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Unification des invariants ADO et Jones colorés pour les noeuds

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Remerciements

Je n'aime pas les traditions. Enfin, c'est un peu fort de dire ça d'un concept qui, à l'instar de la mer, est aussi vague que vaste. Mais disons que de nombreux rites me mettent mal à l'aise. Nombreux sont ceux qui saisissent cet instant suspendu du cours usuel des choses pour y accomplir l'exception, l'inhabituel, appuyés par le confort de l'acceptation unanime qu'offre le rituel. Moi, je m'y sens terriblement étranger. Mes mots et mes actions me semblent alors factices, comme muent par l'idée que je me fais de ce que l'on attend de moi. Bien heureusement, ici le ton est léger, se plier à l'exercice du remerciement écrit est, au pire, une source de perplexité et un peu d'embarras. Essayer d'exprimer ma gratitude d'une manière qui me ressemble, c'est tout de même intéressant.

La thèse - ah oui, c'est bien de ça qu'il s'agit, pas d'un pamphlet autobiographique, oups - est à bien des aspects, une formation. Cette formation nécessite une direction, c'est comme ça qu'elle prend tout son sens. J'ai eu la chance d'en avoir deux. Avant même le début de cette aventure, durant mes quelques mois de stage en M2, mon directeur François Costantino m'a accompagné, initié à la recherche et à cette thématique. Ces cours que tu m'as dispensé furent d'une aide précieuse. Chaque entretien que l'on a eu fut un petit changement de paradigme, que ce soit pour relever la tête avec une ribambelle de nouvelles approches sur une question qui bloque, ou bien pour voir d'un œil neuf un résultat. C'est aussi grâce à toi que j'ai eu l'honneur et le plaisir de rencontrer notre second protagoniste, Bertrand Patureau Mirand. Bertrand, tout d'abord je te remercie d'avoir accepté de diriger ma thèse alors même que nous ne nous connaissions pas. La régularité de nos entretiens et la qualité de tes conseils ont joué un rôle crucial, d'autant plus durant la première année. J'espère que tu ne sous estimes pas la valeur de ce que tu m'as apporté.

Les derniers pas de la thèse, ceux que j'entreprends aujourd'hui, ne se font pas dans la brousse. Une route m'a été tracée, par de formidables rapporteurs qui ont, d'un œil minutieux, rendu compte de mon travail. C'est très réconfortant. Anna, I'm so glad for all our conversations and pondering on the link case. Gwénaël, merci énormément pour cette relecture, ces commentaires et ces précisions. Enfin, merci au jury, qui m'accueille à l'arrivée. Merci à toi Stéphane d'avoir accepté de venir à ma soutenance. Thomas, c'est toi qui m'a enseigné la topologie algébrique, c'est un honneur pour moi de soutenir devant toi.

Je vais en profiter maintenant pour vous parler d'un autre acteur qui fut essentiel, Jules Martel. C'est toi qui m'a aiguillé pour l'interprétation du terme mystère dans la factorisation aux racines de l'unité, et qui m'a permis de réinterpréter la théorie via le point de vue des représentations de tresses. Parcourant le monde, tu as aussi fait connaitre mon travail. Conseiller, collaborateur, mais aussi chargé de communication, quelle vie !

Je rentre désormais dans des considérations plus personnelles. Je vais essayer, au long de ces paragraphes, de rendre compte de l'expérience que j'ai vécu, de ma gratitude pour beaucoup, de mon amertume parfois. Je vais aussi en profiter pour faire de ces mots un voyage personnel sans prétention, une simple promenade dans mes souvenirs. Tant de gens que je remercie, tant de choses que j'ai apprécié, si peu seront présents dans ces mots.

Une brume épaisse recouvre le paysage lorsque je me lève, cette journée s'annonce radieuse. Ce soleil levant inonde de rayons encore froid la plaine autour de moi, chaque foulée est grisante. Puis, la chaleur de l'eau qui ruisselle contre ma peau me laisse propre et apaisé. Les délicats arômes s'échappent de ma tasse lorsque je verse ce thé infusé dans sa fonte et le rythme apaisé de la musique me centre et concentre mon attention sur le travail à faire. Le journal d'une journée quelconque, ma gratitude pour toutes ces petites choses qui constituent mon quotidien.

De mes séjours à l'IMT siège mon bureau comme point de chute. Qui, loin du confort de ma hutte, fut néanmoins central à cette aventure. Si proche du centre de masse où nous gravitâmes avant que commence le pèlerinage vers l'Upsidom. Je lève mon verre d'eau. A Florian, notre collection complète de pissenlits. A JM et Eva, notre échappée à Gruissan. A Valentin, notre dépit devant les notes. A tous les picards, à ces séminaires, ces discutions. Dimitri, j'espère que tu vaincras tes chimères.

Attablé à mon bureau, je parcourais ce monde abstrait qui me plait tant. Cette expédition eut son lot de longues journées devant une feuille désespérément blanche, et ses nombreuses routines où l'on avance sans que rien ne change. Ce fut aussi le théâtre de quelques exultations si jouissives. Ce monde, une fois que l'on a fait l'effort d'y entrer, il est si simple d'y retourner. Il n'y a pas besoin de machines sophistiquées, ou d'armoire magique. On y pense, et l'on s'y trouve transporté. Et l'on y pense. A l'aube comme au couché, lorsque la pensée s'égare. Dans le métro, en marchant, durant tous ses instants où l'esprit n'est pas sollicité. Parfois même, il nous retient, et il faut savoir s'en détacher.

En tout cas ce fut un plaisir de découvrir tant de choses, tous ces auteurs et leurs idées, inscrites dans des livres limpides ou des articles retors. Souvent accessible sur Arxiv, et parfois prêté par notre amie Alexandra. Car oui, dans la publication scientifique le contraste est si saisissant: la beauté du savoir qui se créé d'un coté, et de l'autre, les éditeurs qui ont plus à coeur d'infuser les profits que de diffuser les idées.

Mais que serait cette histoire sans un antagoniste à la hauteur? Du point de vue académique rien à signaler, les gens avec qui j'ai eu l'occasion d'interagir furent tous très agréables, de bons conseils. Je n'ai pas non plus subit le poids écrasant souvent ressenti lorsque l'on débute la recherche. Non, il faut faire un pas sur le coté pour le trouver.

Je remercie le rectorat de Bordeaux pour avoir su me faire découvrir une palette d'émotions jusqu'alors trop peu usitées. Ce sentiment d'injustice devant l'arbitraire et le mépris des conséquences. La rage devant notre déshumanisation et la haine face à leur inhumanité. S'est ensuivi la solitude de ces interminables semaines, et le désespoir de se séparer sans cesse. Un cycle sans fin.

Retenez simplement que si le rectorat était un hôpital, devant sa porte on pourrait lire:

> *"Si à l'avenir vous souffrez de certains maux, assurez vous qu'ils soient inscrits dans le BO."*

Pour m'évader et séparer mon esprit de mes différents travaux, je dois bien remercier mon brave ordinateur. Je prends pour exemple toutes ces heures où ne comptaient que la fluidité de ce leste mouvement vers le sommet. Mais aussi ce plaisir de fumer la radiance. Ou encore ces quelques semaines où baba est thèse. Pourtant le divertissement qu'ils engendrent n'est qu'une conséquence bénéfique de leurs qualités intrinsèques et du plaisir de les découvrir, de les parcourir. C'est le cas pour énormément d'arts et de contenus mais aussi de la pratique du sport qui m'ont enrichis tout au long de cette thèse et tout au long de ma vie.

Bien, c'est maintenant qu'on fait le ptit speech cringe sur le pouvoir de l'amitié. Oui vous mes amis, merci. Des mercis au nombre de nos "différentes" conversations. Parce que ça fait plaisir au quotidien, et quel régal de vous revoir. Bon, quelques pensées viennent à moi.

Gazo, mon ptit gars sûr, souviens toi notre colloc garnie de cartons de pizza, notre graille garantie à prix d'or.

La chaleur des échanges sur des sujets chers, ou à chier, quand s'échine Gach sans chômer.

Sandra, je te sers ces mots sans ciller, les sourires s'ouvre dans ton sillage et je salue, cent fois surement, ton sens de l'hospitalité.

La curiosité de tout ce qui se fait de bien et qui casse ce passé crasse de quelques cakes américains. La classe, Adrien.

Et pour Romain des mots en bien? Nanain ! Mis à part aux laser games, toujours le cœur sur la main.

Enzo grand remerciement, son séant sur scène et plus un pet de peine.

Et chez Mathilde, on s'y attable tant de fois. C'est pour manger ou mener en team nos maintes missions.

Bruno bande les biscotos sur la barre après du bon boulot sur les blocs. Et moi en boule, les mains qui brulent, les bras ballants.

J'en profite aussi pour saluer mes amis que je vois un peu moins au quotidien.

J'aimerai faire un petit coucou à Nicolas, j'espère qu'on se reverra bientôt.

A l'auguste Paul Dintilhac et aux ailes qui volent.

A toi Marie, au plaisir de te revoir dans la vallée ou quand tu passes vers Toulouse. Les souvenirs de nos débuts dans cette ville sont peu précis mais si précieux.

Les petites poulets, faut que je vienne prendre de vos nouvelles.

Tant de souvenirs, de moments marquants, sont comme éclipsés de notre esprit pour ressurgir d'un coup de manière inopinée, sans doute déclenchés par quelques phrases, sons, senteurs ou rêveries. Des souvenirs j'en ai à la pelle, et j'ai la chance d'en avoir tant de si plaisants. J'éprouve ainsi le plus grand plaisir à me remémorer. On prend du recul sur le quotidien quand on fait face à l'altérité de notre passé.

Ma jeunesse à Marcel Paul, où la promiscuité dans un coin de campagne ravit la ribambelle d'enfants libres, se passa à construire des bases dans les bois, ou se balader dans la brousse. Quelques bastons aussi. Les débuts de l'adolescence avec Adam et Julian, nos mondes infinis si clairs dans nos esprits, les dongeons et les raids, et nos discussions dehors, dans le froid de l'hiver. La douleur de la perte est encore si vive.

Avec le lycée, vint la curieuse envie d'apprendre, mais aussi le plaisir de faire fleurir les amitiés. Ces années furent si riches d'émotions détaillées et panachées. C'est aussi là que mon attention se focalisa sur les mathématiques. De la structure, un peu d'ordre et de rigueur avec Fabrice Scanzi et avec Vincent Borderon le plaisir de la découverte, de l'ouverture qu'offrent les maths, la liberté d'explorer ce monde. Je garde aussi des souvenirs fabuleux de l'atelier théâtre, qui fût le centre d'un enrichissement et d'un épanouissement personnel sans limites.

L'esprit, perdu dans les souvenirs, construit des scènes en recollant des éclats de passé.

Petit je marche avec Shane sur le chemin du Boudouissou, c'est une belle journée de printemps. Quelques mètres derrière, papi nous suit, le souffle sifflant tranquille.

Attablé avec mamie au retour du lycée, j'écoute ses angoisses sur l'avenir des proches, quelques bribes de son passé me traversent.

De nouveau petit, j'approche de la maison. Dad est dehors et m'étreint dans ses bras de géant. Lorsque je rentre, Mario est assis sur le canapé en train de fomenter quelques fantasques aventures, Shane est par terre en train de jouer avec des figurines pokémon. Et vers le fond de la pièce, une Jessica encore toute petite à table avec maman. Elles me sourient. La pièce est inondée d'une chaleureuse lumière d'un jour d'hiver magnifique.

Ma famille.

Et ce bonheur au quotidien, l'avenir radieux. Ma vie avec Sophie.

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Introduction

Il y a une vaste portée symbolique au mot "noeud", il peut désigner tout autant l'entrelacement d'une corde, d'un lacet de chaussure, il peut aider à naviguer ou bien nous prendre la tête. Il caractérise aussi bien l'oppression d'une gorge nouée, l'entrave de mains liées que la liberté du point central, de la croisée des chemins où les choix se font. Sa portée et son sens sont à son image, labyrinthique, comme des écouteurs entortillés au grand désarroi de la personne qui les sort de sa poche.

Voyez comme l'on s'y perd, le propos n'a pas commencé qu'il est déja confus. Et bien nous ne prendrons qu'un sens de ce mot, qui sera précisé plus tard, un noeud mathématique. Un concept idéalisé, qui fait écho à notre intuition. On pourrait se l'imaginer comme une ficelle que l'on entrelace et dont on recolle les bouts. Son étude pourrait se fonder sur la question suivante: si l'on prend deux noeuds, deux ficelles entortillées et recollées, peut on déformer l'un en l'autre? C'est à dire, peut on transformer l'un en l'autre sans couper la ficelle? Bien sûr ce n'est qu'une image, qu'une représentation de l'objet mathématique et de son étude. Il est plus dur d'imaginer la ficelle comme n'ayant aucune épaisseur et étant infiniment étirable ou rétractable. Cela nous aide cependant à entrevoir par exemple, la difficulté d'étude d'un objet mathématique en tant que tel.

Un noeud, en mathématiques, c'est le plongement régulier d'un cercle S^1 dans S^3. Son étude c'est l'étude des classes d'isotopie ambiante des noeuds, c'est à dire si l'on peut déformer continuement un plongement en un autre. Si l'on se donne deux noeuds, il n'est pas facile à priori de savoir s'ils sont isotopes où non. Une manière de répondre partiellement à cette interrogation est de trouver un invariant, quelque chose qui ne change pas si deux noeuds sont isotopes. Ainsi, si l'invariant de l'un est différent de l'invariant de l'autre, alors nécessairement ils ne peuvent être isotopes. Bien sur, l'invariant va être intéressant si il arrive à distinguer les différentes classes d'isotopies: un invariant qui serait le même pour tous les noeuds ne nous apprendrait rien. Il est aussi intéressant qu'il soit facilement calculable, ou étudiable. Si l'invariant est un concept aussi complexe que la classe d'isotopie elle même, il n'aidera pas directement à déterminer si deux noeuds sont isotopes. Pourquoi directement? Car il se peut qu'un tel invariant fasse le lien avec un autre domaine des mathématiques ou de la physique, et que par se fait, il ouvre la voie à une nouvelle forme d'étude dans un sens comme dans l'autre. Ce qui est dit ici pour les noeuds est en fait valable pour de nombreux concepts mathématiques, où la recherche et l'étude d'invariants est très utile.

Pourquoi étudier les noeuds? C'est un concept qui intervient de nombreuses fois en physique, car l'idée de croisement de brins vient facilement à l'esprit et se produit souvent de par sa simplicité. On peut penser au tracé de particules se déplacant dans un plan, tournant l'une autour de l'autre, formant un tressage. Son étude a aussi amené à connecter plusieurs domaines des mathématiques et de la physique. La théorie des noeuds est elle même un noeud, un point central. On peut passer de l'étude des noeuds à celle des variétés de dimension 3 en utilisant une opération appellée chirurgie, l'étude de ces variétés permet d'étudier les théories topologiques des champs quantiques, les TQFT. Notre étude des noeuds se basera principalement sur des invariants dits quantiques, issues de l'algèbre.

Les invariants quantiques

Passons au choses sérieuses, on veut des invariants de noeuds. Quelles formes auront ces invariants? Il serait judicieux que ce soit des objets simples à manipuler, comme des nombres, ou des polynômes.

Comment produire de tels invariants? On réduit d'abord l'étude des classes d'isotopies dans S^3 à une étude planaire. Si l'on prends un noeud, qu'on le déforme par isotopies et qu'on le projette sur un plan, de sorte à ce qu'il n'y ait que des croisements transverses entre deux brins, notant quel brin est dessus et quel brin est dessous, on obtient un diagramme du noeud. La question est donc, quels diagrammes représentent la même classe d'isotopie du noeud? La réponse est apportée dans un théoreme de Reidemeister, il prouve que deux diagrammes représentent le même noeud si on peut transformer l'un en l'autre par un nombre finis d'isotopies planes et de mouvements spéciaux, dits de Reidemeister, que l'on précisera plus loin.

Cette réduction de problème est cruciale, elle permet de se restreindre au plan, de comprendre les choses sur des dessins. Surtout, elle met au coeur de l'étude la notion de croisement. Cela va permettre aussi de voir les noeuds comme des diagrammes découpés dans une grille où chaque case correspond a un diagramme élémentaire, comme par exemple un croisement. On voit apparaitre ici de la structure: on peut juxtaposer et concatener des petits morceaux, les croisements obéissent a des règles ressemblant à de l'algèbre non-commutative.

L'idée pour produire les invariants quantiques va être d'exploiter la structure d'algèbre de Hopf enrubannée, plus précisement aux diagrammes élémentaires vont être associés des morphismes de représentations finis de telles algèbres, la juxtaposition et la concaténation seront vu comme composition et produit tensoriel de ces morphismes. La structure d'algèbre de Hopf permet d'avoir une bonne compatibilité du produit tensoriel sur les représentations, l'élément spécial R-matrice va permettre d'encoder ce qu'il se passe lors d'un croisement. Ainsi, a partir d'un noeuds on obtient un scalaire, il peut appartenir à $\mathbb{R}$, $\mathbb{C}$ ou par exemple $\mathbb{Z}[q^{\pm 1}]$.

On va concentrer notre étude sur une algèbre bien particulière, $U_q(\mathfrak{sl}_2)$ ainsi que ses dérivés. C'est une déformation de l'algèbre universelle enveloppante de $\mathfrak{sl}_2$ que l'on va nommer groupe quantique ou algèbre quantique. C'est un peu le modèle d'algèbre de Hopf de base pour les invariants quantiques de par sa richesse et sa simplicité. Il est désormais temps de présenter les protagonistes de cette petite affaire: deux familles de polynômes, toutes deux issues de certaines déclinaisons de notre fameuse algèbre.

Les polynomes de Jones colorés et d'ADO

La première famille se base sur un pilier: le polynôme de Jones. Il a été introduit par Vaughan Jones en 1984 et a permis de distinguer par exemple le noeud de trêfle de son miroir mais c'est Edward Witten, qui interprétant les noeuds comme des trajectoires de particules, a créé un lien étroit avec la physique. Il y a eu depuis de multiple manière de définir ce polynôme et de l'exploiter.

S'il est la figure centrale de cette famille, il y a aussi les autres, les colorés. Pour comprendre qui ils sont, il faut s'intéresser aux représentations de dimension finie de $U_q(\mathfrak{sl}_2)$. Lorsque l'on prend q comme paramètre formel, nous avons une famille de représentations S_N de dimension $N+1$. Le N-ème polynôme de Jones coloré sera alors simplement l'invariant quantique issue de cette représentation. On obtient ainsi une famille de polynômes dans $\mathbb{Z}[q^{\pm 1}]$.

La seconde famille est plus récente. Introduite par Akutsu, Deguchi et Othsuki en 1991, elle se fonde sur un pilier plus ancien, le polynôme d'Alexander, et est parfois appelée polynômes d'Alexander colorés. Introduit en 1923 par James Alexander, le polynôme éponyme est le premier invariant polynomial de noeuds et il restera bien seul durant près de 60 ans.

A la différence de la première famille, on prend ici $q = \zeta_{2r}$ une racine de l'unité. On obtient alors une nouvelle représentation de dimension r, notée V_α dont les scalaires sont $\mathbb{Z}[\zeta_{2r}][A^{\pm 1}]$ et où l'on note $q^\alpha := A$. L'invariant quantique associé sera alors appelé r-ème polynôme ADO. Cette famille de polynômes dans $\mathbb{Z}[\zeta_{2r}][A^{\pm 1}]$ est indexé par r, l'ordre de la racine de l'unité.

Ces deux familles ont une origine, en somme, relativement proche mais diffèrent tout de même fondamentalement, par exemple par leurs variables, respectivement q et A. Il faudra donc se pencher plus en détail sur ce qui les réunit et ce qui les oppose si l'on veut procéder à l'unification de ces deux familles. Il faudra par exemple que l'invariant unifié soit porteur des deux variables q et A.

L'invariant universel et l'invariant unifié

Les travaux au coeur de cette thèse sont ceux de Kazuo Habiro, qui grâce à une complétion d'anneau et d'algèbre va exploiter l'invariant quantique universel du noeud - appartenant à l'algèbre non commutative - et le transposer dans un anneau commutatif où il va unifier les polynômes de Jones colorés. Il permettra aussi d'unifier les invariant de WRT pour les sphères d'homologie entière.

On voudrait procéder de la sorte pour unifier la famille des polynômes ADO. Ce que l'on verra par la suite c'est qu'en plus d'unifier les ADO, l'invariant unifié est en fait équivalent à l'invariant d'Habiro, unifiant par là même les polynômes de Jones. Revenons aux polynômes ADO, pour les unifier il faut comprendre ce qui les distingue, ce qui les rapproche. Ils sont similaires dans leur origine, la seule différence étant l'ordre de la racine considérée. Il faut donc façonner un élément qui ne dépende pas de r. Le point crucial est la structure enrubannée de $U_{\zeta_{2r}}(\mathfrak{sl}_2)$ dont la R-matrice R_r est une troncation de taille r de la R matrice de $U_q(\mathfrak{sl}_2)$, R. On va donc prouver qu'aux racines de l'unité, de la R matrice totale se factorise la R matrice tronquée.

On va ainsi procéder à la construction d'un candidat à l'unification. En utilisant des diagrammes d'états, on va donner une formule des polynômes ADO où la contributions de chaque croisements, donc de chaque R matrice tronquée sera au premier plan. On voudra alors faire de même avec q formel et en prenant la R matrice totale. Si un tel élément existe, on obtiendra aux racines de l'unité une factorisation avec ADO. La seule obstruction viendra de sommes infinies liées à la R-matrice, il suffira de completer l'anneau $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ de sorte à permettre à ces sommes de converger.

Il s'agira alors de prouver que cet élément est bien un invariant de noeuds, c'est le minimum que l'on puisse lui demander. La preuve de cette invariance aurait pu être faite de manière plus pédestre, utilisant les propriétés des R-matrices directement afin de démontrer les mouvements de Reidemeister. Néanmoins, il y a une approche qui va se révéler bien plus riche. On va récuperer cet élément unifiant grâce à l'invariant universel et aux modules de Verma. L'invariant universel a déja été brièvement mentionné plus haut. Il s'agit en réalité d'un élément de l'algèbre, un invariant produit grâce à une algèbre de Hopf enrubannée mais affranchi d'une représentation quelconque. Il a été introduit par Lawrence, et extensivement étudié par Habiro comme mentionné précédemment. Il va nous servir, cet invariant, car la précise mécanique produisant les invariants quantiques nécessite une représentation finie de l'algèbre. Nous, on veut une représentation porteuse des deux variables formelles q et A, et la seule candidate est le module de Verma, une représentation de dimension infinie. L'invariant universel du noeud, étant central par exemple, aura une action scalaire sur ce module de Verma et le scalaire obtenu n'est nul autre que notre élément unifiant, qui, encensé par la puissance de l'invariant universel, obtient le rang d'invariant unifié. De sa variable q se factorise les polynômes ADO aux racines de l'unité, de sa variable A s'obtiennent les polynômes de Jones colorés lorsque prise à q^N.

Propriétés et applications

Bon et bien il faut alors étudier cet invariant. Notre construction nous a amené si proche de celle d'Habiro, on peut y relier désormais nos constructions, notamment les completions d'anneaux dont l'étude poussée va nous permettre d'en extraire de bonnes propriétés. Par exemple, l'intégrité de notre anneau complété, ou encore son injection dans l'anneau h-adique $\mathbb{Q}[[h]]$. Ceci va nous permettre de facilement tirer l'unicité de l'invariant unifié, dans le sens où il est l'unique élément de l'anneau complété dont les évaluations de A en q^N donnent les polynômes de Jones colorés. Et ainsi de fil en aiguille, l'invariant unifié est équivalent à la famille des polynômes de Jones, qui est équivalente à la familles des polynômes ADO. On a une forme d'unicité, les deux familles portent la même informations sur les noeuds, simplement exposée de manière différente.

Cette unicité est cruciale, elle va nous permettre d'obtenir pleins de propriétés sur les polynômes ADO grâce aux propriétés sur les polynômes de Jones colorés. On peut aussi désormais comprendre la factorisation aux racines de l'unité, car si l'on sait que ADO s'y factorise, l'autre terme de la factorisation parait jusqu'alors mystérieux. Il faudra ainsi invoquer le théoreme de Melvin-Morton-Rozansky prouvé par Bar-Natan et Garoufalidis, qui fera apparaitre que le second terme de la factorisation n'est autre que $1/A_{\mathcal{K}}(q^{2r\alpha})$, l'inverse du polynôme d'Alexander. Encore lui.

Parmis les propriétés des polynômes de Jones colorés pour les noeuds, on va pouvoir transférer des propriétés d'holonomie aux polynômes ADO. Une polynôme d'opérateur va venir annuler chacun des polynômes ADO et ce polynôme d'opérateur sera le même que celui qui annule la famille des polynômes de Jones colorés, le A-polynôme quantique. Une autre propriété intéressante que l'on va pouvoir décliner sauce ADO, c'est le developpement en série avec des coefficients qui sont des invariants de types finis, ou invariant de Vassiliev. C'est une propriété des Jones colorés que l'on obtient en effectuant un développement h-adique du polynôme. On peut alors facilement faire de même pour l'invariant unifié. Aux racines de l'unité par contre, le fait qu'il y ait une factorisation de deux termes va venir compliquer un peu la manoeuvre mais on pourra montrer que les ADO sont des invariants de type fini r-adique, une version plus faible d'invariance de type fini.

On peut maintenant choisir de suivre et d'approfondir certaines pistes possibles. Ici, on va partir sur une nouvelle interprétation de l'invariant unifié, en le construisant via des représentations de tresses.

Le point de vue des représentations de tresses: vers une nouvelle direction

On va ici changer un peu de paradigme: on va parler de représentation de tresses. Par exemple, si l'on prend un représentation de notre algèbre favorite, on va pouvoir agir sur le produit tensoriel de n copies de cette algèbre via la R-matrice. Un croisement du i-ème brin avec le suivant sera représenté par l'action de la R matrice sur la i eme composante du produit tensoriel et la suivante, le reste restant inchangé. On peut alors réécrire les polynômes d'ADO, de Jones ou l'invariant unifié comme trace quantique de ces représentations.

L'intérêt est multiple, car la factorisation aux racines de l'unité, par exemple, se comprend bien plus aisément. Nul besoin ici d'utiliser MMR, et le simple usage du théoreme de Mac Mahon, de la pure algèbre linéaire, permet en fait de donner une autre preuve de MMR. On s'affranchit donc facilement de cette boite noire grâce à ce point de vue de représentation de tresses, tout en faisant de celle ci une conséquence de la factorisation. En réalité, la factorisation aux racines de l'unité se produit même au niveau des représentations.

Ce point de vue pourrait permettre de développer une interprétation homologique des invariants ADO et de l'invariant unifié.

Le cas problématique des entrelacs

Bon, nous y voila. Pourquoi nous sommes nous cantonnés aux seuls noeuds? Pourquoi avoir snobbé la richesse des entrelacs? Les entrelacs sont les plongements réguliers de plusieurs cercles. Il ne semble à priori pas si difficile de s'imaginer que ce que l'on vient de faire fonctionne aussi pour des entrelacs. Oui mais voila. En réalité, lorsque l'on a parlé de noeud, on a toujours pris le paradigme suivant: au lieu de le considérer comme tel on le découpe à un endroit et on l'étire par les deux bouts, on appelle cela un noeud long. Cette version du noeud est équivalente à la version normale. Ce n'est pas le cas pour les entrelacs, car ayant plusieurs brin, connaitre celui que l'on découpe est important. Vient alors la débandade.

L'invariant universel, pilier de la théorie, ne peut plus se définir comme un élément de l'algèbre mais comme un élément du produit tensoriel de l copies de l'algèbre, où l est le nombre de composantes, le tout quotienté par les commutateurs. Ce n'est même plus une algèbre. On ne peut plus faire notre petite tambouille.

Les diagrammes d'états, pourtant si prompts a nous fournir de bien pratiques formules pour la factorisations, se heurtent désormais à de sommes infinies qui ne convergent pas. Ces sommes sont issues de la fermeture de brins dans les diagrammes, là où le diagramme d'un noeud long n'en comportait point.

Enfin, les représentations de tresses. Ce point de vue à l'avantage de fonctionner directement au niveau des représentations. Il y a pourtant aussi des problèmes de convergence, car la sommes des traces des représentations ne convergent plus. On a alors le plus grand mal a fournir une formule d'un invariant unifié. On peut néanmoins par certains artifices, dévoiler une factorisation aux racines de l'unité du candidat à l'invariant unifié pour les entrelacs. Le problème c'est que ce candidat n'existe, à ce jour, dans aucun anneau. Voila donc tout ce que l'on peut dire pour l'instant de nos chers entrelacs.

Et après?

Il y a bien des pistes, tout de même, pour continuer le cas des entrelacs. On peut penser aux théories des séries divergentes. Ou peut être faudrait il se contenter d'un invariant d'une algèbre non commutative?

On pourrait aussi se rapprocher d'une formulation de l'invariant unifié pour les noeuds comme déterminant quantique, comme cela a été fait par Huynh et Le pour les polynômes de Jones colorés. Et se rapprocher ainsi de l'étude de la conjecture du volume dans cette direction.

Une fois le cas des entrelacs accomplis on pourrait aussi poursuivre dans la direction des invariants de 3-variétés et TQFT, essayer d'unifier les invariants WRT et les invariants CGP. On pourrait aussi s'aventurer plus en profondeur dans le domaines des invariants de types finis.

La voila donc, cette croisée des chemins. Maintenant, rentrons dans le vif du sujet.

Main results

In [1], Akutsu, Deguchi and Ohtsuki gave a generalisation of the Alexander polynomial, building a colored link invariant at each root of unity. These ADO invariants, also known as colored Alexander's polynomials, can be obtained as the action on 1-1 tangles of the usual ribbon functor on some representation category of a version of quantum $\mathfrak{sl}_2$ at roots of unity (see [5], [11]). On the other hand, we have the colored Jones polynomials, a family of invariants obtained by taking the usual ribbon functor of quantum $\mathfrak{sl}_2$ on finite dimensional representations. It is known ([6]) that given the ADO polynomials of a knot, one can recover the colored Jones polynomials of this knot. One of the results of the present paper is to show the other way around: given the Jones polynomials of a knot, one can recover the ADO polynomials of this knot.
We denote $ADO_r(A, \mathcal{K})$ the ADO invariant at $2r$ root of unity seen as a polynomial in the variable A, $J_n(q, \mathcal{K})$ the n-th colored Jones polynomial in the variable q and $A_{\mathcal{K}}(A)$ the Alexander polynomial in the variable A.

Result 1. *There is a well defined map such that for any knot $\mathcal{K}$ in S^3,*

$$\{J_n(q, \mathcal{K})\}_{n\in\mathbb{N}^*} \mapsto \{ADO_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*},$$

(Detailed version: Theorem 122).

The above result is a consequence of the construction of a unified knot invariant containing both the ADO polynomials and the colored Jones polynomials of the knot. This unified invariant is in fact equal to the two variable colored Jones invariant defined by Habiro in [16] and answers positively the conjectures of its behaviour at roots of unity. Briefly put, we obtain it by looking at the action of the universal invariant (see [19] [20], and also [23]) on some Verma module with coefficients in some ring completion. For the sake of simplicity let's state the result for 0 framed knots.

Result 2. *In some ring completion of $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ equipped with suitable evaluation maps, for any 0-framed knot $\mathcal{K}$ in S^3, there exists a well defined knot invariant $F_\infty(q, A, \mathcal{K})$ such that:*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{ADO_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})}, \quad F_\infty(q, q^n, \mathcal{K}) = J_n(q^2, \mathcal{K})$$

(Detailed version: Theorem 114 and Corollary 98).

A visual representation of the relationship between all these invariants is given at Figure 1.

Let us denote $J_\bullet(q^2, \mathcal{K}) = \{J_n(q, \mathcal{K})\}_{n\in\mathbb{N}^*}$ and call it colored Jones function of $\mathcal{K}$.
The holonomy of the unified invariant and of the ADO polynomials will follow as a simple application of the two previous results and of the q-holonomy of the colored Jones function as shown in [9]. Mainly, there are two operators Q and E on the set of discrete function over $\mathbb{Z}[q^{\pm 1}]$ that forms a quantum plane and for any knot $\mathcal{K}$, there is a two variable polynomial $\alpha_{\mathcal{K}}$ such that $\alpha_{\mathcal{K}}(Q, E)J_\bullet(q^2, \mathcal{K}) = 0$. We say that the colored Jones function is q-holonomic.
This paper gives a proof that the same polynomial $\alpha_{\mathcal{K}}$, in some similar operators as Q and E, annihilates the unified invariant $F_\infty(q, A, \mathcal{K})$ and, at roots of unity, annihilates $ADO_r(A, \mathcal{K})$.

Result 3. *For any 0-framed knot $\mathcal{K}$ and any $r \in \mathbb{N}^*$:*

- *The unified invariant $F_\infty(q, A, \mathcal{K})$ is q-holonomic.*

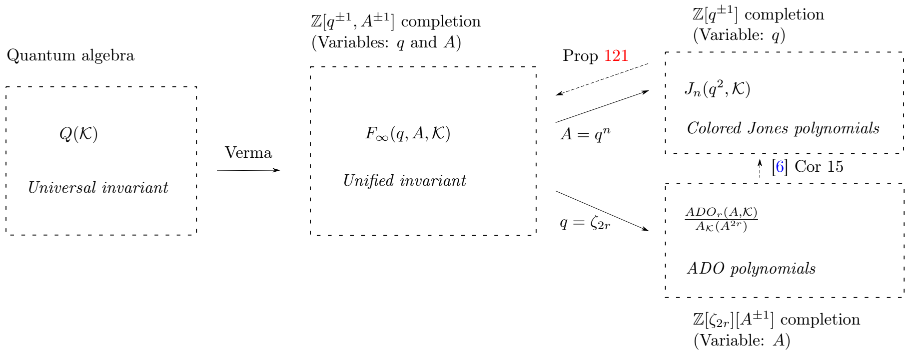


Figure 1: Visual representation of the unified knot invariant.

- *The ADO invariant $ADO_r(A, \mathcal{K})$ is ζ_{2r}-holonomic.*

Moreover they are annihilated by the same polynomial as of the colored Jones function. (Detailed version: Theorems 128 and 130).

In a similar fashion, we can express some Vassiliev expansion formula for the unified invariant using the fact that the colored Jones polynomials have such an expansion.

Result 4. *For any 0-framed knot $\mathcal{K}$, we have :*

$$F_\infty(q, q^\alpha, \mathcal{K}) = \sum_{n,m\geq 0} b_{n,m}(\mathcal{K})(q^2-1)^n(q^{2\alpha}-1)^m$$

where $b_{n,m} \in \mathbb{Z}$ is a Vassiliev invariant of degree at most $n+m$. (Detailed version: Proposition 140)

We can even get such a formula for the ADO polynomials, although the form of the expansion is not unique and thus hard to compute.

