes \widehat{\mathcal{J}_{L_n}} \in \mathrm{HH}_0\left(\widehat{\mathcal{U}^H}^{\sigma \otimes n}\right). \tag{4.3.1}$$

Remark 4.3.1. *1. There is a similar way to define the universal invariant by using the quantum trace* $\mathrm{tr}_q : A \to A/N$ *where* $N = \mathit{Span}_{\mathbb{C}}\{xy - yS^2(x) | \ x, y \in A\}$ *(see [23]).*

2. Product by g^{-1} *(g is pivotal element in A) induces an isomorphism* $HH_0(A) \xrightarrow{\sim} A/N$ *which gives a correspondence between Ohtsuki ([38]) and Habiro ([23]) universal invariant.*

Theorem 4.3.2 (see also Theorem 4.5 [38])**.** $\widehat{\mathcal{J}_L}$ *is a topological invariant of framed links.*

Proof. The proof in the finite dimensional setting apply without change. One can show that $\widehat{\mathcal{J}_{L_j}}$ does not depend on where we start reading the element on the closed components, and $\widehat{\mathcal{J}_L}$ is invariant under the Reidemeister moves for oriented links. This proves $\widehat{\mathcal{J}_L}$ is an invariant of framed links. □

We can similarly define the invariant of the string links by

$$\widehat{\mathcal{J}_T} = \sum \widehat{\mathcal{J}_{T_1}} \otimes \ldots \otimes \widehat{\mathcal{J}_{T_n}} \in \widehat{\mathcal{U}^H}^{\sigma \otimes n} \tag{4.3.2}$$

where T is a string link consisting of n components T_i and $\widehat{\mathcal{J}_{T_i}}$ is determined by reading the elements along the orientation of T_i for $1 \leq i \leq n$. The relation between the invariant of tangles and of its closure is similar as Proposition 7.3 in [23]:

Proposition 4.3.3. *If* T *is a string link, then we have*

$$\widehat{\mathcal{J}_{cl(T)}} = \mathrm{tr}_u^{\otimes n}\left((\phi_0 \otimes \ldots \otimes \phi_0)(\widehat{\mathcal{J}_T})\right) = \mathrm{tr}_u^{\otimes n}\left((\phi_0^{-1} \otimes \ldots \otimes \phi_0^{-1})(\widehat{\mathcal{J}_T})\right)$$

where $cl(T)$ *is the closure of* T*.*

4.3.3 Value of universal invariant of link diagrams

For $x, y \in \mathbb{C}^2 \times \mathbb{C}^2$, call $Q(x,y)$ the polarization of the quadratic form determined by the matrix $B = (b_{ij})$ which is given by $b_{11} = 0,\ b_{12} = b_{21} = -1,\ b_{22} = -2$. Recall that $h_{i,j} = 1 \otimes ... \otimes h_i \otimes ... \otimes 1$ where h_i is in j-th position for $i = 1, 2$ and $j = 1, ..., n$. Let $\mathfrak{H}^{(n)} = Span_{\mathbb{C}}\{h_{i,j}\} \subset \mathcal{U}^{H\otimes n}$ and Q_{ij} be the quadratic form on $\mathfrak{H}^{(n)^*}$ defined by

$$Q_{ij}(h) = Q(h_{[i]}, h_{[j]}) = h_{[i]}^t B h_{[j]}$$

where $h_{[i]}$ is the column matrix $\begin{pmatrix} h_{1,i} \\ h_{2,i} \end{pmatrix}$ for $i = 1, ..., n$. Recall also the formula for the universal $\mathcal{R}$-matrix and its inverse in Equations (4.2.19) and (4.2.20).

Let L be a link diagram consisting of n ordered circle components $L_1, ..., L_n$. Denote by $\mathrm{lk} = (\mathrm{lk}_{ij})$ the linking matrix of the link diagram L and set $Q_L(h) = \sum_{1\le i,j\le n} \mathrm{lk}_{ij}\, Q_{ij}(h)$. We consider the algebraic automorphisms $\varphi_{ij},\ \varphi_{Q_L}$ of $\widehat{\mathcal{U}^H}^{\sigma\otimes n}$ given by

$$\varphi_{ij}(x) = \xi^{-Q_{ij}(h)} x \xi^{Q_{ij}(h)},\ \varphi_{Q_L}(x) = \xi^{-Q_L(h)} x \xi^{Q_L(h)} \text{ for } x \in \widehat{\mathcal{U}^H}^{\sigma\otimes n}. \quad (4.3.3)$$

Remark that φ_{ij} and φ_{Q_L} restrict to an automorphism of $\mathcal{U}^{\sigma\otimes n}$. Indeed, we denote the weight of an element $x \in \mathcal{U}^H$ for h_i by $|x|_i,\ i = 1, 2$, we have that $|x|_i \in \mathbb{Z}$. We also recall that

$$h_i x = x(h_i + |x|_i),\ x h_i = (h_i - |x|_i)x \text{ for } x \in \mathcal{U}^H.$$

These equalities imply that for $x = \bigotimes_{k=1}^n x_k \in \mathcal{U}^{\sigma\otimes n}$ we have

$$\bigotimes_{k=1}^{n} x_k \xi^{h_{1,i}h_{2,j}} = \xi^{1\otimes...\otimes(h_1-|x_i|_1)\otimes...\otimes(h_2-|x_j|_2)\otimes...\otimes 1} \bigotimes_{k=1}^{n} x_k. \quad (4.3.4)$$

Then $\xi^{h_i} = k_i \in \mathcal{U}^\sigma$ implies that $x\xi^{h_{1,i}h_{2,j}} = \xi^{h_{1,i}h_{2,j}} x'$ with $x' \in \mathcal{U}^{\sigma\otimes n}$. This deduces that $\varphi_{ij}(\mathcal{U}^{\sigma\otimes n}) = \mathcal{U}^{\sigma\otimes n}$ for $1 \le i, j \le n$ and $\varphi_{Q_L}(\mathcal{U}^{\sigma\otimes n}) = \mathcal{U}^{\sigma\otimes n}$.

Recall that $\widehat{\mathcal{J}_L} = \mathrm{tr}_u^{\otimes n}(\mathcal{J}_L^b) = \mathcal{J}_L^b + [\widehat{\mathcal{U}^H}^{\sigma\otimes n}, \widehat{\mathcal{U}^H}^{\sigma\otimes n}]$ where $\mathcal{J}_L^b$ depends on the choice of the base points. We have the theorem.

Theorem 4.3.4. *We have $\xi^{-Q_L(h)}\mathcal{J}_L^b \in \mathcal{U}^{\sigma\otimes n}$ and if b' is an other choice of base points then*
$\xi^{-Q_L(h)}\mathcal{J}_L^b - \xi^{-Q_L(h)}\mathcal{J}_L^{b'} \in N_{Q_L}$ *where* $N_{Q_L} = \xi^{-Q_L(h)}[\widehat{\mathcal{U}^H}^{\sigma\otimes n}, \widehat{\mathcal{U}^H}^{\sigma\otimes n}] \cap \mathcal{U}^{\sigma\otimes n}$.

Proof. We fix the base points and represent the value of $\mathcal{J}_L^b$ by the product of two parts, the first one is in $\mathcal{H}(\mathfrak{H}^*)$ and the second one is in the tensor

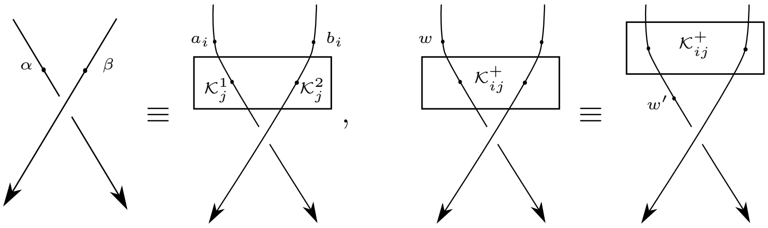


Figure 4.4 – Separation of elements

product of copies of $\mathcal{U}^\sigma$ as follow. For $j = 1, ..., n$ we first put the element of $\widehat{\mathcal{U}^H}^\sigma$ on the strands L_j following the rule depicted in Figure 4.2. By Equality (4.2.19)

$$\mathcal{R}^\sigma = \sum \alpha \otimes \beta = \sum a_i \otimes b_i \sum \mathcal{K}_j^1 \otimes \mathcal{K}_j^2$$

we can separate the elements coming from the Cartan part from the rest. Second, we fix the Cartan parts of the elements at the cross points and then push the rest of the elements to the base point of strand (along the orientation of L_j), see illustration in Figure 4.4 (w and w' related as in Equation (4.3.4)). The product of this part gives an element $w_j \in \mathcal{U}^\sigma$ for $j = 1, ..., n$. At each point of crossing (i, j) between the i-strand and j-strand of L, its Cartan part gives us the element

$$\mathcal{K}_{ij}^{\varepsilon_{ij}} = \xi^{\varepsilon_{ij}(-h_{1,i}h_{2,j} - h_{2,i}h_{1,j} - 2h_{2,i}h_{2,j})} = \xi^{\varepsilon_{ij}Q_{ij}(h)}$$

where $\varepsilon_{ij} = \pm 1$ is the sign of the crossing (i, j). Hence the value of $\mathcal{J}_L^b$ can be written as a product of $\xi^{Q_L(h)}$ and an element of $\mathcal{U}^{\sigma \otimes n}$. This means that $\xi^{-Q_L(h)}\mathcal{J}_L^b \in \mathcal{U}^{\sigma \otimes n}$.
By the definition of the $\widehat{\mathcal{J}_L}$ one has

$$\mathcal{J}_L^b - \mathcal{J}_L^{b'} \in [\widehat{\mathcal{U}^H}^{\sigma \otimes n}, \widehat{\mathcal{U}^H}^{\sigma \otimes n}].$$

It implies that

$$\xi^{-Q_L(h)}\mathcal{J}_L^b - \xi^{-Q_L(h)}\mathcal{J}_L^{b'} \in \xi^{-Q_L(h)}[\widehat{\mathcal{U}^H}^{\sigma \otimes n}, \widehat{\mathcal{U}^H}^{\sigma \otimes n}].$$

We have thus $\xi^{-Q_L(h)}\mathcal{J}_L^b - \xi^{-Q_L(h)}\mathcal{J}_L^{b'} \in \xi^{-Q_L(h)}[\widehat{\mathcal{U}^H}^{\sigma \otimes n}, \widehat{\mathcal{U}^H}^{\sigma \otimes n}] \cap \mathcal{U}^{\sigma \otimes n}$. □

We denote by $\mathcal{J}_L$ any elements $\mathcal{J}_L^b$ which is well defined modulo an element of $\xi^{Q_L(h)}N_{Q_L}$.

Remark 4.3.5. *As* $[\widehat{\mathcal{U}^H}^{\sigma \otimes n}, \widehat{\mathcal{U}^H}^{\sigma \otimes n}] = Span_{\mathbb{C}}\{xy - yx| \ x, y \in \widehat{\mathcal{U}^H}^{\sigma \otimes n}\}$ *then* $\xi^{-Q_L(h)}[\widehat{\mathcal{U}^H}^{\sigma \otimes n}, \widehat{\mathcal{U}^H}^{\sigma \otimes n}] = Span_{\mathbb{C}}\{\xi^{-Q_L(h)}(xy - yx)| \ x, y \in \widehat{\mathcal{U}^H}^{\sigma \otimes n}\} = Span_{\mathbb{C}}\{xy - \varphi_{Q_L}(y)x \mid x, y \in \widehat{\mathcal{U}^H}^{\sigma \otimes n}\}$. *I do not know if the following is true: is* N_{Q_L} *equal to* $Span_{\mathbb{C}}\{xy - \varphi_{Q_L}(y)x \mid x, y \in \mathcal{U}^{\sigma \otimes n}\}$.

Note also that $\mathcal{J}_L$ belongs in $\left(\widehat{\mathcal{U}^H}^{\sigma\otimes n}\right)^{\widehat{\mathcal{U}^H}^{\sigma}}$ where $\left(\widehat{\mathcal{U}^H}^{\sigma\otimes n}\right)^{\widehat{\mathcal{U}^H}^{\sigma}} = \left\{u \in \widehat{\mathcal{U}^H}^{\sigma\otimes n} | u\Delta^{[n]}(x) = \Delta^{[n]}(x)u\right\}$ for all $x \in \widehat{\mathcal{U}^H}^{\sigma}$. A proof of this assertion can be seen in Lemma 6 [3].

4.4 Invariant of 3-manifolds of Hennings type

In the article [26], Hennings proposed a method to construct an invariant of 3-manifolds from a universal invariant of links by using a finite dimensional ribbon algebra with its right integral. The invariant of 3-manifolds is computed from the universal invariant of links. The key point of the construction is the role of a right integral of the Hopf algebra [26]. It is well known that it always exists a right integral on a finite dimensional Hopf algebra. Virelizier generalised this fact by using the notions of a finite type unimodular ribbon Hopf π-coalgebra and the right π-integral to construct an invariant of 3-manifolds with π-structure. Here π is a group and the structure is given by representation of the fundamental group in π (see [48]). When $\pi = G$ is commutative a G-structure reduces to a G-valued cohomology class. In the case of the unrolled quantum algebra $\mathcal{U}^H$, the associated Hopf G-coalgebra can be ribbon but not finite type. However, we show that the associated Hopf G-coalgebra induces a finite type Hopf G-coalgebra by forgetting h_1, h_2. We show that we can still construct an invariant of 3-manifolds of Hennings type by working on the pairs (M, ω) in which M is a 3-manifold and ω is a cohomology class in $H^1(M, G)$. The construction of the invariant uses the discrete Fourier transform and the G-integral for the finite type Hopf G-coalgebra associated with $\mathcal{U}^\sigma$ (see in Section 4.4.1). This invariant is a generalisation of the one in [49] that apply to $\mathcal{U}^H$. We recall some definitions from [37, 49].