Result 5. *Let $\mathcal{K}$ be a 0-framed knot in S^3 and $r = p^l$ be a power of a prime number.*
There exist Vassiliev invariants $c_{n,m}(r, \mathcal{K})$ of degree[1] $n+m$, such that we can write

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0}\sum_{n\geq 0} c_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n(q^{2\alpha}-1)^m$$

in $\mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$.
This means that the ADO polynomials $ADO_r(q^\alpha, \mathcal{K}) \in \mathbb{Z}[\zeta_r, q^\alpha]$ are topological Vassiliev invariants for the filtration $((\zeta_r-1)^n(q^\alpha-1)^m)_{n,m\in\mathbb{N}^*}$.
(Detailed version: Theorem 147)

Nevertheless we can get a unique expansion, but the coefficient will no longer be Vassiliev invariants but r-adic topological Vassiliev invariants, in a sense that will be defined in Section 3.4.

Result 6. *Let $\mathcal{K}$ be a 0-framed knot in S^3 and $r = p^l$ a power of a prime number.*
We can write in a unique way

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0}\sum_{n=0}^{\varphi(r)-1} d_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n(q^{2\alpha}-1)^m$$

in $\mathbb{Z}[\zeta_r, q^\alpha]$, where $d_{n,m}(r, \mathcal{K}) \in \mathbb{Z}$ and we have

1. *$d_{n,m}(r, \mathcal{K}) \mod r^j$ is a Vassiliev invariant of degree[1] $j\varphi(r) + m - 1$.*
2. *In consequence, $d_{n,m}(r, \mathcal{K})$ is a r-adic topological Vassiliev invariant.*

[1] See Nota Bene Section 3.4

3. *Thus,* $ADO_r(q^\alpha, \mathcal{K}) \in \mathbb{Z}[\zeta_r, q^\alpha]$ *is a topological Vassiliev invariant for the filtration* $(r^j(q^\alpha - 1)^m)_{j,m\in\mathbb{N}}$ *topology.*

(Detailed version: Theorem 149)

We can get back the unified invariant using braid representations. This new point of view allows us to prove the factorisation at root of unity theorem using basic linear algebra. The MMR theorem becomes a consequence of the factorisation theorem.

The factorisation at roots of unity even occurs at the level of representations, as described by the following result. (Note that at this point notation has not been fully introduced, in order to understand the following result one can check Chapter 4.)

Result 7. *The isomorphism*

$$\begin{aligned} \Phi : V_n^m &\to V_n^0 \otimes F_r(W_{n,m}) \\ v_{\overline{i+rj}} &\mapsto v_{\overline{i}} \otimes F_r(w_{\overline{j}}), \end{aligned}$$

where $\overline{i+rj} = (i_1 + rj_1, \ldots, i_n + rj_n)$ *with* $i_1, \ldots, i_n \leq r-1$, *is a braid group representation isomorphism.*

In other word, the following diagram commutes:

$$\begin{array}{ccc} V_n^m & \xrightarrow{\varphi_n^m} & V_n^m \\ \Phi \downarrow & & \downarrow \Phi \\ V_n^0 \otimes F_r(W_{n,m}) & \xrightarrow{\varphi_n^0 \otimes (F_r \circ \psi_{n,m})} & V_n^0 \otimes F_r(W_{n,m}) \end{array}$$

where

$$\begin{aligned} F_r : \mathbb{Z}[q^\alpha] &\to \mathbb{Z}[q^\alpha] \\ q^\alpha &\mapsto q^{r\alpha}. \end{aligned}$$

and $\varphi_n^m : B_n \to End(V_n^m)$ *and* $\psi_{n,m} : B_n \to End(W_{n,m})$ *are braid representations defined in section 4.4 and 4.5.*

(Detailed version: Proposition 176)

In a nutshell

The main purpose of this thesis is to construct and study an invariant of knots, unifying both colored Jones and ADO polynomials.

The goal of the first chapter will be to introduce every objects and structures that will be needed after on.

The second chapter will be dedicated to construct the unified invariant. Firstly, we build a good candidate by hand using a states sums approach. Secondly, we prove its invariance using the universal invariant.

We can then proceed to chapter three, the study of this invariant. By making a connection with Habiro's work, we prove a whole bunch of nice properties. Such as the uniqueness of the invariant unifying both colored Jones and ADO, it's relashionship to the inverse of the Alexander polynomial at $q = 1$, some holonomy properties for the ADO polynomials, or even a topological Vassiliev expansion formula for the ADO polynomials.

Chapter four is quite independent from the others. It uses a braid representation approach to build once more the unified invariant. This allows us to give an alternative proof of the Melvin-Morton-Rozanski conjecture (proved by Bar-Natan and Garoufalidis) instead of using it to prove the factorisation at roots of unity.

Finally, the fifth Chapter will simply describe the differents attempts to generalise this work to the case of links. Unfortunately it is an ongoing work and we do not have solid results to show, only obstructions. This means that the proofs will not be much detailled and should be taken with caution.

Chapter 1

Links and quantum invariants

The first chapter of this thesis will introduce the objects and structures that will be needed further on. The idea will be to describe, give or take, only what is necessary to construct a unified quantum invariant and understand its properties.

The main object of focus will be links, and more precisely, in our case, knots. They are topological entities that we study up to isotopy. How does one distinguish one knot from another? Can a given knot be deformed into another one, meaning, are they isotopic? In order to partially address these questions, we could try to understand what doesn't change when we do an isotopy. We would then get a knot invariant.

A knot invariant can have many forms, but they are usually a number or a basic algebraic element, such as a polynomial. In order to produce such invariants, we can, step by step, go from the study of knots modulo isotopy to the study of finite representation of a ribbon Hopf algebra. The first step toward algebra is to reduce the 3-dimensional study of isotopies to a 2-dimensional one. Using planar diagrams of a knot, one only needs to study planar isotopies and the Reidemeister moves in order to understand the isotopy class of the knot. The second step is to construct the right algebraic setup in order to make sense of the elementary diagrams (such as a crossing or a cup or a cap) algebraically. The right structure to do so is the ribbon Hopf algebra structure.

From this construction, we can produce many knots invariants, called quantum invariants. Two important families of quantum invariants are the colored Jones polynomials and the ADO polynomials. The goal of the chapters following this one will be to construct and study an invariant unifying both families.

1.1 Topology and algebra

1.1.1 Knots, links and tangles

Knots and links

Let's introduce the main object of study: knots and links. They are a mathematical description of an intuitive, although idealized, version of real world knots and links. More precisely we considered two knots to be equal if we can deform one into another.

Definition 1. A *knot* is a piecewise smooth embedding $S^1 \to \mathbb{R}^3$. A *link* with l component is a piecewise embedding of a disjoint union l circles in $\mathbb{R}^3$.

We study the isotopy classes invariants, maps going from the set of isotopy classes of knots (or links) to a much simpler set (like $\mathbb{Z}$, $\mathbb{C}$ or polynomials for instance). The ability to distinguish two non isotopic knots, the computability or the connection to other domain of mathematics are important properties of an invariant.

Definition 2. A knot (or link) is *oriented* if the circles are oriented.
A knot (or link) is *framed* if it is endowed with a transverse vector field considered up to isotopy.

Remark 3. We can view a framed link as an embedding of a disjoint union of annuli into $\mathbb{R}^3$.

Diagrams and linking number

One step to produced knots invariants is to reduce the 3 dimensional study to a planar diagram study.

Definition 4. A *knot diagram* is a piecewise smooth immersion $S^1 \to \mathbb{R}^2$ with at most finetely many transversal double points, such that for any double point, one path is assigned to be the under path and the other the over path. Such a double point is called *crossing*.
We define the same way *links diagrams*.

We say that a knot (or link) diagram D is a diagram of a knot (or link) $\mathcal{K}$ if D is obtained from $\mathcal{K}$ via a projection map $\mathbb{R}^3 \to \mathbb{R}^2$. Any knot (or link) is isotopic to a knot (or link) having a diagram. If the knot (or link) is oriented or framed, so will be the diagram.

Diagrams are useful because they allow us to understand 3 dimensional objects with a 2 dimensional picture. There is a fundamental theorem that caracterise isotopy classes of knots (or link) in terms of diagrams.

Theorem 5. *(Reidemeister)*
Let $\mathcal{K}_1$ and $\mathcal{K}_2$ two framed oriented knots (or links) and D_1, D_2 corresponding diagrams.
$\mathcal{K}_1$ and $\mathcal{K}_2$ are isotopic in $\mathbb{R}^3$ iff D_1 is obtained from D_2 using a sequence of planar isotopies, and local moves R1, R2 and R3 presented in Figures 1.1, 1.2, 1.3.

Proof. See Theorem X.3.7 in [18]. □

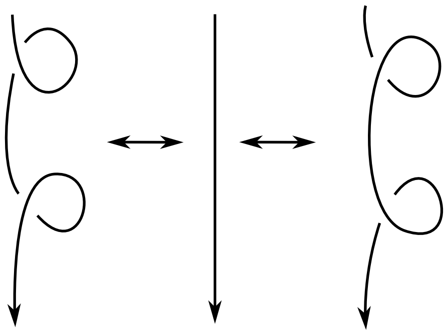
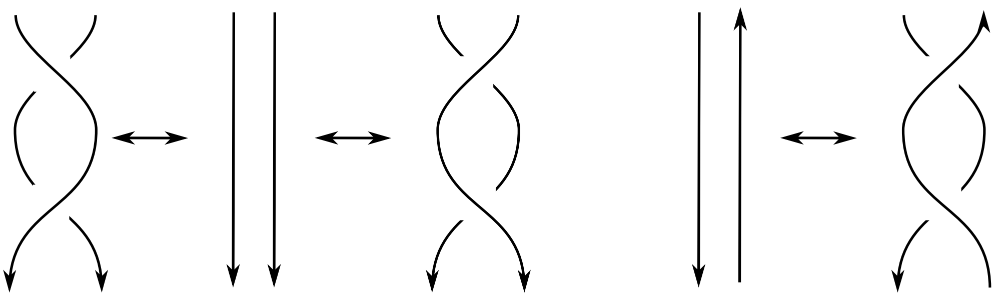


Figure 1.1: R1: First Reidemeister move for framed oriented links



(a) Same orientations , (b) Different orientations

Figure 1.2: R2: Second Reidemeister move for framed oriented links

This means that we can study links diagrams instead of links.

For an oriented link, we distinguish two types of crossings, positive and negatives as shown in Figure 1.4.

Now we have a way to define a link invariant by counting positive and negative crossings between two components of a link.

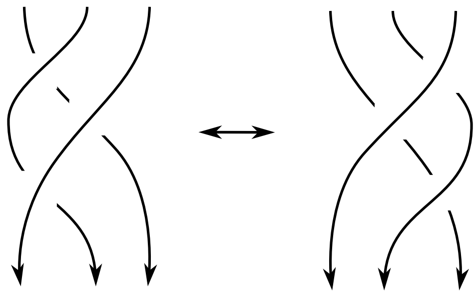


Figure 1.3: R3: Third Reidemeister move for framed oriented links

(a) Positive crossing (b) Negative crossing

Figure 1.4: Positive and negative crossings

Definition 6. Let L_1 and L_2 be two components of an oriented link L. Let D_1, D_2 the two corresponding components of a diagram D of L. We set *pos* the number of positive crossings between D_1 and D_2, and *neg* the number of negative ones.

$$lk(L_1, L_2) := \frac{1}{2}(pos - neg).$$

Proposition 7. *$lk(L_1, L_2)$ is an oriented link invariant of any link L containing $L_1 \cup L_2$. We say that the two components L_1 and L_2 are algebraically split if $lk(L_1, L_2) = 0$.*

Proof.

- Any planar isotopy of a diagram does not change the linking numbers.
- Reidemeister 1 type move only involves one component.
- Reidemeister 2 type moves present two crossings with different signs, canceling out their contributions to the linking number.
- Reidemeister 3 type move doesn't change the number of crossings nor their signs.

□

Definition 8. Let K be a component of a framed oriented link L. Let K' be the curve obtained by pushing points of K along the framing vectors (supposed sufficiently small to be contained in a tubular neighborhood of K disjoint from $L \backslash K'$).
We define the *self linking number* as the linking number between K and K' in $L \cup K'$:

$$lk(K) := lk(K, K').$$

We sometimes denote it $lk(K, K)$.

Proposition 9. *$lk(K)$ is an oriented framed link invariant of any link containing K. We say that K is 0-framed if $lk(K) = 0$.*

Proof. Corollary of Proposition 7, Reidemeister 1 type move adds or subtracts two crossings with differents signs, canceling out their contributions to the linking number. □

Definition 10. For a link $L = \cup_{i=0}^{l} L_i$ with l components, the *linking matrix* is the $l \times l$ matrix with coefficients in $\mathbb{Z}$ defined by:

$$L_k(L) := (lk(L_i, L_j))_{i,j \leq l}.$$

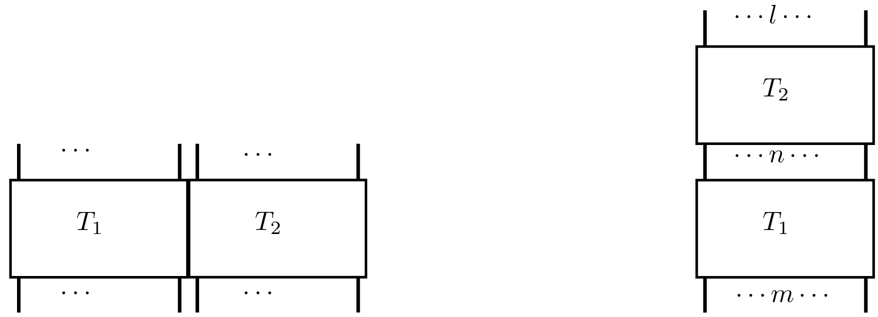
Tangles and elementary tangle diagrams

Now let's introduce the notion of tangle, which will allow us to decompose knot and links diagrams into elementary tangle diagrams.

Definition 11. A *tangle* is a piecewise smooth embedding of n circles S^1 and m arcs $[0,1]$ in $\mathbb{R}^2 \times [0,1]$ such that boundary points lie in $\mathbb{R} \times \{0\} \times \{0,1\}$. We say that two tangles are isotopic if there exists an ambient isotopy between them fixing the boundary points.

A tangle can be oriented and framed the same way knots and links are. An isotopy of framed tangles must also fix boundary vectors. A tangle diagram is defined the same way knots and links diagrams are.

Remark 12. We can concatenate two tangle diagrams as shown in Figure 1.5a. We can also juxtapose two tangles T_1 and T_2 as long as the number boundary points in $\mathbb{R} \times \{0\} \times \{1\}$ of T_1 is equal the number boundary points in $\mathbb{R} \times \{0\} \times \{0\}$ of T_2, as shown in Figure 1.5b.



(a) Concatenation (b) Juxtaposition

Figure 1.5: Tangle juxtaposition and concatenation

Definition 13. A *sliced tangle diagram* is a diagram of a tangle that can be decomposed as concatenations and juxtapositions of *elementary tangles diagrams* defined in Figure 1.6.

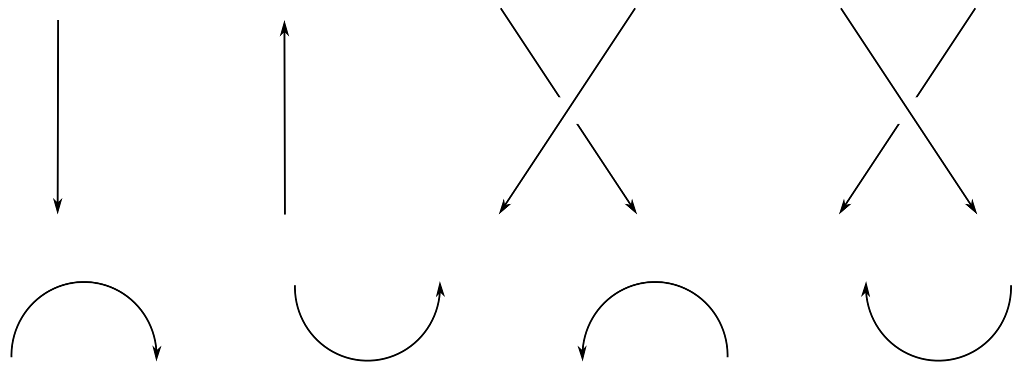


Figure 1.6: Elementary tangle diagrams

Using isotopy if needed, we can henceforth decompose any tangle diagrams - thus any knots and links diagrams - into a sliced tangle diagram. More precisely we have the following statement:

Theorem 14. *Two oriented framed sliced tangle diagrams express the same isotopy class of oriented framed tangle if and only if they are related by a finite sequence of Turaev moves consisting of:*

- *Planar isotopic moves described in Figure 1.7 1.8 1.9 1.10.*
- *Reidemeister moves described in Figures 1.1 1.2 1.3.*

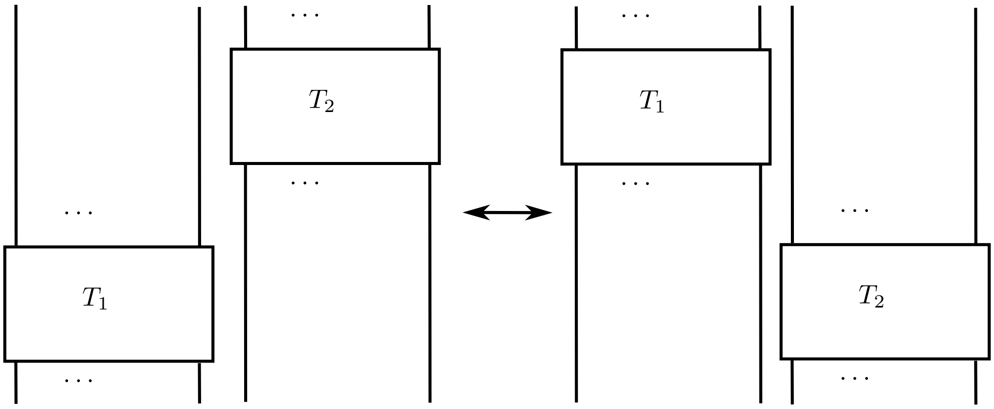


Figure 1.7: Turaev move: first planar move

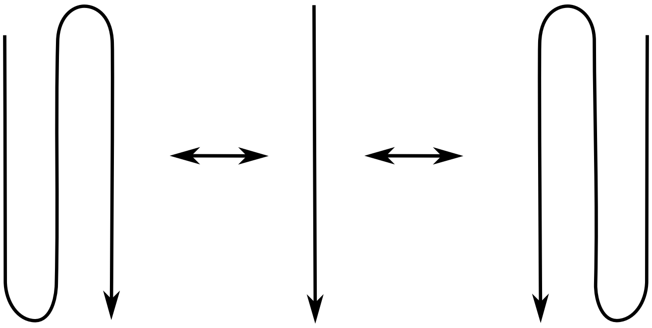


Figure 1.8: Turaev move: second planar move

Proof. See Theorem 3.3 in [23] for more details. It comes down to the fact that taking any diagram, one can get a sliced tangle diagram using planar isotopic moves. □

1.1.2 Ribbon Hopf algebra and quantum invariants

We will now present an algebraic structure, ribbon Hopf algebra, that produces very naturally knots and links invariants.

Ribbon Hopf algebra

We denote K an integral domain.

Definition 15. An *algebra* (A, μ, i) on a ring K is the data of a K-module A along with two linear operators $\mu : A \otimes A \to A$ and $i : K \to A$ such that:

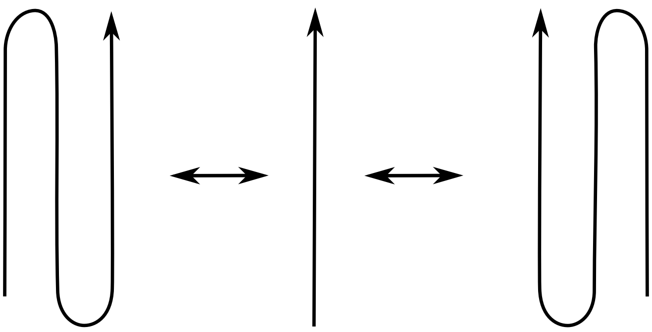


Figure 1.9: Turaev move: third planar move

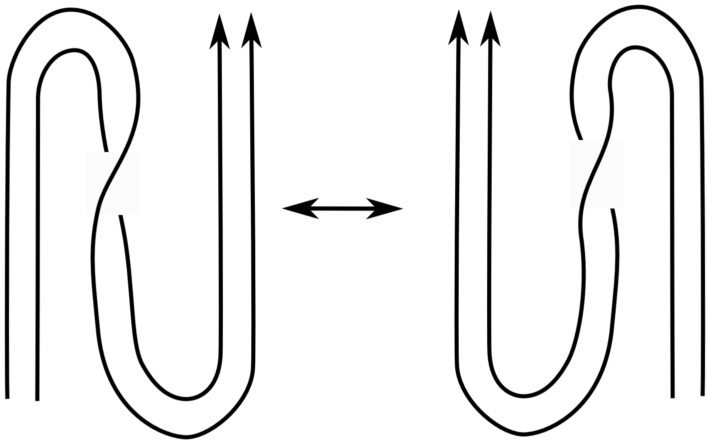


Figure 1.10: Turaev move: fourth planar move

- $\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu)$ (Associativity)
- $\mu \circ (i \otimes id_A) = \mu \circ (id_A \otimes i) = id_A$ (Unitality)

A *bialgebra* $(A, \mu, i, \Delta, \epsilon)$ is the data of an algebra (A, μ, i) along with two algebra morphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to K$ such that:

- $(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta$ (Co-associativity)
- $(\epsilon \otimes id_A) \circ \Delta = (id_A \otimes \epsilon) \circ \Delta = id_A$ (Co-unitality)

(With the following natural identifications $K \otimes A = A \otimes K = A$)

Definition 16. A *Hopf algebra* is a bialgebra endowed with a anti-homomorphism $S : A \to A$ such that:

- $\mu \circ (S \otimes id_A) \circ \Delta = i \circ \epsilon$
- $\mu \circ (id_A \otimes S) \circ \Delta = i \circ \epsilon$

Definition 17. A Hopf algebra is called *quasi-triangular* if there exists an invertible element $R \in A \otimes A$ such that:

- $\tau \circ \Delta(x) = R\Delta(x)R^{-1}\ \forall x \in A,$
- $(\Delta \otimes id_A)(R) = R_{13}R_{23},$
- $(id_A \otimes \Delta)(R) = R_{13}R_{12},$

where $\tau(x \otimes y) = y \otimes x$, $R_{12} = R \otimes 1$ $R_{23} = 1 \otimes R$ et $R_{13} = (\tau \otimes id_A)(1 \otimes R)$.
The element R is called *universal R-matrix.*

Proposition 18. *An R-matrix verifies the Yang-Baxter equation* $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Proof. See Prop 4.2 in [23].

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id_A)(R) \\ &= (\tau \circ \Delta \otimes id_A)(R)R_{12} \\ &= R_{23}R_{13}R_{12} \end{aligned}$$

Using the identification, $(\tau \circ \Delta \otimes id_A)(R) = (\tau \otimes id_A)(R_{13}R_{23}) = R_{23}R_{13}$. □

If $R = \sum_k \alpha_k \otimes \beta_k$ we set

$$u := \sum_k S(\beta_k)\alpha_k \in A.$$

The element u is called *Drinfeld element*.

Definition 19. A *ribbon Hopf algebra* is a quasi-triangular Hopf algebra with an element $v \in A$ such that:

- v is central in A,
- $v^2 = uS(u)$,
- $\Delta(v) = (v \otimes v)(\tau(R)R)^{-1}$,
- $S(v) = v$,
- $\epsilon(v) = 1$.

We say that v is the *ribbon element* and $g := uv^{-1}$ the *pivotal element*.

Let's display some useful properties of R, u and v:

Proposition 20. *Let $R = \sum_k \alpha_k \otimes \beta_k$ be the R-matrix of a ribbon Hopf Algebra A, we have the following properties:*

1. $(\epsilon \otimes id_A)(R) = 1 = (id_A \otimes \epsilon)(R)$,
2. $(S \otimes id_A)(R) = R^{-1} = (id_A \otimes S^{-1})(R)$,
3. $R^{-1} = \sum_k \alpha_k \otimes S^{-1}(\beta_k) = \sum_k S(\alpha_k) \otimes \beta_k$,
4. $(S \otimes S)(R) = R$,
5. $S^2(a) = uau^{-1}$, *in particular,* $S(u)u = uS(u)$,
6. $u^{-1} = \sum_k S^{-2}(\beta_k)\alpha_k = \sum_k S^{-1}(\beta_k)S(\alpha_k)$.

Proof.

1. Recall that $(\Delta \otimes id_A)(R) = R_{13}R_{23}$

$$\begin{aligned}(\epsilon \otimes id_A \otimes id_A)(\Delta \otimes id_A)(R) &= (id_A \otimes id_A)(R)\\ &= R\end{aligned}$$

$$\begin{aligned}(\epsilon \otimes id_A \otimes id_A)(R_{13}R_{23}) &= (\tau \otimes id_A)(id_A \otimes \epsilon \otimes id_A)(R_{23}R_{13})\\ &= ((1 \otimes (\epsilon \otimes id_A)(R))R)\end{aligned}$$

Hence,

$$(\epsilon \otimes id_A)(R) = 1.$$

We proceed similarly for the other identity.

2. Recall that $\mu \circ (id_A \otimes S) \circ \Delta = i \circ \epsilon$,

$$\begin{aligned}R(S \otimes id_A)(R) &= (\mu \otimes id_A)(id_A \otimes S \otimes id_A)(R_{13}R23)\\ &= (\mu \otimes id_A)(id_A \otimes S \otimes id_A)((\Delta \otimes id_A)(R))\\ &= (\mu(id_A \otimes S)\delta \otimes id_A)(R)\\ &= (\epsilon \otimes id_A)(R)\\ &= 1\end{aligned}$$

The other identity is obtained in a similar fashion.

3. Direct application of the precedent item.

4. $S \otimes S(R) = id_A \otimes S(R^{-1}) = R$.

5. We show that $S^2(a)u = ua$, recall that, using sweedler's notation,

$$(R \otimes 1)(a_{(1)} \otimes a_{(2)} \otimes a_{(3)}) = (a_{(2)} \otimes a_{(1)} \otimes a_{(3)})(R \otimes 1).$$

Hence we have:

$$\sum_k \alpha_k a_{(1)} \otimes \beta_k a_{(2)} \otimes a_{(3)} = \sum_k a_{(2)}\alpha_k \otimes a_{(1)}\beta_k \otimes a_{(3)}$$

And hence,

$$\sum_k S^2(a_{(3)})S(a_{(2)})S(\beta_k)\alpha_k a_{(1)} = \sum_k S^2(a_{(3)})S(\beta_k)S(a_{(1)})a_{(2)}\beta_k$$

For the left hand side,

$$\sum_k S^2(a_{(3)})S(a_{(2)})S(\beta_k)\alpha_k a_{(1)} = S^2(a_{(3)})S(a_{(2)})ua_{(1)}$$

Since

$$\begin{aligned} a_{(1)} \otimes a_{(2)}S(a_{(3)}) &= (id_A \otimes \mu)(id_A \otimes id_A \otimes S)(id_A \otimes \Delta)(\Delta(a)) \\ &= (id_A \otimes \epsilon)(\Delta(a)) = a \otimes 1 \end{aligned}$$

We get,

$$\begin{aligned} S^2(a_{(3)})S(a_{(2)})ua_{(1)} &= u\mu(S \otimes id_A)(\tau(a_{(1)} \otimes a_{(2)}S(a_{(3)}))) \\ &= u\mu(S \otimes id_A)(1 \otimes a) \\ &= ua \end{aligned}$$

Similarly for the right hand side we obtain $S^2(a)u$.

6. Let's show that

$$v = \mu \circ (S^{-1} \otimes id)(\tau(R^{-1})) = \sum_k S^{-2}(\beta_k)\alpha_k$$

is the inverse of u.

$$\begin{aligned} uv &= u\sum_k S^{-2}(\beta_k)\alpha_k \\ &= \sum_k \beta_k u \alpha_k \\ &= \sum_{k,i} \beta_k S(\beta_i)\alpha_i\alpha_k \\ &= \mu(S \otimes id_A)\tau(RR^{-1}) \\ &= 1 \end{aligned}$$

□

Quantum invariants

Quantum invariants can be produced via representations of a ribbon Hopf algebra. Using the elementary tangle decomposition, we want to see any tangle as a morphism of tensor products of representations of A, concatenation will be seen as a tensor product and juxtaposition as a composition.

More details and proofs, in a categorical point of view can be found in Chapter 4 and 5 of [4].

For a K-module V, we say that a morphism $\rho : A \to \mathrm{End}(V)$ is a *representation* or *module* of A.

Proposition 21. *Let A be a Hopf algebra over a ring K, V, W be two A-modules, then:*

- *K is a A-module with $A \to End(K)$, $a.s = \epsilon(a)s$ for $a \in A$ $s \in K$,*
- *V^* is a A-module with $A \to End(V^*)$, $a.f(v) = f(S(a)v)$ for $a \in A$ $f \in V^*$ $v \in V$,*
- *$V \otimes_K W$ is a A-module with $A \to End(V \otimes_R W)$, $a.(v \otimes w) = \Delta(a).(v \otimes w)$, for $a \in A$ $v \in V$ $w \in W$.*

Proof. It comes down to the fact that Δ, ϵ are morphisms and S anti-morphism of algebra. □

This allows us to define evaluation and coevaluation maps.

Definition 22. Let A be a Hopf algebra over a ring K, V finite dimensional A-module, $(e_i)_i$ a basis of V.

$$ev_V : V^* \otimes V \to K \qquad\qquad coev_V : K \to V \otimes V^*$$
$$f \otimes v \mapsto f(v) \qquad\qquad 1 \mapsto \sum_i e_i \otimes e_i^*$$

are well defined morphisms of A-modules.

Proof. Let's $a \in A$, $f \in V^*$, $v \in V$, using the definition of the antipode S one gets that

$$S(a_{(1)})a_{(2)} = \epsilon(a)1_A,$$

hence we have:

$$\begin{aligned} a.ev_V(f \otimes v) &= \epsilon(a)f(v) \\ &= f(\epsilon(a)v) \\ &= f(S(a_{(1)})a_{(2)}v) \\ &= (a_{(1)}f)(a_{(2)}v) \\ &= ev_V(a.f \otimes v) \end{aligned}$$

where we use sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Using

$$a_{(1)}S(a_{(2)}) = \epsilon(a)1_A,$$

we obtain the same way:

$$\begin{aligned} coev_V(a.1_k) &= \sum_i \epsilon(a)e_i \otimes e_i^* \\ &= \sum_i (\epsilon(a)e_i) \otimes e_i^* \\ &= \sum_i (a_{(1)}S(a_{(2)})e_i) \otimes e_i^* \end{aligned}$$

Now we set $f_i = S(a_{(2)})e_i$, the dual basis verifies that $a_{(2)}f_i^* = e_i^*$ thus

$$\begin{aligned} coev_V(a.1_k) &= \sum_i (a_{(1)}f_i) \otimes (a_{(2)}f_i^*) \\ &= a.coev_V(1) \end{aligned}$$

□

And their reverse oriented fellows.

Definition 23. Let A be a ribbon Hopf algebra over a ring K, V a finite dimensional A-module.

$$\overleftarrow{ev_V} = ev_V \circ \tau_{V,V^*} \circ (g \otimes 1)id_{V \otimes V^*} : V \otimes V^* \to K$$
$$\overleftarrow{coev_V} = (1 \otimes g^{-1})id_{V^* \otimes V} \circ \tau_{V,V^*} \circ coev_V : K \to V^* \otimes V$$

are well defined morphisms of A-modules.

Proof. We will only show that $\overleftarrow{ev_V}$ is a well defined morphism of A-modules, the $\overleftarrow{coev_V}$ case can be done in the same fashion.

Let $a \in A$, $w \in V$, $f \in V^*$.

$$\begin{aligned}
\overleftarrow{ev_V}(a.(w \otimes f)) &= ev_V \circ \tau \circ (g \otimes 1)(a.(w \otimes f)) \\
&= ev_V \circ \tau(g a_{(1)} w \otimes a_{(2)} f) \\
&= f(S(a_{(2)}) u v^{-1} a_{(1)} w) \\
&= f\left(S^{-1}\left(S(a_{(1)}) v^{-1} S(u) S^2(a_{(2)})\right) w\right) \\
&= f\left(S^{-1}(S(a_{(1)}) v^{-1} S(u) u a_{(2)} u^{-1}) w\right) \\
&= f\left(S^{-1}(S(a_{(1)}) a_{(2)} v u^{-1}) w\right) \\
&= f\left(\epsilon(a) S^{-1}(v u^{-1}) w\right) \\
&= f\left(\epsilon(a) u v^{-1} w\right) \\
&= \epsilon(a) \overleftarrow{ev_V}(w \otimes f)
\end{aligned}$$

Where the fifth equality uses property 5 in Proposition 20, the sixth uses property 6 and the definition of v, and the seventh uses the following fact:

$$S(a_{(1)}) a_{(2)} = \epsilon(a) 1_A.$$

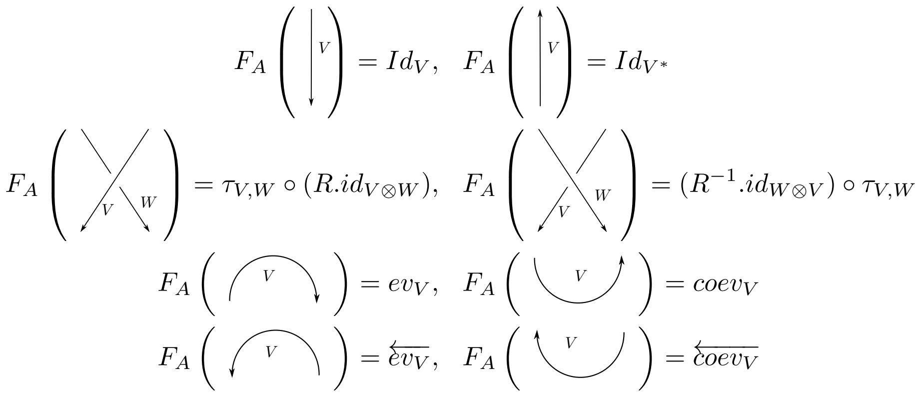
□

Definition 24. Let A be a Hopf Algebra over a ring K and T a tangle. A *color* on T is a function that assigns to each arc and circle of T a finite dimensional A-module V.

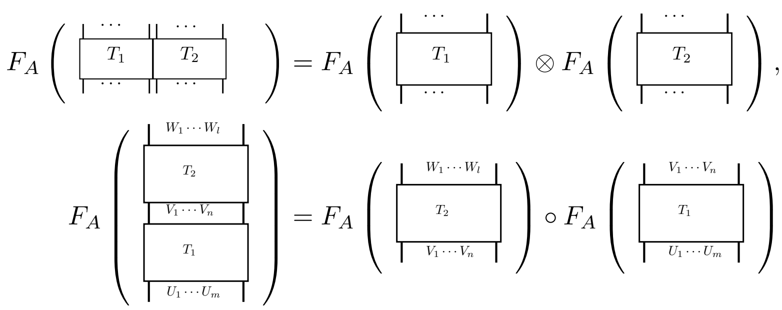
Now let us interpret elementary tangles diagrams and their operations in terms of A-module morphisms.