4.4.1 Hopf G-coalgebra from pivotal Hopf algebra $\mathcal{U}^\sigma$

Definition 4.4.1. *Let π be a group. A π-coalgebra over $\mathbb{C}$ is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of $\mathbb{C}$-spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ of $\mathbb{C}$-linear maps (the coproduct) and a $\mathbb{C}$-linear map $\varepsilon : C_1 \to \mathbb{C}$ (the counit) such that*

1. *Δ is coassociative, i.e. for any $\alpha, \beta, \gamma \in \pi$,*
$$(\Delta_{\alpha,\beta} \otimes \mathrm{Id}_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{Id}_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$
2. *for all $\alpha \in \pi$,* $(\mathrm{Id}_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{Id}_{C_\alpha} = (\varepsilon \otimes \mathrm{Id}_{C_\alpha})\Delta_{1,\alpha}$.

A *Hopf π-coalgebra* is a π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ endowed with a family $S = \{S_\alpha : \ H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\mathbb{C}$-linear maps (the antipode) such that

1. each H_α is an algebra with product m_α and unit element $1_\alpha \in H_\alpha$,
2. $\varepsilon : \ H_1 \to \mathbb{C}$ and $\Delta_{\alpha,\beta} : \ H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra homomorphisms for all $\alpha, \beta \in \pi$,
3. for any $\alpha \in \pi$,

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{Id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(\mathrm{Id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}.$$

A Hopf π-coalgebra is of *finite type* if H_α is finite dimensional algebra for any $\alpha \in \pi$.

Recall that $C = \mathbb{C}[k_1^{\pm\ell}, k_2^{\pm\ell}]$ is the commutative Hopf subalgebra in the center of $\mathcal{U}^\sigma$. Let $G = (\mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}, +) \xrightarrow{\sim} \mathrm{Hom}_{Alg}(C, \mathbb{C})$, $(\overline{\alpha}_1, \overline{\alpha}_2) \mapsto \left(k_i^\ell \mapsto \xi^{\ell\alpha_i}\right)$ for $i = 1, 2$ and let $\mathcal{U}_{\overline{\alpha}}$ be the algebra $\mathcal{U}^\sigma$ modulo the relations $k_i^\ell = \xi^{\ell\alpha_i}$ for $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2) \in G$, $i = 1, 2$.

Proposition 4.4.2. *The family $\mathcal{U}^\sigma = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is a finite type Hopf G-coalgebra.*

Proof. By applying Example 3.2.3 it follows that $\{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is the Hopf G-coalgebra with the coproduct and the antipode determined by the commutative diagrams:

$$\begin{array}{ccc} \mathcal{U}^\sigma & \xrightarrow{\Delta^\sigma} & \mathcal{U}^\sigma \otimes \mathcal{U}^\sigma \\ \downarrow{\scriptstyle p_{\overline{\alpha}+\overline{\beta}}} & & \downarrow{\scriptstyle p_{\overline{\alpha}} \otimes p_{\overline{\beta}}} \\ \mathcal{U}_{\overline{\alpha}+\overline{\beta}} & \xrightarrow{\Delta_{\overline{\alpha},\overline{\beta}}} & \mathcal{U}_{\overline{\alpha}} \otimes \mathcal{U}_{\overline{\beta}} \end{array} \qquad \begin{array}{ccc} \mathcal{U}^\sigma & \xrightarrow{S^\sigma} & \mathcal{U}^\sigma \\ \downarrow{\scriptstyle p_{\overline{\alpha}}} & & \downarrow{\scriptstyle p_{-\overline{\alpha}}} \\ \mathcal{U}_{\overline{\alpha}} & \xrightarrow{S_{\overline{\alpha}}} & \mathcal{U}_{-\overline{\alpha}} \end{array}$$

where $p_{\overline{\alpha}} : \ \mathcal{U}^\sigma \to \mathcal{U}_{\overline{\alpha}}$, $x \mapsto [x]$ is the projection from $\mathcal{U}^\sigma$ to $\mathcal{U}_{\overline{\alpha}}$. For $\overline{\alpha} = \overline{0}$ the Hopf algebra $\mathcal{U}_{\overline{0}}$ is called the restricted quantum $\mathfrak{sl}(2|1)$, i.e. the algebra $\mathcal{U}^\sigma$ modulo the relations $k_i^\ell = 1$ for $i = 1, 2$. Furthermore $\dim(\mathcal{U}_{\overline{\alpha}}) = 32\ell^4$ for $\overline{\alpha} \in G$. This finished the proof. □

Proposition 4.4.3. *The small quantum group $\mathcal{U}_{\overline{0}}$ is unimodular.*

Proof. Call $\mathscr{C}$ the even category of finite dimensional nipotent representations of $\mathcal{U}_\xi\mathfrak{sl}(2|1)$. We claim that the projective cover $P_\mathbb{C}$ of the trivial module is self dual: $P_\mathbb{C} \simeq P_\mathbb{C}^*$. The proof is analogous to Theorem 2.5.1. Furthermore $P_\mathbb{C} \in \mathcal{U}_{\overline{0}}$-mod so the category $\mathcal{U}_{\overline{0}}$-mod is unimodular. By [13, Lemma 4.2.1] confirms that $\mathcal{U}_{\overline{0}}$ is unimodular. □

A consequence of the proposition above is that the Hopf G-coalgebra $\mathcal{U}^\sigma$ is unimodular finite type.

Definition 4.4.4. *A π-trace for a Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is a family of $\mathbb{C}$-linear forms $\mathrm{tr} = \{\mathrm{tr}^\alpha : \ H_\alpha \to \mathbb{C}\}_{\alpha\in\pi}$ which verifies*

$$\mathrm{tr}^\alpha(xy) = \mathrm{tr}^\alpha(yx), \ \ \mathrm{tr}^{\alpha^{-1}}(S_\alpha(x)) = \mathrm{tr}^\alpha(x)$$

for all $\alpha \in \pi$ and $x, y \in H_\alpha$.

It is known that for each finite type Hopf π-coalgebra, there exists a family of linear forms called a family of the right π-integrals ([49]). Call $(\lambda_{\overline{\alpha}})_{\overline{\alpha}\in G}$ the family of right G-integral for the finite type Hopf G-coalgebra $\mathcal{U}^\sigma = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$. This means that the family of $\mathbb{C}$-linear forms $\lambda = (\lambda_{\overline{\alpha}})_{\overline{\alpha}\in G} \in \prod_{\overline{\alpha}\in G} \mathcal{U}^*_{\overline{\alpha}}$ satisfies

$$(\lambda_{\overline{\alpha}} \otimes \mathrm{Id}_{\mathcal{U}_{\overline{\beta}}})\Delta_{\overline{\alpha},\overline{\beta}} = \lambda_{\overline{\alpha}+\overline{\beta}} 1_{\overline{\beta}} \tag{4.4.1}$$

for all $\overline{\alpha}, \overline{\beta} \in G$ (see in Section 3 [49]). Note that $\lambda_{\overline{0}}$ is an usual right integral for the Hopf algebra $\mathcal{U}_{\overline{0}}$. We define a family of $\mathbb{C}$-linear forms $\{\mathrm{tr}^{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ on $\mathcal{U}^\sigma$ determined by

$$\mathrm{tr}^{\overline{\alpha}}(x) := \lambda_{\overline{\alpha}}(G_{\overline{\alpha}} x) \quad \text{for } x \in \mathcal{U}_{\overline{\alpha}}$$

where $G_{\overline{\alpha}} = \sigma\phi_0|_{k_i^\ell = \xi^{\ell\alpha_i}}$ for $i = 1, 2$, i.e. $G_{\overline{\alpha}} = \xi^{-\ell\alpha_1}\sigma k_2^{-2} \mod k_i^\ell - \xi^{\ell\alpha_i}$ for $i = 1, 2$. This family determines a G-trace by proposition below.

Proposition 4.4.5. *The family $\{\mathrm{tr}^{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ above is a G-trace for the unimodular finite type Hopf G-coalgebra $\mathcal{U}^\sigma = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$.*

Proof. As $\mathcal{U}^\sigma = \{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ is a unimodular finite type Hopf G-coalgebra, by Theorem 4.2 and Lemma 6.8 [49] for $\mathcal{U}^\sigma$ one gets

$$\begin{aligned} \lambda_{\overline{\alpha}}(xy) &= \lambda_{\overline{\alpha}}\left(S_{-\overline{\alpha}}S_{\overline{\alpha}}(y)x\right), \\ \lambda_{-\overline{\alpha}}\left(S_{\overline{\alpha}}(x)\right) &= \lambda_{\overline{\alpha}}\left(G_{\overline{\alpha}}^2 x\right) \ \text{ and} \\ S_{-\overline{\alpha}}S_{\overline{\alpha}}(x) &= G_{\overline{\alpha}} x G_{\overline{\alpha}}^{-1} \quad \text{for } x, y \in \mathcal{U}_{\overline{\alpha}}. \end{aligned}$$

By the definition of $\{\mathrm{tr}^{\overline{\alpha}}\}_{\overline{\alpha}\in G}$ we have

$$\begin{aligned} \mathrm{tr}^{\overline{\alpha}}(yx) &= \lambda_{\overline{\alpha}}(G_{\overline{\alpha}} yx) = \lambda_{\overline{\alpha}}\left(S_{-\overline{\alpha}}S_{\overline{\alpha}}(x)G_{\overline{\alpha}} y\right) \\ &= \lambda_{\overline{\alpha}}(G_{\overline{\alpha}} xy) = \mathrm{tr}^{\overline{\alpha}}(xy). \end{aligned}$$

Furthermore, for $x \in \mathcal{U}_{\overline{\alpha}}$

$$\begin{aligned} \lambda_{-\overline{\alpha}}\left(S_{\overline{\alpha}}(x)\right) &= \lambda_{-\overline{\alpha}}\left(S_{\overline{\alpha}}(x)S_{\overline{\alpha}}(G_{\overline{\alpha}})G_{-\overline{\alpha}}\right) \\ &= \lambda_{-\overline{\alpha}}\left(S_{\overline{\alpha}}(G_{\overline{\alpha}} x)G_{-\overline{\alpha}}\right) \\ &= \lambda_{-\overline{\alpha}}\left(S_{\overline{\alpha}}S_{-\overline{\alpha}}(G_{-\overline{\alpha}})S_{\overline{\alpha}}(G_{\overline{\alpha}} x)\right) \\ &= \lambda_{-\overline{\alpha}}\left(G_{-\overline{\alpha}}S_{\overline{\alpha}}(G_{\overline{\alpha}} x)\right) \\ &= \mathrm{tr}^{-\overline{\alpha}}\left(S_{\overline{\alpha}}(G_{\overline{\alpha}} x)\right) \end{aligned}$$

and $\lambda_{\overline{\alpha}}(G^2_{\overline{\alpha}}x) = \mathrm{tr}^{\overline{\alpha}}(G_{\overline{\alpha}}x)$ so $\mathrm{tr}^{-\overline{\alpha}}(S_{\overline{\alpha}}(x)) = \mathrm{tr}^{\overline{\alpha}}(x)$. This implies that the family $\mathrm{tr} = (\mathrm{tr}^{\overline{\alpha}})_{\overline{\alpha}\in G}$ is a G-trace for $\mathcal{U}^\sigma$. □

Note that, since $S_{-\overline{\alpha}}S_{\overline{\alpha}}(G_{\overline{\alpha}}) = G_{\overline{\alpha}}$ for $x \in \mathcal{U}_{\overline{\alpha}}$ then

$$\lambda_{\overline{\alpha}}(G_{\overline{\alpha}}x) = \lambda_{\overline{\alpha}}(S_{-\overline{\alpha}}S_{\overline{\alpha}}(G_{\overline{\alpha}})x) = \lambda_{\overline{\alpha}}(xG_{\overline{\alpha}}).$$

Thus we also have $\mathrm{tr}^{\overline{\alpha}}(x) = \lambda_{\overline{\alpha}}(xG_{\overline{\alpha}})$ for $x \in \mathcal{U}_{\overline{\alpha}}$.

4.4.2 Discrete Fourier transform

For a (partial) map $f : \mathbb{C}^n \to \mathbb{C}$ we define $t_i(f)$ by $t_i(f)(h_1, ..., h_n) = f(h_1, ..., h_i + 1, ..., h_n)$ for $1 \leq i \leq n$. Let $\mathcal{L}_{\overline{\alpha}} = \{(\alpha_1, ..., \alpha_n) + \mathbb{Z}^n\}$ be the lattice of $\mathbb{C}^n$ corresponding to $\overrightarrow{\alpha} = \overline{\alpha} = (\overline{\alpha}_1, ..., \overline{\alpha}_n) \in (\mathbb{C}/\mathbb{Z})^n$. A function $f(h_1, ..., h_n) \in \mathcal{H}(h_1, ..., h_n)$ is called ℓ-periodic in h_i on the lattice $\mathcal{L}_{\overline{\alpha}}$ if it satisfies $f_{|\overline{\alpha}} = t_i^\ell\left(f_{|\overline{\alpha}}\right)$ where $f_{|\overline{\alpha}} := f_{|\mathcal{L}_{\overline{\alpha}}}$. A function $f(h_1, ..., h_n) \in \mathcal{H}(h_1, ..., h_n)$ is ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$ if it is in all variables on $\mathcal{L}_{\overline{\alpha}}$. The functions $\{\xi^{mh_i}\}_{m\in\mathbb{Z}}^{i=1,...,n}$ are ℓ-periodic and $\xi^{\ell h_i} - \xi^{\ell\alpha_i}$ are zero on $\overline{\alpha}$. Let I be the ideal in the ring $R = \mathbb{C}[\xi^{\pm h_1}, ..., \xi^{\pm h_n}]$ generated by $\xi^{\ell h_i} - \xi^{\ell\alpha_i}$ for $1 \leq i \leq n$. Then an element of R/I defines a ℓ-periodic map in all variables on $\mathcal{L}_{\overline{\alpha}}$.