Theorem 25. *Let A be a ribbon Hopf Algebra over a ring K. Le F_A be the map defined on colored framed oriented tangles diagrams via:*

- *On elementary tangles diagrams:*



- *On concatenation and juxtaposition operations:*

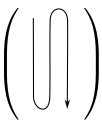


The map F_A is a well defined map on isotopy classes of colored framed oriented tangles and thus a family of invariants of framed oriented tangles indexed on the colors.

Proof. See Theorem 5.3.2 in [4].
In a nutshell, we must prove that the map is invariant via Turaev moves.

1. The first planar move (Figure 1.7) is verified by construction.

2. For the second and third planar moves (Figures 1.8 1.9) we have:

F_A 

$$= (id_V \otimes ev_V) \circ (coev_V \otimes id_V)$$
$$= \sum_i (id_V \otimes ev_V)(e_i \otimes e_i^* \otimes id_V)$$
$$= \sum_i e_i \otimes ev_V(e_i^* \otimes id_V)$$
$$= id_V$$

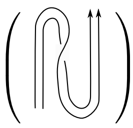
   The other ones are obtained similarly.

3. For the fourth planar move, we color the components with V, W and we set $(e_i)_i$ a basis of V and $(f_j)_j$ a basis of W, $(e_i^*)_i$, $(f_j^*)_j$ the corresponding dual basis.

   Moreover let $\alpha_{k,i,j} \in V$ and $\beta_{k,i,j} \in W$ be such that

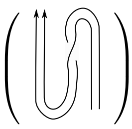
$$Re_i \otimes f_j = \sum_k \alpha_{k,i,j} \otimes \beta_{k,i,j}.$$

   Let $f \in V^*$, $g \in W^*$, we have:

F_A 

$$(f \otimes g) = (ev_v \otimes id_{W^*} \otimes id_{V^*})(id_{V^*} \otimes ev_W \otimes id_V \otimes id_{W^*} \otimes id_{V^*})$$
$$\circ (id_{V^*} \otimes id_{W^*} \otimes (\tau \circ R) \otimes id_{W^*}) \otimes id_{V^*})(id_{V^*} \otimes id_{W^*} \otimes id_V \otimes coev_W \otimes id_{V^*})$$
$$\circ (id_{V^*} \otimes id_{W^*} \otimes coev_V)(f \otimes g)$$
$$= \sum_{i,j} (ev_v \otimes id_{W^*} \otimes id_{V^*})(id_{V^*} \otimes ev_W \otimes id_V \otimes id_{W^*} \otimes id_{V^*})$$
$$\big(f \otimes g \otimes (\tau \circ R(e_i \otimes f_j)) \otimes f_j^* \otimes e_i^*\big)$$
$$= \sum_{i,j,k} (ev_v \otimes id_{W^*} \otimes id_{V^*})(id_{V^*} \otimes ev_W \otimes id_V \otimes id_{W^*} \otimes id_{V^*})$$
$$\big(f \otimes g \otimes \beta_{k,i,j} \otimes \alpha_{k,i,j} \otimes f_j^* \otimes e_i^*\big)$$
$$= \sum_{i,j,k} f(\alpha_{k,i,j}) g(\beta_{k,i,j}) f_j^* \otimes e_i^*$$

   Similarly one gets the same result for the other one:

F_A 

$$(f \otimes g) = \sum_{i,j,k} f(\alpha_{k,i,j}) g(\beta_{k,i,j}) f_j^* \otimes e_i^*$$

4. The first Reidemeister moves (Figure 1.1)

$F_A\Big($  $\Big)$

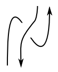
$$\begin{aligned}
&= (id_V \otimes \overleftarrow{ev_V})((\tau \circ R) \otimes id_{V^*})(id_V \otimes coev_V) \\
&= \sum_{k,i} \beta_k e_i \otimes e_i^*(uv^{-1}\alpha_k Id_V) \\
&= \sum_k \beta_k uv^{-1}\alpha_k \\
&= v^{-1}u \sum_k S^{-2}(\beta_k)\alpha_k \\
&= v^{-1}uu^{-1} \\
&= v^{-1}
\end{aligned}$$

where we use property 6 of Proposition 20 in the fourth equality, and property 5 in the fifth. In a similar fashion, we obtain:

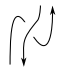
$F_A\Big($  $\Big) = v$

and hence, putting them together, the first Reidemeister move is verified.

5. For the second Reidemeister moves (Figure 1.2), the move described in Figure 1.2a are verified simply because $RR^{-1} = 1$, the other one can be obtained in the following way:

$F_A\Big($  $\Big) = F_A\Big($  $\Big) \circ F_A\Big($  $\Big)$

And we have:

$F_A\Big($  $\Big)(f \otimes w)$

$$\begin{aligned}
&= (ev_V \otimes id_W \otimes id_{V^*})(id_{V^*} \otimes (\tau \circ R) \otimes id_{V^*})(i_d V^* \otimes id_W \otimes coev_V)(f \otimes w) \\
&= \sum_{k,i} f(\beta_k e_i)\alpha_k w \otimes e_i^* \\
&= \sum_k \alpha_k w \otimes S^{-1}(\beta_k) f \\
&= \sum_k (\alpha_k \otimes S^{-1}(\beta_k))(w \otimes f) \\
&= R^{-1}.\tau(f \otimes w)
\end{aligned}$$

$F_A\Big($  $\Big)(w \otimes f)$

$$\begin{aligned}
&= (id_{V^*} \otimes id_W \overleftarrow{ev_V})(id_{V^*} \otimes (R^{-1} \circ \tau) \otimes id_{V^*})(\overleftarrow{coev_V} \otimes id_W \otimes id_{V^*})(w \otimes f) \\
&= \sum_{k,i} f(g\beta_k g^{-1} e_i) e_i^* \otimes S(\alpha_k) w \\
&= \sum_k S^{-1}(g\beta_k g^{-1}) f \otimes S(\alpha_k) w \\
&= \sum_k (S(\beta_k) f \otimes S(\alpha_k) w) \\
&= \tau(R.(w \otimes f))
\end{aligned}$$

where the fourth equality is obtained using property 6 and the fifth is obtained using property 4. We also used the identity of property 3 in order to describe R^{-1}.

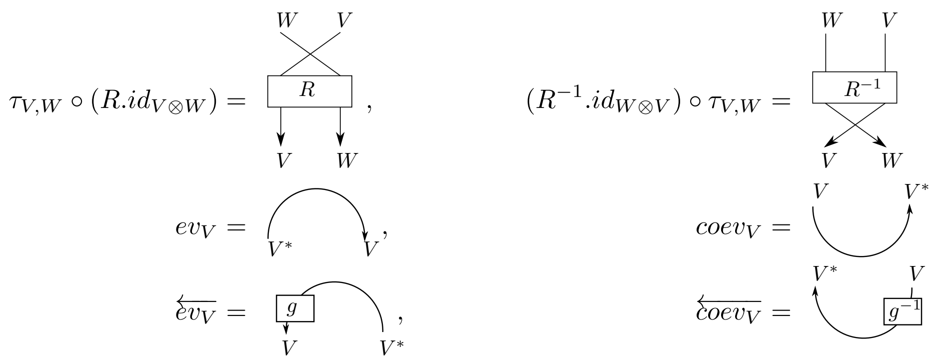
Putting them together, the second Reidemeister move is verified.

6. The third Reidemeister move is just the Yang Baxter equation verified by the R-matrix and described in Proposition 18.

□

Corollary 26. *Let L be a framed oriented link with l components, $L(V_1,\dots,V_l)$ the link L colored with $V_1,\dots,V_l$, then $F_{A;V_1,\dots,V_l}(L) := F_A(L(V_1,\dots,V_l)) \in K$ is a link invariant.*

We can use graphical interpretation of the multiplication by an element of A, the evaluations and coevaluations maps and R-matrix. Now that we have our quantum invariants via elementary tangles, this graphical interpretation will behave the same way as tangles do.



Remark 27. Why did we need all the ribbon Hopf algebra structure?

1. The Hopf algebra structure was necessary to have the tensor product, the duality and the unit K respect the A-module structure.

2. Since the R-matrix satisfies the Yang-Baxter equation, it allows F_A to be invariant under the third Reidemeister move 1.3.

3. If we didn't put the pivotal element g in the definition of $\overleftarrow{ev}_V$ and $\overleftarrow{coev}_V$, the first Reidemeister move 1.1 would not give Id_V but $uS(u)Id_V$.

The $(1,1)$ tangle version

As we will see for the ADO polynomial, we sometime need to open the link at one place and see it as a $(1,1)$-tangle.

Let V a finite dimensional A-module, and T a $(1,1)$-tangle. Let $T(V)$ be the tangle T where the arc is colored by V and the circles by whatever finite dimensional module, then we have:

$$F_A(T(V)) \in \mathrm{End}_A(V).$$

Definition 28. Let V be a A-module, we say that V is *absolutely simple* if

$$\mathrm{End}_A(V) = K.Id_V.$$

Hence if V is a finite dimensional absolutely simple module, $F_A(T(V)) \in K$ is a link invariant. In order to get back the usual quantum invariant, we must take the quantum trace.

Definition 29. We define

$$\begin{aligned} qtr :& \mathrm{End}(V) \to K \\ & f \mapsto \overleftarrow{ev}_V \circ (f \otimes Id_{V^*}) \circ coev_V(1) \end{aligned}$$

We call $qdim(V) := qtr(Id_V)$.

Proposition 30. *Let V be an A-module and T a $(1,1)$-tangle whose closure is L a link, with the open arc colored by V.*

$$F_A(L(V)) = qtr(F_A(T(V)).$$

Moreover if V is absolutely simple,

$$F_A(L(V)) = F_A(T(V)) \times qdim(V).$$

Proof. Straightforward from the definition of the quantum invariant. □

1.2 The colored Jones and the ADO families from quantum group

1.2.1 The colored Jones setup

The usual algebra

The two families of polynomial invariants that we study come from different versions of quantum $\mathfrak{sl}_2$. The idea is to deform the universal enveloping algebra of $\mathfrak{sl}_2$ by a quantum parameter q. Let's describe first the usual Hopf algebra $U_q(\mathfrak{sl}_2)$.

Definition 31. We set $U_q(\mathfrak{sl}_2)$ the $\mathbb{Q}(q)$-algebra generated by $K^{\pm 1}, E, F$ and relations

$$KE = q^2 EK, \; FK = q^2 KF, \; [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

It is endowed with an Hopf algebra structure:

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K & \epsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = K^{-1} \otimes F + F \otimes 1 & \epsilon(F) = 0 & S(F) = -KF \\ \Delta(K) = K \otimes K & \epsilon(K) = 1 & S(K) = K^{-1} \end{array}$$

We can study its representations.

For any variable q, we denote $\{n\}_q = q^n - q^{-n}$, $[n]_q = \frac{\{n\}_q}{\{1\}_q}$, $\{n\}_q! = \prod_{i=1}^n \{i\}_q$, $[n]_q! = \prod_{i=1}^n [i]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]!}$ with convention $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $n < 0$.

For the sake of simplicity, in $\mathbb{Q}(q)[A]$ we will denote $q^\alpha := A$ and use previous notation for quantum numbers. Keep in mind that, here, α is just a notation, not a complex number.
We denote $\{\alpha\}_q = q^\alpha - q^{-\alpha}$, $\{\alpha + k\}_q = q^{\alpha+k} - q^{-\alpha-k}$, $\{\alpha; n\}_q = \prod_{i=0}^{n-1} \{\alpha - i\}_q$.
We use the notation $q^{n\alpha+k} := A^n q^k$.

A *highest weight module* is a $U_q(\mathfrak{sl}_2)$-module generated by an element v such that $Ev = 0$ and $Kv = \lambda v$.

In the infinite dimensional case, we have the Verma modules.

Definition 32. We call *Verma module* V^α the $\mathbb{Q}(q)[A]$-module freely generated by $(v_0, v_1, \dots)$ endowed with the $U_q(\mathfrak{sl}_2)$-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Kv_i = q^{\alpha - 2i} v_i, \quad Fv_i = [i+1]_q [\alpha - i]_q v_{i+1}$$

In the finite dimensional case, we have:

Definition 33. The module S_n is the $(n+1)$-dimensional $\mathbb{Q}(q)$-module freely generated by $(v_0, \dots, v_n)$ and endowed with the $U_q(\mathfrak{sl}_2)$-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Kv_i = q^{n-2i} v_i, \quad Fv_i = [i+1]_q [n-i]_q v_{i+1}$$

Proposition 34. *The module S_n is simple and $S_n \subset V^n$ where V^n is the Verma module setting $A = q^n$.*

Proof. For the simplicity, see VI.3 in [18]. □

We would now like to endow $U_q(\mathfrak{sl}_2)$ with a ribbon structure, hence with an R-matrix and a ribbon element. There are, nonetheless, some issues that need to be addressed.

Let us define an R-matrix:

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{\{1\}^n q^{\frac{n(n-1)}{2}}}{[n]!} E^n \otimes F^n$$

$$R^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n \{1\}^n q^{\frac{-n(n-1)}{2}}}{[n]!} E^n \otimes F^n q^{-\frac{H\otimes H}{2}}$$

Altogether with a pivotal element compatible with the braiding: K.

The R-matrix is not a well defined element of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$. First of all, the sum ranges to infinity. Moreover, the element $q^{\frac{H\otimes H}{2}}$ is undefined. We can nonetheless see it as an operator onto the finite dimensional modules S_n:

$$\begin{aligned} q^{\frac{H\otimes H}{2}} : & S_n \otimes S_m \to S_n \otimes S_m \\ & v_i \otimes v_j \mapsto q^{\frac{(n-2i)(m-2j)}{2}} v_i \otimes v_j \end{aligned}$$

Since the action of E is nilpotent on S_n and S_m, the infinite sum's action is finite. Hence the operator $R : S_n \otimes S_m \to S_n \otimes S_m$ is well defined, and we can compute a link invariant.

Definition 35. Let L be a link with l component, $n_1, \dots, n_l \in \mathbb{N}$, we call $(n_1, \dots, n_l)$ *colored Jones polynomial* the link invariant $F_{U_q(\mathfrak{sl}_2);S_{n_1},\dots,S_{n_l}}(L) \in \mathbb{Q}[q^{\pm 1}]$.
Setting $\overline{n} = (n_1, \dots, n_l)$, we denote it $J^c_{\overline{n}}(q, L)$.

The h-adic version

The algebra $U_q(\mathfrak{sl}_2)$ is not a ribbon algebra, we bypassed this issue by looking only at the action on finite dimensional modules S_n which allow us to produce the colored Jones polynomials. Another way is to complete the algebra in order to make sense of the R-matrix.

Definition 36. We set $U_h(\mathfrak{sl}_2)$ the $\mathbb{Q}[[h]]$-algebra topologically generated by H, E, F and relations

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

where $q = e^h$ and $K = q^H = e^{hH}$.

We will sometime denote it U_h for the sake of simplicity.

It is endowed with an topological Hopf algebra structure:

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K & \epsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = K^{-1} \otimes F + F \otimes 1 & \epsilon(F) = 0 & S(F) = -KF \\ \Delta(H) = 1 \otimes H + H \otimes 1 & \epsilon(H) = 0 & S(H) = -H \end{array}$$

Remark 37. We use the term topological because the co-product has values in $U_h \hat{\otimes} U_h$, the h-adic completion of the tensor product.

It is a ribbon complete Hopf algebra with

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{\{1\}^n q^{\frac{n(n-1)}{2}}}{[n]!} E^n \otimes F^n$$

$$R^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n \{1\}^n q^{\frac{-n(n-1)}{2}}}{[n]!} E^n \otimes F^n q^{-\frac{H\otimes H}{2}}$$

Altogether with a pivotal element compatible with the braiding: K.

Remark 38. Since the family $(e^{nh})_{n\in\mathbb{Z}}$ is free in $\mathbb{Q}[[h]]$, setting $q = e^h$, we have $\mathbb{Z}[q^{\pm 1}] \subset \mathbb{Q}[[h]]$.
Setting $q = e^h$ and $K = e^{hH}$, $U_q(\mathfrak{sl}_2) \subset U_h(\mathfrak{sl}_2)$

Now let us present some representations: In the infinite dimensional case, we have the topological Verma modules.

Definition 39. We call *topological Verma module* V_h^α the $\mathbb{Q}[[h]][\alpha]$-module topologically generated by $(v_0, v_1, \dots)$ endowed with the $U_h(\mathfrak{sl}_2)$-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Hv_i = (\alpha - 2i)v_i, \quad Fv_i = [i+1]_q[\alpha - i]_q v_{i+1}$$

In the finite dimensional case, we have:

Definition 40. The module S_n^h is the $(n+1)$-dimensional $\mathbb{Q}[[h]]$-module freely generated by $(v_0, \dots, v_n)$ and endowed with the $U_h(\mathfrak{sl}_2)$-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Hv_i = (n - 2i)v_i, \quad Fv_i = [i+1]_q[n - i]_q v_{i+1}$$

Proposition 41. *When we restrict them to $U_q(\mathfrak{sl}_2)$-modules, we have:*

$$S_n^h \cong S_n \otimes \mathbb{Q}[[h]].$$

Proof. The proof is straightforward, the modules are finite, freely generated and have the same $U_q(\mathfrak{sl}_2)$ module structure. □

Corollary 42. *Let L be a link with l component and $\overline{n} = (n_1, \dots, n_l)$, $J^c_{\overline{n}}(q, L) = F_{U_h(\mathfrak{sl}_2);S^h_{n_1},\dots,S^h_{n_l}}(L)$.*

The open version

We can also look a the version of the colored Jones polynomial with an open component.

Proposition 43. *The modules S_n are absolutely simple*

Proof. Let $f \in End_{U_q(\mathfrak{sl}_2)}(S_n)$.
$Kf(v_i) = f(Kv_i) = q^{n-2i}f(v_i)$, thus $\exists\lambda_i \in \mathbb{Q}(q)$ such that $f(v_i) = \lambda_i v_i$.
Now since $Ef(v_{i+1}) = f(Ev_{i+1}) = f(v_i)$, then $\lambda_{i+1}v_i = \lambda_i v_i$ hence we define $\lambda := \lambda_i$ and we have $f = \lambda Id_{S_n}$. □

Let T a $(1,1)$-tangle whose closure is L, colored with $S_{n_1}, \dots, S_{n_l}$, and S_{n_1} on the open component we set $J^{(n_1)}_{\overline{n}}(q, L)Id_{S_{n_1}} = F_{U_h(\mathfrak{sl}_2);S_{n_1},\dots,S_{n_l}}(T) \in \mathbb{Q}(q)$.

Remark 44. In order to get back the colored Jones polynomial, we take the quantum trace. Since the operator is scalar, we get a scalar multiplication by the quantum dimension :

$$J^c_{\overline{n}}(q, L) = [n_1 + 1]_q J^{(n_1)}_{\overline{n}}(q, L).$$

The knot case

In the case of a knot $\mathcal{K}$ we can use the open version.

Definition 45. If $\mathcal{K}$ is a knot colored by S_n,

$$J_n(q, \mathcal{K}) := J^{(n)}_n(q, \mathcal{K}).$$

1.2.2 The ADO setup

Now the goal is to take a version of quantum $\mathfrak{sl}_2$ at roots of unity, in order to get more finite dimensional module and thus more quantum invariants.

The algebra and modules

Definition 46. Let $q = e^{\frac{i\pi}{r}} = \zeta_{2r}$ be a root of unity.
We work at roots of unity with $U_{\zeta_{2r}}(\mathfrak{sl}_2)$ defined as a $\mathbb{Q}(\zeta_{2r})$ algebra:
Generators:

$$E, F, K, K^{-1}$$

Relations:

$$KK^{-1} = K^{-1}K = 1 \quad KE = \zeta_{2r}^2 EK \quad KF = \zeta_{2r}^{-2} FK \quad [E, F] = \tfrac{K-K^{-1}}{\zeta_{2r}-\zeta_{2r}^{-1}}$$

It has a Hopf algebra structure:

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K & \epsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = K^{-1} \otimes F + F \otimes 1 & \epsilon(F) = 0 & S(F) = -KF \\ \Delta(K) = K \otimes K & \epsilon(K) = 1 & S(K) = K^{-1} \\ \Delta(K^{-1}) = K^{-1} \otimes K^{-1} & \epsilon(K^{-1}) = 1 & S(K^{-1}) = K \end{array}$$

Evaluating at root of unity allows us to have more finite dimensional modules:

Definition 47. We call V_α the $\mathbb{Q}(\zeta_{2r})[A]$-module freely generated by $(v_0, \ldots, v_{r-1})$ endowed with the $U_{\zeta_{2r}}(\mathfrak{sl}_2)$-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Kv_i = \zeta_{2r}^{\alpha-2i} v_i, \quad Fv_i = [i+1]_{\zeta_{2r}} [\alpha - i]_{\zeta_{2r}} v_{i+1}$$

where we set $\zeta_{2r}^{\alpha} := A$ as a notation.

Proposition 48. *The module V_α is absolutely simple.*

Proof. Let $f \in End_{U_q(\mathfrak{sl}_2)}(V_\alpha)$.
$Kf(v_i) = f(Kv_i) = q^{\alpha-2i} f(v_i)$, thus $\exists \lambda_i \in \mathbb{Q}(q)$ such that $f(v_i) = \lambda_i v_i$.
Now since $Ef(v_{i+1}) = f(Ev_{i+1}) = f(v_i)$, then $\lambda_{i+1} v_i = \lambda_i v_i$ hence we define $\lambda := \lambda_i$ and we have $f = \lambda Id_{V_\alpha}$. □

The idea is to compute quantum invariant from this module, we will hence get a polynomial in $\mathbb{Q}(\zeta_{2r})[A]$. As opposed to the colored Jones polynomial where the free parameter q comes from the algebra, here the free parameter A will be coming from the module.

As we did before, we can equip this module with a ribbon structure using the R-matrix operator:

$$R_r = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{r-1} \zeta_{2r}^{\frac{n(n-1)}{2}} \frac{\{1\}_q^n F^n}{[n]_q!} E^n \otimes F^n$$

$$R_r^{-1} = (\sum_{n=0}^{r-1} (-1)^n \zeta_{2r}^{-\frac{n(n-1)}{2}} \frac{\{1\}_q^n F^n}{[n]_q!} E^n \otimes F^n) q^{-\frac{H \otimes H}{2}}.$$

And a pivotal element compatible with the braiding : K^{1-r}.

Remark 49. There is still a problem in the definition of the operator $q^{\frac{H \otimes H}{2}}$:

$$\begin{aligned} q^{\frac{H \otimes H}{2}} : & V_\alpha \otimes V_\beta \to q^{\frac{\alpha\beta}{2}} V_\alpha \otimes V_\beta \\ & v_i \otimes v_j \mapsto q^{\frac{(\alpha-2i)(\beta-2j)}{2}} v_i \otimes v_j \end{aligned}$$

We can see $q^{\frac{\alpha\beta}{2}}$ as an independent variable.

The ADO invariants

Now we would like to compute the quantum invariant associated to V_α.

Remark 50. The issues is that $qdim(V_\alpha) = [r]_{\zeta_{2r}} = 0$ hence if L is a link and T a $(1,1)$ tangle whose closure is L:

$$F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(L(V_\alpha)) = F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(T(V_\alpha))qdim(V_\alpha) = 0.$$

In order to have a non zero link invariant, we need to multiply $F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(T(V_\alpha))$ by a modified quantum dimension that is non zero and make it so the invariant does not depend which strand is cut.

Proposition 51. *If L a link and T is any $(1,1)$-tangle whose closure is L such that the open component is colored with V_α. We say that $d(V_\alpha) = \frac{\{\alpha\}}{\{r\alpha\}}$ is the modified quantum dimension of V_α.*
Then

$$ADO_r^c(L) := d(V_\alpha)F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(T(V_\alpha))$$

is a link invariant such that if T' is another $(1,1)$-tangle with the open component colored by V_β then

$$ADO_r^c(L) = d(V_\beta)F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(T'(V_\beta)).$$

Proof. For more details and proof see [5]. □

Definition 52. Let L be a link with l components with linking matrix L_k and colored with $(V_{\alpha_i})_{1\leq i\leq l}$. Set $\overline{\alpha} = (\alpha_1, \dots, \alpha_l)$.

$$ADO_r^c(L) \in q^{\frac{\overline{\alpha}^T L_k \overline{\alpha}}{2}} \times \mathbb{Q}(\zeta_{2r})[q^{\pm\alpha_1}, \dots, q^{\pm\alpha_l}]$$

is called *ADO polynomial* or *coloured Alexander polynomial* of the link L.

The knot case

In the case of a knot $\mathcal{K}$, since it doesn't matter where the strand is open, we can use the open version.

Definition 53. We define

$$ADO_r(A, \mathcal{K}) := F_{U_{\zeta_{2r}}(\mathfrak{sl}_2)}(T(V_\alpha))$$

where T is any $(1,1)$-tangle whose closure is $\mathcal{K}$.

Corollary 54. *We have $ADO_r(A, \mathcal{K}) \in q^{\frac{f\alpha^2}{2}}\mathbb{Q}(\zeta_{2r})[q^{\pm\alpha}]$ where f is the framing of the knot.*

1.2.3 An "integral" version

One major issue in order to unify the two families for knots is that we need to take values in a ring with two variables q and A along with evaluation maps at roots of unity for q and at q^n for A. We will also need to complete this ring for a family of ideals. This can be done using a ring such as $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$.

Definition 55. Let $\mathcal{U} := U_q^D(\mathfrak{sl}_2)$ the $\mathbb{Z}[q^{\pm 1}]$ algebra generated by $K^{\pm 1}, E, F^{(n)}$ where

$$F^{(n)} = \frac{\{1\}_q^n F^n}{[n]_q!}.$$

Since setting $q = e^h$, $\mathcal{U} \subset U_h$, it is endowed with the Hopf algebra structure coming from U_h. In particular:

$$\Delta(F^{(n)}) = \sum_{k=0}^{n} q^{k(n-k)}F^{(k)}K^{k-n} \otimes F^{(n-k)}, \quad S(F^{(n)}) = (-1)^n q^{-n(n-1)}F^{(n)}K^n$$

Remark 56. Inspired by Lusztig quantum algebra of $\mathfrak{sl}_2$ where both generators E and F are taken with divided powers, this version only take divided powers of F and have coefficients in an "integral" ring $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$. This is enough to define the leading term of the R-matrix.

We can rewrite the R-matrix:

$$R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$$

We can now look at the action of these "integral" algebra on the finite dimensional modules S_N generated by $(v_0, \dots, v_N)$:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Kv_i = q^{N-2i} v_i, \quad F^{(n)} v_i = \begin{bmatrix} n+i \\ n \end{bmatrix}_q \{N-i; n\}_q v_{i+1}$$

Hence S_N can be seen as a $\mathbb{Z}[q^{\pm 1}]$-module endowed with a $\mathcal{U}$-action.

Corollary 57. *If $\mathcal{K}$ is a knot colored by S_n:*

$$J_n(q, \mathcal{K}) \in \mathbb{Z}[q^{\pm 1}].$$

We can proceed similarly with the ADO setup and we get:

Corollary 58. *If $\mathcal{K}$ is a knot colored by V_α with framing f:*

$$ADO_r(A, \mathcal{K}) \in q^{\frac{f\alpha^2}{2}} \mathbb{Z}[\zeta_{2r}][q^{\pm \alpha}].$$

1.3 A classical invariant: the Alexander polynomial

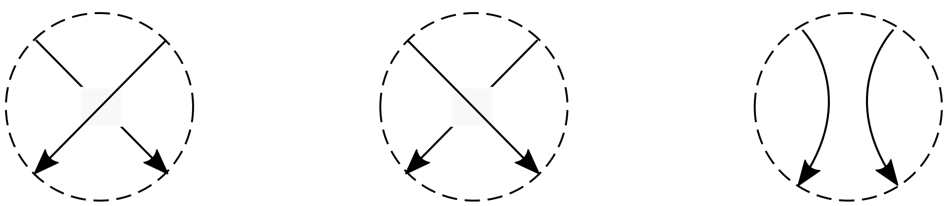
Although quantum invariants are our main study, we will need this classical invariant fellow as it will appear in the factorisation at root of unity of the - yet to be constructed- unified invariant.

1.3.1 The Alexander polynomial

We can define a polynomial in $\mathbb{Z}[t^{\pm 1}]$ using inductive relations. Let L, L' be two isotopic unframed oriented links, there exist a unique polynomial $A_L(t^2)$ satisfying:

$$A_L(t^2) = A_{L'}(t^2)$$
$$A_{L_+}(t^2) - A_{L_-}(t^2) = (t - t^{-1}) A_{L_0}(t^2)$$
$$A_U(t^2) = 1$$

where U is the unknot, $L_{+,-,0}$ are defined in Figure 1.11.



(a) L_+ (b) L_- (c) L_0

Figure 1.11: Local changes for a link L

Theorem 59. *The function $L \mapsto A_L(t^2)$ exists and is unique, we call $A_L(t^2)$ the Alexander polynomial of the link L.*

Proof. See section 4 of [26]. □

Proposition 60. *Let L be a link and U the unknot,*

$$A_{L\cup U}(t^2) = 0$$

Proof. Let L^+, L^- be the diagram of the link L with framing $+1$ or -1,

$$A_{L^+}(t^2) - A_{L^-}(t^2) = A_{L\cup U}(t^2)$$

Since, by reidemeister 1 move for unframed link, $A_{L^+}(t^2) = A_{L^-}(t^2)$, we have the desired identity. □

1.3.2 The multivariable Alexander polynomial

We can be more general and define an unframed oriented link invariant with value in $\mathbb{Q}(t_1,\dots,t_c)$ where c is the number of components of the link.

Let $L = L_1 \cup \dots \cup L_c$ an unframed oriented link in S^3, we define an element of $\mathbb{Q}(t_1,\dots,t_c)$ as follows:

$$\begin{aligned}
A_L(t_1^2,\dots,t_c^2) &= A_{L'}(t_1^2,\dots,t_c^2)\\
A_U(t^2) &= \frac{1}{t-t^{-1}}\\
A_L(t_1^2,\dots,t_c^2) &\in \mathbb{Z}[t_1,\dots,t_c] \ \text{ if } c \geq 2\\
A_{L_+}(t^2,\dots,t^2) - A_{L_-}(t^2,\dots,t^2) &= (t-t^{-1})A_{L_0}(t^2,\dots,t^2)\\
A_C(t_1^2,\dots,t_c^2) &= (T_i + T_i^{-1})A_L(t_1,\dots,t_{i-1}^2,t_i^4,t_{i+1}^2,\dots t_c^2)
\end{aligned}$$

where

1. L and L' are isotopic,
2. U is the unknot,
3. L_+, L_-, L_0 are defined as in Figure 1.11,
4. C is obtained from L by $(2,1)$-cabling the component L_i,
5. $T_i = t_i \prod_{j\neq i} t_j^{lk(L_i,L_j)}$

Theorem 61. *The function $L \mapsto A_L(t_1^2,\dots,t_c^2)$ exists and is unique, we call $A_L(t_1^2,\dots,t_c^2)$ the multivariable Alexander polynomial of the link L.*

Proof. See section 4 of [26]. □

Corollary 62. *Let L be an unframed oriented link in S^3,*

$$A_L(t^2) = (t-t^{-1})A_L(t^2,\dots,t^2).$$

Chapter 2

Unified ADO and colored Jones invariant of a knot

This chapter will be dedicated to construct a two variables invariant unifying both the colored Jones polynomials and the ADO polynomials. The first section seeks to generalise a formula of the ADO polynomials using state diagrams with the truncated R-matrix to a formula with untruncated R-matrix. In order to do so we construct ring completions similar as Habiro's where quantum factorials become small and the R-matrix contributions converge. But this new element is not a knot invariant, a priori.

The second section will prove the knot invariance. To do so, we use the universal quantum invariant and an infinite Verma type module in two variables in order to recover the unifying element and prove that it is a knot invariant. Furthermore, we show that the colored Jones polynomials can be recovered from it the same way it can be recovered from the universal quantum invariant.

2.1 Unifying the ADO polynomials of a knot

The approach here will be to unify the invariants: using completions of rings and algebras, we will explicit an integral invariant in some variable q that can be evaluated at any root of unity, recovering ADO invariants defined previously.

The first subsection introduces states diagram and how to see the ADO invariants in terms of states diagrams.

The second subsection will create a completion of the ring of integral Laurent polynomials in two variables q, A, where we can define a *unified* element $F_\infty(q, A, D)$ using state diagrams with the non truncated R-matrix. Note that at this point, the defined element is not a knot invariant, as it *a priori* depends on the diagram D of the knot.

The third subsection will make the bridge between the first two sections. By evaluating the unified element at roots of unity ζ_{2r} with $r \in \mathbb{N}^*$, we factor out the ADO invariant. We will then explicit a map sending the unified form of a knot to the corresponding ADO invariants. This will show that the ADO invariants are contained in the unified element and that we can recover them from it.

2.1.1 The ADO polynomial via state diagram

In order to calculate a useful form of this invariant one may look at *state diagram* of a knot. For any knot seen as a $(1,1)$ tangle, take a diagram D, label the top and bottom strands 0 and starting from the bottom strand, label the strand after the k-th crossing encountered with the rule described in Figure 2.1. The resulting diagram is called a *state diagram* of D.

Let $\mathcal{K}$ be a knot and D a diagram of the knot seen as a $(1,1)$ tangle. Suppose the diagram has N crossings. We denote $\zeta_{2r}^\alpha := q^\alpha = A$.

(a) Positive crossing. (b) Negative crossing.

Figure 2.1: The two possibilities for the k-th crossing in D.

Now for any state diagram of D we can associate an element:

$$\begin{aligned} D_r(i_1,\dots,i_N) =& \left(\prod_{j=1}^{S} \zeta_{2r}^{\pm(r-1)(\alpha-2\epsilon_j)}\right) \prod_{k\in pos} \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - a_k; i_k\}_{\zeta_{2r}} \\ & \times \zeta_{2r}^{-(a_k+b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in neg} (-1)^{i_k} \zeta_{2r}^{-\frac{i_k(i_k-1)}{2}} \\ & \times \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{(a_k+b_k)\alpha} \zeta_{2r}^{-2a_k b_k} \end{aligned}$$

where $neg \cup pos = [|1,N|]$ and $k \in pos$ if the k-th crossing of D is positive, else $k \in neg$. a_k, b_k are the strands labels at the k-th crossing of the state diagram (see Figure 2.1), S is the number of $\smile$ + $\frown$ appearing in the diagram, and ϵ_j the strand label at the j-th $\smile$ or $\frown$, the $\pm$ sign is positive for $\smile$ and negative for $\frown$.