Proposition 4.4.6 (Discrete Fourier transform). *Let $f = f(h_1, ..., h_n) \in \mathcal{H}(h_1, ..., h_n)$ be a ℓ-periodic function on $\mathcal{L}_{\overline{\alpha}}$. Then there is a unique element $\mathcal{F}_{\overrightarrow{\alpha}}(f) \in R/I$ which coincides with f on $\mathcal{L}_{\overline{\alpha}}$ and is given by*

$$\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{m_1,...,m_n=0}^{\ell-1} a_{m_1...m_n}\xi^{m_1h_1+...+m_nh_n}. \tag{4.4.2}$$

The coefficients $a_{m_1...m_n}$ (Fourier coefficients) are determined by

$$a_{m_1...m_n} = \frac{1}{\ell^n}\sum_{i_1,...,i_n=0}^{\ell-1}\xi^{-m_1(\alpha_1+i_1)-...-m_n(\alpha_n+i_n)}f(\alpha_1 + i_1, ..., \alpha_n + i_n).$$

Proof. We consider first the function $f(h_1) \in \mathcal{H}(h_1)$ is ℓ-periodic on $\mathcal{L}_{\overline{\alpha}_1}$ for $\overline{\alpha}_1 \in \mathbb{C}/\mathbb{Z}$ which is denoted by $f_{|\overline{\alpha}_1}$. The set of such functions is a ℓ-dimensional vector space. The family $\{\xi^{m_1h_1}\}_{m_1=0}^{\ell-1}$ of linearly independent ℓ-periodic functions on $\mathcal{L}_{\overline{\alpha}_1}$ is a basis of this space, so we can write

$$f_{|\overline{\alpha}_1} = \sum_{m_1=0}^{\ell-1} a_{m_1}\xi^{m_1h_1}.$$

To determine $(a_{m_1})_{m_1}$ we evaluate the function at $\alpha_1 + i_1$ for $i_1 = 0, ..., \ell - 1$, we have a linear system of ℓ variables a_{m_1} with $m_1 = 0, .., \ell - 1$

$$\sum_{m_1=0}^{\ell-1} a_{m_1} \xi^{m_1(\alpha_1+i_1)} = f(\alpha_1 + i_1) \text{ for } i_1 = 0, ..., \ell - 1.$$

The matrix of this linear system is

$$A = \begin{bmatrix} 1 & \xi^{\alpha_1} & \dots & \xi^{(\ell-1)\alpha_1} \\ 1 & \xi^{\alpha_1+1} & \dots & \xi^{(\ell-1)(\alpha_1+1)} \\ & \dots & \dots & \dots \\ 1 & \xi^{\alpha_1+k} & \dots & \xi^{(\ell-1)(\alpha_1+k)} \\ & \dots & \dots & \dots \\ 1 & \xi^{\alpha_1+\ell-1} & \dots & \xi^{(\ell-1)(\alpha_1+\ell-1)} \end{bmatrix}.$$

Note that $\sum_{k=0}^{\ell-1} \xi^{k(i-j)} = \ell \delta_j^i$, so we have

$$A^{-1} = \frac{1}{\ell} \begin{bmatrix} 1 & 1 & \dots & 1 \\ \xi^{-\alpha_1} & \xi^{-(\alpha_1+1)} & \dots & \xi^{-(\alpha_1+\ell-1)} \\ \vdots & \vdots & \dots & \vdots \\ \xi^{-(\ell-1)\alpha_1} & \xi^{-(\ell-1)(\alpha_1+1)} & \dots & \xi^{-(\ell-1)(\alpha_1+\ell-1)} \end{bmatrix}.$$

This implies that $a_{m_1} = \frac{1}{\ell} \sum_{i_1=0}^{\ell-1} \xi^{-m_1(\alpha_1+i_1)} f(\alpha_1 + k_1)$ for $m_1 = 0, .., \ell - 1$. Then by induction on i for $1 \le i \le n$ we have a similar affirmation for the ℓ-periodic functions on $\mathcal{L}_{\overline{\alpha}}$ with $\overline{\alpha} \in (\mathbb{C}/\mathbb{Z})^n$. □

Denote $\mathcal{U}_{\otimes \overrightarrow{\alpha}} := \mathcal{U}_{\overline{\alpha}_1} \otimes ... \otimes \mathcal{U}_{\overline{\alpha}_n}$ for $\overrightarrow{\alpha} \in ((\mathbb{C}/\mathbb{Z})^2)^n$ in which $\overline{\alpha}_j = (\overline{\alpha}_{1j}, \overline{\alpha}_{2j}) \in (\mathbb{C}/\mathbb{Z})^2$ and $\mathcal{U}^0_{\otimes \overrightarrow{\alpha}}$ the subalgebra of $\mathcal{U}_{\otimes \overrightarrow{\alpha}}$ generated by $k_{i,j}^{\pm 1} = \xi^{\pm h_{i,j}}$ for $i = 1, 2$ and $j = 1, ..., n$ (see Equation (4.2.13)).

Corollary 4.4.7. *Let $f = f(h_{i,j}) \in \mathcal{H}(h_{i,j})$ be a ℓ-periodic function on $\mathcal{L}_{\overline{\alpha}}$. Then there is a unique element of $\mathcal{U}^0_{\otimes \overrightarrow{\alpha}}$ which coincides with f on $\mathcal{L}_{\overline{\alpha}}$ and it is given by*

$$\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{i_1,...,i_n,\ j_1,...,j_n=0}^{\ell-1} a_{i_1...i_n j_1...j_n} \prod_{s=1}^{n} k_{1,s}^{i_s} k_{2,s}^{j_s} \in \mathcal{U}^0_{\otimes \overrightarrow{\alpha}}$$

where

$$a_{i_1...i_n j_1...j_n} = \frac{1}{\ell^{2n}} \sum_{s_1,...,s_n,\ t_1,...,t_n=0}^{\ell-1} \xi^{-\sum_{m=1}^{n} i_m(\alpha_{1m}+s_m)+j_m(\alpha_{2m}+t_m)}.$$
$$f(\alpha_{11} + s_1, \alpha_{21} + t_1, ..., \alpha_{1n} + s_n, \alpha_{2n} + t_n).$$

Proof. By Proposition 4.4.6 we have

$$\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{i_1,\ldots,i_n,\ j_1,\ldots,j_n=0}^{\ell-1} a_{i_1\ldots i_n j_1\ldots j_n} \xi^{\sum_{s=1}^{n} i_s h_{1,s} + j_s h_{2,s}}.$$

Since $\xi^{h_{i,j}} = k_{i,j}$ for $i = 1, 2$ and $j = 1, \ldots, n$ then

$$\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{i_1,\ldots,i_n,\ j_1,\ldots,j_n=0}^{\ell-1} a_{i_1\ldots i_n j_1\ldots j_n} \prod_{s=1}^{n} k_{1,s}^{i_s} k_{2,s}^{j_s} \in \mathcal{U}^0_{\otimes\overrightarrow{\alpha}}.$$

Proposition 4.4.6 gives the formula determining the coefficients $a_{i_1\ldots i_n j_1\ldots j_n}$. □

Example 4.4.8. *The function $\mathcal{K} = \xi^{-h_1\otimes h_2 - h_2\otimes h_1 - 2h_2\otimes h_2}$ is ℓ-periodic on $\mathcal{L}_{\overline{0}}$ and we have*

$$\mathcal{F}_{\overrightarrow{0}}(\mathcal{K}) = \frac{1}{\ell^2} \sum_{i_1,i_2,j_1,j_2=0}^{\ell-1} \xi^{i_1 j_2 + i_2 j_1 - 2 i_1 i_2} k_1^{i_1} k_2^{j_1} \otimes k_1^{i_2} k_2^{j_2} \in \mathcal{U}_{\overline{0}} \otimes \mathcal{U}_{\overline{0}}. \tag{4.4.3}$$

Indeed, by Corollary 4.4.7 one has

$$\mathcal{F}_{\overrightarrow{0}}(\mathcal{K}) = \sum_{i_1,i_2,j_1,j_2=0}^{\ell-1} a_{i_1 i_2 j_1 j_2} k_1^{i_1} k_2^{j_1} \otimes k_1^{i_2} k_2^{j_2}.$$

The coefficients $a_{i_1 i_2 j_1 j_2}$ are computed as below

$$\begin{aligned}
a_{i_1 i_2 j_1 j_2} &= \frac{1}{\ell^4} \sum_{s_1,s_2,t_1,t_2=0}^{\ell-1} \xi^{-i_1 s_1 - j_1 t_1 - i_2 s_2 - j_2 t_2} \xi^{-s_1 t_2 - t_1 s_2 - 2 t_1 t_2} \\
&= \frac{1}{\ell^4} \sum_{t_1,t_2=0}^{\ell-1} \xi^{-j_1 t_1 - j_2 t_2 - 2 t_1 t_2} \sum_{s_1,s_2=0}^{\ell-1} \xi^{-i_1 s_1 - i_2 s_2 - s_1 t_2 - t_1 s_2} \\
&= \frac{1}{\ell^4} \sum_{t_1,t_2=0}^{\ell-1} \xi^{-j_1 t_1 - j_2 t_2 - 2 t_1 t_2} \sum_{s_1,s_2=0}^{\ell-1} \xi^{-(i_1+t_2)s_1 - (i_2+t_1)s_2} \\
&= \frac{1}{\ell^4} \sum_{t_1,t_2=0}^{\ell-1} \xi^{-j_1 t_1 - j_2 t_2 - 2 t_1 t_2} \sum_{s_1=0}^{\ell-1} \xi^{-(i_1+t_2)s_1} \sum_{s_2=0}^{\ell-1} \xi^{-(i_2+t_1)s_2} \\
&= \frac{1}{\ell^4} \sum_{t_1,t_2=0}^{\ell-1} \xi^{-j_1 t_1 - j_2 t_2 - 2 t_1 t_2} \ell \delta^0_{i_1+t_2 \bmod \ell\mathbb{Z}} \ell \delta^0_{i_2+t_1 \bmod \ell\mathbb{Z}} \\
&= \frac{1}{\ell^2} \sum_{t_1=0}^{\ell-1} \xi^{-j_1 t_1} \delta^0_{i_2+t_1 \bmod \ell\mathbb{Z}} \sum_{t_2=0}^{\ell-1} \xi^{-j_2 t_2 - 2 t_1 t_2} \delta^0_{i_1+t_2 \bmod \ell\mathbb{Z}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\ell^2} \sum_{t_1=0}^{\ell-1} \xi^{-j_1 t_1} \delta^0_{i_2+t_1 \, mod \, \ell\mathbb{Z}} \xi^{-j_2(-i_1)-2t_1(-i_1)} \\
&= \frac{1}{\ell^2} \xi^{-j_1(-i_2)} \xi^{-j_2(-i_1)-2(-i_2)(-i_1)} \\
&= \frac{1}{\ell^2} \xi^{j_1 i_2 + j_2 i_1 - 2 i_1 i_2}.
\end{aligned}$$

For $\overline{\alpha}_i \in (\mathbb{C}/\mathbb{Z})^2$ we call $\widehat{\mathcal{U}^H}^{\mathrm{per}}_{\overline{\alpha}_i}$ the subalgebra of $\widehat{\mathcal{U}^H}^{\sigma}$ generated by elements forms $u = \sum_j f_{ij}(h_1, h_2) w_j$ where $w_j \in \sigma^m \mathfrak{B}_+ \mathfrak{B}_-$ for $m = 0, 1$ and $f_{ij}(h_1, h_2) \in \mathcal{H}(h_1, h_2)$ are ℓ-periodic on $\mathcal{L}_{\overline{\alpha}_i}$. Denote $\widehat{\mathcal{U}^H}^{\mathrm{per}}_{\otimes\overrightarrow{\alpha}} = \widehat{\mathcal{U}^H}^{\mathrm{per}}_{\overline{\alpha}_1} \otimes \ldots \otimes \widehat{\mathcal{U}^H}^{\mathrm{per}}_{\overline{\alpha}_n}$. We extend linearly $\mathcal{F}_{\overrightarrow{\alpha}}$ to a map $\widehat{\mathcal{U}^H}^{\mathrm{per}}_{\otimes\overrightarrow{\alpha}} \to \mathcal{U}_{\otimes\overrightarrow{\alpha}}$ by the rule $\sum_m f_m(h_{1,i}, h_{2,j}) w_m \mapsto \sum \mathcal{F}_{\overrightarrow{\alpha}}(f_m(h_{1,i}, h_{2,j})) w_m$.