Remark 63. Note that the a_k and b_k appearing are defined in terms of i_j. You can find some examples of state diagrams in Appendix A.

Proposition 64. *If D is a diagram of $\mathcal{K}$ seen as a a 1-1 tangle we have:*

$$\begin{aligned} ADO_r(A,\mathcal{K}) &= q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{r-1} D_r(i_1,\dots,i_N) \\ &= q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{r-1} \left(\prod_{j=1}^{S} \zeta_{2r}^{\pm(r-1)(\alpha-2\epsilon_j)}\right) \prod_{k\in pos} \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \\ & \times \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{-(a_k+b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in neg} (-1)^{i_k} \\ & \times \zeta_{2r}^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{(a_k+b_k)\alpha} \zeta_{2r}^{-2a_k b_k} \end{aligned}$$

where $\bar{i} = (i_1,\dots,i_N)$, N is the number of crossings, S the number of $\smile$ + $\frown$ and f is the framing of the knot.

Proof. Notice that $q^{\frac{f\alpha^2}{2}} D_r(i_1,\dots,i_N)$ is the element obtained by adding to the k-th crossing a coupon labeled with: $q^{\frac{H\otimes H}{2}} q^{\frac{i_k(i_k-1)}{2}} E^{i_k} \otimes F^{(i_k)}$ if positive and $q^{\frac{-H\otimes H}{2}} q^{\frac{-i_k(i_k-1)}{2}} E^{i_k} \otimes F^{(i_k)}$ if negative. Then add a coupon to $\smile$ labeled K^{r-1} and $\frown$ labeled K^{1-r}. We get an element of $U_{\zeta_{2r}}(\mathfrak{sl}_2)$, its action on $v_0 \in V_{\alpha+r-1}$ the highest weight vector gives the element $q^{\frac{f\alpha^2}{2}} D_r(i_1,\dots,i_N)$. Summing them over i_k for all k gives the ADO polynomial. □

2.1.2 Ring completion and unified element

Let's lay the groundwork for an unified form to exist. It must be a ring in which infinite sums previously mentioned converge.

Let $R = \mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$, we will construct a completion of that ring. For the sake of simplicity, we will denote $q^\alpha := A$ and use previous notation for quantum numbers.

Definition 65. Let I_n be the ideal of R generated by the following set $\{\{\alpha + l; n\}_q,\ l \in \mathbb{Z}\}$.

Lemma 66. *I_n is generated by elements of the form $\{n; i\}\{\alpha; n-i\}$, $i \in \{0, \dots n\}$.*

Proof. The proof can be found at Proposition 5.1 in [15], replacing K (resp. K^{-1}) by q^α (resp. $q^{-\alpha}$), one gets the proof of this lemma.

In a nutshell, one can prove by induction that :

$$\{\alpha + m; n\}_q = \sum_{i=0}^{n} q^{(n-i)m} q^{-i\alpha} \{n; i\}_q \{\alpha; n-i\}_q$$

Moreover, for any fixed n, one can also prove by induction on i that $\{n; i\}_q\{\alpha; n-i\}_q \in I_n$ using the previous formula. □

We then have a projective system :

$$\hat{I} : I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$$

We can define the completion of R, taking the projective limit:

Definition 67. Let $\widehat{\mathbb{Z}[q, q^\alpha]} = \lim_{\substack{\leftarrow \\ n}} \dfrac{R}{I_n} = \{(a_n)_{n\in\mathbb{N}^*} \in \prod_{i=1}^{\infty} \frac{R}{I_n} \mid \rho_n(a_{n+1}) = a_n\}$ where $\rho_n : \frac{R}{I_{n+1}} \to \frac{R}{I_n}$ is the projection map.

This completion is a bigger ring containing R:

Proposition 68. *The canonical projection map induces an injective map $R \hookrightarrow \widehat{\mathbb{Z}[q, q^\alpha]}$*

Proof. It is sufficient to prove that $\bigcap_{n\in\mathbb{N}^*} I_n = \{0\}$.

Since $R = \mathbb{Z}[q^{\pm 1}][A^{\pm 1}]$, it is a Laurent polynomial ring. Let's define $\deg_q(x)$, $val_q(x)$ the degree and valuation of x in the variable q.

Let $f_k : \mathbb{Z}[q^{\pm 1}, A^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$, $A \mapsto q^k$. We have $f_k(I_n) \subset \{n\}!\mathbb{Z}[q^{\pm 1}]$ because I_n is generated by elements of the form $\{n; i\}\{\alpha; n-i\}$ that maps to $\{n; i\}\{k; n-i\}$, which is divisible by $\{n\}!$.

Hence if $x \in \bigcap_{n\in\mathbb{N}^*} I_n$, $f_k(x) \in \{n\}!\mathbb{Z}[q^{\pm 1}]$ for all n, since $\mathbb{Z}[q^{\pm 1}]$ is factorial, so $f_k(x) = 0$ for all k.

Take $x \in \bigcap_{n\in\mathbb{N}^*} I_n$, written $x = \sum a_n A^n$ with $a_n \in \mathbb{Z}[q^{\pm 1}]$. Take N such that $\deg_q(x) < N$ and $val_q(x) > -N$.

This implies that $deg_q(a_n) < N$ and $val_q(a_n) > -N$ (since it is the case for x and any higher or lower terms could not compensate since the power of A is different before each a_n).

Thus since $f_{2N}(x) = 0$, $\sum a_n q^{2Nn} = 0$, we have $\deg_q(a_n q^{2Nn}) < N(1+2n)$ and $val_q(a_n q^{2Nn}) > N(2n-1)$, then all the terms $a_n q^{2Nn}$ must be 0. Hence $a_n = 0$ for all n, meaning that $x = 0$. □

Remark 69. If $b_0 \in R$ and $b_n \in I_{n-1}$ for $n \geq 1$, the partial sums $\sum_{n=0}^{N} b_n$ converges in $\widehat{\mathbb{Z}[q, q^\alpha]}$ as N goes to infinity.

We denote the limit $\sum_{n=0}^{+\infty} b_n := (\overline{\sum_{n=0}^{N} b_n})_{N\in\mathbb{N}^*}$.

Conversely, if $a = (\overline{a_N})_{N\in\mathbb{N}^*} \in \widehat{\mathbb{Z}[q, q^\alpha]}$, let $a_n \in R$ be any representative of $\overline{a_n}$ in R, then $a = \sum_{n=0}^{+\infty} b_n$ where $b_0 = a_1$ and $b_n = a_{n+1} - a_n$ for $n \in \mathbb{N}^*$.

We proceed similarly as in the paragraph preceding Proposition 64. Let $\mathcal{K}$ be a knot seen as a $(1,1)$ tangle and D a diagram of it. For a state diagram of D we define:

$$
\begin{aligned}
D(i_1,\dots,i_N) =& (\prod_{j=1}^{S} q^{\mp(\alpha-2\epsilon_j)}) \prod_{k\in pos} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \{\alpha-a_k;i_k\}_q \\
&\times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in neg} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \\
&\times \{\alpha-a_k;i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_k b_k}
\end{aligned}
$$

where $neg \cup pos = [|1,N|]$ and $k\in pos$ if the k-th crossing of D is positive, else $k\in neg$. a_k, b_k are the strands labels at the k-th crossing of the state diagram (see Figure 2.1), S is the number of $\overleftarrow{\cup}$ + $\overleftarrow{\cap}$ appearing in the diagram, and ϵ_j the strand label at the j-th $\overleftarrow{\cup}$ or $\overleftarrow{\cap}$, the $\mp$ sign is negative for $\overleftarrow{\cup}$ and positive for $\overleftarrow{\cap}$.

Remark 70. Note that the a_k and b_k appearing are defined in terms of i_j. As mentioned previously, you can find some examples of state diagrams in Appendix A..

Definition 71. Let $\mathcal{K}$ be a knot and T 1-1 tangle whose closure is $\mathcal{K}$. Let D be a diagram of T. We define:

$$
\begin{aligned}
F_\infty(q,A,D) :=& q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{+\infty} D(i_1,\dots,i_N) \\
=& q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{+\infty} (\prod_{j=1}^{S} q^{\mp(\alpha-2\epsilon_j)}) \prod_{k\in pos} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \{\alpha-a_k;i_k\}_q \\
&\times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in neg} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \\
&\times \{\alpha-a_k;i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_k b_k}
\end{aligned}
$$

where $\bar{i} = (i_1,\dots,i_N)$, N is the number of crossings, S the number of $\overleftarrow{\cup}$ + $\overleftarrow{\cap}$ and f is the framing of the knot.

We have that $q^{-\frac{f\alpha^2}{2}} F_\infty(q,A,D)$ is a well defined element of $\widehat{\mathbb{Z}[q,q^\alpha]}$.

Note that it is not clear that this element is a knot invariant, it could depend *a priori* on the diagram D and we will have to prove later that it does not.

2.1.3 Recovering the ADO polynomials

In this subsection we will see how to evaluate at a root of unity an element of $\widehat{\mathbb{Z}[q,q^\alpha]}$. We will first need some useful lemma.

Let r be any integer, $R_r = \mathbb{Z}[\zeta_{2r}, A^{\pm 1}]$, we use the same previous notations and $q^\alpha := A$.

Lemma 72. *For any k, $\{\alpha-k;r\}_{\zeta_{2r}} = (-1)^k \zeta_{2r}^{\frac{-r(r-1)}{2}} \{r\alpha\}_{\zeta_{2r}}$.*

Proof. Since $\{\alpha-k-r-1\} = -\{\alpha-k-1\}$, we have:

$$
\begin{aligned}
\{\alpha-k;r\} &= \{\alpha-k\}\dots\{\alpha-k-r+1\} \\
&= (-1)\{\alpha-k+1\}\dots\{\alpha-k-r+2\} \\
&= (-1)^k\{\alpha\}\dots\{\alpha-r+1\} \\
&= (-1)^k\{\alpha;r\}
\end{aligned}
$$

$$
\begin{aligned}
\{\alpha;r\} &= \prod_{j=0}^{r-1}(\zeta_{2r}^{\alpha-j} - \zeta_{2r}^{-\alpha+j}) \\
&= \zeta_{2r}^{\frac{-r(r-1)}{2}} \zeta_{2r}^{-r\alpha} \prod_{j=0}^{r-1}(\zeta_{2r}^{2\alpha} - \zeta_{2r}^{2j}) \\
&= \zeta_{2r}^{\frac{-r(r-1)}{2}} \zeta_{2r}^{-r\alpha}(\zeta_{2r}^{2r\alpha} - 1) \\
&= \zeta_{2r}^{\frac{-r(r-1)}{2}} \{r\alpha\}
\end{aligned}
$$

where the third equality is just developing the factorized form of $X^r - 1$ at $X = \zeta_{2r}^{2\alpha}$.

□

Let $I = \{r\alpha\}_{\zeta_{2r}} R_r$ and we build the I-adic completion of R_r:

Definition 73. $\hat{R_r^I} = \lim\limits_{\overleftarrow{n}} \frac{R_r}{I^n} = \{(a_n)_{n\in\mathbb{N}^*} \in \prod_{i=1}^{\infty} \frac{R_r}{I^n} \mid \rho'_n(a_{n+1}) = a_n\}$ where $\rho'_n : \frac{R_r}{I^{n+1}} \to \frac{R_r}{I^n}$ is the projection map.

This completion is a bigger ring containing R_r:

Proposition 74. *The canonical projection maps induce an injective map* $R_r \hookrightarrow \hat{R_r^I}$

Proof. It is sufficient to prove that $\bigcap\limits_{n\in\mathbb{N}^*} I^n = \{0\}$.
Since $R_r = \mathbb{Z}[\zeta_{2r}][A^{\pm 1}]$, it is a Laurent polynomial ring. Hence, any non zero element x can be uniquely written $x = \sum_{i=l}^{n} a_i A^i$ where $a_k \in \mathbb{Z}[\zeta_{2r}], \forall k \in \{l, l+1, \dots, n-1, n\}$ and $a_n, a_l \neq 0$.
Let's define $len(x) = n - l$. We have that $len(xy) = len(x) + len(y)$.
Thus if $x \in \bigcap\limits_{n\in\mathbb{N}^*} I^n$ is non zero, of length k, $\exists y \in R_r$ such that $x = \{r\alpha\}^k y$ hence $len(x) \in [2rk, +\infty[$, contradiction.

□

Let's now define the evaluation map from $\widehat{\mathbb{Z}[q, q^\alpha]}$ to $\hat{R_r^I}$.
At the level of R and R_r we have a well defined evaluation map, $ev_{\zeta_{2r}} : R \to R_r,\ q \mapsto \zeta_{2r}$. We will extend this map to the completions.

Proposition 75. $ev_{\zeta_{2r}}(I_{rn}) = I^n$

Proof. Direct application of Lemma 72. □

Hence, ev_r factorize into maps $\psi_n : R/I_{rn} \to R_r/I^n$, we can then define the map extension:

Proposition 76. *We have a well defined map:*

$$ev_r : \widehat{\mathbb{Z}[q, q^\alpha]} \to \hat{R_r^I}$$

such that, if $(a_n)_{n\in\mathbb{N}^*} \in \widehat{\mathbb{Z}[q, q^\alpha]}$, $ev_r((a_n)_{n\in\mathbb{N}^*}) = (\psi_n(a_{rn}))_{n\in\mathbb{N}^*}$.

Proof. If we denote $\lambda_n : R/I_{r(n+1)} \to R/I_{rn}$ the projection maps, the statement follows from the fact that the following diagram is commutative:

$$\begin{array}{ccc} R/I_{r(n+1)} & \xrightarrow{\psi_{n+1}} & R_r/I^{n+1} \\ \lambda_n \downarrow & & \downarrow \rho'_n \\ R/I_{rn} & \xrightarrow{\psi_n} & R_r/I^n \end{array}$$

□

It is now time to study the element $F_\infty(\zeta_{2r}, A, D) := ev_r(F_\infty(q, A, D))$, we will see that the ADO invariant $ADO_r(A, \mathcal{K})$ can be factorized from it.
In order to do so, we will need some useful computations:

Lemma 77. *We have the following factorizations:*

1. $\zeta_{2r}^{\frac{(i+rl)(i+rl-1)}{2}} = (-1)^{il} \zeta_{2r}^{\frac{rl(rl-1)}{2}} \zeta_{2r}^{\frac{i(i-1)}{2}}$,
2. $\{\alpha - a - ru; i + rl\}_{\zeta_{2r}} = (-1)^{al+rul+ui+li} \zeta_{2r}^{\frac{-rl(r-1)}{2}} \zeta_{2r}^{\frac{-rl(l-1)}{2}} \{r\alpha\}_{\zeta_{2r}}^l \{\alpha - a; i\}_{\zeta_{2r}}$,
3. $\begin{bmatrix} a+i+r(u+l) \\ i+rl \end{bmatrix}_{\zeta_{2r}} = (-1)^{al+rul+ui} \binom{u+l}{l} \begin{bmatrix} a+i \\ i \end{bmatrix}_{\zeta_{2r}}$

4. $\zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}} = \zeta_{2r}^{\frac{-rl(rl-1)}{2}}$

Proof.

1. It's obtained by developing the product.

2. It's an application of Lemma 72.
First $\{\alpha - a - ru; i + rl\} = \zeta_{2r}^{(i+rl)ru}\{\alpha - a; i + rl\} = (-1)^{iu}(-1)^{rul}\{\alpha - a; i + rl\}$.
Then,

$$\begin{aligned}\{\alpha - a; i + rl\} &= \{\alpha - a; rl\}\{\alpha - a - rl; i\}\\ &= (-1)^{al}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\{\alpha; rl\}\{\alpha - a - rl; i\}\\ &= (-1)^{al}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\zeta_{2r}^{\frac{-rl(r-1)}{2}}\{r\alpha\}^{l}\{\alpha - a - rl; i\}\end{aligned}$$

Finally $\{\alpha - a - rl; i\} = (-1)^{li}\{\alpha - a; i\}$.
Put together, we get

$$\{\alpha - a - ru; i + rl\}_{\zeta_{2r}} = (-1)^{al+rul+ui+li}\zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\{r\alpha\}_{\zeta_{2r}}^{l} \times \{\alpha - a; i\}_{\zeta_{2r}}.$$

3. It follows from the fact that $ev_{\zeta_{2r}}(\frac{\{rk\}_q}{\{r\}_q}) = (-1)^{1-k}k$. In $\left[{a+i+r(u+l) \atop i+rl}\right]_{\zeta_{2r}}$ seen as $\frac{\{a+i+r(u+l)\}!}{\{a+ru\}!\{i+rl\}!}$, taking only the terms $\{rk\}$, we extract $(-1)^{ul}\binom{u+l}{l}$. Now we only have to deal with non multiples of $\{r\}$. We use the equality $\{t + r\} = (-1)\{t\}$ in order to have consecutive terms in the denominators (excepted from multiple of r), indeed $\{a + ru\}! = \{ru\}!\{a + ru; a\}$ and $\{i + rl\})(-1)^{u(i+rl)}\{i + rl + ru; i + rl\}$, hence $\frac{\{a+i+r(u+l)\}!}{\{a+ru\}!\{i+rl\}!} = (-1)^{u(i+rl)}\frac{\{a+i+r(u+l);a\}}{\{a+ru;a\}} = (-1)^{u(i+rl)}(-1)^{au}(-1)^{a(u+l)}\frac{\{a+i;a\}}{\{a;a\}} = (-1)^{ui}(-1)^{rul}(-1)^{al}\left[{a+i \atop i}\right]_{\zeta_{2r}}$.
Putting things together with the quantum r multiple part, we get the desired result.

4. It's obtained as follow:

$$\zeta_{2r}^{\frac{-rl(rl-1)}{2}} = \prod_{k=0}^{rl-1}\zeta_{2r}^{-k} = \prod_{j=0}^{l}\prod_{k=0}^{r-1}\zeta_{2r}^{-k-rj} = \prod_{j=0}^{l}\zeta_{2r}^{-rj}\prod_{k=0}^{r-1}\zeta_{2r}^{-k} = \zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}}.$$

□

We proceed similarly as in the paragraph preceding Definition 71 and define an element for each state diagram of D that will be used to factorise $F_\infty(q, A, D)$. Let $\mathcal{K}$ be a knot seen as a $(1, 1)$ tangle and D a diagram of it. For a state diagram of D we define:

$$\begin{aligned}D_{C,r}(l_1, \dots, l_N) =&(\prod_{j=1}^{S}\zeta_{2r}^{\mp r\alpha})\prod_{k\in pos}\binom{u_k + l_k}{l_k}\{r\alpha\}_{\zeta_{2r}}^{l_k}\zeta_{2r}^{-(u_k+v_k)r\alpha}\\ &\times\prod_{k\in neg}(-1)^{l_k}\binom{u_k + l_k}{l_k}\{r\alpha\}_{\zeta_{2r}}^{l_k}\zeta_{2r}^{(u_k+v_k)r\alpha}\end{aligned}$$

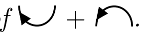
where $neg \cup pos = [|1, N|]$ and $k \in pos$ if the k-th crossing of D is positive, else $k \in neg$, $a_k, b_k \in [|0, \dots, r-1|]$, $a_k + ru_k, b_k + rv_k$ are the strands labels at the k-th crossing of the state diagram (see Figure 2.2), S is the number of + appearing in the diagram, and ϵ_j the strand label at the j-th or , the $\mp$ sign is negative for and positive for .

Proposition 78.
For a knot $\mathcal{K}$ and a diagram of the knot D, $r \in \mathbb{N}^*$, we have the following factorization in $\hat{R}_r^I$:

$$F_\infty(\zeta_{2r}, A, D) = C_\infty(r, A, D) \times ADO_r(A, \mathcal{K})$$

where :

$$\begin{aligned}
C_\infty(r,A,D) &= \sum_{\bar{l}=0}^{+\infty} D_{C,r}(l_1,\dots,l_N)\\
&= \sum_{\bar{l}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}\Big) \prod_{k\in pos} \binom{u_k+l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{-(u_k+v_k)r\alpha}\\
&\qquad \times \prod_{k\in neg} (-1)^{l_k} \binom{u_k+l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{(u_k+v_k)r\alpha} \in \hat{R}_r^I
\end{aligned}$$

where $\bar{l}=(l_1,\dots,l_N)$, N is the number of crossings and S the number of *.*

Proof. For the sake of simplicity, we will only consider positive crossings in the following proof. We factorize as follows:

$$\begin{aligned}
F_\infty(\zeta_{2r},A,D) &= q^{\frac{f\alpha^2}{2}} \sum_{\bar{s}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp(\alpha-2\epsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{s_k(s_k-1)}{2}} \begin{bmatrix} z_k+s_k \\ s_k \end{bmatrix}_{\zeta_{2r}}\\
&\times \{\alpha - z_k; s_k\}_{\zeta_{2r}} \zeta_{2r}^{(-z_k-y_k)\alpha} \zeta_{2r}^{2(z_k+s_k)(y_k-s_k)}\\
&= q^{\frac{f\alpha^2}{2}} \sum_{i+rl=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp(\alpha-2\epsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{(i_k+rl_k)(i_k+rl_k-1)}{2}}\\
&\times \begin{bmatrix} a_k+i_k+r(u_k+l_k) \\ i_k+rl_k \end{bmatrix}_{\zeta_{2r}} \{\alpha-(a_k+ru_k); i_k+rl_k\}_{\zeta_{2r}}\\
&\times \zeta_{2r}^{(-(a_k+ru_k)-(b_k+rv_k))\alpha} \zeta_{2r}^{2((a_k+ru_k)+(i_k+rl_k))(b_k+rv_k-(i_k+rl_k))}\\
&= q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{r-1} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\pm(r-1)(\alpha-2\epsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}}\\
&\times \{\alpha-a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{(-a_k-b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)}\\
&\times \sum_{\bar{l}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}\Big) \prod_{k=1}^{N} \binom{u_k+l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{(-u_k-v_k)r\alpha}
\end{aligned}$$

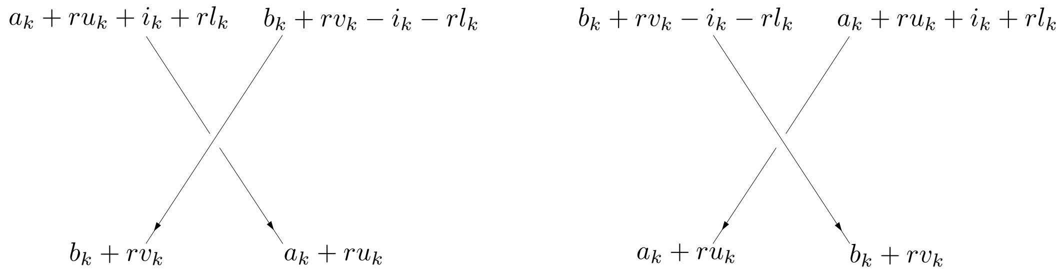
The second equality is obtained by changing variables $s_k = i_k + rl_k$ $0 \le i_k \le r-1$ and writing the strands labels at crossings z_k as $z_k = a_k + ru_k$ $0 \le a_k \le r-1$ and y_k as $y_k = b_k + rv_k$ $0 \le b_k \le r-1$.
Note that a_k, b_k solely depends on i_k and u_k, b_k on l_k. This relies on the fact that $\begin{bmatrix} n+m \\ n \end{bmatrix}_q = 0$ at $q=\zeta_{2r}$ if $n,m \le r-1$ and $n+m \ge r$.

The third one is obtained by replacing each term with its factorization given by Lemma 77, the crossed terms between i_k and l_k are just signs, that eventually compensate. Hence, we have the factorization. □

In order to get back $ADO_r(A,\mathcal{K})$ from $F_\infty(\zeta_{2r},A,D)$, we need to prove that $C_\infty(r,A,D)$ is a unit in $\hat{R}_r^I$.

Proposition 79. *If $a=(a_n)_{n\in\mathbb{N}^*} \in \hat{R}_r^I$ and $a_1 \in R_r/I$ is a unit, then a is a unit in $\hat{R}_r^I$.*

Proof. Let $a=(a_n)_{n\in\mathbb{N}^*} \in \hat{R}_r^I$ be such that a_1 is a unit of R_r/I.
Let's prove that a_n is also a unit in R_r/I^n. Indeed, if y is an element of R_r/I^n such that $a_n y = a_1 y = 1 \mod I$ then $\exists z \in I.R_r/I^n$ such that $a_n y = 1+z$, $z = a_n y - 1$ thus $0 = z^n = (a_n y-1)^n$, which proves that a_n is invertible.
Hence, $a^{-1} = (a_n^{-1})_{n\in\mathbb{N}^*}$ is the inverse of a in $\hat{R}_r^I$. □



(a) Positive crossing. (b) Negative crossing.

Figure 2.2: The two possibilities for the k-th crossing in D when factorizing.

Since $C_\infty(r, A, D) = (\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}) \bmod \{r\alpha\}_{\zeta_{2r}}$, therefore equal to an invertible element of R_r/I, then $C_\infty(r, A, D)$ is a unit of $\hat{R}_r^I$.

Corollary 80.

$$ADO_r(A, \mathcal{K}) = F_\infty(\zeta_{2r}, A, D)C_\infty(r, A, D)^{-1}$$

Finally, one can recover $C_\infty(r, A, D)$ from $F_\infty(q, A, D)$, this will prove that not only that ADO is contained in $F_\infty(q, A, D)$ but that it's possible to extract them with the sole datum of $F_\infty(q, A, D)$.

For $r = 1$, one gets

$$\begin{aligned} ev_1(F_\infty(q, A, D)) &= F_\infty(\zeta_2, A, D) \\ &= C_\infty(1, A, D) \times ADO_1(A, \mathcal{K}) \\ &= q^{\frac{f\alpha^2}{2}} C_\infty(1, A, D) \end{aligned}$$

Remark 81. Note that $ADO_1(A, \mathcal{K})$ is only defined as the case $r = 1$ in Prop 64, which is well defined. Nevertheless, the algebraic setup for ADO fails at $r = 1$ since $[E, F]$ is not well defined.

But then $C_\infty(1, A, D) \in \widehat{\mathbb{Z}[A^{\pm 1}]^{\{\alpha\}}} := \varprojlim_n \frac{\mathbb{Z}[A^{\pm 1}]}{\{\alpha\}^n}$, hence for each r we have a well defined map:

$$g_r : \widehat{\mathbb{Z}[A^{\pm 1}]^{\{\alpha\}}} \to \widehat{\mathbb{Z}[A^{\pm 1}]^{\{r\alpha\}}}, \quad q^\alpha \mapsto q^{r\alpha}$$

with

$$\widehat{\mathbb{Z}[A^{\pm 1}]^{\{r\alpha\}}} \hookrightarrow \hat{R}_r^I$$

via the completion of the inclusion $\mathbb{Z}[A^{\pm 1}] \subset \mathbb{Z}[\zeta_{2r}, A^{\pm 1}]$.

This map is such that $g_r(C_\infty(1, A, D)) = C_\infty(r, A, D)$.

This proves the following proposition:

Proposition 82. *For all r, we have a well defined map $FC_r = g_r \circ ev_1 : \widehat{\mathbb{Z}[q, q^\alpha]} \to \widehat{\mathbb{Z}[A^{\pm 1}]^{\{r\alpha\}}}$ and for any knot $\mathcal{K}$ and any diagram D of the knot, $F_\infty(q, A, D) \mapsto C_\infty(r, A, D)$.*

Corollary 83. *For all r, we have a well defined map $ev_r \times \frac{1}{FC_r} : (\widehat{\mathbb{Z}[q, q^\alpha]})^\times \to (\hat{R}_r^I)^\times$ and for any knot $\mathcal{K}$ and any diagram D of the knot, $F_\infty(q, A, D) \mapsto ADO_r(A, \mathcal{K})$.*

Proof. Let $x \in (\widehat{\mathbb{Z}[q, q^\alpha]})^\times$ an invertible element, since FC_r is a ring morphism, $FC_r(x)$ is invertible. Then $Id \times \frac{1}{FC_r}(F_\infty(q, A, D)) = F_\infty(\zeta_{2r}, A, D) \times C_\infty(r, A, D)^{-1} = ADO_r(A, \mathcal{K})$. □

2.2 Knot invariance via the universal invariant

2.2.1 Algebra completion and universal invariant

Universal invariant in U_h

The universal quantum invariant is also a knot invariant coming from a ribbon Hopf algebra, but it doesn't require a finite dimensional representation, since the invariant itself lies in the algebra and not in a scalar ring. This allows us to look a the action of this algebra element upon infinite dimensional modules, such as Verma module and to recover our unified element. The general construction of the universal invariant can be found in [23].

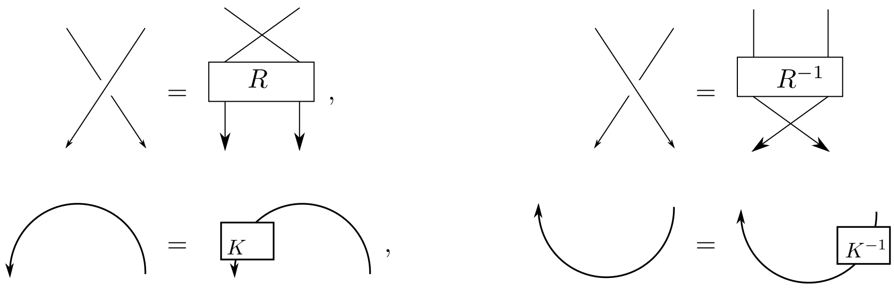
For a knot $\mathcal{K}$ and the ribbon Hopf algebra U_h we can construct the universal invariant $Q^{U_h}(\mathcal{K}) \in U_h$ as follows.

We consider a $(1,1)$ tangle T whose closure is a knot, we graphically represent elements of U_h as coupons on T that can slide along it, and satisfying:

(a) Inductive rule (b) No coupons left in T

Figure 2.3: Definition of coupons tangles as elements of U_h

If we put coupons on T using the following rules we get $Q^{U_h}(\mathcal{K})$:



Proposition 84. *$Q^{U_h}(\mathcal{K}) \in U_h$ is a knot invariant called universal invariant.*

Proof. See more detail in Section 4.2 of [23]. □

Integral algebra completion

Recall the integral version of $U_q(\mathfrak{sl}_2)$ defined in Subsection 1.2.3 as:

Definition 85. Let $\mathcal{U} := U_q^D(\mathfrak{sl}_2)$ the $\mathbb{Z}[q^{\pm 1}]$ algebra generated by $K^{\pm 1}, E, F^{(n)}$ where

$$F^{(n)} = \frac{\{1\}_q^n F^n}{[n]_q!}.$$

If we set $q = e^h$ we have an injective map $\mathcal{U} \subset U_h$ and hence, $\mathcal{U}$ is endowed with the Hopf algebra structure coming from U_h.

We would like to make sense of the universal invariant of a knot in terms of an algebra over $\mathbb{Z}[q^{\pm 1}]$. The main issue with $\mathcal{U}$ is that the R-matrix does not belong to the algebra:

$$R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$$

we need to control the terms $q^{\frac{H\otimes H}{2}}$ and allow the infinite sum to converge in our algebra.

One way to proceed is to complete the algebra in order to make such sums converge.

We denote $\{H+m\}_q = K^m q^m - K^{-m} q^{-m}$, $\{H+m; n\}_q = \prod_{i=0}^{n-1} \{H+m-i\}_q$.

Definition 86. Let L_n be the $\mathbb{Z}[q^{\pm 1}]$ ideal generated by $\{n\}!$.
Let J_n be the $\mathcal{U}$ two sided ideal generated by the following elements:

$$F^{(i+k)}\{H+m; n-i\}_q$$

where $m \in \mathbb{Z}$, $i \in \{0, \ldots, n\}$ and $k \in \mathbb{N}$.

Lemma 87. *J_n is generated by elements of the form $F^{(i+k)}\{n-i;j\}\{H;n-i-j\}$, for all $j \in \{0, \ldots, n-i\}$, $i \in \{0, \ldots, n\}$ and $k \in \mathbb{N}$.*

Proof. The proof can be found in Habiro's article [15], Proposition 5.1. □

Following the completion described by Habiro in his article [15] Section 4. We have:

1. $J_{n+1} \subset J_n$,
2. $L_n \subset J_n$ (see Prop 5.1 in Habiro's article [15]),
3. $\Delta(J_n) \subset \sum\limits_{i+j=n} J_i \otimes J_j$,
4. $\epsilon(J_n) \subset L_n$,
5. $S(J_n) \subset J_n$.

Thus we can define the completion

$$\hat{\mathcal{U}} := \lim_{\overleftarrow{n}} \frac{\mathcal{U}}{J_n}$$

as a

$$\widehat{\mathbb{Z}[q^{\pm 1}]} := \lim_{\overleftarrow{n}} \frac{\mathbb{Z}[q^{\pm 1}]}{L_n}$$

algebra and we say that $\widehat{\mathbb{Z}[q^{\pm 1}]}$ is Habiro's ring.

It is endowed with a complete Hopf algebra structure:

$$\hat{\Delta} : \hat{\mathcal{U}} \to \hat{\mathcal{U}} \hat{\otimes} \hat{\mathcal{U}}, \quad \hat{\epsilon} : \hat{\mathcal{U}} \to \widehat{\mathbb{Z}[q^{\pm 1}]}, \quad \hat{S} : \hat{\mathcal{U}} \to \hat{\mathcal{U}}$$

where : $\hat{\mathcal{U}} \hat{\otimes} \hat{\mathcal{U}} = \lim\limits_{\overleftarrow{k,l}} \dfrac{\hat{\mathcal{U}} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \hat{\mathcal{U}}}{\hat{\mathcal{U}} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \overline{J_k} + \overline{J_l} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \hat{\mathcal{U}}}$ and $\overline{J_n}$ is the closure of J_n in $\hat{\mathcal{U}}$.

This completion contains $\mathcal{U}$:

Proposition 88. *The canonical projection map induces an injective map $\mathcal{U} \hookrightarrow \hat{\mathcal{U}}$.*

Proof. Take the projective maps $j_n : \mathcal{U} \to \mathcal{U}/J_n$, they induce a map $j : \mathcal{U} \to \hat{\mathcal{U}}$. This map in injective because if $j(x) = 0$ then $x \in \bigcap\limits_{n \in \mathbb{N}^*} J_n$. But since $J_n \subset h^n U_h$ then $\bigcap\limits_{n \in \mathbb{N}^*} J_n \subset \bigcap\limits_{n \in \mathbb{N}^*} h^n U_h$. It is a well known fact that $\bigcap\limits_{n \in \mathbb{N}^*} h^n U_h = \{0\}$. □

Moreover, since $J_n \subset h^n U_h$ regarding $\mathcal{U}$ as a subalgebra of U_h, we have a map $i : \hat{\mathcal{U}} \to U_h$. Since we do not know if this map is injective, we consider $\tilde{\mathcal{U}} := i(\hat{\mathcal{U}})$ the image in U_h. It is also an Hopf algebra.

Remark 89. $\sum\limits_{i=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)} \in \tilde{\mathcal{U}} \hat{\otimes} \tilde{\mathcal{U}}$

Universal invariant from the integral completion

Now it is time to see the universal invariant as an element of $\tilde{\mathcal{U}}$.

We will need a lemma to compute some commutation rules.

Lemma 90.
$(E\otimes 1)\times q^{\frac{H\otimes H}{2}} = q^{\frac{H\otimes H}{2}}\times(E\otimes 1)\times(1\otimes K)$
$(F^{(n)}\otimes 1)\times q^{\frac{H\otimes H}{2}} = q^{\frac{H\otimes H}{2}}\times(F^{(n)}\otimes 1)\times(1\otimes K^{-n})$

Proof. Notice that since $EH^n=(H+2)^nE$ and $q^{\frac{H\otimes H}{2}}=\sum(\frac{h^n}{2^n n!})H^n\otimes H^n$, then

$$(E\otimes 1)\times q^{\frac{H\otimes H}{2}} = q^{\frac{(H+2)\otimes H}{2}}\times(E\otimes 1) = q^{\frac{H\otimes H}{2}}\times(1\otimes K)\times(E\otimes 1).$$

The same can be done for $F^{(n)}$. □

We have the following proposition:

Proposition 91. *If $\mathcal{K}$ is a knot and D a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$, then:*

$$Q^{U_h}(\mathcal{K}) = q^{f\frac{H^2}{2}}Q^{\tilde{\mathcal{U}}}(D)$$

where $Q^{\tilde{\mathcal{U}}}(D)\in\tilde{\mathcal{U}}$ and f is the writhe of the diagram.

Proof. The construction of the universal invariant can be followed step by step in the Example 92 of the trefoil knot.
Using coupons and Lemma 90, we can factorise the tangle containing only coupons $q^{\pm\frac{H\otimes H}{2}}$ coming from the R matrices. Using the identity $q^{\pm\frac{H\otimes H}{2}}=\sum(\frac{(\pm h)^n}{2^n n!})H^n\otimes H^n$, one get $q^{f\frac{H^2}{2}}$. □

Example 92. Let's do the example of the trefoil.

- First we put coupons (Figure 2.4).
- Then we factorise the quadratic part (Figure 2.5).
- Finally we compute the quadratic part (Figure 2.6).

2.2.2 Completed Verma modules and unified invariant

Now that we have a universal invariant $q^{-f\frac{H^2}{2}}Q^{U_h}(\mathcal{K})$ which lies in $\tilde{\mathcal{U}}$, we can construct a topological Verma module on which the scalar action of this invariant will be $F_\infty(q,A,D)$.

Let V^α be a R-module freely generated by vectors $\{v_0,v_1,\dots\}$, and we endow it with a $\mathcal{U}$-module structure:

$$Ev_0=0,\quad Ev_{i+1}=v_i,\quad Kv_i=q^{\alpha-2i}v_i,\quad F^{(n)}v_i=\begin{bmatrix}n+i\\ i\end{bmatrix}_q\{\alpha-i;n\}_q v_{n+i}$$

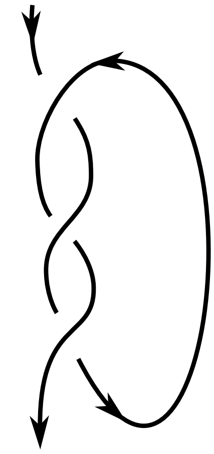
We define the *completed Verma module* as the $\widehat{\mathbb{Z}[q,q^\alpha]}$-module $\widehat{V^\alpha}=\lim\limits_{\overleftarrow{n}}\frac{V^\alpha}{I_nV^\alpha}$ where I_nV^α is the ideal generated by elements of the form $\lambda\times v$, $\lambda\in I_n$, $v\in V^\alpha$.

Since $J_nV^\alpha\subset I_nV^\alpha$, we can naturally endow it with a $\tilde{\mathcal{U}}$ module structure.

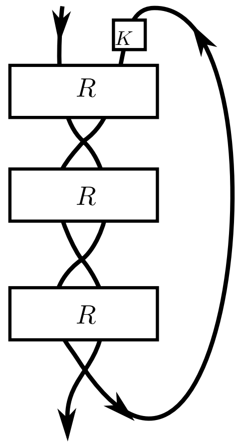
Moreover since $\bigcap I_n=0$ and V_α is R-free, then $\bigcap I_nV^\alpha=0$ and thus $V^\alpha\subset\widehat{V^\alpha}$ as a R-module.

Proposition 93. *The $\tilde{\mathcal{U}}$-endomorphism of $\widehat{V^\alpha}$ are scalars i.e.*

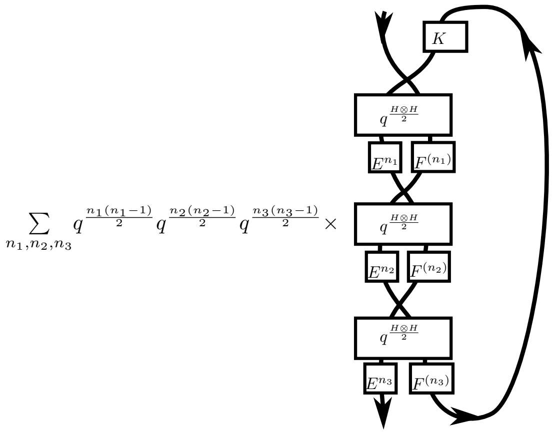
$$End_{\tilde{\mathcal{U}}}(\widehat{V^\alpha})=\widehat{\mathbb{Z}[q,q^\alpha]}Id_{\widehat{V^\alpha}}.$$



(a) The trefoil knot.

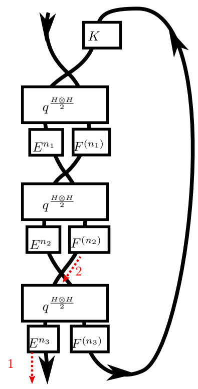


(b) The universal invariant.

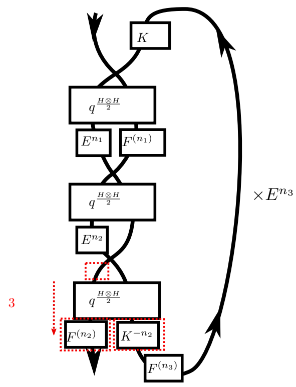


(c) R-matrices seen has sums.

Figure 2.4: The example of the trefoil knot.



(a) The first two steps, sliding coupons.

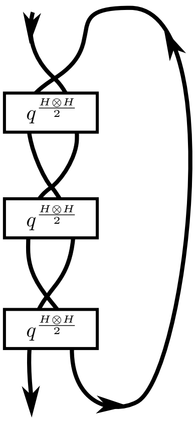


(b) The third step, passing through the (2,2) coupons.

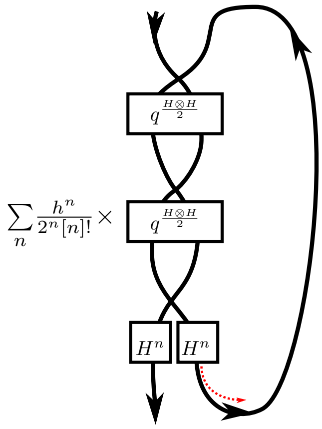
Figure 2.5: Sliding coupons in the example of the trefoil knot.

Proof. Let $f \in End_{\tilde{\mathcal{U}}}(\widehat{V^\alpha})$.
$Kf(v_i) = f(Kv_i) = q^{\alpha-2i}f(v_i)$, thus $\exists \lambda_i \in \widehat{\mathbb{Z}[q, q^\alpha]}$ such that $f(v_i) = \lambda_i v_i$.
Now since $Ef(v_{i+1}) = f(Ev_{i+1}) = f(v_i)$, then $\lambda_{i+1}v_i = \lambda_i v_i$ hence we define $\lambda := \lambda_i$ and we have $f = \lambda Id_{\widehat{V^\alpha}}$. □

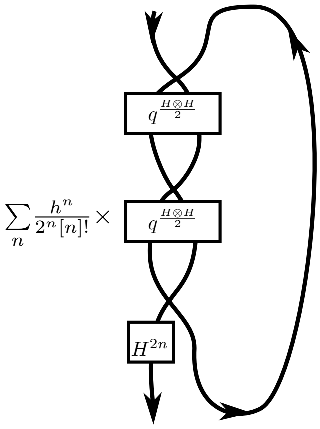
We set $q^{\pm\frac{H^2}{2}} v_i = q^{\pm\frac{(\alpha-2i)^2}{2}} v_i \in q^{\pm\frac{\alpha^2}{2}} \widehat{V^\alpha}$.



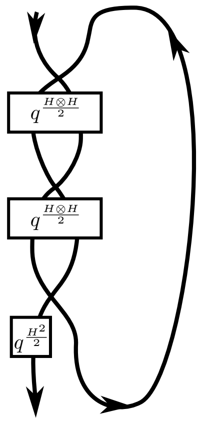
(a) Quadratic part of trefoil universal invariant.



(b) Illustration of quadratic simplification: first step.



(c) Illustration of quadratic simplification: second step.



(d) Illustration of quadratic simplification: third step.

Figure 2.6: Quadratic factorization and simplification for the trefoil knot.

Proposition 94. *$Q^{U_h}(\mathcal{K})$ is in the center of U_h.*

Proof. See [14] Proposition 8.2. □

Proposition 95. *If $\mathcal{K}$ is a knot and D is a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$, then:*

$$Q^{U_h}(\mathcal{K})v_0 = q^{f\frac{H^2}{2}} Q^{\tilde{\mathcal{U}}}(D)v_0 = F_\infty(q, A, D)v_0.$$

Hence, in particular, $F_\infty(q, A, D)$ is independent of the choice of the diagram and we denote it $F_\infty(q, A, \mathcal{K})$.

Proof. The identity comes from the definition of the state diagrams contribution $D(i_1, \ldots, i_N)$ and the definition of the action of the R matrices, where i_j is the index of the sum corresponding to the R-matrix at the j-th crossing. □

2.2.3 Recovering the colored Jones polynomials

Knowing that the unified invariant comes from the universal invariant, we can use this fact to recover the colored Jones polynomials. When we evaluate $A = q^\alpha$ at q^n in $F_\infty(q, A, \mathcal{K})$, we obtain the n-colored Jones polynomial denoted $J_n(q, \mathcal{K})$ (with normalization $J_n(q, \mathit{unknot}) = 1$).

The evaluation takes place in a completion of $\mathbb{Z}[q^{\pm 1}]$ called Habiro's ring:

$$\widehat{\mathbb{Z}[q]} := \lim_{\overleftarrow{n}} \frac{\mathbb{Z}[q^{\pm 1}]}{(\{n\}_q!)}$$

Remark that, since $\bigcap_{n\in\mathbb{N}^*} (\{n\}_q!) = \{0\}$,

$$\mathbb{Z}[q^{\pm 1}] \subset \widehat{\mathbb{Z}[q]}.$$

Moreover, since $\forall n \in \mathbb{N}^*$, $(\{n\}_q) \subset I_n$, we have a well defined evaluation map for each $N \in \mathbb{N}$:

$$\begin{aligned} \widehat{\mathbb{Z}[q,q^\alpha]} &\to \widehat{\mathbb{Z}[q]} \\ A &\mapsto q^N \end{aligned}$$

The following lemma and proposition are a reformulation of the proof that the universal invariant contains the colored Jones polynomials as found in [16] Subsection 7.1.

Lemma 96. *We can endow S_n with a $\tilde{\mathcal{U}}$ module structure and if we denote $\widehat{V^n}$ the Verma module of highest weight q^n and highest weight vector v_0, then $S_n \cong \tilde{\mathcal{U}} v_0 \subset \widehat{V^n}$.*

Proof. Since $F^{(n+1)}v_0 = \{n; n+1\}v_{n+1} = 0 \times v_{n+1} = 0$, we have

$$\tilde{\mathcal{U}} v_0 =< v_0, \dots, v_n >,$$

thus $S_n \cong \tilde{\mathcal{U}} v_0 \subset \widehat{V^n}$. □

This proves the following proposition:

Proposition 97. *If $\widehat{V^n}$ is the Verma of highest weight q^n and highest weight v_0,*

$$Q^{U_h}(\mathcal{K})v_0 = J_n(q^2, \mathcal{K})v_0.$$

This allows us to state the following corollary.

Corollary 98. $F_\infty(q, q^n, \mathcal{K}) = J_n(q^2, \mathcal{K}) \in \mathbb{Z}[q^{\pm 1}]$.

Hence, $F_\infty(q, A, \mathcal{K})$ plays a double role in this dance, evaluating its first variable q at a root of unity ζ_{2r}, gives us the r-th ADO polynomial multiplied by this $C_\infty(r, A, \mathcal{K})$ element. But if one evaluates the second variable A at q^n, one gets the n-th colored Jones polynomial.

Chapter 3

Unified invariants properties and applications

In this chapter, the common theme will be to study the unified invariant and to find some applications. The first thing to do would be to understand the unified invariant using Habiro's tools. The completion rings and algebra are very alike, and the extensive studies of them by Habiro will allow us to use powerful properties. One consequence will be the uniqueness of the unified invariant, meaning that there is a unique element in our completion ring that specifies at integer weight into the colored Jones polynomials.

This will have nice applications, first of all it will prove that the family of colored Jones polynomials of a knot and the family of ADO polynomials of this knot are equivalent. Moreover, using Melvin-Morton-Rozansky conjecture proved by Bar-Natan and Garoufalidis we can recognize the Alexander polynomial in the term C_∞ when we factorise at root of unity.

It will also allow us to transfer properties of the colored Jones family to the ADO family, such as the holonomic properties or the expansion as Vassiliev invariants.

3.1 The h-adic point of view and the Habiro setup

In this section we will embed our setup in the h-adic setup, allowing us to get nice properties from it. One of them is that Habiro's algebra setup in [15] can also be used to produce the unified invariant.

Once this is done, we will show that $F_\infty(q, A, \mathcal{K})$ is in fact the two variable Jones invariant $J_{\mathcal{K}}(q^\alpha, q)$ defined by Habiro in [16] (Section 7).

Remark 99. Recall that the variable q in this paper corresponds to the variable v in [15] [16].

Let's recall that U_h is the h-adic algebra defined in Definition 36. Similarly $\mathcal{U}$ is defined as a subalgebra of U_h in Definition 55. We use the notation $\{H\} = K - K^{-1}$.

Recall also the completion ring we used all along to define the unified invariant, $\widehat{\mathbb{Z}[q, q^\alpha]}$, defined as a completion of $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ in Definition 67. Let us define U_h^0 as the $\mathbb{Q}[[h]]$-subalgebra of U_h topologically generated by H. Let $\mathcal{U}^0$ be the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $\mathcal{U}$ generated by K. We can complete the algebra into

$$\hat{\mathcal{U}}^0 = \lim_{\overleftarrow{n}} \frac{\mathcal{U}^0}{< \{H+m; n\},\ m \in \mathbb{Z} >}.$$

We then have the following proposition.

Proposition 100. *Let $\mathbb{Q}[\alpha]$ the polynomial ring with the formal variable α.*

Using the identification $H \to \alpha$ and $K \to A$ we get:
$U_h^0 \cong \mathbb{Q}[\alpha][[h]]$ *the h-adic completion of the polynomial ring $\mathbb{Q}[\alpha]$,*
$\hat{\mathcal{U}}^0 \cong \widehat{\mathbb{Z}[q, q^\alpha]}$.

Proof. The first statement is the definition of U_h^0 replacing H with the formal variable α. The second statement comes from the fact that, replacing K by A, $\mathcal{U}^0 \cong \mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ and $\{H + m; n\} \cong I_n$. □

Now, elements in $\widehat{\mathbb{Z}[q, q^\alpha]}$ can be uniquely expressed. This fact comes from Corollary 5.5 in [15]. Recall that $q^\alpha := A$ and let $\{\alpha; n\}' = \prod_{i=0}^{n-1}(q^{2\alpha} - q^i)$ we have the following proposition:

Proposition 101. *We have the following isomorphism:*

$$\widehat{\mathbb{Z}[q, q^\alpha]} \cong \lim_{\overleftarrow{n}} \frac{\widehat{\mathbb{Z}[q]}[A]}{(\{\alpha; n\}')}.$$

Moreover, any element $t \in \widehat{\mathbb{Z}[q, q^\alpha]}$ *can be uniquely written as* $\sum_{n=0}^{\infty} t_n \{\alpha; n\}'$ *where* $t_n \in \widehat{\mathbb{Z}[q]} + \widehat{\mathbb{Z}[q]}A$.

Proof. See Corollary 5.5 in [15]. □

Remark 102. This means that the unified invariant $F_\infty(q, A, \mathcal{K})$ can be uniquely written as a series $\sum_{n=0}^{\infty} t_n \{\alpha; n\}'$ where $t_n \in \widehat{\mathbb{Z}[q]} + \widehat{\mathbb{Z}[q]}A$.

Moreover, one can use Proposition 6.8 and 6.9 in Habiro's article [15] and we have:

Proposition 103. *We have an injective map* $\widehat{\mathbb{Z}[q, q^\alpha]} \to \mathbb{Q}[\alpha][[h]]$ *where* $q \mapsto e^h$ *and* $A \mapsto e^{h\alpha}$*, and thus* $\widehat{\mathbb{Z}[q, q^\alpha]}$ *is an integral domain.*

Proof. A quick sketch of the proof. We use Proposition 6.8 of [15] to prove the injectivity of the map:

$$i : \widehat{\mathbb{Z}[q, q^\alpha]} \to \mathbb{Z}[[q - 1, A - 1]]$$

where $\mathbb{Z}[[q - 1, A - 1]]$ is the formal power series ring in the two variables $q - 1$ and $A - 1$.

Take $t \in \widehat{\mathbb{Z}[q, q^\alpha]}$ such that $i(t) = 0$. Let's use Proposition 101 to write in a unique way:

$$t = \sum_{l=0}^{\infty} t_l \{\alpha; l\}'_q$$

where $t_l \in \widehat{\mathbb{Z}[q, q^\alpha]} + \widehat{\mathbb{Z}[q, q^\alpha]}A$.

One can show that

$$\{\alpha; l\}' = \sum_{j=1}^{l} b_{l,j}(q^2 - 1)^{l-j}(A^2 - 1)^j$$

where $b_{l,j} \in \widehat{\mathbb{Z}[q]}$ and $b_{l,l} = 1$.

Hence,

$$\begin{aligned} i(t) &= t_0 + \sum_{l=0}^{+\infty} t_l \sum_{j=1}^{l} b_{l,j}(q^2 - 1)^{l-j}(A^2 - 1)^j \\ &= t_0 + \sum_{j=0}^{+\infty} c_j (A^2 - 1)^j \end{aligned}$$

where $c_j = \sum_{l=1}^{+\infty} t_l b_j (A^2 - 1)^{l-j} \in \mathbb{Z}[[q - 1]] + \mathbb{Z}[[q - 1]]A$.

Since $i(t) = 0$ then $t_0 = 0$, $c_j = 0$ for all $j \in \mathbb{N}^*$.

Using $b_{l,l} = 1$ for all $l \in \mathbb{N}$, one can show that $(A^2 - 1)^m | t_j$ for all $j \in \mathbb{N}^*$, $m \in \mathbb{N}$. Hence $t_j = 0$ for all $j \in \mathbb{N}$, meaning that $t = 0$. The map i is thus injective.

To finish the proof, we use the standard inclusion $\mathbb{Z}[[q - 1, A - 1]] \subset \mathbb{Q}[\alpha][[h]]$ where $q = e^h$, $A = e^{h\alpha}$. □

Now let us present the quantum algebra setup used by Habiro. Our algebra and completion was defined for the sole purpose of getting a nice form for our unified invariant, allowing us to factorize it at each roots of unity. Habiro's quantum algebra setup has been studied more in detail, and thus we know more of its proprieties.

Let $\mathcal{U}_{Hab}$ be the $\mathbb{Z}[q^{\pm 1}]$ subalgebra of U_h generated by elements $K^{\pm 1},\ e,\ F^{[n]}$ where $e = \{1\}E$ and $F^{[n]} = \frac{F^n}{\{n\}!}$. Let $\tilde{J}_n$ be the ideal generated by elements $e^i\{H+m; n-i\}$ for all $m \in \mathbb{Z}$.

We denote $\hat{\mathcal{U}}_{Hab} := \lim\limits_{\overleftarrow{n}} \dfrac{\mathcal{U}_{Hab}}{\tilde{J}_n}$.

Remark 104. In Habiro's article [15], $\mathcal{U}_{Hab}$ correspond to his notation $\mathcal{U}$ and $\hat{\mathcal{U}}_{Hab}$ to $\hat{\mathcal{U}}$. The filtration $\tilde{J}_n$ is denoted $\mathcal{U}_n$.

Remark 105. The differences between $\mathcal{U}_{Hab}$ and $\mathcal{U}$ is only a change of variable $e = \{1\}E$ and $F^{[n]} = \frac{F^n}{\{n\}!}$ nevertheless, the filtration for the completion differs since we use powers of e instead of divided powers of F.

Using Prop 6.8 and 6.9 in[15], we have:

Proposition 106. *We have an injective map*

$$\hat{\mathcal{U}}_{Hab} \to U_h.$$

Now that his algebra setup is stated, let us make the connection with our unified invariant $F_\infty(q, A, \mathcal{K})$. To do so, we will build a unified invariant with Habiro's setup and prove that it is in fact $F_\infty(q, A, \mathcal{K})$.

First remark that we have a similar property for the universal invariant of U_h in the Habiro setup:

Remark 107. If $\mathcal{K}$ is a knot and D a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$, then:

$$Q^{U_h}(\mathcal{K}) = q^{f\frac{H^2}{2}} Q^{\tilde{\mathcal{U}}}(D)$$

with $Q^{\tilde{\mathcal{U}}}(D) \in \hat{\mathcal{U}}_{Hab}$ and f is the writhe of the diagram.

Recall the Verma module V^α defined in 2.2.2 as a $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$-module and its completed version $\widehat{V^\alpha}$.

We can endow V^α with a $\mathcal{U}_{Hab}$-module structure and we denote it V^α_{Hab}:

$$ev_0 = 0, \quad ev_{i+1} = \{\alpha - i\}_q v_i, \quad Kv_i = q^{\alpha - 2i} v_i, \quad F^{[n]} v_i = \begin{bmatrix} n+i \\ i \end{bmatrix}_q v_{n+i}.$$

Moreover since $\tilde{J}_n V^\alpha_{Hab} \subset I_n V^\alpha_{Hab}$, we can naturally endow $\widehat{V^\alpha}$ with a $\hat{\mathcal{U}}_{Hab}$-module structure and we denote it $\widehat{V^\alpha}_{Hab}$.

Then, following the same steps as in Subsection 2.2.2, we have the subsequent construction.

Remark 108. Let $\mathcal{K}$ be a knot and D is a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$. We define the element $F^{Hab}_\infty(q, A, \mathcal{K})$ as the scalar action of the universal invariant on the previously defined Verma module. In other words,

$$Q^{U_h}(\mathcal{K}) v_0 = q^{f\frac{H^2}{2}} Q^{\tilde{\mathcal{U}}}(D) v_0 = F^{Hab}_\infty(q, A, \mathcal{K}) v_0$$

and $q^{-f\frac{\alpha^2}{2}} F^{Hab}_\infty(q, A, \mathcal{K}) \in \widehat{\mathbb{Z}[q, q^\alpha]}$.

Let $V^\alpha_h = V^\alpha \otimes_R \mathbb{Q}[\alpha][[h]]$.

We define the *h-adic completed Verma module* $\widehat{V^\alpha_h} = \lim\limits_{\overleftarrow{n}} \dfrac{V^\alpha_h}{h^n V^\alpha_h}$.

In other words, $\widehat{V^\alpha_h}$ is the $\mathbb{Q}[\alpha][[h]]$-module topologically generated by vectors $\{v_0, v_1, \dots\}$, and we endow it with a U_h-module structure:

$$Ev_0 = 0, \quad Ev_{i+1} = v_i, \quad Hv_i = (\alpha - 2i) v_i, \quad Fv_i = [\alpha - i]_q v_{i+1}.$$

We can also use another useful topological basis $\{w_0, w_1, \dots\}$ such that $w_i = [\alpha; i]_q v_i$ and get hence :

$$Ew_0 = 0, \quad Ew_{i+1} = [\alpha - i]_q w_i, \quad Hw_i = (\alpha - 2i) w_i, \quad Fw_i = w_{i+1}.$$

Proposition 109. *$\widehat{V^\alpha} \subset \widehat{V^\alpha_h}$ as $\tilde{\mathcal{U}}$-modules and $\widehat{V^\alpha}_{Hab} \subset \widehat{V^\alpha_h}$ as $\hat{\mathcal{U}}_{Hab}$-modules.*

Proof. We have a well defined $\tilde{\mathcal{U}}$-modules map $i : \widehat{V^\alpha} \to \widehat{V^\alpha_h}$, $v_i \mapsto v_i$ since $I_n V^\alpha \subset h^n V^\alpha_h$ and the action of $\tilde{\mathcal{U}}$ coincide on the topological basis. Moreover,

$$\begin{aligned}\widehat{V^\alpha} &= \{(\overline{u_k})_{k\in\mathbb{N}^*} \in \frac{V^\alpha}{I_k V^\alpha} |\ \overline{u_{k+1}} = \overline{u_k} \mod I_k V^\alpha\} \\ &= \{(\sum_n \overline{a_{k,n}} v_n)_{k\in\mathbb{N}^*} \in \frac{V^\alpha}{I_k V^\alpha} |\ \overline{a_{k,n}} \in \frac{R}{I_k},\ (\overline{a_{k,n}})_{n\in\mathbb{N}} \text{ have finite support}, \\ &\qquad \overline{a_{k+1,n}} = \overline{a_{k,n}} \mod I_k\}.\end{aligned}$$

Note that $a_n := (a_{k,n})_{k\in\mathbb{N}^*} \in \widehat{\mathbb{Z}[q,q^\alpha]}$ by definition.

If $u = (\overline{u_k})_{k\in\mathbb{N}^*} = (\sum_n \overline{a_{k,n}} v_n)_{k\in\mathbb{N}^*} \in \widehat{V^\alpha}$ is such that $i(u) = 0$ then, using $j : \widehat{\mathbb{Z}[q,q^\alpha]} \hookrightarrow \mathbb{Q}[\alpha][[h]]$,

$$\forall n \in \mathbb{N},\ j(a_n) = 0$$

Hence, by injectivity,

$$\forall n \in \mathbb{N},\ a_n = 0$$

Thus, $u = 0$ and the map i is injective.

We proceed in the same fashion with $i : \widehat{V^\alpha}_{Hab} \to \widehat{V^\alpha_h}$, $v_i \mapsto w_i$. □

Then $F^{Hab}_\infty(q, A, \mathcal{K}) = F_\infty(q, A, \mathcal{K})$ as elements of $\mathbb{Q}[\alpha][[h]]$.

Furthermore they both belong to $\widehat{\mathbb{Z}[q,q^\alpha]} \subset \mathbb{Q}[\alpha][[h]]$, hence $F^{Hab}_\infty(q, A, \mathcal{K}) = F_\infty(q, A, \mathcal{K})$ as elements of $\widehat{\mathbb{Z}[q,q^\alpha]}$.

The unified invariant and the two variable colored Jones invariant are the same

In [16] section 7, Habiro defines a two-variable colored Jones invariant unifying the colored Jones polynomials.

Remark 110. Recall that the notation v (resp q) in [16] correspond to the notation q (resp q^2) in this article.

This two-variable colored Jones invariant is defined as the universal invariant of a long knot seen in a completion ring

$$\varprojlim_k \mathbb{Z}[q, q^{-1}, t, t^{-1}] / \left(\prod_{-k<i<k} (t - q^{2i}) \right)$$

where t is identified, using the Casimir element $C = \{1\}^2 FE + qK + q^{-1}K^{-1}$, to

$$C^2 = t + t^{-1} + 2.$$

Now the Casimir element is in the center of $\mathcal{U}_{Hab}$ and we have $Cv_0 = (q^{\alpha+1} + q^{-\alpha-1})v_0$ where v_0 is the highest weight vector of V^α. Hence we get an action :

$$\varprojlim_k \mathbb{Z}[q, q^{-1}, t, t^{-1}] / \left(\prod_{-k<i<k} (t - q^{2i}) \right) \to \mathrm{End}(\widehat{V^\alpha}_{Hab})$$

with $t.v_0 = q^{2(\alpha+1)} v_0$.

Let's come back to the unified invariant. For a 0-framed knot $\mathcal{K}$, we have :

$$Q^{U_h}(\mathcal{K}) \in Z(\hat{\mathcal{U}}_{Hab}).$$

We also have, using Theorem 9.1 and Subsection 9.2 in [16],

$$Z(\hat{\mathcal{U}}_{Hab}) \hookrightarrow \hat{\mathcal{U}}^0.$$

Since $\hat{\mathcal{U}}^0 \cong \widehat{\mathbb{Z}[q,q^\alpha]}$, we have the following equality in $\widehat{\mathbb{Z}[q,q^\alpha]}$:

$$f(Q^{U_h}(\mathcal{K})) = F_\infty(q, q^\alpha, \mathcal{K})$$

where $f : Z(\hat{\mathcal{U}}_{Hab}) \to \widehat{\mathbb{Z}[q,q^\alpha]}$ is the scalar action of central elements on $\widehat{V^\alpha}_{Hab}$.

In other words, the two-variable colored Jones invariant $J_\mathcal{K}(t,q)$ defined in [16] (section 7) verifies:

$$J_\mathcal{K}(q^{2(\alpha+1)}, q^2) = F_\infty(q, q^\alpha, \mathcal{K})$$

as elements in $\widehat{\mathbb{Z}[q,q^\alpha]}$.

3.2 Study of C_∞ and uniqueness of the unified invariant

3.2.1 Study of C_∞

To study the factorization of ADO polynomials, we will use the evaluation into colored Jones polynomials along with h-adic properties. Indeed, the Melvin-Morton-Rozansky conjecture proved by Bar-Natan and Garoufalidis in [2] makes the junction between the inverse of the Alexander polynomial and the colored Jones polynomials. We will use the h-adic version that we state below in Theorem 111 (see [8] Theorem 2).

We denote by $A_\mathcal{K}(t)$ the Alexander polynomial of the knot $\mathcal{K}$, with normalisation $A_{unknot}(t) = 1$ and $A_\mathcal{K}(1) = 1$.

Theorem 111. *(Bar-Natan, Garoufalidis)*
For $\mathcal{K}$ a knot, we have the following equality in $\mathbb{Q}[[h]]$:

$$\lim_{n\to\infty} J_n(e^{h/n}) = \frac{1}{A_\mathcal{K}(e^h)}.$$

For the sake of simplicity, let's assume that the knot $\mathcal{K}$ is 0-framed so $f = 0$.

Now, note that since $F_\infty(q, q^n, \mathcal{K}) = J_n(q^2, \mathcal{K}) \in \mathbb{Z}[q^{\pm 1}]$ and $\mathbb{Z}[q^{\pm1}] \subset \mathbb{Q}[[h]]$ via $q \mapsto e^h$, we have a map $\mathbb{Q}[[h]] \to \mathbb{Q}[[h]]$, $h \mapsto \frac{h}{n}$ that sends $F_\infty(q, q^n, \mathcal{K}) \mapsto F_\infty(q^{1/n}, q, \mathcal{K})$ as elements of $\mathbb{Q}[[h]]$. But now $F_\infty(q^{1/n}, q, \mathcal{K})$ converges to $F_\infty(1, q, \mathcal{K})$ in the sense stated in [8] below Theorem 2, namely:

$$\begin{aligned} &\lim_{n\to\infty} F_\infty(q^{1/n}, q, \mathcal{K}) = F_\infty(1, q, \mathcal{K}) \\ \iff &\lim_{n\to\infty} \text{coeff}(F_\infty(q^{1/n}, q, \mathcal{K}), h^m) = \text{coeff}(F_\infty(1, q, \mathcal{K}), h^m),\ \forall m \in \mathbb{N}, \end{aligned}$$

where, for any analytic function f, $\text{coeff}(f(h), h^m) = \frac{1}{m!}\frac{d^m}{dh^m} f(h)|_{h=0}$.

By Theorem 111, $F_\infty(1, q, \mathcal{K}) = \frac{1}{A_\mathcal{K}(q^2)}$ in $\mathbb{Q}[[h]]$.

On the other hand, if we denote $\widehat{\mathbb{Z}[A^{\pm1}]^{\{1\}_A}}$ the ring completion of $\mathbb{Z}[A^{\pm1}]$ by ideals $((A - A^{-1})^n)$, then $F_\infty(1, A, \mathcal{K}) = C_\infty(1, A, \mathcal{K})$ in $\widehat{\mathbb{Z}[A^{\pm1}]^{\{1\}_A}}$. Indeed setting $q = 1$ in Definition 71 and looking at the definition of $C_\infty(1, A, \mathcal{K})$ in Proposition 78, one gets $F_\infty(1, A, \mathcal{K}) = C_\infty(1, A, \mathcal{K})$.

Since $\widehat{\mathbb{Z}[A^{\pm1}]^{\{1\}_A}} \hookrightarrow \mathbb{Q}[[h]]$ via $A \mapsto e^h$ (see [15] Proposition 6.1 and [13] Corollary 4.1), then we have the following proposition:

Proposition 112. *If $\mathcal{K}$ is 0-framed, $C_\infty(1, A, \mathcal{K}) = \frac{1}{A_\mathcal{K}(A^2)}$.*

By the discussion in the paragraph preceding Proposition 82, we have:

Corollary 113. *If $\mathcal{K}$ is 0-framed, $C_\infty(r, A, \mathcal{K}) = \frac{1}{A_\mathcal{K}(A^{2r})}$.*

This allows us to state a factorization theorem:

Theorem 114. *(Factorization)*
For a knot $\mathcal{K}$ and an integer $r \in \mathbb{N}^*$, we have the following factorization in $\hat{R}^I_r$:

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{A^{rf} \times ADO_r(A, \mathcal{K})}{A_\mathcal{K}(A^{2r})}$$

where f is the framing of the knot.