Lemma 4.4.9. *The map $\mathcal{F}_{\overrightarrow{\alpha}} : \widehat{\mathcal{U}^H}^{per}_{\otimes\overrightarrow{\alpha}} \to \mathcal{U}_{\otimes\overrightarrow{\alpha}}$ is an algebra map.*

Proof. By the unicity in Proposition 4.4.6, as $fg|_{\mathcal{L}_{\overline{\alpha}}} = \mathcal{F}_{\overrightarrow{\alpha}}(f)\mathcal{F}_{\overrightarrow{\alpha}}(g)$ we have

$$\mathcal{F}_{\overrightarrow{\alpha}}(fg) = \mathcal{F}_{\overrightarrow{\alpha}}(f)\mathcal{F}_{\overrightarrow{\alpha}}(g)$$

for the ℓ-periodic functions f, g on $\mathcal{L}_{\overline{\alpha}}$.
Consider the elements $f(h_1, h_2)w_1, g(h_1, h_2)w_2 \in \widehat{\mathcal{U}^H}^{\mathrm{per}}_{\overline{\alpha}_i}$ where f, g are ℓ-periodic on $\mathcal{L}_{\overline{\alpha}_i}$ and $w_1, w_2 \in \sigma^m \mathfrak{B}_+ \mathfrak{B}_-$ for $m = 0, 1$. By Remark 4.2.4 one has

$$\begin{aligned}
(f(h_1, h_2)w_1)(g(h_1, h_2)w_2) &= f(h_1, h_2)(w_1 g(h_1, h_2))w_2 \\
&= f(h_1, h_2) g(h_1 + |w_1|_1, h_2 + |w_1|_2) w_2
\end{aligned}$$

where $(|w_1|_1, |w_1|_2)$ is the weight of w_1 for (h_1, h_2). So we have

$$\begin{aligned}
\mathcal{F}_{\overrightarrow{\alpha}}(f w_1 g w_2) &= \mathcal{F}_{\overrightarrow{\alpha}}(f(h_1, h_2) g(h_1 + |w_1|_1, h_2 + |w_1|_2) w_1 w_2) \\
&= \mathcal{F}_{\overrightarrow{\alpha}}(f g(h_1 + |w_1|_1, h_2 + |w_1|_2)) w_1 w_2 \\
&= \mathcal{F}_{\overrightarrow{\alpha}}(f) \mathcal{F}_{\overrightarrow{\alpha}}(g(h_1 + |w_1|_1, h_2 + |w_1|_2)) w_1 w_2 \\
&= \mathcal{F}_{\overrightarrow{\alpha}}(f) w_1 \mathcal{F}_{\overrightarrow{\alpha}}(g(h_1 + |w_1|_1 - |w_1|_1, h_2 + |w_1|_2 - |w_1|_2)) w_2 \\
&= \mathcal{F}_{\overrightarrow{\alpha}}(f) w_1 \mathcal{F}_{\overrightarrow{\alpha}}(g) w_2 \\
&= \mathcal{F}_{\overrightarrow{\alpha}}(f w_1) \mathcal{F}_{\overrightarrow{\alpha}}(g w_2).
\end{aligned}$$

□

Lemma 4.4.10. *Assume $x \in \widehat{\mathcal{U}^H}^{per}_{\otimes\overrightarrow{\alpha}}$ is a commutator in $\widehat{\mathcal{U}^H}^{\sigma\otimes n}$ then $\mathcal{F}_{\overrightarrow{\alpha}}(x)$ is a commutator in $\mathcal{U}_{\otimes\overrightarrow{\alpha}}$.*

Proof. We consider an extension of the discrete Fourier transform on the lattice $\mathcal{L}_{\overline{\alpha}}$ which denoted by $\mathcal{F}'$. The extension will depend on $(\alpha_i)_i \in \mathbb{C}^{2n}$ and coincide with $\mathcal{F}_{\overrightarrow{\alpha}}$ on elements ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$. The transformation is defined as follow: let $f = f(h_{1,i}, h_{2,j})$ be a holomorphic function of $\mathcal{H}(h_{1,i}, h_{2,j})$ then if f is ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$ we define $\mathcal{F}'_{\overrightarrow{\alpha}}(f) = \mathcal{F}_{\overrightarrow{\alpha}}(f)$; if f is not ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$ we define $\mathcal{F}'(f) = \mathcal{F}_{\overrightarrow{\alpha}}(f\mid_{Rec})$ given by the right hand of Equation (4.4.2) where $Rec = \{(\alpha_1+i_1,\ \dots,\ \alpha_n+i_n),\ 0 \le i_1, \dots,\ i_n < \ell\}$. Then $\mathcal{F}'(f)$ is the unique element of R/I (see Section 4.4.2) which coincides with f on Rec. The map $\mathcal{F}'$ is also an algebra map. A proof is similar as the one of Lemma 4.4.9. As x is a commutator in $\widehat{\mathcal{U}^H}^{\sigma\otimes n}$ we write

$$x = fx_1gx_2 - gx_2fx_1$$

where $f,\ g \in \mathcal{H}(h_{1,i}, h_{2,j})$ and $x_1, x_2 \in \mathcal{U}_{\otimes\overrightarrow{\alpha}}$. Applying $\mathcal{F}'$ to the above equality one gets

$$\mathcal{F}'(x) = \mathcal{F}'(f)x_1\mathcal{F}'(g)x_2 - \mathcal{F}'(g)x_2\mathcal{F}'(f)x_1 \in [\mathcal{U}_{\otimes\overrightarrow{\alpha}},\ \mathcal{U}_{\otimes\overrightarrow{\alpha}}].$$

Furthermore $x \in \widehat{\mathcal{U}^H}^{\mathrm{per}}_{\otimes\overrightarrow{\alpha}}$ then $\mathcal{F}_{\overrightarrow{\alpha}}(x) = \mathcal{F}'(x)$. Thus $\mathcal{F}_{\overrightarrow{\alpha}}(x)$ is a commutator in $\mathcal{U}_{\otimes\overrightarrow{\alpha}}$. □

Lemma 4.4.11. *Let $\overline{\beta}, \overline{\gamma} \in (\mathbb{C}/\mathbb{Z})^2$ and let $\overrightarrow{\alpha} = \overline{\alpha} = \overline{\beta}+\overline{\gamma}$. Assume $f(h_1, h_2)$ is a ℓ-periodic entire function on $\mathcal{L}_{\overline{\alpha}}$. Then $\Delta(f)$ is ℓ-periodic on $\mathcal{L}_{(\overline{\beta},\overline{\gamma})}$ and*

$$\Delta_{\overline{\beta},\overline{\gamma}}\mathcal{F}_{\overrightarrow{\alpha}}(f) = \mathcal{F}_{(\overline{\beta},\overline{\gamma})}\left(\Delta(f)\right).$$

Proof. First, by Proposition 4.4.6 we have

$$\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{m_1,m_2=0}^{\ell-1} a_{m_1m_2}\xi^{m_1h_1+m_2h_2}$$

where $a_{m_1m_2} = \dfrac{1}{\ell^2}\sum_{i_1,i_2=0}^{\ell-1} \xi^{-m_1(\alpha_1+i_1)-m_2(\alpha_2+i_2)} f(\alpha_1+i_1, \alpha_2+i_2)$. Then

$$\Delta_{\overline{\beta},\overline{\gamma}}\mathcal{F}_{\overrightarrow{\alpha}}(f) = \sum_{m_1,m_2=0}^{\ell-1} a_{m_1m_2}\xi^{m_1(h_1\otimes 1+1\otimes h_1)+m_2(h_2\otimes 1+1\otimes h_2)}.$$

Second, the algebra homomorphism Δ gives us $\Delta f(h_1, h_2) = f(h_1\otimes 1 + 1\otimes h_1, h_2\otimes 1 + 1\otimes h_2)$. Applying the discrete Fourier transform one gets

$$\mathcal{F}_{(\overline{\beta},\overline{\gamma})}\left(\Delta(f)\right) = \sum_{n_1,n_2,n_3,n_4=0}^{\ell-1} b_{n_1n_2n_3n_4}\xi^{n_1(h_1\otimes 1)+n_2(h_2\otimes 1)+n_3(1\otimes h_1)+n_4(1\otimes h_2)}$$

where the Fourier coefficient

$$b_{n_1n_2n_3n_4} = \frac{1}{\ell^4}\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1} \xi^{-n_1(\beta_1+i_1)-n_2(\beta_2+i_2)-n_3(\gamma_1+j_1)-n_4(\gamma_2+j_2)}. \\ f(\beta_1+i_1+\gamma_1+j_1, \beta_2+i_2+\gamma_2+j_2).$$

By $\overline{\alpha} = \overline{\beta} + \overline{\gamma}$, one has

$$b_{n_1n_2n_3n_4} = \frac{1}{\ell^4}\xi^{-n_1\beta_1-n_2\beta_2-n_3\gamma_1-n_4\gamma_2}\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1} \xi^{-n_1i_1-n_2i_2-n_3j_1-n_4j_2}. \\ f(\alpha_1+i_1+j_1, \alpha_2+i_2+j_2).$$

Since $f(h_1,h_2)$ is ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$, setting $s = i_1 + j_1$ and $t = i_2 + j_2$ then

$$b_{n_1n_2n_3n_4} = \frac{1}{\ell^4}\xi^{-n_1\beta_1-n_2\beta_2-n_3\gamma_1-n_4\gamma_2}\sum_{i_1,i_2,s,t=0}^{\ell-1} f(\alpha_1+s,\alpha_2+t). \\ \xi^{-n_1i_1-n_2i_2-n_3(s-i_1)-n_4(t-i_2)}.$$

$$b_{n_1n_2n_3n_4} = \frac{1}{\ell^4}\xi^{-n_1\beta_1-n_2\beta_2-n_3\gamma_1-n_4\gamma_2}\sum_{s,t=0}^{\ell-1} f(\alpha_1+s,\alpha_2+t)\xi^{-n_3s-n_4t}. \\ \sum_{i_1,i_2=0}^{\ell-1}\xi^{(n_3-n_1)i_1+(n_4-n_2)i_2}.$$

Since

$$\begin{aligned}\sum_{i_1,i_2=0}^{\ell-1}\xi^{(n_3-n_1)i_1+(n_4-n_2)i_2} &= \sum_{i_1=0}^{\ell-1}\xi^{(n_3-n_1)i_1}\sum_{i_2=0}^{\ell-1}\xi^{(n_4-n_2)i_2} \\ &= \ell^2\delta_{n_3}^{n_1}\delta_{n_4}^{n_2}\end{aligned}$$

then $b_{n_1n_2n_3n_4} = 0$ if $(n_1,n_2) \neq (n_3,n_4)$ and when $(n_1,n_2) = (n_3,n_4)$ then $b_{n_1n_2n_1n_2}$ is computed

$$\begin{aligned}b_{n_1n_2n_1n_2} &= \frac{1}{\ell^2}\xi^{-n_1(\beta_1+\gamma_1)-n_2(\beta_2+\gamma_2)}\sum_{s,t=0}^{\ell-1} f(\alpha_1+s,\alpha_2+t)\xi^{-n_1s-n_2t} \\ &= \frac{1}{\ell^2}\xi^{-n_1\alpha_1-n_2\alpha_2}\sum_{s,t=0}^{\ell-1} f(\alpha_1+s,\alpha_2+t)\xi^{-n_1s-n_2t} \\ &= \frac{1}{\ell^2}\sum_{s,t=0}^{\ell-1}\xi^{-n_1(\alpha_1+s)-n_2(\alpha_2+t)}f(\alpha_1+s,\alpha_2+t) \\ &= a_{n_1n_2}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{F}_{(\overline{\beta},\overline{\gamma})}(\Delta(f)) &= \sum_{n_1,n_2=0}^{\ell-1} b_{n_1n_2n_1n_2}\xi^{n_1(h_1\otimes 1)+n_2(h_2\otimes 1)+n_1(1\otimes h_1)+n_2(1\otimes h_2)} \\ &= \sum_{n_1,n_2=0}^{\ell-1} a_{n_1n_2}\xi^{n_1(h_1\otimes 1+1\otimes h_1)+n_2(h_2\otimes 1+1\otimes h_2)} \\ &= \Delta_{\overline{\beta},\overline{\gamma}}\mathcal{F}_{\overrightarrow{\alpha}}(f).\end{aligned}$$

□

Remark 4.4.12. *As $S(h_i) = -h_i$ for $i = 1, 2$, by the similar calculations as in Lemma 4.4.11 then*

$$\mathcal{F}_{-\overrightarrow{\alpha}}S(f) = S_{\overline{\alpha}}\mathcal{F}_{\overrightarrow{\alpha}}(f).$$

A consequence of Lemma 4.4.11 is that $\mathcal{R}^{\overline{0}} = R_1\check{\mathcal{R}}\mathcal{F}_{\overrightarrow{0}}(\mathcal{K})$ is the universal R-matrix of $\mathcal{U}_{\overline{0}}$ with $\mathcal{R}^{\overline{0}} = \mathcal{F}_{\overrightarrow{0}}(\mathcal{R}_q)$ is given by

$$\mathcal{R}^{\overline{0}} = \frac{1}{\ell^2}R_1\sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{\rho,\delta=0}^{1}\frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!}\xi^{i_1j_2+i_2j_1-2i_1i_2}. \\ e_1^ie_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\otimes f_1^if_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2} \quad (4.4.4)$$

where $R_1 = \frac{1}{2}(1\otimes 1 + \sigma\otimes 1 + 1\otimes\sigma - \sigma\otimes\sigma)$ (see Section 4.2.3). Indeed the relations satisfied by the $\mathcal{R}$-matrix $\mathcal{R}_q$ (see [31], [51]) translate to the relations for $\mathcal{R}^{\overline{0}}$.

4.4.3 Invariant of 3-manifolds of Hennings type

Let L be a framed link in S^3 consisting of n components (still denote by L its link diagram), M be a 3-manifold obtained by surgery along the link L. Let ω be an element of the cohomology group $H^1(M, G)$ (see Section 2 [8]). The value of the invariant of link $\mathcal{J}_L^b$ is in $\xi^{Q_L(h)}\mathcal{U}^{\sigma\otimes n}$. Let $\overline{\alpha}_j = \omega(m_j) = (\overline{\alpha}_{1j}, \overline{\alpha}_{2j})$ here m_j is a meridian of the j-th component of L. Denote $\overline{\alpha} = (\overline{\alpha}_1, ..., \overline{\alpha}_n)$. Since ω is an element of the cohomology group $H^1(M, G)$ it vanishes on longitudes of L, this implies the relation $\sum_{j=1}^n \mathrm{lk}_{ij}\,\overline{\alpha}_j = \overline{0}$, $\forall i = 1, ..., n$. We have

Proposition 4.4.13. *The function $f(h_{1,i}, h_{2,j}) = \xi^{Q_L(h)}$ is ℓ-periodic on $\mathcal{L}_{\overline{\alpha}}$.*

Proof. We denote $h_{1,i}+\ell = 1\otimes \ldots \otimes (h_1+\ell)\otimes \ldots \otimes 1$ where $h_1+\ell$ is in i-th position. We have

$$\begin{aligned} f(h_{1,i}+\ell, h_{2,j}) &= \xi^{-\sum_{i,j=1}^{n} \mathrm{lk}_{ij}((h_1+\ell)_i h_{2,j}+h_{2,i}h_{1,j}+2h_{2,i}h_{2,j})} \\ &= \xi^{\sum_{i,j=1}^{n} \mathrm{lk}_{ij}\, Q_{ij}(h)} \xi^{-\sum_{i,j=1}^{n} \mathrm{lk}_{ij}\, \ell h_{2,j}} \\ &= f(h_{1,i}, h_{2,j}) \xi^{-\ell \sum_{i,j=1}^{n} \mathrm{lk}_{ij}\, \alpha_{2j}}. \end{aligned}$$

The equalities $\sum_{j=1}^{n} \mathrm{lk}_{ij}\, \overline{\alpha}_j = \overline{0}$ imply that $\sum_{i,j=1}^{n} \mathrm{lk}_{ij}\, \alpha_{2j} \in \mathbb{Z}$. Hence we get $f(h_{1,i}+\ell, h_{2,j}) = f(h_{1,i}, h_{2,j})$. The computation is similar for the variables $h_{2,j}$. □

Lemma 4.4.6 implies that $\mathcal{F}_{\overrightarrow{\alpha}}\left(\xi^{Q_L(h)}\right) \in \mathcal{U}_{\otimes \overrightarrow{\alpha}}$. We define

$$\mathcal{J}_L^{\omega} = \mathcal{F}_{\overrightarrow{\alpha}}\left(\mathcal{J}_L\right) \in \mathrm{HH}_0(\mathcal{U}_{\otimes \overrightarrow{\alpha}}) \tag{4.4.5}$$

thanks to Theorem 4.3.4 and Lemma 4.4.10. Let $\theta_{\overline{0}}$ be the ribbon element of the small quantum group $\mathcal{U}_{\overline{0}}$.