Corollary 115. *As an immediate consequence, Conjecture 7.5 and subsequent paragraph in [16] are verified.*
In other words:

$$J_\mathcal{K}(t, -1) = \frac{A_\mathcal{K}(-t)}{A_\mathcal{K}(t^2)}$$

where $t = q^{2(\alpha+1)}$.
Or in terms of unified invariant:

$$F_\infty(i, q^\alpha, \mathcal{K}) = \frac{A_\mathcal{K}(q^{2\alpha})}{A_\mathcal{K}(q^{4\alpha})}.$$

Proof. Recall that $ADO_2(q^\alpha, \mathcal{K}) = A_\mathcal{K}(q^{2\alpha})$.
At $r = 2$, using the identification $t = q^{2(\alpha+1)}$, we have the following:

$$\begin{aligned} J_\mathcal{K}(t, -1) &= F_\infty(i, q^\alpha, \mathcal{K}) \\ &= \frac{ADO_2(q^\alpha, \mathcal{K})}{A_\mathcal{K}(q^{4\alpha})} \\ &= \frac{A_\mathcal{K}(q^{2\alpha})}{A_\mathcal{K}(q^{4\alpha})} \\ &= \frac{A_\mathcal{K}(-t)}{A_\mathcal{K}(t^2)}. \end{aligned}$$

□

Putting all the pieces together,

Theorem 116. *(Unified invariant) Let $\mathcal{K}$ be a 0-framed knot, the unified invariant $F_\infty(q, q^\alpha, \mathcal{K})$ verifies:*

$$F_\infty(q, q^N, \mathcal{K}) = J_N(q^2, \mathcal{K})$$

where J_N is the N-th colored Jones polynomial associated to the $N+1$ dimensional simple $U_q(\mathfrak{sl}_2)$-module normalised by $J_N(q, unknot) = 1$. Furthermore,

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \frac{ADO_r(q^\alpha, \mathcal{K})}{A_\mathcal{K}(q^{2r\alpha})}$$

where ADO_r is the r-th ADO polynomial associated to quantum $\mathfrak{sl}_2$ at root of unity $q = \zeta_{2r}$ normalised by $ADO_r(q^\alpha, unknot) = 1$ and $A_\mathcal{K}$ is the Alexander polynomial with normalisation $A_{unknot}(t) = 1$ and $A_\mathcal{K}(1) = 1$.

3.2.2 The equivalence between the Jones and ADO families

One may ask what is the relationship between ADO invariants and the colored Jones polynomials. Does one family of polynomial determines completely the other? For the sake of simplicity the knot $\mathcal{K}$ is supposed 0 framed in this paragraph.

One way is already known. Knowing the ADO polynomials $\{ADO_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*}$ allow to find the colored Jones polynomials $\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}$. This result was stated in [6] Corollary 15.
With our setup, we can get back this result as follows.

Remark 117. Notice that $F_\infty(\zeta_{2r}, \zeta_{2r}^N, \mathcal{K}) = J_N(\zeta_r, \mathcal{K}) = \frac{ADO_r(\zeta_{2r}^N, \mathcal{K})}{A_{\mathcal{K}}(1)} = ADO_r(\zeta_{2r}^N, \mathcal{K})$, $(A_r(1) = 1)$.

Since J_N is a polynomial knowing an infinite number of values of it determines it. Given the family of polynomials $\{ADO_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*}$ we then know each value of J_N at any root of unity hence we know J_N entirely.

But we can also have the other way around. Knowing only the colored Jones polynomials $\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}$, one recovers the ADO polynomials $\{ADO_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*}$. We will prove it by seeing that the colored Jones polynomials determine the unified invariant $F_\infty(q, A, \mathcal{K})$.

Remark 118. Alternatively, The previous assertion may also be proved using the identification with the two-variable colored Jones invariant and Habiro's work of it being equivalent to the universal invariant.

Let $\forall k \in \mathbb{N}$, $f_k : \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[[h]]$, $\alpha \mapsto k$ the evaluation map.

Proposition 119. $\underset{k\in\mathbb{N}}{\cap} ker(f_k) = \{0\}$.

Proof. Let $x \in \underset{k\in\mathbb{N}}{\cap} ker(f_k)$, we write $x = \sum_n g_n(\alpha)h^n$ where $g_n(\alpha) \in \mathbb{Q}[\alpha]$.
Then, for each $k \in \mathbb{N}$, we have that $g_n(k) = 0\ \forall n$. Since g_n are polynomials that vanish at an infinite number of points, they are 0.
Hence $\underset{k\in\mathbb{N}}{\cap} ker(f_k) = \{0\}$. □

Let $f : \mathbb{Q}[\alpha][[h]] \to \prod_{k\in\mathbb{N}} \mathbb{Q}[[h]]$, $x \mapsto (f_k(x))_{k\in\mathbb{N}}$. $ker(f) = \underset{k\in\mathbb{N}}{\cap} ker(f_k) = \{0\}$, hence f is injective.

Remark 120. For any knot $\mathcal{K}$, $f(F_\infty(q, A, \mathcal{K})) = \{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}$.

Proposition 121. *For any knot $\mathcal{K}$, $F_\infty(q, A, \mathcal{K}) = f^{-1}(\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*})$.*

Setting $g : \widehat{\mathbb{Z}[q, q^\alpha]} \to \prod_{r\in\mathbb{N}^*} \hat{R}_r^I$, $x \mapsto (ev_r \times \frac{1}{FC_r}(x))_{r\in\mathbb{N}^*}$, we get the following theorem:

Theorem 122. *The map $\mu = g \circ f^{-1} : Im(f|_{\widehat{\mathbb{Z}[q,q^\alpha]}}) \to \prod_{r\in\mathbb{N}^*} \hat{R}_r^I$ is such that for every knot $\mathcal{K}$,*

$$\{ADO_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*} = \mu(\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}).$$

Theorem 123. *For a knot $\mathcal{K}$ with framing f, there exists a unique element $a \in \widehat{\mathbb{Z}[q, q^\alpha]}$ such that:*

- *For an infinite number of k, $f_k(a) = J_n(q^2, \mathcal{K})$,*
- *Or, for an infinite number of r, $ev_r(a) = \frac{A^{rf} \times ADO_r(A,\mathcal{K})}{A_{\mathcal{K}}(A^{2r})}$.*

This element is $F_\infty(q, A, \mathcal{K})$.

3.2.3 Symmetry of the unified invariant

We can show a symmetry of the unified invariant by studying the the Verma module at integer weight.

Lemma 124. *For $N \in \mathbb{N}^*$, we have the isomorphism of $\tilde{\mathcal{U}}$ modules:*

$$V^{-N-2} \cong V^N / S_N.$$

Proof. If we set $(v_i)_i$ the basis of V^N, $(\overline{v_i})_i$ the basis of the quotient V^N/S_N, we get :

$$\begin{aligned} E\overline{v_{N+1}} &= 0 \\ E\overline{v_{N+1+i+1}} &= \overline{v_{N+1+i}} \\ K\overline{v_{N+1+i}} &= q^{-N-2-2i}\overline{v_{N+1+i}} \\ F^{(n)}\overline{v_{N+1+i}} &= \begin{bmatrix} n+N+1+i \\ n \end{bmatrix}_q \{-i-1; n\}_q \overline{v_{N+1+n+i}} \end{aligned}$$

We can transform a bit the last equation using:

$$\{n+N+1+i;n\}_q\{-i-1;n\}_q = \{i+n;n\}_q\{-N-2-i;n\}_q$$

Hence,

$$F^{(n)}\overline{v_{N+1+i}} = \begin{bmatrix} n+i \\ n \end{bmatrix}_q \{-N-2-i;n\}_q\overline{v_{N+1+n+i}}$$

Setting $w_i := \overline{v_{N+1+i}}$ one gets the exact definition of the Verma type $\tilde{\mathcal{U}}$ module at $q^\alpha = q^{-N-2}$. □

Using this lemma, we can prove the following fact:

Proposition 125. *Let $\mathcal{K}$ be a 0 framed knot,*

$$F_\infty(q, q^\alpha, \mathcal{K}) = F_\infty(q, q^{-\alpha-2}, \mathcal{K}).$$

Proof. The element $Q^{\tilde{\mathcal{U}}}(\mathcal{K})$ being central, we get on V^N the following identities:

$$Q^{\tilde{\mathcal{U}}}(\mathcal{K})v_0 = F_\infty(q, q^N, \mathcal{K})v_0$$

and

$$Q^{\tilde{\mathcal{U}}}(\mathcal{K})v_{N+1} = F_\infty(q, q^N, \mathcal{K})v_{N+1}.$$

But using Lemma 124, we also have :

$$Q^{\tilde{\mathcal{U}}}(\mathcal{K})v_{N+1} = F_\infty(q, q^{-N-2}, \mathcal{K})v_{N+1}.$$

Thus for all $N \in \mathbb{N}^*$, we have

$$F_\infty(q, q^N, \mathcal{K}) = F_\infty(q, q^{-N-2}, \mathcal{K}).$$

Using Proposition 119, we have the equality at formal weight q^α:

$$F_\infty(q, q^\alpha, \mathcal{K}) = F_\infty(q, q^{-\alpha-2}, \mathcal{K}).$$

□

Corollary 126. *Let $\mathcal{K}$ be a 0 framed knot,*

$$ADO_r(q^\alpha, \mathcal{K}) = ADO_r(q^{-\alpha-2}, \mathcal{K}).$$

Proof. At $q = \zeta_{2r}$, we have the factorisation:

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \frac{A^{rf} \times ADO_r(A, \mathcal{K})}{A_\mathcal{K}(A^{2r})}$$

and since

$$A_\mathcal{K}(A^{2r}) = A_\mathcal{K}(A^{-2r})$$

one gets the desired identity. □

3.3 Holonomy and loop expansion formula

3.3.1 The ADO and unified invariant are holonomic

The fact that the colored Jones polynomials are q-holonomic was proved in [9]. Let us state what it means and then let's prove that the unified invariant and the ADO polynomials verify the same holonomic rule. For the sake of simplicity we will work with 0-framed knot.

Let $Q : \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*} \to \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*}$ and $E : \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*} \to \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*}$ such that:

$$(Qf)(n) = q^{2n}f(n), \quad (Ef)(n) = f(n+1).$$

Note that these operators can be extended to operators on $\mathbb{Q}[[h]]^{\mathbb{N}^*}$.

Let us denote $J_\bullet(q^2,\mathcal{K}) = \{J_n(q^2,\mathcal{K})\}_{n\in\mathbb{N}^*}$ the colored Jones function. Now, from Theorem 1 in [9], for any knot $\mathcal{K}$ there exists a polynomial $\alpha_\mathcal{K}(Q,E,q^2)$ such that $\alpha_\mathcal{K}(Q,E,q^2)J_\bullet(q^2,\mathcal{K}) = 0$. We say that $J_\bullet(q^2,\mathcal{K})$ is q-holonomic.

We define similar operators on $\mathbb{Q}[\alpha][[h]]$ and show that the same polynomial $\alpha_\mathcal{K}$, taken in terms of those new operators, annihilates $F_\infty(q,q^\alpha,\mathcal{K})$.

Let $\tilde{Q} : \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[\alpha][[h]]$ and $\tilde{E} : \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[\alpha][[h]]$ such that if we take $x(\alpha) := \sum_{k=0}^{+\infty} x_k(\alpha)h^k \in \mathbb{Q}[\alpha][[h]]$ with $x_k(\alpha) \in \mathbb{Q}[\alpha]$:

$$\tilde{Q}(x(\alpha)) = q^{2\alpha}x(\alpha), \quad \tilde{E}(x(\alpha)) = x(\alpha+1).$$

where $x(\alpha+1) := \sum_{k=0}^{+\infty} x_k(\alpha+1)h^k$

Remark 127. Here, $\tilde{Q}$ is just the multiplication of any element with $q^{2\alpha}$.

Notice that if you take the injective map $f : \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[[h]]^{\mathbb{N}^*}$, $x(\alpha) \mapsto (x(k))_{k\in\mathbb{N}^*}$ defined in Subsection 3.2.2 , you have:

$$f \circ \tilde{Q} = Q \circ f, \quad f \circ \tilde{E} = E \circ f.$$

Hence, $f \circ \alpha_\mathcal{K}(\tilde{Q},\tilde{E},q^2) = \alpha_\mathcal{K}(Q,E,q^2)\circ f$. Since $\alpha_\mathcal{K}(Q,E,q^2)J_\bullet(q^2,\mathcal{K}) = 0$ and $f(F_\infty(q,q^\alpha,\mathcal{K})) = J_\bullet(q^2,\mathcal{K})$ we obtain $f \circ \alpha_\mathcal{K}(\tilde{Q},\tilde{E},q^2)(F_\infty(q,q^\alpha,\mathcal{K})) = 0$. The injectivity of f gives the following theorem.

Theorem 128. *For any* 0*-framed knot* $\mathcal{K}$*,* $\alpha_\mathcal{K}(\tilde{Q},\tilde{E},q^2)(F_\infty(q,q^\alpha,\mathcal{K})) = 0$.

Now let us look at what happens at roots of unity. To do so we must restrict ourselves to a ring allowing evaluation at roots of unity such as $\widehat{\mathbb{Z}[q,q^\alpha]}$.

Since $\tilde{Q}(I_n) \subset I_n$ and $\tilde{E}(I_n) \subset I_n$, using Proposition 103 we can restrict the operators $\tilde{Q}$ and $\tilde{E}$ to $\widehat{\mathbb{Z}[q,q^\alpha]}$, for the sake of simplicity we will still write them $\tilde{Q} : \widehat{\mathbb{Z}[q,q^\alpha]} \to \widehat{\mathbb{Z}[q,q^\alpha]}$ and $\tilde{E} : \widehat{\mathbb{Z}[q,q^\alpha]} \to \widehat{\mathbb{Z}[q,q^\alpha]}$.

Now let $r \in \mathbb{N}^*$, let $\overline{Q} : \hat{R}_r^I \to \hat{R}_r^I$ and $\overline{E} : \hat{R}_r^I \to \hat{R}_r^I$ such that if we take $x(\alpha) = \sum_{k=0}^{\infty} x_k(\alpha)\{r\alpha\}^k \in \hat{R}_r^I$ with $x_k(\alpha) \in \mathbb{Z}[\zeta_{2r}, A]$ (recall that we denote $q^\alpha := A$):

$$\overline{Q}(x(\alpha)) = \zeta_{2r}^{2\alpha}x(\alpha), \quad \overline{E}(x(\alpha)) = x(\alpha+1)$$

where $x(\alpha+1) = \sum_{k=0}^{\infty} x_k(\alpha+1)(-1)^k\{r\alpha\}^k$.

Since $ev_r \circ \tilde{Q} = \overline{Q} \circ ev_r$ and $ev_r \circ \tilde{E} = \overline{E} \circ ev_r$, the same formula holds:

$$\alpha_\mathcal{K}(\overline{Q},\overline{E},\zeta_{2r}^2)(F_\infty(\zeta_{2r},q^\alpha,\mathcal{K})) = 0.$$

By Theorem 114, $\alpha_\mathcal{K}(\overline{Q},\overline{E},\zeta_{2r}^2)(\frac{ADO_r(q^\alpha,\mathcal{K})}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})}) = 0$.

Remark 129. We have the following identities:

$\overline{Q}(\frac{ADO_r(q^\alpha,\mathcal{K})}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})}) = q^{2\alpha}\frac{ADO_r(q^\alpha,\mathcal{K})}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})} = \frac{\overline{Q}(ADO_r(q^\alpha,\mathcal{K}))}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})}$

$\overline{E}(\frac{ADO_r(q^\alpha,\mathcal{K})}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})}) = \frac{\overline{E}(ADO_r(q^\alpha,\mathcal{K}))}{\overline{E}(A_\mathcal{K}(\zeta_{2r}^{2r\alpha}))} = \frac{\overline{E}(ADO_r(q^\alpha,\mathcal{K}))}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})}$ (because $\zeta_{2r}^{2r(\alpha+1)} = \zeta_{2r}^{2r\alpha}$)

Hence $\frac{\alpha_\mathcal{K}(\overline{Q},\overline{E},\zeta_{2r}^2)(ADO_r(q^\alpha,\mathcal{K}))}{A_\mathcal{K}(\zeta_{2r}^{2r\alpha})} = 0$, which proves the following theorem:

Theorem 130. *For any* 0*-framed knot* $\mathcal{K}$*,* $\alpha_\mathcal{K}(\overline{Q},\overline{E},\zeta_{2r}^2)(ADO_r(q^\alpha,\mathcal{K})) = 0$.

3.3.2 The loop expansion formula of the unified invariant

Let's first introduce the loop expansion of the colored Jones polynomials (see section 2 in [8]). We can write the colored Jones polynomials as an expansion (see [25] for more details):

$$J_n(e^{2h}, \mathcal{K}) = \sum_{k=0}^{+\infty} \frac{P_k(e^{2nh})}{A_{\mathcal{K}}(e^{2nh})^{2k+1}} h^k$$

where $P_k(X) \in \mathbb{Q}[X, X^{-1}]$.

Hence we get an element:

$$J_\alpha(q^2, \mathcal{K}) = \sum_{k=0}^{+\infty} \frac{P_k(e^{2\alpha h})}{A_{\mathcal{K}}(e^{2\alpha h})^{2k+1}} h^k \in \mathbb{Q}[\alpha][[h]]$$

that is such that $f(J_\alpha(q^2, \mathcal{K})) = J_\bullet(q^2, \mathcal{K})$.
This means that it evaluates into the colored Jones at $\alpha = n$, we call it *loop expansion of the colored Jones function.*

Proposition 131. *For any knot $\mathcal{K}$, we have the following identity in $\mathbb{Q}[\alpha][[h]]$:*

$$J_\alpha(q^2, \mathcal{K}) = F_\infty(q, q^\alpha, \mathcal{K}).$$

Proof. The fact that f is injective proves the proposition. □

Remark 132. Putting everything together, this subsection implies that the unified invariant $F_\infty(q, A, \mathcal{K})$ is an integral version of the colored Jones function, built in a ring allowing evaluations at roots of unity. The integrality and the existence of evaluation maps allow us to recover the ADO polynomials, the fact that the completion ring is a subring of an h-adic ring allows us to connect it to other notions of colored Jones function/ invariants.

Another approach, described by Gukov and Manolescu in [12], would be to see the unified invariant as a power series in q, A (as opposed to a quantum factorial expansion as we have here). This would be another integral version of it.
Indeed, because it verifies Proposition 131 and Theorem 128, the unified invariant $F_\infty(q, A, \mathcal{K})$ except being a power series, also verifies conjecture 1.5 and 1.6 in [12].
Thus, if $F_\infty(q, A, \mathcal{K})$ could be written as a power serie, it would fully verify the conjectures.
This is the case for positive braid knots, as show by Park in [24]. This means that for a positive braid knot, the unified invariant and the GM power series coincide.

3.4 Vassiliev invariants and the ADO polynomials

3.4.1 Vassiliev invariants

Let K be the free abelian group generated by the oriented knots in S^3. Let K_d be the subgroup generated by singular knots with d double points using the identification :

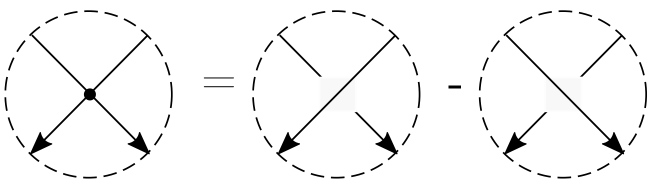


Figure 3.1: Relation embedding singular knots in K

where, using a diagram of the singular knot, the pictures in Figure 3.1 correspond to a local change at a neighborhood of the double point.
The knots in K are endowed with the 0 framing.

A linear map $\nu : \mathsf{K} \to \mathbb{Q}$ is a *Vassiliev invariant of degree d* if $\nu|_{\mathsf{K}_{d+1}} = 0$.

Nota bene. In this definition, a Vassiliev invariant of degree d is also of degree n for all $n \geq d$. This means that, in the rest of the article, a Vassiliev invariant of degree d must be understood as a Vassiliev invariant *of degree d or less.*
We do not give the smallest degree for any invariants.

Remark 133. Instead of free abelian groups, we could also consider $\mathbb{Q}$-vector spaces.

Example 134. The colored Jones polynomials, written as h-adic power series by setting $q = e^h$, have Vassiliev invariants coefficients. More precisely, if $\mathcal{K}$ is an oriented knot, we endow it with the 0 framing and we have:

$$J_N(q^2, \mathcal{K}) = \sum_d a_{N,d} h^d$$

where $a_{N,d}(\mathcal{K}) \in \mathbb{Q}$ and the map $\mathsf{K} \to \mathbb{Q}$, $\mathcal{K} \mapsto a_{N,d}(\mathcal{K})$ is a Vassiliev invariant of degree[1] d (see Corollary 7.5 in [23]).

Now let's define a broader notion: topological Vassiliev invariants.

Definition 135. Let G be an abelian group, $\mathcal{T}$ a topology on G.
We say that $v : \mathsf{K} \to G$ is a $\mathcal{T}$*-topological Vassiliev invariant* if $\forall V$ neighborhood of 0, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $v(\mathsf{K}_n) \subset V$.

Example 136.

- A Vassiliev invariant $v : \mathsf{K} \to \mathbb{Z}$ is a discrete topological Vassiliev invariant.
- The colored Jones invariant $J_N(e^{2h}, \mathcal{K})$ is a h-adic topological Vassiliev invariant.

3.4.2 Integral power series approach

We now transpose the unified invariant into a power series ring intermediary between $\widehat{\mathbb{Z}[q, q^\alpha]}$ and $\mathbb{Q}[[h]]$, allowing us on one hand to keep some integral features and evaluate eventually at root of unity, and on the other hand to see it as a series with Vassiliev invariant coefficients.

Proposition 137. *The natural map* $\widehat{\mathbb{Z}[q, q^\alpha]} \hookrightarrow \mathbb{Z}[[q-1, q^\alpha - 1]]$ *is injective.*

Proof. See proof of Proposition 103. □

Moreover from Theorem 116, we have $F_\infty(q, q^N, \mathcal{K}) = J_N(q^2, \mathcal{K})$, thus we can write the unified invariant in q^2:

$$F_\infty(q, q^\alpha, \mathcal{K}) = \sum_{n,m \geq 0} b_{n,m}(\mathcal{K})(q^2-1)^n(q^{2\alpha}-1)^m$$

where $b_{n,m} \in \mathbb{Z}$.

Remark 138. We can write $q^{-\alpha} = \sum_{m \geq 0}(1-q^\alpha)^m$.

Remark 139. We have for all $N \in \mathbb{N}$, $b_{0,0}(\mathcal{K}) = a_{N,0}(\mathcal{K})$.

Proposition 140. *The coefficient* $b_{n,m}(\mathcal{K})$ *is a Vassiliev invariant of degree*[1] $n + m$.

Proof. Since $F_\infty(q, q^N, \mathcal{K}) = J_N(q^2, \mathcal{K})$, we proceed by induction on $d = n + m$:
- $b_{0,0}(\mathcal{K}) = a_{N,0}(\mathcal{K})$.
- Let $d \in \mathbb{N}$, suppose that for every n, m such that $n + m < d$, $b_{n,m}$ is a Vassiliev invariant of degree[1] $n + m$.

Now let us understand $a_{N,d}$ in terms of $b_{n,m}$. Remark that if $n + m > d$ then $h^{d+1} | (q^2 - 1)^n(q^{2N}-1)^m$, thus $b_{n,m}$ doesn't contribute to $a_{N,d}$ whenever $n + m > d$. Moreover, if we have $n + m = d$, only the first term of the exponential expansion of $(q^2-1)^n(q^{2\alpha}-1)^m$ will contribute to $a_{N,d}$, which is $b_{n,m}2^{n+m}N^m$ for each n, m such that $n + m = d$.

[1] See Nota Bene Section 3.4
[1] See Nota Bene Section 3.4

We don't even have to look at the contribution coming from $n+m<d$, indeed on an element $K \in \mathsf{K}_{d+1}$, $b_{n,m}(K) = 0$ when $n+m<d$ by induction hypothesis. Thus, for all $N \in \mathbb{N}$, $0 = a_{N,d}(K) = \sum_{n+m=d} b_{n,m} 2^{n+m} N^m$ since $a_{N,d}(K)$ is a Vassiliev invariant of degree d.
Since this equality holds for all $N \in \mathbb{N}^*$ and since $a_{N,d}(K)$ is a polynomial in N, this means that $b_{n,m}(K) = 0$ $\forall n, m$ such that $n+m=d$. Hence they are Vassiliev invariants of degree $n+m$. □

Remark 141. As a corollary, the unified invariant $F_\infty(q, q^\alpha, \mathcal{K})$ is a topological Vassiliev invariant for the $((q-1)^n(q^\alpha-1)^m)_{n,m\in\mathbb{N}^*}$ filtration topology.

3.4.3 Recovering ADO

Now we must transpose these results to the *ADO* polynomials.

In the $\widehat{\mathbb{Z}[q,q^\alpha]}$ setup we could recover *ADO* by evaluating at roots of unity. Recall that we have the result from Theorem 123:

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \frac{ADO_r(q^\alpha, \mathcal{K})}{A_\mathcal{K}(q^{2r\alpha})}$$

where $A_\mathcal{K}(q^{2r\alpha})$ is the Alexander polynomial of $\mathcal{K}$.

We can also evaluate at roots of unity in the ring $\mathbb{Z}[[q-1, q^\alpha-1]]$, but we must be much more careful. Indeed, the codomain of such an evaluation is a (ζ_r-1)-adic completion, which is trivial if $\zeta_r - 1$ is invertible, hence we will need some conditions on r.

That being said, we will be able to study the factorisation at roots of unity in this setup and prove that the ADO polynomials are $\zeta_r - 1$ adic topological Vassiliev invariants.

We will then study the modulo r case, and show some asymptotic behaviour when r grows.

The study of the Alexander polynomial:
First of all we need to study the the product $F_\infty(q, A, \mathcal{K}) \times A_\mathcal{K}(q^{2r\alpha})$ as power series with Vassiliev invariant coefficients. We focus this first paragraph on the term $A_\mathcal{K}(q^{2r\alpha})$.

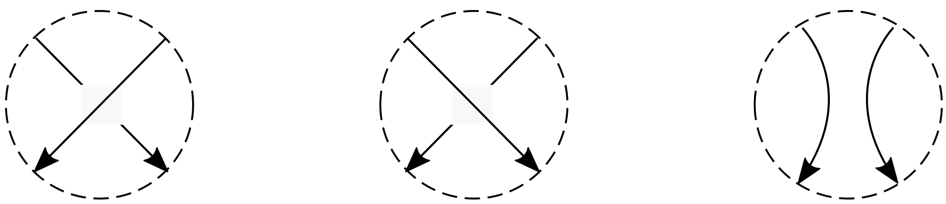
Let us write $A_\mathcal{K}(q^{2r\alpha})$ as a power series (it exists since it is a Laurent polynomial).

$$A_\mathcal{K}(q^{2r\alpha}) = \sum_m \lambda_m(\mathcal{K})(q^{2r\alpha}-1)^m = \sum_m \tilde{\lambda}_m(r,\mathcal{K})(q^{2\alpha}-1)^m.$$

The fact that $\lambda_m(\mathcal{K})$ are Vassiliev invariants comes from the skein relation verified by the Alexander Polynomial:

$$A_{\mathcal{K}_+}(q^{2\alpha}) - A_{\mathcal{K}_-}(q^{2\alpha}) = \{\alpha\} A_{\mathcal{K}_0}(q^{2\alpha})$$

where $\mathcal{K}_{+,-,0}$ are defined in Figure 3.2.



(a) $\mathcal{K}_+$ (b) $\mathcal{K}_-$ (c) $\mathcal{K}_0$

Figure 3.2: Local changes for a knot $\mathcal{K}$

Proposition 142. *The coefficient $\lambda_m(\mathcal{K})$ is a Vassiliev invariant of degree*[1] *m.*

Proof. From the skein relation we get that for any K_{d+1} singular knot with $d+1$ double points, $\{\alpha\}^{d+1} | A_{K_{d+1}}(q^{2\alpha})$. Hence $\lambda_d(K_{d+1}) = 0$, $\forall K_{d+1} \in \mathsf{K}_{d+1}$.

□

[1] See Nota Bene Section 3.4

Now we can prove that the $\tilde{\lambda}_m(r,\mathcal{K})$ are also Vassiliev invariants of degree m.

Lemma 143. *The $\tilde{\lambda}_m(r,\mathcal{K})$ are Vassiliev invariants of degree*[1] *m.*

Proof. Since $(q^{2\alpha}-1)^k|(q^{2r\alpha}-1)^k$, $\tilde{\lambda}_m(r,\mathcal{K})$ can be written as a linear combination of λ_k for $k\leq m$, and since they all vanish on K_{m+1}, $\tilde{\lambda}_m(\mathcal{K})$ is a Vassiliev invariant of degree m. □

The study of the product:
Now let's study the product $F_\infty(q,A,\mathcal{K})\times A_{\mathcal{K}}(q^{2r\alpha})$ using the multiplicativity of the Vassiliev invariants.

Lemma 144. *If μ and ν are Vassiliev invariants of degree n and m respectively, then $\mu\nu$ defined by $\mu\nu(\mathcal{K})=\mu(\mathcal{K})\nu(\mathcal{K})$ for any knot $\mathcal{K}$ is a Vassiliev invariant of degree $n+m$.*

Proof. See Corollary 3 in [28]. □

This means that the coefficients of the product $A_{\mathcal{K}}(q^{2r\alpha})\times F_\infty(q,q^\alpha,\mathcal{K})$ are Vassiliev invariants. More precisely we have:

$$\begin{aligned}A_{\mathcal{K}}(q^{2r\alpha})\times F_\infty(q,q^\alpha,\mathcal{K}) &= \sum_{m,n}\Big(\sum_{k=0}^{m}\tilde{\lambda}_k(r,\mathcal{K})b_{n,m-k}(\mathcal{K})\Big)(q^2-1)^n(q^{2\alpha}-1)^m\\ &= \sum_{m,n}c_{n,m}(r,\mathcal{K})(q^2-1)^n(q^{2\alpha}-1)^m\end{aligned}$$

where $c_{n,m}=\sum_{k=0}^{m}\tilde{\lambda}_k b_{n,m-k}$ are Vassiliev invariant of degree $n+m$.

The ADO polynomial:
Let's define the factorisation at roots of unity. First we need to define the evaluation map. Let

$$\mathbb{Z}[[\zeta_r-1]] := \lim_{\substack{\leftarrow\\ n}} \mathbb{Z}[\zeta_r]/(\zeta_r-1)^n.$$

The natural map:

$$j:\mathbb{Z}[\zeta_r]\to\mathbb{Z}[[\zeta_r-1]]$$

is injective iff $\bigcap_{n\in\mathbb{N}^*}(\zeta_r-1)^n=\{0\}$.

Moreover, we have a well defined surjective map

$$ev_r:\mathbb{Z}[[q-1,q^\alpha-1]]\to\mathbb{Z}[[\zeta_r-1,q^\alpha-1]]$$

where $\mathbb{Z}[[\zeta_r-1,q^\alpha-1]]:=\mathbb{Z}[[\zeta_r-1]][[q^\alpha-1]]$.

Then, one gets the equality :

$$j(ADO_r(q^\alpha,\mathcal{K}))=ev_r(A_{\mathcal{K}}(q^{2r\alpha})F_\infty(q,q^\alpha,\mathcal{K}))$$

It is essential to study for which r the map j is injective.

Lemma 145. *(ζ_r-1) is not invertible in $\mathbb{Z}[\zeta_r]$ if and only if r is a power of a prime number. More precisely, if $r=p^l$ is a power of a prime number $((\zeta_r-1)^{\varphi(r)})=(r)$ as ideals in $\mathbb{Z}[\zeta_r]$, where $\varphi(r)$ is the Euler phi function.*

Proof. See Lemma 1.4 for the case $r=p$ and the proof of Prop 2.8 for the case $r=p^a$, $a\in\mathbb{N}^*$ in [27] for the equivalence.

We denote $\Phi_d(X)$ is the d-th cyclotomic polynomial.

Using the case $n=p$ and the well know fact that, for all $1\leq m\leq l$, $\Phi_{p^l}(X)=\Phi_{p^m}(X^{p^{l-m}})$, we get $\Phi_{p^m}(1)=r$.

We get $\Phi_{p^l}(1)=r$, and since $\frac{1-\zeta_r^j}{1-\zeta_r}$ is invertible whenever $\gcd(j,r)=1$, $((\zeta_r-1)^{\varphi(r)})=(r)$. □

Corollary 146.

- *$\mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$ is non trivial iff r is a power of a prime number.*
- *j is injective iff r is a power of a prime number.*

Proof. The second point comes from the fact that, by Lemma 145, r is a power of a prime iff $\bigcap_{n\in\mathbb{N}^*} (\zeta_r - 1)^n = \{0\}$. □

Let us suppose now that $r = p^l$ is a power of a prime number, we have $j : \mathbb{Z}[\zeta_r] \subset \mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$, we will omit the j.

Hence we have the following equality in $\mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$:

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0} \sum_{n\geq 0} c_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n (q^{2\alpha} - 1)^m$$

and $c_{n,m}(r, \mathcal{K})$ is a Vassiliev invariant of degree $n + m$.

Thus we have the following theorem.

Theorem 147. *Let $\mathcal{K}$ be a 0-framed knot in S^3 and $r = p^l$ be a power of a prime number.*

There exist Vassiliev invariants $c_{n,m}(r, \mathcal{K}) \in \mathbb{Z}$ of degree[1] $n + m$, such that we can write

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0} \sum_{n\geq 0} c_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n (q^{2\alpha} - 1)^m$$

in $\mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$.

This means that the ADO polynomials $ADO_r(q^\alpha, \mathcal{K}) \in \mathbb{Z}[\zeta_r, q^\alpha]$ are topological Vassiliev invariants for the filtration $((\zeta_r - 1)^n (q^\alpha - 1)^m)_{n,m\in\mathbb{N}^*}$.

Nevertheless, as we show in the next paragraph, the coefficients of the expansion are not unique and it takes place in the completion ring $\mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]]$.

The r-adic form: We can compute an expansion for the ADO polynomials with unique coefficients. These coefficients will then be r-adic topological invariants.

Let φ be the Euler phi function.

For a power of a prime $r = p^l$, we have

$$\varphi(r) = (p - 1)p^{l-1}.$$

The ADO polynomial can be uniquely written as:

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0} \sum_{n=0}^{\varphi(r)-1} d_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n (q^{2\alpha} - 1)^m$$

where $d_{n,m} \in \mathbb{Z}$.

This comes from the fact that since $1, \zeta_r, \dots, \zeta_r^{\varphi(r)-1}$ form a basis of $\mathbb{Z}[\zeta_r]$ as a $\mathbb{Z}$ module, so do $1, \zeta_r - 1, \dots, (\zeta_r - 1)^{\varphi(r)-1}$.

Let's now define $\mathbb{Z}_r := \varprojlim_n \mathbb{Z}/(r)^n$ the ring of r adic integers.

Since $((\zeta_r - 1)^{n(\varphi(r)-1)}) = (r^n)$, we have an isomorphism

$$i : \mathbb{Z}[[\zeta_r - 1, q^\alpha - 1]] \to \mathbb{Z}_r[\zeta_r][[q^\alpha - 1]].$$

Now, if we write

$$\begin{aligned} ADO_r(q^\alpha, \mathcal{K}) &= \sum_{m,n\geq 0} c_{n,m}(r, \mathcal{K})(\zeta_r - 1)^n (q^{2\alpha} - 1)^m \\ &= \sum_{m\geq 0} \sum_{i=0}^{\varphi(r)-1} \sum_{j\geq 0} CL_{j,i,m}(r, \mathcal{K}) r^j (\zeta_r - 1)^i (q^{2\alpha} - 1)^m \end{aligned}$$

[1] See Nota Bene Section 3.4

where $CL_{j,i,m}(r,\mathcal{K})$ is a linear combination of $(c_{n,m}(r,\mathcal{K}))_n$.