Lemma 4.4.14. *There exists a normalization of $(\lambda_{\overline{\alpha}})_{\overline{\alpha}\in G}$ such that*

$$\lambda_{\overline{0}}(\theta_{\overline{0}}) = \lambda_{\overline{0}}(\theta_{\overline{0}}^{-1}) = 1.$$

Proof. The proof is thanks to Lemma 4.4.20. □

Theorem 4.4.15.

$$\mathcal{J}(M,\omega) = \bigotimes_{j=1}^{n} \mathrm{tr}^{\overline{\alpha}_j}\left(\mathcal{J}_L^{\omega}\right) \tag{4.4.6}$$

is a topological invariant of the pairs (M,ω) where n is the number of components of the surgery link L.

Remark 4.4.16. *Usual quantum surgery invariants are renormalized thanks to the signature. There is no need of renormalisation here thanks to Lemma 4.4.14.*

We use a result on the equivalence of 3-manifolds obtained by surgery along a link to prove Theorem 4.4.15, that is the theorem below.

Theorem 4.4.17 ([32])**.** *Let M_1 and M_2 be oriented 3-manifolds and $f: M_1 \to M_2$ be an orientation preserving diffeomorphism. Any two surgery presentations L_1 and L_2 of M_1 and M_2, respectively can be connected by a sequence of handle-slides, blow-up moves and blow-down moves such that the induced diffeomorphism between $M_1 = S^3_{L_1}$ and $M_2 = S^3_{L_2}$ is isotopic to f.*

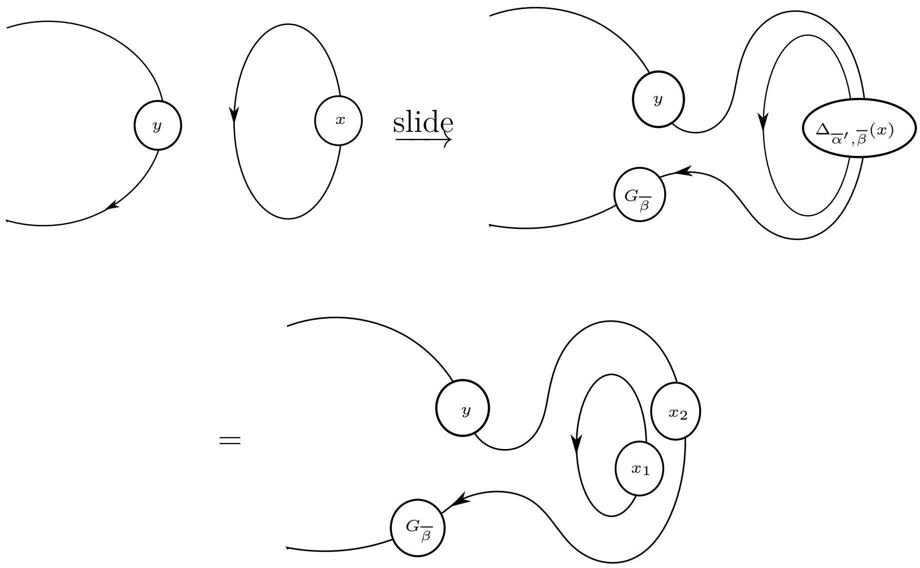


Figure 4.5 – Second Kirby's move

Proof of Theorem 4.4.15. We need to show that $\mathcal{J}(M,\omega)$ does not change under two Kirby's moves. In the case of handle slide (the second Kirby's move), we can assume that the algebraic element on the strands are already concentrated as illustrated in the first component of Figure 4.5 where $x \in \mathcal{U}_{\overline{\alpha}}$, $y \in \mathcal{U}_{\overline{\beta}}$ are given by the discrete Fourier transform (see Equation (4.4.5)). The associated invariant of 3-manifolds will be computed by

$$\operatorname{tr}^{\overline{\alpha}}(x)y = \lambda_{\overline{\alpha}}(G_{\overline{\alpha}}x)y \quad x \in \mathcal{U}_{\overline{\alpha}}, y \in \mathcal{U}_{\overline{\beta}}.$$

After sliding, by the commutativity of the Fourier transform with the coproduct in Lemma 4.4.11 and by the property of the element $\mathcal{R}$-matrix we replace x by $\Delta_{\overline{\alpha}',\overline{\beta}}(x) = x_1 \otimes x_2$ for $x \in \mathcal{U}_{\overline{\alpha}}$, $x_1 \in \mathcal{U}_{\overline{\alpha}'}$, $x_2 \in \mathcal{U}_{\overline{\beta}}$ as in the second and third component of Figure 4.5. Note that the relation between the homology classes of the meridians is $m_{x_1} + m_y = m_x$, i.e. $\omega(m_{x_1}) + \omega(m_y) = \omega(m_x) \Leftrightarrow \overline{\alpha}' + \overline{\beta} = \overline{\alpha}$. The invariant is determined by

$$\operatorname{tr}^{\overline{\alpha}'}(x_1)yx_2G_{\overline{\beta}} = \lambda_{\overline{\alpha}'}(G_{\overline{\alpha}'}x_1)yx_2G_{\overline{\beta}}.$$

Furthermore, the definition of the right G-integral $(\lambda_{\overline{\alpha}})_{\overline{\alpha}\in G}$ implies that

$$\left(\lambda_{\overline{\alpha}'} \otimes \operatorname{Id}_{\mathcal{U}_{\overline{\beta}}}\right) \Delta_{\overline{\alpha}',\overline{\beta}}(xG_{\overline{\alpha}}) = \lambda_{\overline{\alpha}}(xG_{\overline{\alpha}})1_{\overline{\beta}}$$

then

$$\lambda_{\overline{\alpha}'}(x_1G_{\overline{\alpha}'})x_2G_{\overline{\beta}} = \lambda_{\overline{\alpha}}(xG_{\overline{\alpha}})1_{\overline{\beta}}$$

and finally

$$\lambda_{\overline{\alpha}'}(x_1 G_{\overline{\alpha}'}) y x_2 G_{\overline{\beta}} = \lambda_{\overline{\alpha}}(x G_{\overline{\alpha}}) y,$$

i.e. $\mathcal{J}(M,\omega)$ does not change under the second Kirby's move. Changing the orientation of a component changes $\mathcal{J}_L$ by applying an antipode (see [38]), Proposition 4.4.5 and Remark 4.4.12 imply $\mathcal{J}(M,\omega)$ does not depend on the orientation. For the first Kirby's move, the blowing up and blowing down, it is easy to see that $\omega(m) = \overline{0}$ for m the meridian of ± 1-framed loops and the two ± 1-framed loops evaluate as $\lambda_{\overline{0}}(\theta_{\overline{0}})$ and $\lambda_{\overline{0}}(\theta_{\overline{0}}^{-1})$, respectively. □

Recall that the Hopf algebra $\mathcal{U}_{\overline{0}}$ has a PBW basis $\{f_1^i f_3^\rho f_2^\delta e_1^{i'} e_3^{\rho'} e_2^{\delta'} k_1^{j_1} k_2^{j_2} \sigma^m : 0 \leq \rho, \delta, \rho', \delta', m \leq 1,\ 0 \leq i, i', j_1, j_2 \leq \ell - 1\}$. To prove Lemma 4.4.14 we need the proposition below.

Proposition 4.4.18. *The linear form $\lambda_{\overline{0}} : \mathcal{U}_{\overline{0}} \to \mathbb{C}$ determined by*

$$\lambda_{\overline{0}}(f_1^i f_3^\rho f_2^\delta e_1^{i'} e_3^{\rho'} e_2^{\delta'} k_1^{j_1} k_2^{j_2} \sigma^m) = \eta \delta_{\ell-1}^i \delta_1^\rho \delta_1^\sigma \delta_{\ell-1}^{i'} \delta_1^{\rho'} \delta_1^{\sigma'} \delta_0^{j_1} \delta_{\ell-2}^{j_2} \delta_0^m \tag{4.4.7}$$

is a right integral of $\mathcal{U}_{\overline{0}}$ where $\eta \in \mathbb{C}^*$ is a constant and δ_j^i is Kronecker symbol.

Proof. See in Appendix A.2. □

By Equation (4.4.1) we have the remark.

Remark 4.4.19. *For $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}$ then*

$$\lambda_{\overline{\alpha}}(f_1^i f_3^\rho f_2^\delta e_1^{i'} e_3^{\rho'} e_2^{\delta'} k_1^{j_1} k_2^{j_2} \sigma^m) = \eta \xi^{\ell(\alpha_1+\alpha_2)} \delta_{\ell-1}^i \delta_1^\rho \delta_1^\sigma \delta_{\ell-1}^{i'} \delta_1^{\rho'} \delta_1^{\sigma'} \delta_0^{j_1} \delta_{\ell-2}^{j_2} \delta_0^m$$

is a right G-integral for the Hopf G-coalgebra $\{\mathcal{U}_{\overline{\alpha}}\}_{\overline{\alpha} \in G}$.

By using Proposition 4.4.18 one gets the lemma.

Lemma 4.4.20. *We have*

$$\lambda_{\overline{0}}(\theta_{\overline{0}}) = \lambda_{\overline{0}}(\theta_{\overline{0}}^{-1}) = \frac{\{1\}^{\ell+1}(1-\xi)^{\ell-1}}{\ell(\ell-1)} \eta.$$

Proof. See in Appendix A.3. □

Appendix A

Computations in $\mathcal{U}_\xi\mathfrak{sl}(2|1)$

A.1 Proof of Lemma 2.3.3

To prove Lemma 2.3.3 we need the lemma below. Set $e_{\max} = e_2e_3e_1^{\ell-1}$, $f_{\max} = f_2f_3f_1^{\ell-1}$.

Lemma A.1.1. *1. $e_{max}f_{max} \neq 0$.*

2. The set $\{e_{max}k_1^sk_2^tf_{max}\ s,t \in \mathbb{Z}\}$ is free over $\mathbb{C}$.

Proof. First, let V_μ be a typical module with the highest weight $\mu = (\mu_1, \mu_2)$. We show $e_{\max}f_{\max} \neq 0$ by considering its action on V_μ. We have

$$e_{\max}f_{\max}w_{0,0,0} = e_2e_3e_1^{\ell-1}f_2f_3f_1^{\ell-1}w_{0,0,0} = e_2e_3e_1^{\ell-1}w_{1,1,\ell-1}.$$

Using the representation of V_μ determined in (2.4.5) one gets

$$e_1^{\ell-1}w_{1,1,\ell-1} = \xi^{\mu_1+\mu_2+2}\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]w_{1,1,0}$$

and by (2.4.6)

$$\begin{aligned}
e_2e_3e_1^{\ell-1}w_{1,1,\ell-1} &= \xi^{\mu_1+\mu_2+2}\left(\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]\right)e_2e_1e_2w_{1,1,0}\\
&= \xi^{\mu_1+\mu_2+2}\left(\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]\right)\left(-\xi^{\mu_1+1}[\mu_2+1][\mu_2]-\xi^{-\mu_2}[\mu_1][\mu_2]\right)w_{0,0,0}\\
&= -\xi^{\mu_1+\mu_2+2}\left(\prod_{i=1}^{\ell-1}[i][\mu_1+1-i][\mu_2]\right)\left(\xi^{\mu_1+1}[\mu_2+1]+\xi^{-\mu_2}[\mu_1]\right)w_{0,0,0}\\
&= -\xi^{\mu_1+\mu_2+3}\left(\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]\frac{[\mu_2]}{\xi-\xi^{-1}}\right)\left(\xi^{\mu_1+\mu_2+1}-\xi^{-\mu_1-\mu_2-1}\right)w_{0,0,0}
\end{aligned}$$

$$= -\xi^{\mu_1+\mu_2+3}\left(\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]\right)[\mu_2][\mu_1+\mu_2+1]w_{0,0,0}.$$

As V_μ is the typical module then $\prod_{i=1}^{\ell-1}[\mu_1+1-i].[\mu_2][\mu_1+\mu_2+1]\neq 0$ (see Remark 2.4.1). This implies that $e_{\max}f_{\max}w_{0,0,0}\neq 0$, i.e. $e_{\max}f_{\max}\neq 0$.

Second, one has $e_2e_3e_1^{\ell-1}k_i^m = \xi^{-2ma_{i2}}k_i^m e_2e_3e_1^{\ell-1}$ where $(a_{ij})_{1\le i,j\le 2}$ is the Cartan matrix in Definition 2.3.1, then one can write

$$e_2e_3e_1^{\ell-1}k_1^sk_2^tf_2f_3f_1^{\ell-1} = \xi^{2s}k_1^sk_2^te_2e_3e_1^{\ell-1}f_2f_3f_1^{\ell-1}.$$

By (2.4.1) and (2.4.2) we get

$$k_1^sk_2^te_{\max}f_{\max}w_{0,0,0} = \xi^{s\mu_1+t\mu_2}c(\mu_1,\mu_2)w_{0,0,0}$$

where $c(\mu_1,\mu_2) = -\xi^{\mu_1+\mu_2+3}[\mu_2][\mu_1+\mu_2+1]\prod_{i=1}^{\ell-1}[i][\mu_1+1-i]$.