Since, $\forall n \geq (j+1)\varphi(r)$, $c_{n,m}(r,\mathcal{K})(\zeta_r-1)^n \in r^{j+1}\mathbb{Z}[\zeta_r]$, then $CL_{j,i,m}(r,\mathcal{K})$ is a linear combination of $(c_{n,m}(r,\mathcal{K}))_{n<(j+1)\varphi(r)}$.

Hence, $CL_{j,i,m}(r,\mathcal{K})$ is a Vassiliev invariant of degree $(j+1)\varphi(r)+m-1$.

Remark 148. Note that $\mathbb{Z}[\zeta_r, q^{\pm\alpha}] \subset \mathbb{Z}_r[\zeta_r][[q^\alpha - 1]]$.

Theorem 149. *Let $\mathcal{K}$ be a 0-framed knot in S^3 and $r = p^l$ a power of a prime number.*

We can write in a unique way

$$ADO_r(q^\alpha,\mathcal{K}) = \sum_{m\geq 0}\sum_{n=0}^{\varphi(r)-1} d_{n,m}(r,\mathcal{K})(\zeta_r-1)^n(q^{2\alpha}-1)^m$$

in $\mathbb{Z}[\zeta_r, q^{\pm\alpha}]$, where $d_{n,m}(r,\mathcal{K}) \in \mathbb{Z}$, and we have

1. *The reduction modulo r^j, $d_{n,m}(r,\mathcal{K}) \mod r^j$, is a Vassiliev invariant of degree*[1] *$j\varphi(r)+m-1$.*
2. *As a consequence, $d_{n,m}(r,\mathcal{K})$ is a r-adic topological Vassiliev invariant.*
3. *Thus, $ADO_r(q^\alpha,\mathcal{K}) \in \mathbb{Z}[\zeta_r, q^{\pm\alpha}]$ is a topological Vassiliev invariant for the filtration $(r^j(q^\alpha - 1)^m)_{j,m\in\mathbb{N}}$ topology.*

Proof.

1. $d_{n,m}(r,\mathcal{K}) = \sum_{l=0}^{j-1} CL_{l,n,m}(r,\mathcal{K})r^l \mod r^j$ by the uniqueness of the decomposition, hence is a Vassiliev invariant of degree $j\varphi(r)+m-1$.
2. Let $m\in\mathbb{N}$ and $j\in\mathbb{N}$, if $K \in \mathsf{K}_d$ for $d > (j+1)\varphi(r)+m$, $d_{n,m}(r,K) \in r^j\mathbb{Z}$, hence $d_{n,m}(r,K)$ is an r-adic topological Vassiliev invariant.
3. Let $m, j \in \mathbb{N}$, using the last point, $\forall K \in \mathsf{K}_d$ for $d > (j+1)\varphi(r)+m$, $\forall l \leq m$ $d_{n,l}(r,K) \in r^j\mathbb{Z}$. Hence, one gets:
$$ADO_r(q^\alpha, K) \in r^j\mathbb{Z}_r[\zeta_r][[q^\alpha-1]] + (q^\alpha-1)^m\mathbb{Z}_r[\zeta_r][[q^\alpha-1]].$$

□

Here we got a unique expansion, but the price to pay is that the coefficients are, a priori, no longer Vassiliev invariant but r adic topological Vassiliev invariants.

3.4.4 Study of $\overline{d_{n,m}(r,\mathcal{K})}$ and ADO asymptotic behavior mod r

Let's focus on the previous setup but modulo r. We will study the dependence of $c_{n,m}$ and $d_{n,m}$ in r, and deduce some asymptotic behavior of ADO modulo r.

We denote $\bar{a}$ the reduction modulo r of an element $a \in \mathbb{Z}$.

Remark 150. Since $\overline{(\zeta_{2r}-1)^n} = 0$, $\forall n > \phi(r)-1$, we have

$$\sum_{m\geq 0}\sum_{n=0}^{+\infty} \overline{c_{n,m}(\mathcal{K})}(\zeta_r-1)^n(q^{2\alpha}-1)^m = \sum_{m\geq 0}\sum_{n=0}^{\varphi(r)-1} \overline{c_{n,m}(\mathcal{K})}(\zeta_r-1)^n(q^{2\alpha}-1)^m.$$

In the ring $(\mathbb{Z}[\zeta_{2r}]/(r))[[q^\alpha-1]]$ we have the following equality:

$$\sum_{m\geq 0}\sum_{n=0}^{\varphi(r)-1} \overline{c_{n,m}(\mathcal{K})}(\zeta_r-1)^n(q^{2\alpha}-1)^m = \sum_{m\geq 0}\sum_{n=0}^{\varphi(r)-1} \overline{d_{n,m}(r,\mathcal{K})}(\zeta_r-1)^n(q^{2\alpha}-1)^m$$

where $\overline{c_{n,m}}, \overline{d_{n,m}} \in \mathbb{Z}/r\mathbb{Z}$.

[1] See Nota Bene Section 3.4

Proposition 151. *Let $r = p^l$ be a power of a prime number.*
Recall that we can write in a unique way

$$ADO_r(q^\alpha, \mathcal{K}) = \sum_{m\geq 0} \sum_{n=0}^{\varphi(r)-1} d_{n,m}(r,\mathcal{K})(\zeta_r - 1)^n (q^{2\alpha} - 1)^m.$$

And, for any $m \in \mathbb{N}$ and any $1 \leq n \leq \varphi(r) - 1$, the map

$$\begin{aligned} \nu_{n,m} : \mathsf{K} &\to \mathbb{Z}/r\mathbb{Z} \\ \mathcal{K} &\mapsto \overline{d_{n,m}(r,\mathcal{K})} \end{aligned}$$

is a Vassiliev invariant of degree[1] *$n + m$.*

Proof. Using the uniqueness of the coefficients in the expansion, since $c_{n,m}(\mathcal{K})$ are Vassiliev invariants of degree[1] $n + m$, so are the $\overline{d_{n,m}(\mathcal{K})}$. □

We may now study the dependence of $d_{n,m}(r, \mathcal{K})$ in the variable r.
Recall that :

$$F_\infty(q, q^\alpha, \mathcal{K}) = \sum_{n,m\geq 0} b_{n,m}(\mathcal{K})(q^2 - 1)^n (q^{2\alpha} - 1)^m,$$

$$A_{\mathcal{K}}(q^{2r\alpha}) = \sum_m \lambda_m(\mathcal{K})(q^{2r\alpha} - 1)^m = \sum_m \tilde{\lambda}_m(r,\mathcal{K})(q^{2\alpha} - 1)^m.$$

Remark 152. $b_{n,m}(\mathcal{K})$ and $\lambda_m(\mathcal{K})$ don't depend on r, but $\tilde{\lambda}_m(r,\mathcal{K})$ does.

Let's compute $\tilde{\lambda}_m(r,\mathcal{K})$:

$$\begin{aligned} A_{\mathcal{K}}(q^{2r\alpha}) &= \sum_{m\geq 0} \lambda_m(\mathcal{K})(q^{2r\alpha} - 1)^m \\ &= \sum_{m\geq 0} \lambda_m(\mathcal{K}) \sum_{k\geq 0} \binom{m}{k} (-1)^{m-k} q^{2rk\alpha} \\ &= \sum_{m\geq 0} \lambda_m(\mathcal{K}) \sum_{k\geq 0} \binom{m}{k} (-1)^{m-k} \sum_{j\geq 0} \binom{rk}{j} (q^{2\alpha} - 1)^j \\ &= \sum_{j\geq 0} (\sum_{m\geq 0} \lambda_m(\mathcal{K}) \sum_{k\geq 0} \binom{m}{k} (-1)^{m-k} \binom{rk}{j})(q^{2\alpha} - 1)^j. \end{aligned}$$

Lemma 153. $\forall m > j$, $\sum_{k\geq 0} \binom{m}{k}(-1)^{m-k}\binom{rk}{j} = 0$.

Proof. Let $F_j(X, m) = \sum_{k\geq 0} \binom{m}{k}(-1)^{m-k}\binom{rk}{j} X^{rk-j}$, we study the X derivatives:

$$\begin{aligned} \frac{F_0^{(n)}(X,m)}{(n)!} &= \sum_{k\geq 0} \binom{m}{k} (-1)^{m-k} \frac{(rk)!}{(rk-n)!(n)!} X^{rk-n} \\ &= F_n(X,m). \end{aligned}$$

Since

$$F_0(X, m) = \sum_{k\geq 0} \binom{m}{k} (-1)^{m-k} X^{rk} = (X^r - 1)^m,$$

then

$$F_0^{(j)}(1, m) = 0 \ \forall m > j$$

Hence, $\forall m > j$, $F_j(1, m) = 0$. □

[1] See Nota Bene Section 3.4

We can thus write $\tilde{\lambda}_j(r,\mathcal{K}) = \sum_{m\geq 0} \lambda_m \sum_{k\geq 0} \binom{m}{k}(-1)^{m-k}\binom{rk}{j}$.
Moreover, $\binom{rk}{j} \in r\mathbb{Z}$ if $0<j<r$ (recall that $\binom{a}{b}=0$ if $b>a$).
In particular, $\tilde{\lambda}_j(r,\mathcal{K}) \in r\mathbb{Z}$, $\forall 0<j<r$.

Proposition 154. *Let $r=p^l$ be a power of a prime number, $\mathcal{K}$ a 0 framed knot in S^3.*

1. *$\forall n\in\mathbb{N}$, $\forall m<r$, $\overline{d_{n,m}(r,\mathcal{K})} = \overline{b_{n,m}(\mathcal{K})}$ in $\mathbb{Z}/r\mathbb{Z}$,*
2. *$Val_{q^\alpha-1}(\overline{F_\infty(\zeta_r,q^\alpha,\mathcal{K}) - ADO_r(q^\alpha,\mathcal{K})}) \underset{r\to+\infty}{\longrightarrow} +\infty$ where*

$$Val_{q^\alpha-1}(\sum_m a_m(q^\alpha-1)^m) = \min\{m\in\mathbb{N}|\ a_m\neq 0\}.$$

Proof. The first point comes from the fact that

$$A_{\mathcal{K}}(q^{2r\alpha})\times F_\infty(q,q^\alpha,\mathcal{K}) = \sum_{m,n\geq 0}(\sum_{k=0}^{m}\tilde{\lambda}_k(r,\mathcal{K})b_{n,m-k}(\mathcal{K}))(q^2-1)^n(q^{2\alpha}-1)^m,$$

hence for all $m<r$

$$\overline{d_{n,m}(r,\mathcal{K})} = \overline{\sum_{k=0}^{m}\tilde{\lambda}_k(r,\mathcal{K})b_{n,m-k}(\mathcal{K})} = \overline{\tilde{\lambda}_0(r,\mathcal{K})b_{n,m}(\mathcal{K})}$$

since $\tilde{\lambda}_j(r,\mathcal{K})\in r\mathbb{Z}$, $\forall 0<j<r$. Since $A_{\mathcal{K}}(1)=1$, then $\tilde{\lambda}_0(r,\mathcal{K})=1$.

The second is a direct application of the first, since for any $m<r$, $\overline{d_{n,m}(r,\mathcal{K})} = \overline{b_{n,m}(\mathcal{K})}$,

$$\overline{F_\infty(\zeta_r,q^\alpha,\mathcal{K}) - ADO_r(q^\alpha,\mathcal{K})}) \in (q^{2\alpha}-1)^r(\mathbb{Z}[\zeta_{2r}]/(r))[[q^\alpha-1]].$$

□

As a corollary of Proposition 154, we have that $\overline{d_{n,m}(r,\mathcal{K})}$ are not all trivial Vassiliev invariants.

Corollary 155. *If $b_{n,m}(\mathcal{K})\neq 0$ then for any big enough $r=p^l$ power of a prime,*

$$\overline{d_{n,m}(r,\mathcal{K})}\neq 0 \text{ in } \mathbb{Z}/r\mathbb{Z}.$$

Chapter 4

Braid representation point of view

This chapter introduces another perspective to construct and understand the unified invariant. Seeing knots as the closure of a braid, we can recover the unified invariant from quantum braid representation. This will allow us to get back the theorem of factorisation at root of unity with simple linear algebra tools, and give an alternative proof of the Melvin-Morton-Rozansky conjecture initially proved by Bar Natan and Garoufalidis.

This will lay the groundwork for the homological study of the Verma module and the unified invariant.

This chapter is an excerpt of a joint work with Jules Martel.

4.1 States diagrams and unified invariant

Let us briefly recall how the unified invariant $F_\infty(q, q^\alpha, \mathcal{K})$ is defined using states diagrams of the knot.

For any knot seen as a $(1,1)$-tangle, take a diagram D and $\bar{i} = (i_1, \ldots, i_N) \in \mathbb{N}^N$ where N is the number of crossings of D.

Label the top and bottom strands 0 and starting from the bottom strand, label the strand after the k-th crossing encountered with the rule described in Figure 4.1. The resulting labeled diagram is called a *state diagram* of D, we denote it $D_{\bar{i}}$.

(a) Positive crossing. (b) Negative crossing.

Figure 4.1: The two possibilities for the k-th crossing in D.

Let $D_{\bar{i}}$ a state diagram of D, we define:

$$\begin{aligned} D(i_1, \ldots, i_N) =& q^{\frac{f\alpha^2}{2}} \Big(\prod_{j=1}^{S} q^{\mp(\alpha - 2\epsilon_j)}\Big) \prod_{k \in pos} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha - a_k; i_k\}_q \\ &\times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k \in neg} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \\ &\times \{\alpha - a_k; i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_k b_k} \end{aligned}$$

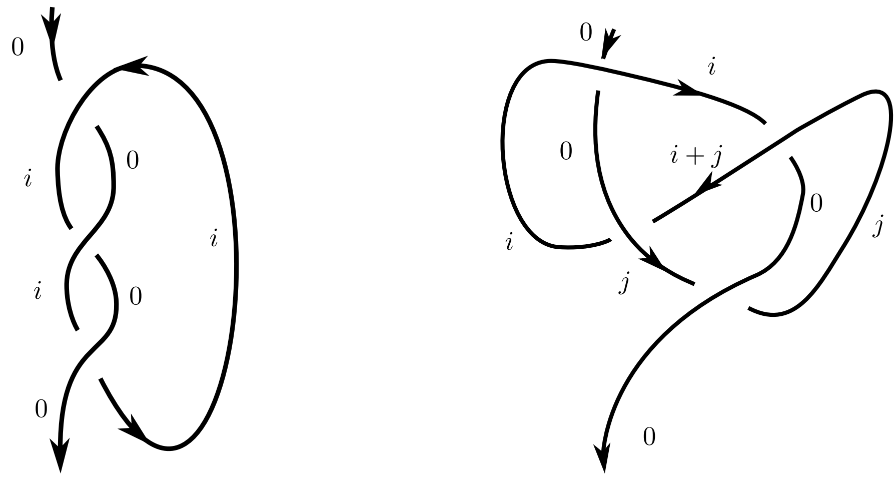
where :

- f is the writhe of D,
- $neg \cup pos = [|1, N|]$ is the set of crossings and $k \in pos$ if the k-th crossing of D is positive, else $k \in neg$,
- a_k, b_k are the strands labels at the k-th crossing of the state diagram (see Figure 2.1),
- S is the number of $\smile + \frown$ appearing in the diagram, and ϵ_j the strand label at the j-th $\smile$ or $\frown$, the $\mp$ sign is negative if $\smile$ and positive if $\frown$.

Remark 156. $D(i_1, \ldots, i_N)$ is the scalar one gets by considering only the $E^{i_k} \otimes F^{(i_k)}$ term in the the R-matrix action of the k-th crossing of D, where R can be found in Definition 55:

$$R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}.$$

Example 157. See Figure 4.2 for some examples of state diagrams.



(a) The trefoil knot. (b) The figure eight knot.

Figure 4.2: Examples of state diagrams to compute the invariants.

Definition 158. By definition, $F_\infty(q, A, \mathcal{K}) := \sum_{\bar{i}=0}^{+\infty} D(i_1, \ldots, i_N) \in \widehat{\mathbb{Z}[q, q^\alpha]}$.

4.2 Quantum braid representation

The braid group with n strands is defined as

$$B_n := \langle \sigma_1, \ldots, \sigma_{n-1} |\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i,\ 0 \leq i \leq n-2,\ i+2 \leq j \leq n-1 \rangle$$

We can visualise braids as (n, n) tangles without closed components, seeing the braid group operation as juxtaposition of tangles, and sending σ_i to the elementary positive crossing of the i-th strand with the $(i+1)$-th.

Theorem 159. *(Alexander) Any knot can be represented as the closure of a braid.*

The Reshetikin-Turaev functor restricted to braids provides quantum braid representations on the Verma module.

Definition 160. We define
$$\varphi_n(q^\alpha,.) : B_n \to \mathrm{End}((V^\alpha)^{\otimes n})$$
by
$$\varphi_n(q^\alpha,\sigma_i) = 1^{\otimes i-1} \otimes (\tau \circ R) \otimes 1^{\otimes n-i-2}$$
where $\tau(v \otimes w) = w \otimes v$ and V^α is defined in Subsection 2.2.2.

Remark 161. Even if V^α is only a $\mathcal{U}$ module, we can endow it with a B_n structure using the fact that E is nilpotent, making the action of the R-matrix finite.

Definition 162. Let $\beta \in B_n$ whose closure is a knot,
$$\mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta)) := (\sum_{\bar{j}=0}^{+\infty}[((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta))v_{\bar{j}})]_{v_{\bar{j}}} \in \widehat{\mathbb{Z}[q,q^\alpha]}$$
where

- $\bar{j} = (0, j_2, \dots, j_n) \in \mathbb{N}^n$,
- $[((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta))v_{\bar{j}})]_{v_{\bar{j}}} \in \mathbb{Z}[q^\pm, q^{\pm\alpha}]$ is the projection of $(1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta)v_{\bar{j}}$ on $v_{\bar{j}}$.

4.3 Combining states diagrams and trace decomposition

Let $\mathcal{K}$ be a long knot, $\beta \in B_n$ whose closure is $\mathcal{K}$. Let D^β be the diagram associated with $\mathcal{K}$ seen as the closure of β.

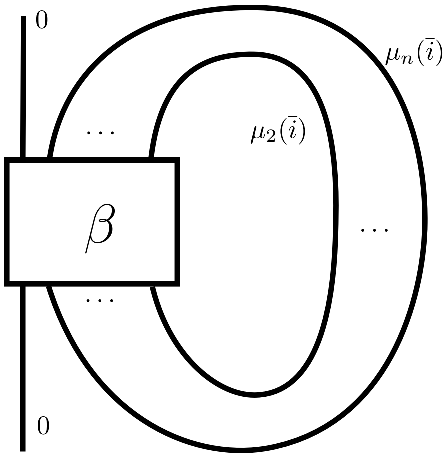


Figure 4.3: State diagram of a braid partial closure.

We denote $\mu_2(\bar{i}),\dots,\mu_n(\bar{i})$ the labels of the closure strands of the state diagram $D^\beta_{\bar{i}}$. Let $\mu(\bar{i}) = (0,\mu_2(\bar{i}),\dots,\mu_n(\bar{i}))$.

We can then write,
$$[((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta))v_{\bar{j}})]_{v_{\bar{j}}} = \sum_{\substack{\bar{i}=0\\ \mu(\bar{i})=\bar{j}}}^{+\infty} D(i_1,\dots,i_N),$$
where $[((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta))v_{\bar{j}})]_{v_{\bar{j}}} \in \mathbb{Z}[q^\pm, q^{\pm\alpha}]$ is the coefficient of $(1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta)v_{\bar{j}}$ on $v_{\bar{j}}$.

Hence,
$$\sum_{\bar{j}=0}^{+\infty}[((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta))v_{\bar{j}})]_{v_{\bar{j}}} = \sum_{\bar{i}=0}^{+\infty} D(i_1,\dots,i_N).$$

Finally we get,

$$\mathrm{Tr}_{2,\ldots,n}((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)) = F_\infty(q, q^\alpha, \mathcal{K}),$$

concluding the proof of the following proposition.

Proposition 163. *Let $\mathcal{K}$ be a knot in S^3 and $\beta \in B_n$ a braid whose closure is $\mathcal{K}$, then we have*

$$F_\infty(q, q^\alpha, \mathcal{K}) = Tr_{2,\ldots,n}((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)) \in \widehat{\mathbb{Z}[q, q^\alpha]}$$

4.4 Weight sub-representations

Elements of same weight in the tensor product of Verma form sub-representations of the braid group.

Definition 164. Let

$$V_{n,m}(q, q^\alpha) :=< v_{i_1} \otimes \cdots \otimes v_{i_n} | \sum_{k=1}^{n} i_k = m >$$

and the associated representation

$$\varphi_{n,m}(q, q^\alpha, .) : B_n \to \mathrm{End}(V_{n,m}).$$

When there is no ambiguity, we will write $V_{n,m} := V_{n,m}(q, q^\alpha)$ and $\varphi_{n,m}(\beta) := \varphi_{n,m}(q, q^\alpha, \beta)$.

Remark 165. When q is formal, we have $V_{n,m}(q, q^\alpha) = \{v \in V_\alpha^{\otimes n} | \; Kv = q^{\alpha-2m}v\}$. At roots of unity, since $q^{-2r} = 1$, the equality does not stand.

Using Proposition 163, we can then write the unified invariant using these sub-representations.

Corollary 166. *Let $\mathcal{K}$ be a knot in S^3 and $\beta \in B_n$ a braid whose closure is $\mathcal{K}$, then we have*

$$F_\infty(q, q^\alpha, \mathcal{K}) = \sum_m Tr_{2,\ldots,n}((1 \otimes K^{\otimes n-1})\varphi_{n,m}(\beta))$$

We treat the $q = 1$ case slightly differently, removing the quadratic part:

Definition 167.

$$W_{n,m} := V_{n,m}(1, q^\alpha)$$

$$\psi_{n,m}(q^\alpha, \beta) := q^{\frac{-f\alpha^2}{2}} \varphi_{n,m}(1, q^\alpha, \beta)$$

where f is the number of positive crossings minus the number of negative crossings in the braid β.

Moreover we denote $w_i := v_i$ at $q = 1$.

A nice property of the $q = 1$ case is that the m weight level representation can be obtained using symmetric power of the weight level one.

Proposition 168. *Let $\beta \in B_n$, then :*

$$\psi_{n,m}(q^\alpha, \beta) = Sym^m(\psi_{n,1}(q^\alpha, \beta)).$$

Proof. First we need to consider a small change of bases. We set $u_j := j!w_j$. Let's take a basis of $V_{n,1}$, $e_k := u_0 \otimes \cdots \otimes u_1 \otimes \cdots \otimes u_0$ where the only u_1 is located at the k-th position. We can identify higher weight tensors with symmetric powers of the e_k:

$$u_{j_1} \otimes \cdots \otimes u_{j_n} = \prod_{k=1}^{n} e_k^{j_k}.$$

Now, in the basis e_k, we have

$$\psi_{n,1}(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 1-q^{-2\alpha} & q^{-\alpha} & 0 \\ & q^{-\alpha} & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1} \end{array}\right)$$

Hence we compute the symmetric power on the basis u_j,

$$\begin{aligned} \mathrm{Sym}^m(\psi_{n,1}(\sigma_i))u_{\bar{j}} &= \prod_{k=0}^{n} (\psi_{n,1}(\sigma_i)e_k)^{j_k} \\ &= ((1-q^{-2\alpha})e_i + q^{-\alpha}e_{i+1})^{j_i} \times (q^{-\alpha}e_i)^{j_{i+1}} \\ &= \sum_{l=0}^{j_i} \binom{j_i}{l} \{\alpha\}^l q^{-(j_i+j_{i+1})\alpha} u_{j_1} \otimes \cdots \otimes u_{j_{i+1}+l} \otimes u_{j_i-l} \otimes \cdots \otimes u_{j_n} \end{aligned}$$

If we transpose it back in the basis w_j we get:

$$\begin{aligned} Sym^m(\psi_{n,1}(\sigma_i))w_{\bar{j}} &= \sum_{l=0}^{j_i} \binom{j_{i+1}+l}{l} \{\alpha\}^l q^{-(j_i+j_{i+1})\alpha} w_{j_1} \otimes \cdots \otimes w_{j_{i+1}+l} \otimes w_{j_i-l} \otimes \cdots \otimes w_{j_n} \\ &= \psi_{n,m}(q^\alpha, \sigma_i) w_{\bar{j}} \end{aligned}$$

where the second equality comes from the computation of the action of the R-matrix. □

4.5 At root of unity: r-part sub-representations

In this subsection we set $q = \zeta_{2r}$. You can recall if needed the action on the Verma in Subsection 2.2.2.

Definition 169. The r-part of a tensor $v = v_{i_1+rj_1} \otimes \cdots \otimes v_{i_n+rj_n}$ where $i_1, \ldots i_n \leq r-1$ is defined by

$$\mathrm{rp}(v) := \sum_{k=0}^{n} j_k.$$

Definition 170. We define the following $\mathbb{Z}[\zeta_{2r}, q^\alpha]$ modules:

$$V_n^m(\zeta_{2r}, q^\alpha) :=< v |\ \mathrm{rp}(v) = m >$$

$$V_n^{\leq m}(\zeta_{2r}, q^\alpha) := \bigoplus_{i=0}^{m} V_n^i(\zeta_{2r}, q^\alpha)$$

when there is no ambiguity, we will write $V_n^m := V_n^m(\zeta_{2r}, q^\alpha)$ and $V_n^{\leq m} := V_n^{\leq m}(\zeta_{2r}, q^\alpha)$.

Proposition 171. *$V_n^{\leq m}$ is a sub-representation of braids defined by $\varphi_n^{\leq m}(\beta)$.*

Proof. First remark that

$$\mathrm{rp}(E^r \otimes F^{(r)}(v_i \otimes v_j)) = \mathrm{rp}(v_i \otimes v_j).$$

Moreover $F^{(i+rj)}v_{a+ru} = 0$ if $i, a \leq r-1$ and $a + i \geq r$, hence

$$\mathrm{rp}(E^n \otimes F^{(n)}(v_i \otimes v_j)) \leq \mathrm{rp}(v_i \otimes v_j).$$

This implies that,

$$\mathrm{rp}(R.(v_i \otimes v_j)) \leq \mathrm{rp}(v_i \otimes v_j),$$

and thus,

$$\mathrm{rp}(\varphi_n(\beta).(v_i \otimes v_j)) \leq \mathrm{rp}(v_i \otimes v_j),$$

Hence, $V_n^{\leq m}$ is invariant via the action of the R-matrix and its inverse. □

Remark 172. We have a filtration $(V_n^{\leq m})_{m\in\mathbb{N}}$ of V_n invariant via B_n.

This allows us to have a quotient representation on V_n^m:

Proposition 173. *Let $\rho_n^m : V_n^{\leq m} \to V_n^m$ the surjective map associated with the usual basis, then $V_n^m \cong V_n^{\leq m}/V_n^{\leq m-1}$ is a representation of braids with*

$$\varphi_n^m := \rho_n^m \circ \varphi_n^{\leq m}|_{V_n^m}.$$

Proof. Since $V_n^{\leq m-1}$ is a sub-representation, if $v \in V_n^{\leq m-1}$ we have $\varphi_n^{\leq m-1}(\beta_1)v \in V_n^{\leq m-1}$ and hence $\rho_n^m \circ \varphi_n^{\leq m}(\beta_1)v = 0$.

This means that

$$\rho_n^m \circ \varphi_n^{\leq m}(\beta_1) \circ \rho_n^m \circ \varphi_n^{\leq m}(\beta_2)|_{V_n^m} = \rho_n^m \circ \varphi_n^{\leq m}(\beta_1) \circ \varphi_n^{\leq m}(\beta_2)|_{V_n^m}.$$

Finally $\varphi_n^m(\beta_1\beta_2) = \varphi_n^m(\beta_1) \circ \varphi_n^m(\beta_2)$. □

Remark 174. As braid group representation, $V_n^0 \cong V_n^{\leq 0}$, meaning that $\varphi_n^0 = \varphi_n^{\leq 0}$

We may now state the factorisation of r-part representation, first we define the Frobenius map :

$$\begin{aligned} F_r : \mathbb{Z}[q^\alpha] &\to \mathbb{Z}[q^\alpha] \\ q^\alpha &\mapsto q^{r\alpha}. \end{aligned}$$

Remark 175. $F_r(W_{n,m}) := V_{n,m}(1, q^{r\alpha})$.

Proposition 176. *The isomorphism*

$$\begin{aligned} \Phi : V_n^m &\to V_n^0 \otimes F_r(W_{n,m}) \\ v_{\overline{i+rj}} &\mapsto v_{\overline{i}} \otimes F_r(w_{\overline{j}}), \end{aligned}$$

where $\overline{i+rj} = (i_1 + rj_1, \ldots, i_n + rj_n)$ with $i_1, \ldots, i_n \leq r-1$, and $F_r(W_{n,m}) = V_{n,m}(1, q^{r\alpha})$, is a braid group representation isomorphism.

In other words, the following diagram commutes:

$$\begin{array}{ccc} V_n^m & \xrightarrow{\quad \varphi_n^m(\beta) \quad} & V_n^m \\ \Phi \downarrow & & \downarrow \Phi \\ V_n^0 \otimes F_r(W_{n,m}) & \xrightarrow{\varphi_n^0(\beta) \otimes (F_r \circ \psi_{n,m})(\beta)} & V_n^0 \otimes F_r(W_{n,m}) \end{array} \quad \text{with } \beta \in B_n.$$

Proof. Using Lemma 77 we can factorise the action of the R-matrix as follows.

Let $0 \leq a, b, i \leq r-1$ such that $0 \leq a + i \leq r-1$ and $0 \leq b - i \leq r-1$, we have:

$$\begin{aligned} q^{\frac{H\otimes H}{2}}(q^{\frac{(i+rj)(i+rj-1)}{2}} E^{i+rj} \otimes F^{(i+rj)}).v_{b+rv} \otimes v_{a+ru} &= q^{\frac{\alpha^2}{2}} q^{\frac{(i+rj)(i+rj-1)}{2}} \begin{bmatrix} i+rj+a+ru \\ i+rj \end{bmatrix}_q \\ &\times \{\alpha - a - ru; i+rj\}_q q^{-(a+ru+b+rv)\alpha} \\ &\times q^{2(a+ru+i+rj)(b+rv-i-rj)} v_{b+rv-i-rj} \otimes v_{a+ru+i+rj} \\ &= q^{\frac{\alpha^2}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} a+i \\ i \end{bmatrix}_q \\ &\times \{\alpha - a; i\}_q q^{-(a+b)\alpha} q^{2(a+i)(b-i)} v_{b-i} \otimes v_{a+i} \\ &\otimes F_r\left(\binom{u+j}{j} \{\alpha\}^j q^{-(u+v)\alpha} w_{v-j} \otimes w_{u+j}\right) \end{aligned}$$

Hence we have,

$$\Phi\left(\rho_2^{u+v}(R.(v_{b+rv} \otimes v_{a+ru}))\right) = (R.(v_b \otimes v_a)) \otimes F_r(R.(w_v \otimes w_u))$$

Finally,

$$\Phi\left(\varphi_n^m(\beta).v_{\overline{i+rj}}\right) = \varphi_n^0(\beta).v_{\bar{i}} \otimes F_r\left(\psi_{n,m}(\beta).w_{\bar{j}}\right)$$

□

Example 177. Figure 4.4 illustrates the weight level pyramid at $n = 2$ and $q = \zeta_6$ where we denote

$$v_{a,b} = v_a \otimes v_b.$$

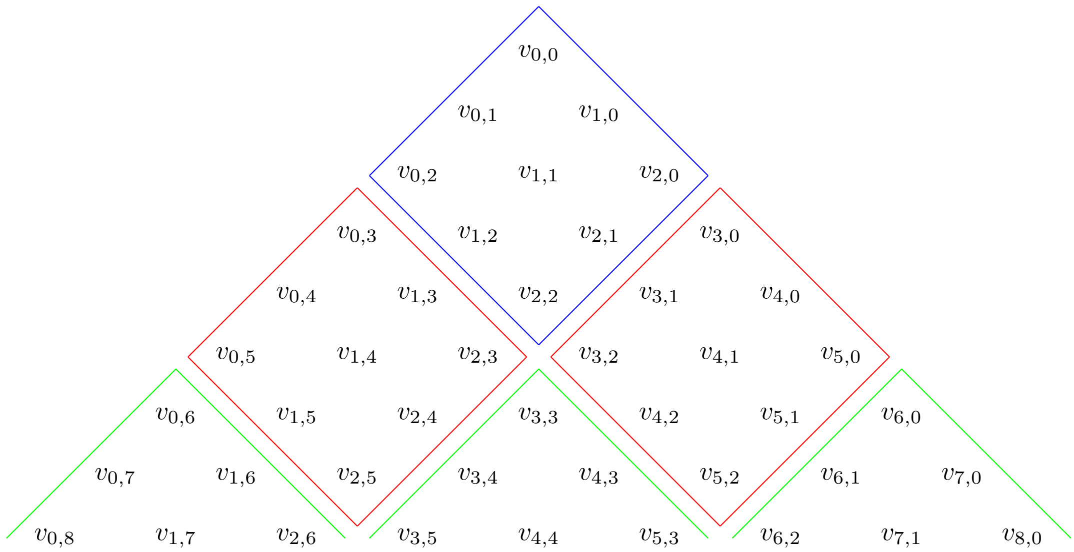


Figure 4.4: Weight level pyramid factorisation at root of unity

The blue square delimits generators of V_n^0, the red squares of V_n^1, etc. Each square corresponds to a tensor $w_{i,j} := w_i \otimes w_j$ in the pyramid at $q = 1$ as shown in Figure 4.5.

$w_{0,0}$

$w_{0,1}$ $w_{1,0}$

$w_{0,2}$ $w_{1,1}$ $w_{2,0}$

Figure 4.5: Weight level pyramid at $q = 1$

4.6 At root of unity: factorisation of the unified invariant

Now we can finally factorise the unified invariant at roots of unity using braid representations. This will allow us to give a proof of this factorisation without Bar-Natan and Garoufalidis MMR theorem and hence a braid proof of MMR. In this subsection, we will also assume $q = \zeta_{2r}$.

First of all we can obtain ADO polynomials with the 0 r-part representation.

Proposition 178.

$$ADO_r(q^\alpha, \mathcal{K}) = Tr_{2,\dots,n}((1 \otimes (K^{1-r})^{\otimes n-1})\varphi_n^0(\beta))$$

Proof. This is the same proof than for the unified invariant in Subsection 4.3, setting $q = \zeta_{2r}$, using truncated R-matrix and K^{r-1} as pivotal element. □

Now let's see a factorisation proposition at roots of unity, first recall that:

$$F_r : \mathbb{Z}[q^\alpha] \to \mathbb{Z}[q^\alpha]$$

sending $F_r(q^\alpha) = q^{r\alpha}$.