The expression $\xi^{s\mu_1+t\mu_2}c(\mu_1,\mu_2)$ determines a complex function $f_{st}(\mu_1,\mu_2)$ of two variables μ_1,μ_2. As the set of the functions $\{f_{st}:\ s,t\in\mathbb{Z}\}$ is linearly independent then the set $\{k_1^sk_2^te_{\max}f_{\max}:\ s,t\in\mathbb{Z}\}$ is free over $\mathbb{C}$. Thus we have the second affirmation. □

Proof of Lemma 2.3.3. We consider the superalgebra $\mathcal{U}=\mathcal{U}_\xi\mathfrak{sl}(2|1)/(e_1^\ell,f_1^\ell)$ as the one generated by generators e_i,f_i,k_i,k_i^{-1} and the relations as in Definition 2.3.1 with additional relations $e_1^\ell = f_1^\ell = 0$. From $e_3 = e_1e_2 - \xi^{-1}e_2e_1$, $f_3 = f_2f_1-\xi f_1f_2$ one gets

$$[e_1,f_3] = -\xi f_2k_1,\ [e_3,f_1] = -e_2k_1^{-1}$$
$$e_2f_3+f_3e_2 = \xi^{-1}f_1k_2^{-1},\ e_3f_2+f_2e_3 = \xi^{-1}e_1k_2.$$

Define the length on generators by

$$\mathrm{l}(e_i)=\mathrm{l}(f_i)=1,\ \mathrm{l}(k_i)=0 \text{ for } i=1,2$$

then the above relations imply that one can reorder the monomials in $\mathcal{U}$ up to elements of smaller length. This implies by induction on length that the set $\{e_2^\rho e_3^\sigma e_1^p k_1^sk_2^t f_2^{\rho'}f_3^{\sigma'}f_1^{p'}\ \rho,\sigma,\rho',\sigma'\in\{0,1\},\ p,p'\in\{0,1,...,\ell-1\},\ s,t\in\mathbb{Z}\}$ is a generating set for $\mathcal{U}$ (see [6]).

To prove the linear independence of the vectors we consider the relation

$$\sum x_{\rho,\sigma,p,s,t,\rho',\sigma',p'}e_2^\rho e_3^\sigma e_1^pk_1^sk_2^tf_2^{\rho'}f_3^{\sigma'}f_1^{p'} = 0 \tag{A.1.1}$$

where $\rho,\sigma,\rho',\sigma'\in\{0,1\}$, $p,p'\in\{0,1,...,\ell-1\}$, $s,t\in\mathbb{Z}$. The sum in Equation (A.1.1) contains four blocs associated with (ρ,σ) and can rewrite

$$\text{LHS of (A.1.1)} = \sum x_{0,0,p,s,t,\rho',\sigma',p'}e_1^pk_1^sk_2^tf_2^{\rho'}f_3^{\sigma'}f_1^{p'} \tag{A.1.2}$$
$$+\sum x_{1,0,p,s,t,\rho',\sigma',p'}e_2e_1^pk_1^sk_2^tf_2^{\rho'}f_3^{\sigma'}f_1^{p'} \tag{A.1.3}$$
$$+\sum x_{0,1,p,s,t,\rho',\sigma',p'}e_3e_1^pk_1^sk_2^tf_2^{\rho'}f_3^{\sigma'}f_1^{p'} \tag{A.1.4}$$
$$+\sum x_{1,1,p,s,t,\rho',\sigma',p'}e_2e_3e_1^pk_1^sk_2^tf_2^{\rho'}f_3^{\sigma'}f_1^{p'}. \tag{A.1.5}$$

As $e_2^2 = e_3^2 = 0$ then the three last blocs (A.1.3) - (A.1.5) are zero after the left multiplication at Equation (A.1.1) by e_2e_3 and one gets

$$\sum x_{0,0,p,s,t,\rho',\sigma',p'} e_2e_3e_1^p k_1^s k_2^t f_2^{\rho'} f_3^{\sigma'} f_1^{p'} = 0. \tag{A.1.6}$$

By $(e_2e_3)e_1 = e_1(e_2e_3)$ and $e_1^\ell = 0$, using the left multiplication at Equation (A.1.6) by $e_1^{\ell-1}$ we get

$$\sum x_{0,0,0,s,t,\rho',\sigma',p'} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_2^{\rho'} f_3^{\sigma'} f_1^{p'} = 0. \tag{A.1.7}$$

Using the right multiplication Equation (A.1.7) by $f_1^{\ell-1}$ one gets

$$\sum x_{0,0,0,s,t,\rho',\sigma',0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_2^{\rho'} f_3^{\sigma'} f_1^{\ell-1} = 0. \tag{A.1.8}$$

Now we write the left hand side of Equation (A.1.8) as the sum of four blocs.

$$\begin{aligned} \text{LHS of (A.1.8)} = &\sum x_{0,0,0,s,t,0,0,0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_1^{\ell-1} &\text{(A.1.9)}\\ &+ \sum x_{0,0,0,s,t,1,0,0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_2 f_1^{\ell-1} &\text{(A.1.10)}\\ &+ \sum x_{0,0,0,s,t,0,1,0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_3 f_1^{\ell-1} &\text{(A.1.11)}\\ &+ \sum x_{0,0,0,s,t,1,1,0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_2 f_3 f_1^{\ell-1}. &\text{(A.1.12)} \end{aligned}$$

As $f_1(f_2f_3) = (f_2f_3)f_1$ and $f_2^2 = f_3^2 = 0$ then Equation (A.1.8) gives

$$\sum x_{0,0,0,s,t,0,0,0} e_2e_3e_1^{\ell-1} k_1^s k_2^t f_2 f_3 f_1^{\ell-1} = 0.$$

By second statement of Lemma A.1.1 one deduces that $x_{0,0,0,s,t,0,0,0} = 0$ for $s, t \in \mathbb{Z}$. Now the left hand side of Equation (A.1.8) remains three blocs (A.1.10) - (A.1.12). Similarly, we deduce that $x_{0,0,0,s,t,1,0,0} = x_{0,0,0,s,t,0,1,0} = x_{0,0,0,s,t,1,1,0} = 0$.
Thus we see that from Equation (A.1.7) we get $x_{0,0,0,s,t,\rho',\sigma',0} = 0$ for $0 \le \rho', \sigma' \le 1,\ s, t \in \mathbb{Z}$. Repeating the calculations gives $x_{0,0,0,s,t,\rho',\sigma',p'} = 0$ for $0 \le \rho', \sigma' \le 1,\ 1 \le p' \le \ell - 1,\ s, t \in \mathbb{Z}$.

Applying similar calculations we get

$$x_{\rho,\sigma,p,s,t,\rho',\sigma',p'} = 0$$

for $0 \le \rho, \sigma, \rho', \sigma' \le 1,\ 1 \le p, p' \le \ell - 1,\ s, t \in \mathbb{Z}$.
Hence, $\{e_2^\rho e_3^\sigma e_1^p k_1^s k_2^t f_2^{\rho'} f_3^{\sigma'} f_1^{p'}\ \rho, \sigma, \rho', \sigma' \in \{0, 1\},\ p, p' \in \{0, 1, ..., \ell - 1\},\ s, t \in \mathbb{Z}\}$ is a basis of $\mathcal{U}$. □

A.2 Proof of Proposition 4.4.18

It is necessary to check $\lambda_{\bar{0}}$ satisfies the condition (4.4.1), i.e.

$$(\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})\Delta(x) = \lambda_{\bar{0}}(x)1 \tag{A.2.1}$$

for all $x \in \mathcal{U}_{\bar{0}}$. We check Equation (A.2.1) for the elements in PBW basis. This equation holds true for all elements $f_1^i f_3^\rho f_2^\delta e_1^{i'} e_3^{\rho'} e_2^{\delta'} k_1^{j_1} k_2^{j_2} \sigma^m$ in which $(i, \rho, \delta, i', \rho', \delta', j_1, j_2, m) \neq (\ell - 1, 1, 1, \ell - 1, 1, 1, 0, \ell - 2, 0)$.
For $(i, \rho, \delta, i', \rho', \delta', j_1, j_2, m) = (\ell - 1, 1, 1, \ell - 1, 1, 1, 0, \ell - 2, 0)$ we have the right hand side of Equation (A.2.1) at $w = f_1^{\ell-1} f_3 f_2 e_1^{\ell-1} e_3 e_2 k_2^{\ell-2}$ is equal to $\eta 1$. The left hand side of Equation (A.2.1) at w is computed as follows. First, one has

$$\Delta(e_3) = e_3 \otimes 1 + k_1^{-1} k_2^{-1} \sigma \otimes e_3 + (\xi - \xi^{-1}) e_2 k_1^{-1} \otimes e_1,$$
$$\Delta(f_3) = f_3 \otimes k_1 k_2 + \sigma \otimes f_3 + (\xi^{-1} - \xi) f_1 \sigma \otimes k_1 f_2$$

and one can write

$$\Delta(e_1)^{\ell-1} = e_1^{\ell-1} \otimes 1 + k_1^{\ell-1} \otimes e_1^{\ell-1} + \sum_{u,v<\ell-1} c_{uv} e_1^u k_1^{-v} \otimes e_1^v,$$
$$\Delta(f_1)^{\ell-1} = f_1^{\ell-1} \otimes k_1^{\ell-1} + 1 \otimes f_1^{\ell-1} + \sum_{u',v'<\ell-1} c'_{u'v'} f_1^{u'} \otimes k_1^{u'} f_1^{v'}$$

where $c_{uv}, c'_{u'v'}$ are the coefficients in $\mathbb{C}$ and the powers of e_1, f_1 and k_1 are less then $\ell - 1$.
Then we have the decomposition

$$\begin{aligned}
\Delta(w) &= \Delta(f_1)^{\ell-1}\Delta(f_3)\Delta(f_2)\Delta(e_1)^{\ell-1}\Delta(e_3)\Delta(e_2)\Delta(k_2^{\ell-2}) \\
&= (f_1^{\ell-1} \otimes k_1^{\ell-1})(f_3 \otimes k_1 k_2)(f_2 \otimes k_2)(e_1^{\ell-1} \otimes 1)(e_3 \otimes 1)(e_2 \otimes 1)(k_2^{\ell-2} \otimes k_2^{\ell-2}) \\
&\quad + \sum c_{i,\rho,\delta,j,\rho',\delta',j_1,j_2,m}^{i',\rho_1,\delta_1,j',\rho'_1,\delta'_1,j'_1,j'_2} f_1^i f_3^\rho f_2^\delta e_1^j e_3^{\rho'} e_2^{\delta'} k_1^{j_1} k_2^{j_2} \sigma^m \otimes f_1^{i'} f_3^{\rho_1} f_2^{\delta_1} e_1^{j'} e_3^{\rho'_1} e_2^{\delta'_1} k_1^{j'_1} k_2^{j'_2}
\end{aligned}$$

where the terms in the sum satisfy $(i, \rho, \delta, j, \rho', \delta', j_1, j_2, m) \neq (\ell - 1, 1, 1, \ell - 1, 1, 1, 0, \ell - 2, 0)$. By Equation (4.4.7) and $k_i^\ell = 1$ for $i = 1, 2$ the decomposition above implies that

$$\begin{aligned}
(\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})\Delta(w) &= (\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})((f_1^{\ell-1} \otimes k_1^{\ell-1})(f_3 \otimes k_1 k_2)(f_2 \otimes k_2). \\
&\qquad (e_1^{\ell-1} \otimes 1)(e_3 \otimes 1)(e_2 \otimes 1)(k_2^{\ell-2} \otimes k_2^{\ell-2})) \\
&= (\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})(f_1^{\ell-1} f_3 f_2 e_1^{\ell-1} e_3 e_2 k_2^{\ell-2} \otimes k_1^{\ell-1} k_1 k_2 k_2 k_2^{\ell-2}),
\end{aligned}$$

i.e.

$$\begin{aligned}
(\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})\Delta(w) &= (\lambda_{\bar{0}} \otimes \mathrm{Id}_{\mathcal{U}_{\bar{0}}})(f_1^{\ell-1} f_3 f_2 e_1^{\ell-1} e_3 e_2 k_2^{\ell-2} \otimes 1) \\
&= \lambda_{\bar{0}}(f_1^{\ell-1} f_3 f_2 e_1^{\ell-1} e_3 e_2 k_2^{\ell-2})1 \\
&= \eta 1.
\end{aligned}$$

Thus the linear form $\lambda_{\bar{0}}$ is a right integral of $\mathcal{U}_{\bar{0}}$.

A.3 Proof of Lemma 4.4.20

Firstly, we represent the decomposition of $\theta_{\bar{0}}^{-1}$ in a PBW basis of $\mathcal{U}_{\bar{0}}$. By Equation (4.2.15) the ribbon element $\theta_{\bar{0}}$ of $\mathcal{U}_{\bar{0}}$ is determined by

$$\theta_{\bar{0}} = \phi_0^{\sigma}.(m \circ \tau^s \circ (\mathrm{Id} \otimes S_{\bar{0}})(\mathcal{R}^{\bar{0}}))^{-1},$$

i.e.

$$\theta_{\bar{0}}^{-1} = m \circ \tau^s \circ (\mathrm{Id} \otimes S_{\bar{0}})(\mathcal{R}^{\bar{0}}).(\phi_0^{\sigma})^{-1}. \tag{A.3.1}$$

In Equation (A.3.1) the terms are determined by

$$(\phi_0^{\sigma})^{-1} = \phi_0^{-1}\sigma^{-1} = k_2^2\sigma$$

and

$$\mathcal{R}^{\bar{0}} = \frac{1}{\ell^2} R_1 \sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1} \sum_{\rho,\delta=0}^{1} \frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!} \xi^{i_1 j_2 + i_2 j_1 - 2 i_1 i_2}. \\ e_1^i e_3^\rho e_2^\delta k_1^{i_1} k_2^{j_1} \sigma^{\rho+\delta} \otimes f_1^i f_3^\rho f_2^\delta k_1^{i_2} k_2^{j_2}$$

where $R_1 = \frac{1}{2}(1\otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma) = \frac{1}{2}\sum_{m,n=0}^{1}(-1)^{mn}\sigma^m \otimes \sigma^n$, i.e.

$$\mathcal{R}^{\bar{0}} = \frac{1}{2\ell^2} \sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1} \sum_{m,n,\rho,\delta=0}^{1} (-1)^{mn} \frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!} \xi^{i_1 j_2 + i_2 j_1 - 2 i_1 i_2}. \\ \sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1} k_2^{j_1} \sigma^{\rho+\delta} \otimes \sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2} k_2^{j_2}.$$

Since

$$\begin{aligned}
&S_{\bar{0}}(\sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2} k_2^{j_2}) \\
&\quad = S_{\bar{0}}(k_2^{j_2}) S_{\bar{0}}(k_1^{i_2}) S_{\bar{0}}(f_2^\delta) S_{\bar{0}}(f_3^\rho) S_{\bar{0}}(f_1^i) S_{\bar{0}}(\sigma^n) \\
&\quad = k_2^{-j_2} k_1^{-i_2} (-1)^\delta \sigma^\delta f_2^\delta k_2^{-\delta} \sigma^\rho \left((-1)^\rho \xi^{-2\rho} f_3^\rho + (\xi^{-2\rho}-1) f_2^\rho f_1^\rho\right) k_1^{-\rho} k_2^{-\rho} (-f_1 k_1^{-1})^i \sigma^n \\
&\quad = (-1)^{\delta+i} k_2^{-j_2} k_1^{-i_2} \sigma^\delta f_2^\delta k_2^{-\delta} \sigma^\rho \left((-1)^\rho \xi^{-2\rho} f_3^\rho + (\xi^{-2\rho}-1) f_2^\rho f_1^\rho\right) k_1^{-\rho} k_2^{-\rho} \xi^{i(i-1)} f_1^i k_1^{-i} \sigma^n
\end{aligned}$$

where in the second equality we used

$$S_{\bar{0}}(f_3^\rho) = \sigma^\rho \left((-1)^\rho \xi^{-2\rho} f_3^\rho + (\xi^{-2\rho}-1) f_2^\rho f_1^\rho\right) k_1^{-\rho} k_2^{-\rho},$$

then

$$\begin{aligned}
&(\mathrm{Id} \otimes S_{\bar{0}})(\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1} k_2^{j_1} \sigma^{\rho+\delta} \otimes \sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2} k_2^{j_2}) \\
&= (-1)^{\delta+\rho+i} \xi^{i(i-1)-2\rho} \sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1} k_2^{j_1} \sigma^{\rho+\delta} \otimes k_2^{-j_2} k_1^{-i_2} \sigma^\delta f_2^\delta k_2^{-\delta} \sigma^\rho f_3^\rho k_1^{-\rho} k_2^{-\rho} f_1^i k_1^{-i} \sigma^n \\
&+ (-1)^{\delta+i} (\xi^{-2\rho}-1) \xi^{i(i-1)} \sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1} k_2^{j_1} \sigma^{\rho+\delta} \otimes k_2^{-j_2} k_1^{-i_2} \sigma^\delta f_2^\delta k_2^{-\delta} \sigma^\rho f_2^\rho f_1^\rho k_1^{-\rho} k_2^{-\rho} f_1^i k_1^{-i} \sigma^n.
\end{aligned}$$

We have

$$\begin{aligned}
&m\circ\tau^s\circ(\mathrm{Id}\otimes S_{\overline{0}})(\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\otimes\sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2})\\
&=(-1)^{\delta+\rho+i}\xi^{i(i-1)-2\rho}(-1)^{\rho+\delta}k_2^{-j_2}k_1^{-i_2}\sigma^\delta f_2^\delta k_2^{-\delta}\sigma^\rho f_3^\rho k_1^{-\rho}k_2^{-\rho}f_1^i k_1^{-i}\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&+(-1)^{\delta+i+\delta+\rho}(\xi^{-2\rho}-1)\xi^{i(i-1)}k_2^{-j_2}k_1^{-i_2}\sigma^\delta f_2^\delta k_2^{-\delta}\sigma^\rho f_2^\rho f_1^\rho k_1^{-\rho}k_2^{-\rho}f_1^i k_1^{-i}\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&=(-1)^i\xi^{i(i-1)-2\rho}k_2^{-j_2}k_1^{-i_2}\sigma^\delta f_2^\delta k_2^{-\delta}\sigma^\rho f_3^\rho k_1^{-\rho}k_2^{-\rho}f_1^i k_1^{-i}\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&+(-1)^{i+\rho}(\xi^{-2\rho}-1)\xi^{i(i-1)}k_2^{-j_2}k_1^{-i_2}\sigma^\delta f_2^\delta k_2^{-\delta}\sigma^\rho f_2^\rho f_1^\rho k_1^{-\rho}k_2^{-\rho}f_1^i k_1^{-i}\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&=X_1+X_2.
\end{aligned}$$

Since $k_if_j=\xi^{-a_{ij}}f_jk_i,\ k_ie_j=\xi^{a_{ij}}e_jk_i,\ k_if_3=\xi^{-(a_{i1}+a_{i2})}f_3k_i,\ k_ie_3=\xi^{a_{i1}+a_{i2}}e_3k_i$ for $i,j=0,1$ and $\sigma x=(-1)^{\deg x}x\sigma$ then

$$\begin{aligned}
X_1&=(-1)^i\xi^{i(i-1)-2\rho}\xi^{-(j_2+\delta)\rho-(j_2+\delta+\rho)i+(j_2+\delta+\rho)i+(j_2+\delta+\rho)\rho}\cdot\\
&\quad\xi^{-i_2\delta+i_2\rho+2(i_2+\rho)i-2(i+i_2+\rho)i-(i+i_2+\rho)\rho+(i+i_2+\rho)\delta}\cdot\\
&\quad\sigma^\delta f_2^\delta\sigma^\rho f_3^\rho f_1^i\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\delta-\rho}\sigma^{\rho+\delta}\\
&=(-1)^i\xi^{-i-i^2-i\rho+i\delta+\rho\delta-2\rho}\sigma^\delta f_2^\delta\sigma^\rho f_3^\rho f_1^i\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\delta-\rho}\sigma^{\rho+\delta}\\
&=(-1)^{i+\delta\delta+(\rho+\delta)\rho+(\rho+\delta+m+n)(\rho+\delta)}\xi^{-i-i^2-i\rho+i\delta+\rho\delta-2\rho}\cdot\\
&\quad f_2^\delta f_3^\rho f_1^i e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\delta-\rho}\sigma^{2(\rho+\delta)+m+n}\\
&=(-1)^{i+\delta\rho+(m+n)(\delta+\rho)}\xi^{-i-i^2-i\rho+i\delta+\rho\delta-2\rho}f_2^\delta f_3^\rho f_1^i e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\delta-\rho}\sigma^{m+n}
\end{aligned}$$

and

$$\begin{aligned}
X_2&=(-1)^{i+\rho+\rho\delta}(\xi^{-2\rho}-1)\xi^{i(i-1)+i\rho}k_2^{-j_2}k_1^{-i_2}\sigma^\delta k_2^{-\delta}\sigma^\rho f_2^{\rho+\delta}f_1^{\rho+i}k_2^{-\rho}k_1^{-i-\rho}\cdot\\
&\quad\sigma^{m+n}e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&=(-1)^{i+\rho+\rho\delta+(m+n)(\rho+\sigma)}(\xi^{-2\rho}-1)\xi^{-i-i^2-i\rho+\rho\delta+i\delta}\cdot\\
&\quad f_2^{\rho+\delta}f_1^{\rho+i}e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\rho-\delta}\sigma^{m+n}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\theta_{\overline{0}}^{-1}=m\circ\tau^s\circ(\mathrm{Id}\otimes S_{\overline{0}})(\mathcal{R}^{\overline{0}})(\phi_0^\sigma)^{-1}\\
&=\frac{1}{2\ell^2}\sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n,\rho,\delta=0}^{1}(-1)^{mn}\frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!}\xi^{i_1j_2+i_2j_1-2i_1i_2}(X_1+X_2)k_2^2\sigma.
\end{aligned}$$

Since

$$\begin{aligned}
X_2k_2^2\sigma&=(-1)^{i+\rho+\rho\delta+(m+n)(\rho+\sigma)}(\xi^{-2\rho}-1)\xi^{-i-i^2-i\rho+\rho\delta+i\delta}\cdot\\
&\quad f_2^{\rho+\delta}f_1^{\rho+i}e_1^i e_3^\rho e_2^\delta k_1^{i_1-i_2-i-\rho}k_2^{j_1-j_2-\rho-\delta+2}\sigma^{m+n+1}
\end{aligned}$$

then by Proposition 4.4.18 implies that $\lambda_{\bar{0}}(X_2k_2^2\sigma)=0$. Hence we have

$$\begin{aligned}\lambda_{\bar{0}}(\theta_{\bar{0}}^{-1}) = -\frac{1}{2\ell^2}\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n=0}^{1}\frac{\{1\}^{\ell-1}(-\{1\})^{1+1}}{(\ell-1)_\xi!(1)_\xi!(1)_\xi!}(-1)^{mn}\xi^{-1+i_1j_2+i_2j_1-2i_1i_2}\\ \lambda_{\bar{0}}(f_2f_3f_1^{\ell-1}e_1^{\ell-1}e_3e_2k_1^{i_1-i_2-(\ell-1)-1}k_2^{j_1-j_2-1-1+2}\sigma^{m+n+1})\\ = -\frac{1}{2\ell^2}\frac{\{1\}^{\ell+1}\xi^{-1}}{(\ell-1)_\xi!}\eta'\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n=0}^{1}\xi^{i_1j_2+i_2j_1-2i_1i_2}.\\ \delta^0_{i_1-i_2 \bmod \ell\mathbb{Z}}\delta^{\ell-2}_{j_1-j_2 \bmod \ell\mathbb{Z}}\delta^0_{m+n+1 \bmod 2\mathbb{Z}}\end{aligned}$$

where $\eta'=\lambda_{\bar{0}}(f_2f_3f_1^{\ell-1}e_1^{\ell-1}e_3e_2k_2^{\ell-2})$, i.e.

$$\begin{aligned}\lambda_{\bar{0}}(\theta_{\bar{0}}^{-1}) &= -\frac{1}{2\ell^2}\frac{\{1\}^{\ell+1}\xi^{-1}}{(\ell-1)_\xi!}\eta'\sum_{i_2,j_2=0}^{\ell-1}\xi^{i_2j_2+i_2(j_2+\ell-2)-2i_2i_2}\sum_{m,n=0}^{1}(-1)^{mn}\delta^0_{m+n+1 \bmod 2\mathbb{Z}}\\ &= -2\frac{1}{2\ell^2}\frac{\{1\}^{\ell+1}\xi^{-1}}{(\ell-1)_\xi!}\eta'\sum_{i_2=0}^{\ell-1}\xi^{-2i_2^2-2i_2}\sum_{j_2=0}^{\ell-1}\xi^{2i_2j_2}\\ &= -\frac{1}{\ell^2}\frac{\{1\}^{\ell+1}\xi^{-1}}{(\ell-1)_\xi!}\eta'\sum_{i_2=0}^{\ell-1}\xi^{-2i_2^2-2i_2}\ell\delta^0_{i_2 \bmod \ell\mathbb{Z}}\\ &= -\frac{\{1\}^{\ell+1}\xi^{-1}}{\ell(\ell-1)_\xi!}\eta'.\end{aligned}$$

To computer $\lambda_{\bar{0}}(\theta_{\bar{0}})$ we use the equality

$$\theta_{\bar{0}} = \phi_0^\sigma.(m\circ\tau^s\circ(S_{\bar{0}}^2\otimes \mathrm{Id})(\mathcal{R}^{\bar{0}})).$$

Since

$$\begin{aligned}S_{\bar{0}}^2(e_1^i) &= \phi_0^\sigma e_1^i(\phi_0^\sigma)^{-1} = \sigma k_2^{-2}e_1^ik_2^2\sigma = \xi^{2i}e_1^i,\\ S_{\bar{0}}^2(e_3^\rho) &= (-1)^\rho\xi^{2\rho}e_3^\rho,\\ S_{\bar{0}}^2(e_2^\delta) &= (-1)^\delta e_2^\delta,\\ S_{\bar{0}}^2(k_1^{i_1}) &= k_1^{i_1},\\ S_{\bar{0}}^2(k_2^{j_1}) &= k_2^{j_1},\\ S_{\bar{0}}^2(\sigma^{\rho+\delta}) &= \sigma^{\rho+\delta}\end{aligned}$$

then

$$\begin{aligned}S_{\bar{0}}^2(\sigma^me_1^ie_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}) &= S_{\bar{0}}^2(\sigma^m)S_{\bar{0}}^2(e_1^i)S_{\bar{0}}^2(e_3^\rho)S_{\bar{0}}^2(e_2^\delta)S_{\bar{0}}^2(k_1^{i_1})S_{\bar{0}}^2(k_2^{j_1})S_{\bar{0}}^2(\sigma^{\rho+\delta})\\ &= (-1)^{\rho+\delta}\xi^{2(i+\rho)}\sigma^me_1^ie_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}.\end{aligned}$$

It implies that

$$\begin{aligned}
&(S_{\bar 0}^2\otimes \mathrm{Id})(\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\otimes \sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2})\\
&\qquad = (-1)^{\rho+\delta}\xi^{2(i+\rho)}\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\otimes \sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2}
\end{aligned}$$

then

$$\begin{aligned}
&m\circ\tau^s\circ(S_{\bar 0}^2\otimes \mathrm{Id})(\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\otimes \sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2})\\
&\quad = (-1)^{\rho+\delta+\rho+\delta}\xi^{2(i+\rho)}\sigma^n f_1^i f_3^\rho f_2^\delta k_1^{i_2}k_2^{j_2}\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1}k_2^{j_1}\sigma^{\rho+\delta}\\
&\quad = \xi^{2(i+\rho)}\xi^{i_2(2i+\rho-\delta)}\xi^{-j_2(i+\rho)}\sigma^n f_1^i f_3^\rho f_2^\delta\sigma^m e_1^i e_3^\rho e_2^\delta k_1^{i_1+i_2}k_2^{j_1+j_2}\sigma^{\rho+\delta}\\
&\quad = (-1)^{(2n+m)(\rho+\delta)}\xi^{2(i+\rho)+i_2(2i+\rho-\delta)-j_2(i+\rho)} f_1^i f_3^\rho f_2^\delta e_1^i e_3^\rho e_2^\delta k_1^{i_1+i_2}k_2^{j_1+j_2}\sigma^{m+m+\rho+\delta}\\
&\quad = (-1)^{m(\rho+\delta)}\xi^{2(i+\rho)+i_2(2i+\rho-\delta)-j_2(i+\rho)} f_1^i f_3^\rho f_2^\delta e_1^i e_3^\rho e_2^\delta k_1^{i_1+i_2}k_2^{j_1+j_2}\sigma^{m+m+\rho+\delta}.
\end{aligned}$$

So we have

$$\begin{aligned}
\theta_{\bar 0} &= \sigma k_2^{-2}\frac{1}{2\ell^2}\sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n,\rho,\delta=0}^{1}(-1)^{mn}\frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!}\xi^{i_1j_2+i_2j_1-2i_1i_2}\cdot\\
&\quad (-1)^{m(\rho+\delta)}\xi^{2(i+\rho)+i_2(2i+\rho-\delta)-j_2(i+\rho)} f_1^i f_3^\rho f_2^\delta e_1^i e_3^\rho e_2^\delta k_1^{i_1+i_2}k_2^{j_1+j_2}\sigma^{m+m+\rho+\delta}\\
&= \frac{1}{2\ell^2}\sum_{i,i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n,\rho,\delta=0}^{1}\frac{\{1\}^i(-\{1\})^{\rho+\delta}}{(i)_\xi!(\rho)_\xi!(\delta)_\xi!}\xi^{2(i+\rho)+i_2(2i+\rho-\delta)-j_2(i+\rho)}\cdot\\
&\quad (-1)^{m(\rho+\delta)+mn}\xi^{i_1j_2+i_2j_1-2i_1i_2} f_1^i f_3^\rho f_2^\delta e_1^i e_3^\rho e_2^\delta k_1^{i_1+i_2}k_2^{j_1+j_2-2}\sigma^{m+n+\rho+\delta+1}.
\end{aligned}$$

By Proposition 4.4.18 one has

$$\begin{aligned}
\lambda_{\bar 0}(\theta_{\bar 0}) &= \frac{1}{2\ell^2}\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1}\sum_{m,n=0}^{1}\frac{\{1\}^{\ell-1}(-\{1\})^{1+1}}{(\ell-1)_\xi!}(-1)^{mn}\xi^{-2i_2+i_1j_2+i_2j_1-2i_1i_2}\eta.\\
&\qquad \delta^0_{i_1+i_2 \bmod \ell\mathbb{Z}}\delta^0_{j_1+j_2-2 \bmod \ell\mathbb{Z}}\delta^0_{m+n+1 \bmod 2\mathbb{Z}}\\
&= \frac{1}{2\ell^2}\frac{\{1\}^{\ell+1}}{(\ell-1)_\xi!}\eta\sum_{i_1,i_2,j_1,j_2=0}^{\ell-1}\xi^{-2i_2+i_1j_2+i_2j_1-2i_1i_2}\delta^0_{i_1+i_2 \bmod \ell\mathbb{Z}}\delta^0_{j_1+j_2-2 \bmod \ell\mathbb{Z}}\cdot\\
&\qquad\qquad \sum_{m,n=0}^{1}(-1)^{mn}\delta^0_{m+n+1 \bmod 2\mathbb{Z}}
\end{aligned}$$

where $\eta = \lambda_{\bar 0}(f_1^{\ell-1}f_3f_2e_1^{\ell-1}e_3e_2k_2^{\ell-2})$, i.e.

$$\begin{aligned}
\lambda_{\bar 0}(\theta_{\bar 0}) &= \frac{1}{\ell^2}\frac{\{1\}^{\ell+1}}{(\ell-1)_\xi!}\eta\sum_{i_1,j_1=0}^{\ell-1}\xi^{-2(\ell-i_1)+i_1(\ell-j_1+2)+(\ell-i_1)j_1-2i_1(\ell-i_1)}\\
&= \frac{1}{\ell^2}\frac{\{1\}^{\ell+1}}{(\ell-1)_\xi!}\eta\sum_{i_1=0}^{\ell-1}\xi^{2i_1^2+4i_1}\sum_{j_1=0}^{\ell-1}\xi^{-2i_1j_1}
\end{aligned}$$

$$= \frac{1}{\ell^2} \frac{\{1\}^{\ell+1}}{(\ell-1)_\xi!} \eta \sum_{i_1=0}^{\ell-1} \xi^{2i_1^2+4i_1} \ell \delta^0_{i_1 \bmod \ell\mathbb{Z}}$$
$$= \frac{\{1\}^{\ell+1}}{\ell(\ell-1)_\xi!} \eta.$$

On the other hand

$$f_2 f_3 f_1 = \xi^{-1} f_2 f_1 f_3 = \xi^{-1}(f_3 + \xi f_1 f_2) f_3 = f_1 f_2 f_3.$$

Since $f_2 f_3 = -\xi f_3 f_2$ we get

$$\begin{aligned} \eta' &= \lambda_{\bar{0}}(f_2 f_3 f_1^{\ell-1} e_1^{\ell-1} e_3 e_2 k_2^{\ell-2}) \\ &= -\xi \lambda_{\bar{0}}(f_1^{\ell-1} f_3 f_2 e_1^{\ell-1} e_3 e_2 k_2^{\ell-2}) \\ &= -\xi \eta. \end{aligned}$$

Thus we have

$$\lambda_{\bar{0}}(\theta_{\bar{0}}^{-1}) = -\frac{\{1\}^{\ell+1}\xi^{-1}}{\ell(\ell-1)_\xi!} \eta' = \frac{\{1\}^{\ell+1}}{\ell(\ell-1)_\xi!} \eta = \lambda_{\bar{0}}(\theta_{\bar{0}}).$$

Since $(\ell-1)_\xi! = \prod_{i=1}^{\ell-1} \frac{1-\xi^i}{1-\xi} = \frac{\ell-1}{(1-\xi)^{\ell-1}}$ then $\lambda_{\bar{0}}(\theta_{\bar{0}}) = \frac{\{1\}^{\ell+1}(1-\xi)^{\ell-1}}{\ell(\ell-1)}\eta$.

Bibliography

[1] B. Abdesselam, D. Arnaudon, and M. Bauer. Centre and representations of $\mathcal{U}_q\mathfrak{sl}(2|1)$ at roots of unity. *J.Phys. A:Math.Gen.*, pages 867–880, 1997.

[2] A. Beliakova, C. Blanchet, and A. M. Gainutdinov. Modified trace is a symmetrised integral. *arXiv:1801.00321*, 2017.

[3] A. Beliakova, C. Blanchet, and N. Geer. Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$. *arXiv:1705.03083*, 2017.

[4] C. Blanchet, F. Costantino, N. Geer, and B. Patureau-Mirand. Non-semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants. *Advances in Mathematics*, 301:1–78, 2016.

[5] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quatum field theories derived from the Kauffman bracket. *Topology*, 34, no.4:883–927, 1995.

[6] C. De Concini and V. G. Kac. *Representations of quantum groups at roots of 1. In: Progress in Math.* Birkhäuser, 1990.

[7] C. De Concini, V.G. Kac, and C. Procesi. Quantum coadjoint action. *J. Amer. Math. Soc.*, 5, No. 1:151–189, 1992.

[8] F. Costantino, N. Geer, and B. Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *Journal of Topology*, pages 1005–1053, 2014.

[9] F. Costantino, N. Geer, and B. Patureau-Mirand. Some remarks on the unrolled quantum group of $\mathfrak{sl}(2)$. *J. Pure Appl. Algebra 219*, no. 8:3238–3262, 2015.

[10] P. Freyd, J. Hoste, W.B.R. Lickorish, K. Millet, A. Ocneanu, and D. Yetter. A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc.*, 12:239–246, 1985.

[11] A. M. Gainutdinov and I. Runkel. Projective objects and the modified trace in factorisable finite tensor categories. *arXiv:1703.00150*, 2017.

[12] N. Geer, J. Kujawa, and B. Patureau-Mirand. Generalized trace and modified dimension functions on ribbon categories. *Selecta Math.*, pages 453–504, 2011.

[13] N. Geer, J. Kujawa, and B. Patureau-Mirand. Ambidextrous objects and trace fuctions for nonsemisimple categories. *Proceedings of the American Mathematical Society 141*, 2013.

[14] N. Geer and B. Patureau-Mirand. Multivariable link invariants arising from $\mathfrak{sl}(2|1)$ and the Alexander polynomial. *J. Pure Appl. Algebra*, pages 283–298, 2007.

[15] N. Geer and B. Patureau-Mirand. An invariant supertrace for the category of representations of Lie superalgebras. *Pacific J. Math.*, pages 331–348, 2008.

[16] N. Geer and B. Patureau-Mirand. Topological invariants from unrestricted quantum groups. *Algebraic & Geometric Topology*, 2013.

[17] N. Geer, B. Patureau-Mirand, and V. Turaev. Modified quantum dimensions and re-normalized links invariants. *Compositio Mathematica*, pages 196–212, 2009.

[18] N. Geer, B. Patureau-Mirand, and A. Virelizier. Traces on ideals in pivotals categories. *Quantum Topology*, 4, No. 1:91–124, 2013.

[19] A. Grothendieck. Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires. *Annales de l'Institut Fourrier*, tome 4:p. 73–112, 1952.

[20] N. P. Ha. A Hennings type invariant of 3-manifolds from a topological Hopf superalgebra. *arXiv:1806.08277*, pages 1–35, 2018.

[21] N. P. Ha. Modified trace from pivotal Hopf G-coalgebra. *arXiv: 1804.02416*, pages 1–25, 2018.

[22] N. P. Ha. Topological invariants from quantum group $\mathcal{U}_\xi\mathfrak{sl}(2|1)$ at roots of unity. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 88(1):163–188, 2018.

[23] K. Habiro. Bottom tangles and universal invariants. *Algebraic & Geometric Topology*, 6(3):1113–1214, 2006.

[24] K. Habiro and Thang T Q. Lê. Unified quantum invariants for integral homology spheres associated with simple Lie algebras. *Geometry & Topology*, 20:2687–2835, 2006.

[25] I. Heckenberger. *Nichols algebras of diagonal type and arithmetic root systems.* Habilitationsarbeit, Leipzig, Dezember 2005.

[26] M. Hennings. Invariants of links and 3-manifolds obtained from Hopf algebras. *Journal of the London Mathematical Society*, 54:594–624, 1996.

[27] V. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc.*, (N.S.) 12, no. 1:103–111, 1985.

[28] V. G. Kac. Lie superalgebra. *Advances Math.*, pages 8–96, 1977.

[29] C. Kassel. *Quantum Groups.* Springer-Verlag, 1995.

[30] L. Kauffman and D. E. Radford. Oriented quantum algebras, categories and invariants of knots and links. *Journal of Knot Theory and Its Ramifications*, 10(07):1047–1084, 2001.

[31] S. M. Khoroshkin and V. N. Tolstoy. Universal $\mathcal{R}$-matrix for quantized (super)algebras. *Commun. Math.*, pages 599–617, 1991.

[32] R. Kirby. A calculus for framed links. *Invent. Math.*, 45:35–56, 1978.

[33] S. Mac Lane. *Categories for the Woking Mathematician.* Springer, 1998.

[34] R. G. Larson and M. E. Sweedler. An associative orthogonal bilinear form for Hopf algebras. *American Journal of Mathematics*, Vol. 91, No. 1 (Jan.):75–94, 1969.

[35] S. Lentner and D. Nett. New $\mathcal{R}$-matrices for small quantum groups. *Algebras and Representation Theory*, 18(6):1649–1673, 2015.

[36] S. Majid. Cross products by braided groups and bosonization. *Journal of algebra*, 163:165–190, 1994.

[37] T. Ohtsuki. Colored ribbon Hopf algebras and universal invariants of framed links. *J. Knot Theory Ramifications 2 (2)*, pages 211–232, 1993.

[38] T. Ohtsuki. *Quantum invariants.* World Scientific Publishing Co. Pte. Ltd, 2002.

[39] B. Patureau-Mirand. *Invariants topologiques quantiques non semi-simples.* Habilitation à diriger des recherches, Université de Bretagne Sud, 2012.

[40] Markus J. Pflauma and Martin Schottenloher. Holomorphic deformation of Hopf algebras and applications to quantum groups. *Journal of Geometry and Physics*, Volume 28, Issues 1-2:31–44, 1998.

[41] David E Radford. *Hopf Algebras.* World Scientific, 2012.

[42] M. De Renzi. Non-semisimple extended topological quantum field theories. *ArXiv:1703.07573*, 2017.

[43] N. Reshetikhin and V. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys*, 127:1–26, 1990.

[44] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. math. 103*, pages 547–597, 1991.

[45] F. Trèves. *Topological vector spaces, Distributions and Kernels*, volume 25. Academic Press Inc., 1967.

[46] V. Turaev. *Homotopy Quantum Field Theory.* European Mathematical Society, 2010.

[47] V. G. Turaev. Quantum invariants of knots and 3-manifolds. *Studies in Mathematiques*, 1994.

[48] A. Virelizier. *Algèbres de Hopf graduées et fibrés plats sur les 3-variétés.* Thèse de doctorat, Université Louis Pasteur, 2001.

[49] A. Virelizier. Hopf group-coalgebra. *Journal of Pure and Applied Algebra*, 171:75–122, 2002.

[50] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121, vol.3:351–399, 1989.

[51] H. Yamane. Quantized enveloping algebras associatied with simple Lie superalgebras and their universal $\mathcal{R}$-matrices. *Publ. RIMS, Kyoto Univ.*, pages 15–87, 1994.