As a direct corollary of Propositions 176 and 178,

Corollary 179.

$$Tr_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n^m(\beta)) = ADO_r(q^\alpha, \mathcal{K}) \times F_r(Tr_{2,\dots,n}((1 \otimes K^{\otimes n-1})\psi_{n,m}(\beta))$$

Now let us cite the Mac-Mahon master theorem:

Theorem 180. *(Mac-Mahon)*
Consider $A = (a_{i,j})_{i,j \le n}$ an $n \times n$ matrix with coefficients in some commutative ring K.

In $K[t_1, \dots t_n]$ we set

$$T_j := \sum_{i=0}^{n} a_{i,j} t_i$$

and $G(m_1, \dots, m_n)$ to be the coefficient of $t_1^{m_1} \dots t_n^{m_n}$ in $\prod_{i=0}^{n} T_i^{m_i}$.

We have the following indentity:

$$\sum_{m_1,\dots,m_n \ge 0} G(m_1, \dots, m_n) t_1^{m_1} \dots t_n^{m_n} = \frac{1}{\det\left(I_n - \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} A\right)}$$

Remark 181. At $t_1 = \dots = t_n$, we get

$$\sum_{m \ge 0} \mathrm{Tr}(\mathrm{Sym}^m(A)) t^m = \frac{1}{\det(I_n - tA)}$$

We now need to connect the determinant of the Burau matrix:

Proposition 182.

$$\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)\psi_{n,1}(\beta)) = q^{(n-1-f)\alpha} A_{\mathcal{K}}(q^{2\alpha}).$$

Proof. First of all we need to relate $\psi_{n,1}(q^\alpha, \sigma_i)$ to the Burau matrix. Recall that if we take a basis of $V_{n,1}$, $e_k := u_0 \otimes \dots \otimes u_1 \otimes \dots \otimes u_0$ where the only u_1 is located at the k-th position. We can identify higher weight tensors with symmetric powers of the e_k:

$$u_{j_1} \otimes \dots \otimes u_{j_n} = \prod_{k=1}^{n} e_k^{j_k}.$$

Now, in the basis e_k, we have

$$\psi_{n,1}(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1 - q^{-2\alpha} & q^{-\alpha} & 0 \\ & q^{-\alpha} & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array}\right)$$

Now if we take the change of basis $f_k = q^{-k\alpha} e_k$, we obtain the Burau matrix :

$$B(q^{-\alpha}) := \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1-q^{-2\alpha} & q^{-2\alpha} & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array}\right)$$

We thus have the equality :

$$\det\left(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)\psi_{n,1}(q^\alpha,\beta)\right) = \det\left(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)B(q^{-\alpha})\right)$$

Moreover, $\det\left(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)B(q^{-\alpha})\right)$ is the $(n-1)\times(n-1)$ minor of $I_n - B(q^{-\alpha})$ where we remove the first column and the first row.

Using Prop 3.10 in [3] and the last paragraph in VIII.3 of [7], we get that :

$$\det\left(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)B(q^{-\alpha})\right) = uA_{\mathcal{K}}(q^{2\alpha})$$

where u is an invertible element of $\mathbb{Z}[q^{\pm\alpha}]$ and hence of the form $u = \pm q^{n\alpha}$ with $n \in \mathbb{N}$.

One known property of the Alexander polynomial is that it is symmetrical, hence :

$$A_{\mathcal{K}}(q^{2\alpha}) = A_{\mathcal{K}}(q^{-2\alpha})$$

Using Proposition 125 at $q = 1$, one gets that :

$$D(q^\alpha) := q^{(-(n-1)+f)\alpha}\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)\psi_{n,1}(\beta))$$

satisfies

$$D(q^\alpha) = D(q^{-\alpha}).$$

Moreover, $A_{\mathcal{K}}(1) = 1$ and so does $D(1) = 1$, we hence have

$$u = q^{(n-1-f)\alpha}.$$

□

We can now use Mac-Mahon master theorem to prove the following proposition:

Proposition 183. *For $\beta \in B_n$ whose closure is a knot $\mathcal{K}$, then :*

$$\sum_m Tr_{2,\dots,n}((1\otimes K^{\otimes n-1})\psi_{n,m}(\beta)) = \frac{q^{f\alpha}}{A_{\mathcal{K}}(q^{2\alpha})}.$$

Proof. Using Mac-mahon master theorem (180) we have

$$\sum_m [\mathrm{Sym}^m(\psi_{n,1}(\beta))v_{\bar{i}}]_{v_{\bar{i}}} t_1^{i_1}\dots t_n^{i_n} = \frac{1}{\det\left(I_n - \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}\psi_{n,1}(\beta)\right)}$$

Now if one takes $t_1 = 0$ and $t_i = 1$ for $i \neq 1$, we have the following equality:

$$\sum_m \mathrm{Tr}_{2,\dots,n}(\mathrm{Sym}^m(\psi_{n,1}(\beta))) = \frac{1}{\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)\psi_{n,1}(\beta))}$$

And, using Proposition 182:

$$\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)\psi_{n,1}(\beta)) = q^{(n-1-f)\alpha}A_{\mathcal{K}}(q^{2\alpha}).$$

Since, using Proposition 168,

$$\sum_m \mathrm{Tr}_{2,\dots,n}((1\otimes K^{\otimes n-1})\psi_{n,m}(\beta)) = q^{(n-1)\alpha}\sum_m \mathrm{Tr}_{2,\dots,n}(\mathrm{Sym}^m(\psi_{n,1}(\beta)))$$

we get,

$$\sum_m \mathrm{Tr}_{2,\dots,n}((1\otimes K^{\otimes n-1})\psi_{n,m}(\beta)) = \frac{q^{f\alpha}}{A_{\mathcal{K}}(q^{2\alpha})}.$$

□

Now we can prove the factorisation theorem.
Since $(V_n^{\leq m})_{m\in\mathbb{N}}$ is a filtration of V_n we have:

$$\begin{aligned} F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) &= \mathrm{Tr}_{2,\ldots,n}((1\otimes K^{\otimes n-1})\varphi_n(q^\alpha,\beta)) \\ &= \sum_m \mathrm{Tr}_{2,\ldots,n}((1\otimes K^{\otimes n-1})\varphi_n^m(q^\alpha,\beta)) \end{aligned}$$

where the first equality is detailed in Subsection 4.3.

Using Corollary 179 and Proposition 183, we get back the factorisation theorem :

Theorem 184. *(Factorization)*
For a knot $\mathcal{K}$ and an integer $r\in\mathbb{N}^*$, we have the following factorization in $\hat{R}_r^I$:

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \frac{q^{rf\alpha}\times ADO_r(q^\alpha,\mathcal{K})}{A_{\mathcal{K}}(q^{2r\alpha})}$$

where f is the framing of the knot.

We recall the Melvin–Morton–Rozanski conjecture:

Theorem 185. *(Bar-Natan, Garoufalidis)*
For $\mathcal{K}$ a knot, we have the following equality in $\mathbb{Q}[[h]]$:

$$\lim_{n\to\infty} J_n(e^{h/n}) = \frac{1}{A_{\mathcal{K}}(e^h)}$$

in the sense that, $\forall m\in\mathbb{N}$,

$$\lim_{n\to\infty} \mathit{coeff}\left(J_n(e^{h/n}), h^m\right) = \mathit{coeff}\left(\frac{1}{A_{\mathcal{K}}(e^h)}, h^m\right)$$

where, for any analytic function f, coeff$(f(h),h^m) = \frac{1}{m!}\frac{d^m}{dh^m}f(h)|_{h=0}$.

Remark 186. Let $\mathcal{K}$ be a 0 framed knot.
From the unified invariant,

- on one hand we get the colored Jones polynomials back

$$F_\infty(q,q^n,\mathcal{K}) = J_n(q^2,\mathcal{K}),$$

- on the other hand, instead of using Theorem 185 theorem as we did in Chapter 3, we can recover the Alexander polynomial using Theorem 184 at $r=1$, and we get

$$F_\infty(1,A,\mathcal{K}) = \frac{1}{A_{\mathcal{K}}(A^2)}$$

Using the identification $q=e^h$ and $q^\alpha = e^{\alpha h}$, we have an injective map (see Prop 6.8, 6.9 in [15])

$$\widehat{\mathbb{Z}[q,q^\alpha]} \to \mathbb{Q}[\alpha][[h]]$$

Hence, as elements in $\mathbb{Q}[\alpha][[h]]$, we have the following limit (in the sense defined in Theorem 185)

$$\lim_{n\to\infty} F_\infty(q^{\frac{1}{n}}, q, \mathcal{K}) = F_\infty(1,q,\mathcal{K})$$

Thus we get

$$\begin{aligned} \lim_{n\to\infty} J_n(e^{\frac{2h}{n}}) &= \lim_{n\to\infty} F_\infty(q^{\frac{1}{n}}, q, \mathcal{K}) \\ &= F_\infty(1,q,\mathcal{K}) \\ &= \frac{1}{A_{\mathcal{K}}(e^{2h})} \end{aligned}$$

giving us a proof of Theorem 185.

Chapter 5

Toward the link case

Now that all this work is done for knots, one may wonder if we can generalise to the link case. This happens to be difficult, and the purpose of this chapter is to shed some light on the obstructions that one can encounter following different approaches, to give some partial results and to present some perspective for further research.

Disclaimer: The following chapter presents recent or unfinished work. The first subsection is very technical and yield no real results. The second one follows Chapter 4 for multicolored braids, the proofs displayed are sketchy.

5.1 Obstructions and approaches to the unification for links

Throughout this thesis we only focused on the case of knots in S^3. Why? Because we cannot apply neither the state diagram computation nor the universal invariant scalar action on Verma modules.

As we will see in this section, we have convergence issues or obstructions. The braid representation approach seems to yield better results, yet some convergence issues still need to be addressed.

5.1.1 State sum diagrams approach

We could try to construct a unified link invariant trough state diagrams. We have the R-matrix action on $V_\alpha \otimes V_\beta$:

$$R.v_a \otimes v_b = q^{\frac{\alpha\beta}{2}} \sum_n q^{\frac{n(n-1)}{2}} q^{-a\beta - b\alpha} q^{2(a-n)(b+n)} \begin{bmatrix} b+n \\ n \end{bmatrix}_q \{\beta - b; n\}_q v_{a-n} \otimes v_{b+n}$$

And we have the state diagrams contributions,

$$\begin{aligned} D(p_1, \dots, p_l; i_1, \dots, i_N) =& (\prod_{j=1}^{S} q^{\mp(\alpha_j - 2\epsilon_j)}) \prod_{k \in pos} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha_k - a_k; i_k\}_q \\ & \times q^{-a_k\beta_k - b_k\alpha_k} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k \in neg} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \\ & \times \{\alpha_k - a_k; i_k\}_q q^{a_k\beta_k + b_k\alpha_k} q^{-2a_k b_k} \end{aligned}$$

where the k-th crossing involves $V_{\beta_k} \otimes V_{\alpha_k}$, $p_1, \dots, p_l$ correspond to colors on the different components of the link.

Summing these contributions over every state sum diagram would construct an analog of F_∞ for links.

But the main problem is that since we do not have a long knot, the link case will have some closed components. Hence, the state sum diagram contribution doesn't only depend on the index of the crossings. An unfortunate consequence is that the state sum diagram contribution $D(p_1, \dots, p_l; i_1, \dots, i_N)$ is not small in $\widehat{\mathbb{Z}[q, q^\alpha]}$ when p_1 or ... or p_l are big. Since we need to sum all the state diagrams contributions we will have a convergence issue.

5.1.2 Universal invariant approach

Introduction

When we work with a link $L = L_1 \cup \cdots \cup L_l$, we can still construct a universal invariant.

Let T be a $l-l$ tangle with no closed components and whose closure is L. We use the same construction as for knots in order to produce an element:

$$Q^{U_h}(T) \in U_h^{\otimes l}$$

This is not a link invariant. The $l-l$ tangles with no closed components are not in a one to one correspondence with the links (contrarily to the knot case).

In order to produce a link invariant, let us define the space of commutators

$$I = \{xy - yx |\ x, y \in U_h\}.$$

This is not an ideal, and U_h/I is not an algebra, only a $\mathbb{Q}[[h]]$ module.

We thus have the previous element seen in the quotient space:

$$Q^{U_h}(L) \in (U_h/I)^n$$

is called the universal invariant of the link L.

In order to produce an invariant from the universal invariant of the tangle L, we need the map to be invariant through commutators.

Let's lay down an algebraic setup in order to try to produce such an invariant. We will see that we can produce the ADO polynomials and the colored Jones polynomials from the universal invariant of T, passing through a commutative ring, the map sending the colored Jones polynomials even annihilates the commutators. Hence, we have a map sending the universal invariant of L to the colored Jones polynomials. The same cannot be easily done for the ADO polynomials, we will display some obstructions to it.

Algebraic setup

Let's recall the algebraic setup:

Let U be a subalgebra of U_h generated by $K^{\pm1}, E, F$ over $\mathbb{Q}(q)$. This algebra is graded by assigning

$$\deg(K^{\pm1}) = 0, \quad \deg(E) = 1 \quad \text{and} \quad \deg(F) = -1.$$

Let us denote by U_0 its degree zero subalgebra. By the standard PBW argument

$$U_0 = \mathrm{Span}_{\mathbb{Q}(q)}\{K^j E^i F^i \,|\, j \in \mathbb{Z}, i \in \mathbb{N}\}.$$

The center of U is generated by the quantum Casimir element

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}.$$

For any $k \in \mathbb{Z}$ and $n \geq 0$ we denote $\{k\} := q^k - q^{-k}$, $[k] := \frac{\{k\}}{\{1\}}$ and $[n]! = [n] \dots [1]$. In addition, for any $a \in \mathbb{C} + \mathbb{Z}H$ and $m \in \mathbb{N}$ let

$$\{a; n\}_q = \{a\}\{a-1\} \dots \{a-n+1\} \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]!}{[m]![n-m]!}.$$

We will be interested in the $\mathbb{Z}[q^{\pm 1}]$-subalgebra $\mathcal{U}$ of U generated by

$$E,\ F^{(n)},\ \text{and}\ \ K^{\pm 1} \quad \text{where} \quad F^{(n)} = \frac{\{1\}^n F^n}{[n]!}$$

with the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF^{(n)} = q^{-2n}F^{(n)}K, \tag{5.1}$$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)} \quad (n, m \geq 1), \tag{5.2}$$

$$EF^{(n)} = F^{(n)}E + F^{(n-1)}\left(q^{-n+1}K - q^{n-1}K^{-1}\right) \tag{5.3}$$

Let us define the R-matrix

$$R = q^{\frac{H\otimes H}{2}} \sum_{i=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}.$$

The completion

$$\widehat{\mathbb{Z}[q]} := \varprojlim_n \frac{\mathbb{Z}[q^{\pm 1}]}{\{n\}!}$$

is known as Habiro's ring. Denote by J_n the two sided ideal of $\mathcal{U}$ generated by the following elements:

$$F^{(i)}\{H + m; n - i\}_q$$

where $m \in \mathbb{Z}$ and $i \in \{0, \ldots, n\}$. By Prop 5.1 in [15] the ideal $\{H + m; n\}_q$ is generated by the elements

$$\{n; i\}\{\alpha - H_0; n - i\} \quad \text{for} \quad i = 0, \ldots, n.$$

This gives the completion

$$\hat{\mathcal{U}} := \varprojlim_n \frac{\mathcal{U}}{J_n}$$

the structure of an algebra over $\widehat{\mathbb{Z}[q]}$. Denote $\hat{\mathcal{U}}_0$ the degree zero subalgebra of $\hat{\mathcal{U}}$. By Corollary 5.11 in [15]

$$\hat{\mathcal{U}}_0 = \mathrm{Span}_{\widehat{\mathbb{Z}[q^{\pm 1}]}}\{K^{\varepsilon}E^i F^{(i)}\{H; j\}' \,|\, \varepsilon \in \{0, 1\}, i, j \in \mathbb{N}\}$$

where $\{H; j\}' = (K^2 - 1)\ldots(K^2 - q^{2(j-1)})$. Since $E^i F^{(i)}$ can be expressed as a polynomial in $\{1\}^2 C$ and K, the above generators commute with each other.

Embedding

We show that $\hat{\mathcal{U}}_0$ is actually a subalgebra of some completion of the polynomial algebra defined below. Let $\mathcal{A} := \mathbb{Z}[\widehat{q, q^{\alpha}}, K_0]\left[\left[\begin{smallmatrix} H_0+m \\ n \end{smallmatrix}\right]_q, n \in \mathbb{N}\right]$ be such that

$$\mathbb{Z}[\widehat{q, q^{\alpha}}, K_0] = \varprojlim_n \frac{\mathbb{Z}[q^{\pm 1}, q^{\pm\alpha}K_0]}{(\{\alpha - H_0 + m; n\}_q, m \in \mathbb{Z})}$$

and $\left[\begin{smallmatrix} H_0+m \\ n \end{smallmatrix}\right]_q$ verifies $\{n\}!\left[\begin{smallmatrix} H_0+m \\ n \end{smallmatrix}\right]_q = \{H_0 + m; n\}_q$. By the same argument as above, $\mathcal{A}$ is an algebra over the Habiro ring.

Proposition 187. *There is an injective algebra homomorphism*

$$\varphi : \hat{\mathcal{U}}_0 \to \mathcal{A} \quad \textit{sending}$$

$$K \mapsto q^{\alpha}K_0^{-2} = q^{\alpha}q^{-2H_0},$$

$$E^n F^{(n)} \mapsto \begin{bmatrix} H_0 + n \\ n \end{bmatrix}_q \{\alpha - H_0; n\}_q$$

$$F^{(n)}E^n \mapsto \begin{bmatrix} H_0 \\ n \end{bmatrix}_q \{\alpha - H_0 + n; n\}_q.$$

Proof. Let us first show that φ is injective. For any

$$x = \sum_{i,j} a^{\epsilon}_{ij} K^{\varepsilon} E^i F^{(i)} \{H; j\}' \in \hat{\mathcal{U}}_0$$

with $a^{\varepsilon}_{ij} \in \widehat{\mathbb{Z}[q]}$ we have

$$\varphi(K^{\varepsilon} E^i F^{(i)} \{H; j\}') = q^{\varepsilon\alpha} K_0^{-2\varepsilon} \begin{bmatrix} H_0 + i \\ i \end{bmatrix}_q \{\alpha - H_0; i\}_q \{\alpha - 2H_0; j\}'.$$

Since the coefficient of $q^{\alpha(\varepsilon+i+2j)}$ in this expression is

$$\begin{bmatrix} H_0 + i \\ i \end{bmatrix}_q K_0^{-(2\varepsilon+i+4j)} q^{-\binom{i}{2}} q^{-j(j-1)}$$

the summands of $\varphi(x)$ with different i, j, ε do not cancel.

It remains to check that φ preserves the relations. For any $n \geq 0$, we have

$$[n+1] E^{n+1} F^{(n+1)} = E^n E F^{(n)} F^{(1)} = E^n F^{(n)} \left(E F^{(1)} + [n](q^{-n-1} K - q^{n+1} K^{-1}) \right),$$

where we used (5.3), (5.1), (5.2). Since $\{1\}\varphi$ sends the expression in the brackets to $\{H_0 + n + 1\}\{\alpha - H_0 - n\}$, we verify inductively that φ respects the relations. The case of $F^{(n)} E^n$ follows similarly. Finally, we verify inductively that the relations

$$E^n F^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F^{(n-k)} \{H - 2n + 2k; k\}_q E^{n-k}$$

holds in the image. □

Remark that $\{1\}^2 \varphi(C) = q^{\alpha+1} + q^{-\alpha-1}$. Now let's define the linear translation operator:

$$\begin{aligned} M : \mathcal{A} &\to \mathcal{A} \\ K_0 &\mapsto q K_0 \\ \begin{bmatrix} H_0 + m \\ n \end{bmatrix} &\mapsto \begin{bmatrix} H_0 + m + 1 \\ n \end{bmatrix} \end{aligned}$$

and set $P_n = \varphi(E^n F^{(n)})$ and $Q_n = \varphi(F^{(n)} E^n)$.

Lemma 188. *For any $n, m \in \mathbb{N}$ we have $M^n Q_n = P_n$ and $M^{-n} P_n = Q_n$,*

$$P_n M^n(P_m) = \begin{bmatrix} n+m \\ n \end{bmatrix}_q P_{n+m} \quad \text{and} \quad Q_n P_m = \begin{bmatrix} n+m \\ n \end{bmatrix}_q M^{-n}(P_{n+m}).$$

Proof. The first three formulas follow by a straightforward verification using Proposition 187. The last formula is obtained by applying M^{-n} to the previous one. □

From here on, we identify $\hat{\mathcal{U}}_0$ with its image in $\mathcal{A}$. Proposition 187 and the operators $M^{\pm 1}$ are useful to compute the commutators, e.g.

$$E^n P_m Q_k F^{(n)} = M^n(P_m Q_k) E^n F^{(n)} \quad \text{or} \quad F^{(n)} P_m Q_k E^n = M^{-n}(P_m Q_k) F^{(n)} E^n$$

Traces

Let us introduce the following evaluation maps

$$f_n : \mathcal{A} \to \widehat{\mathbb{Z}[q, K_0]} \left[\begin{bmatrix} H_0 + m \\ m \end{bmatrix}_q, m \in \mathbb{N} \right], \; q^{\alpha} \mapsto q^n,$$

$$ev_r : \mathcal{A} \to \mathbb{Z}[\widehat{\zeta_{2r}, q^\alpha}, K_0]\left[\begin{bmatrix} H_0+m \\ m \end{bmatrix}_{\zeta 2r}, m \in \mathbb{N}\right],\ q \to \zeta_{2r}.$$

We denote indistinctly g the map that sends H_0 to 0, for example it sends:

$$\mathcal{A} \to \hat{R} \text{ (completion ring)},$$

$$\widehat{\mathbb{Z}[q, K_0]}\left[\begin{bmatrix} H_0+m \\ m \end{bmatrix}_q, m \in \mathbb{N}\right] \to \widehat{\mathbb{Z}[q]} \text{ (Habiro's ring)},$$

$$\mathbb{Z}\left[\widehat{\zeta_{2r}, q^\alpha}, K_0\right]\left[\begin{bmatrix} H_0+m \\ m \end{bmatrix}_{\zeta 2r}, m \in \mathbb{N}\right] \to \hat{R}_r \text{ (completion ring at root of unity)}.$$

Let I (respectively I_0) be the $\widehat{\mathbb{Z}[q]}$ submodule of $\hat{\mathcal{U}}$ (respectively $\hat{\mathcal{U}}_0$) generated by commutators, i.e. elements of the form $[x, y] = xy - yx$ for any $x, y \in \hat{\mathcal{U}}$. The universal trace on $\hat{\mathcal{U}}$ is the quotient $\hat{\mathcal{U}}/I$. The following operator

$$\mathrm{Tr}_N = \sum_{i=0}^{N-1} M^i$$

will play the role of the trace on an $N+1$-dimensional irreducible representation. Let us show that combined with appropriate evaluations it kills the commutators.

Lemma 189. *For any $a \in I_0$,*

- $g \circ f_N \circ Tr_{N+1}(a) = 0,$

Proof. It is enough to prove the claims on generators of I_0, that are commutators

$$\left[K^m E^{n_1} F^{(l_1)}, E^{n_2} F^{(l_2)}\right] \quad \text{for any} \quad m, n_i, l_i \quad \text{with} \quad n_1+n_2 = l_1+l_2, n_1 \geq l_1.$$

Set $n = n_1 - l_1 = l_2 - n_2$. Using Lemma 188 it is easy to see that

$$\begin{bmatrix} l_2 \\ n_2 \end{bmatrix}_q \left[K^m E^{n_1} F^{(l_1)}, E^{n_2} F^{(l_2)}\right] = q^{2mn}(M^n - Id) K^m P_{n_2} Q_n P_{l_1}.$$

But now, $\mathrm{Tr}_{N+1} \circ (M^n - Id) = \mathrm{Tr}_n \circ \left(M^{N+1} - Id\right)$, and hence

$$\begin{aligned} g \circ f_N \circ \mathrm{Tr}_{N+1} \circ (M^n - Id)(Q_n) &= g \circ f_N \circ \mathrm{Tr}_n \circ \left(M^{N+1} - Id\right)(Q_n) \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} N+k+1 \\ n \end{bmatrix}_q \{n-k-1; n\} - \begin{bmatrix} k \\ n \end{bmatrix}_q \{N-k+n; n\} \\ &= 0, \end{aligned}$$

where the third equality comes from the fact that $\{n-k-1; n\} = \left[\begin{smallmatrix} k \\ n \end{smallmatrix}\right]_q = 0$, $\forall k \leq n-1$. Thus,

$$g \circ f_N \circ \mathrm{Tr}_{N+1}\left(\begin{bmatrix} l_2 \\ n_2 \end{bmatrix}_q \left[K^m E^{n_1} F^{(l_1)}, E^{n_2} F^{(l_2)}\right]\right) =$$
$$= g \circ f_N \circ \mathrm{Tr}_{N+1} \circ (M^n - Id)\left(q^{2mn} K^m P_{n_2} Q_n P_{l_1}\right) = 0.$$

□

Remark 190. If we try to prove the root of unity case $g \circ ev_r \circ Tr_r(a)$, $a \in I_0$. We can rewrite

$$\left[K^m E^{n_1} F^{(l_1)}, E^{n_2} F^{(l_2)}\right] = q^{2mn}(M^n - Id) K^m M^{-n}(P_{l_2}) P_{l_1}$$

using the last formula in Lemma 188.

$$ev_r \circ \mathrm{Tr}_r\left(\left[K^m E^{n_1} F^{(l_1)}, E^{n_2} F^{(l_2)}\right]\right) = ev_r \circ \mathrm{Tr}_n \circ (M^r - Id)\left(q^{2mn} K^m M^{-n}(P_{l_2}) P_{l_1}\right).$$

But $ev_r \circ (M^r - Id) \neq 0$ since

$$g \circ ev_r \circ (M^r - Id)\begin{bmatrix} H_0+m \\ n \end{bmatrix}_q = \begin{bmatrix} r+m \\ n \end{bmatrix}_q - \begin{bmatrix} m \\ n \end{bmatrix}_q \neq 0$$

Tensor products

Let us introduce a completed tensor product

$$\hat{\mathcal{U}}\hat{\otimes}\hat{\mathcal{U}} = \lim_{\overleftarrow{k,l}} \frac{\hat{\mathcal{U}} \otimes_{\mathbb{Z}[q^{\pm 1}]} \hat{\mathcal{U}}}{(\overline{J}_k \otimes_{\mathbb{Z}[q^{\pm 1}]} \hat{\mathcal{U}} + \hat{\mathcal{U}} \otimes_{\mathbb{Z}[q^{\pm 1}]} \overline{J}_l)}$$

where $\overline{J}_k$ is a closure of J_k in $\hat{\mathcal{U}}$. All construction we made can be extended to tensor products $\hat{\mathcal{U}}^{\hat{\otimes} l}$, and rings with more variables $q^{\alpha_i}, K_{0,i}, M_i$.

We still call

$$\mathcal{A} := \mathbb{Z}\,\widehat{[q, q^{\alpha_i}, K_{0,i}; 1 \leq i \leq l]} \left[\begin{bmatrix} H_{0,i} + m \\ n \end{bmatrix}_q, m \in \mathbb{Z}, n \in \mathbb{N} \right]$$

and have an injective algebra homomorphism

$$\begin{aligned} (\hat{\mathcal{U}}_0)^{\hat{\otimes} l} &\to A \\ 1 \otimes \cdots \otimes K \otimes \cdots \otimes 1 &\mapsto q^{\alpha_i} K_{0,i}^{-2} = q^{\alpha_i} q^{-2H_{0,i}}, \\ 1 \otimes \cdots \otimes E^n F^{(n)} \otimes \cdots \otimes 1 &\mapsto \begin{bmatrix} H_{0,i} + n \\ n \end{bmatrix}_q \{\alpha - H_{0,i}; n\}_q, \end{aligned}$$

where K and $E^n F^{(n)}$ are in the i-th position of tensor product.

Let $\overline{N} = (N_1, \ldots, N_l)$, we have:

$$M_i : A \to A,\ K_{0,i} \mapsto qK_{0,i},$$

$$\mathrm{Tr}_{\overline{N}} = \prod_{i=1}^{l} \sum_{k=0}^{N_i - 1} M_i^k.$$

$$f_{\overline{N}} : A \to \mathbb{Z}\,\widehat{[q, K_{0,i}; 1 \leq i \leq l]} \left[\begin{bmatrix} H_{0,i} + m \\ m \end{bmatrix}_q, m \in \mathbb{N} \right],\ q^{\alpha_i} \mapsto q^{N_i},$$

$$ev_r : A \to \mathbb{Z}\,\widehat{[\zeta_{2r}, \zeta_{2r}^{\alpha_i}, K_{0,i} : 1 \leq i \leq l]} \left[\begin{bmatrix} H_{0,i} + m \\ m \end{bmatrix}_{\zeta_{2r}}, m \in \mathbb{N} \right],\ q \to \zeta_{2r}.$$

Moreover g is the map that sends any $H_{0,i}$ to 0.

ADO and Jones from the universal invariant

Let L be a 0-framed algebraically split link with l components, and T a $(l-l)$ tangle whose closure is L.

We denote J_T the universal invariant associated to this tangle (with pivotal element added at the top of the strands), and J_L the universal invariant of link. We have that $J_T \in \hat{\mathcal{U}}^{\otimes l}$ and $J_L \in (\hat{\mathcal{U}}/I)^{\otimes l}$ where I is the $\widehat{\mathbb{Z}[q]}$ submodule generated by elements $xy - yx$ for any $x, y \in \hat{\mathcal{U}}$. Finally, let $J_L^{(i)} \in (\hat{\mathcal{U}}/I)^{\otimes i-1} \otimes \hat{\mathcal{U}} \otimes (\hat{\mathcal{U}}/I)^{\otimes l-i}$ be the universal invariant of the link L with the i-th strand open.

We denote I_0 the 0 weight part of I, $J_{T,0}, J_{L,0}, J_{L,0}^{(i)}$ the 0 weight parts of these elements.

(If we'd decided not to put the pivotal element at the top of the strands of T, we would have needed to consider the pivotal twisted commutators $[E, F]_q = EF - K^{-1}FKE$ instead of the standard ones and everything would have worked the same.)

Theorem 189 implies that there exists well defined maps $j_n : (\hat{\mathcal{U}}/I) \to \widehat{\mathbb{Z}[q]}$ and $a_r : (\hat{\mathcal{U}}/I) \to \hat{R}_r$.

Lemma 191. *Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_l)$, $\overline{n} = (n_1, \ldots, n_l)$ and $\overline{r} = (r, \ldots r)$.*

- $g \circ f_{\overline{n}} \circ Tr_{\overline{n+1}}(J_{T,0}) = J_{\overline{n}}(q, L)$,
- $g \circ ev_r \circ Tr_{\overline{r}}^{(i)}(J_{T,0}) = C_\infty(r, \overline{\zeta_{2r}^{r\alpha}}, L) ADO_r(\overline{q^{\alpha}}, L) \frac{\{r\alpha_i\}}{\{\alpha_i\}}$,

where $J_{\overline{n}}(q,L)$ is the colored Jones polynomial of L and $ADO_r(\overline{q^\alpha},L)$ is the r-th ADO polynomial of L.

Putting the two lemma together we obtain:

Proposition 192. *There are well defined maps $j_{\overline{n}} : (\hat{\mathcal{U}}/I)^{\otimes l} \to \widehat{\mathbb{Z}[q]}$ and $a_r^{(i)} : (\hat{\mathcal{U}}/I)^{\otimes i-1} \otimes \hat{\mathcal{U}} \otimes (\hat{\mathcal{U}}/I)^{\otimes l-i} \to \hat{R}_r$ such that :*

- $j_{\overline{n}}(J_L) = J_{\overline{n}}(q,L)$,

Remark 193. Because of remark 190, we cannot formulate a priori the same proposition for the root of unity case.

Proof. (of Lemma 191)
Let $v_{i_k} \in V_{\alpha_k}$ of weight $\alpha_k - 2i_k$, we denote $< J_T v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} >$ be the coefficient before $v_{i_1} \otimes \cdots \otimes v_{i_l}$ in the basis decomposition of $J_T v_{i_1} \otimes \cdots \otimes v_{i_l}$.

1) For $\alpha_i = n_i$, the colored Jones ponlynomial associated to L can be written :

$$J_{\overline{n}}(q,L) = \sum_{\overline{i}=0}^{\overline{n}} < J_T v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} >$$

where $\overline{i} = (i_1, \dots, i_l)$ and $\overline{n} = (n_1, \dots, n_l)$.

Let $M_{\overline{i}} = M_1^{i_1} \times \cdots \times M_l^{i_l}$, we have the relation :

$$g \circ f_{\overline{n}} \circ M_{\overline{i}}(J_{T,0}) =< J_T v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} > .$$

This means that :

$$g \circ f_{\overline{n}} \circ \mathrm{Tr}_{\overline{n+1}}(J_{T,0}) = J_{\overline{n}}(q,L).$$

2)Let J_T^r the universal invariant using truncated R-matrix instead of the infinite R-matrix, and using pivotal element K^{r-1} instead of K^{-1}. Then, we can write the ADO polynomial as:

$$\frac{\{r\alpha_k\}}{\{\alpha_k\}} ADO_r(q^{\overline{\alpha}}) = \sum_{\overline{i}-i_k=0}^{\overline{r-1}} < J_T^r v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} >$$

where $q^{\overline{\alpha}} = q^{\alpha_1}, \dots, q^{\alpha_l}$.

Moreover using the factorisation lemma for $\overline{i} \le \overline{r-1}$,

$$< J_T v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} >= C_\infty(r, q^{\overline{r\alpha}}, L) \times < J_T^r v_{i_1} \otimes \cdots \otimes v_{i_l}, v_{i_1} \otimes \cdots \otimes v_{i_l} > .$$

Note that $C_\infty(r, q^{\overline{r\alpha}}, L)$ does not depend on $\overline{i}$.

Hence, $g \circ ev_r \circ Tr_{\overline{r}}^{(k)}(J_{T,0}) = C_\infty(r, \overline{q^{r\alpha}}, L) ADO_r(\overline{q^\alpha}, L) \frac{\{r\alpha_k\}}{\{\alpha_k\}}$. □

As a conclusion of this subsection, one can easily get the colored Jones polynomial from the universal invariant, passing through a commutative ring. But the ADO case presents some obstructions.

5.1.3 Braid representation approach

We will try in this subsection to use the braid representation factorisation at roots of unity showed in Proposition 176 in order to understand what happens if we try to take the quantum trace on the tensor product of Verma module at root of unity. An obstruction to this approach will be similar to the obstruction of the state sum diagram approach: the trace does not converge in the type of completed rings presented before.

Colored braid representation

Let's generalise the braid representations to multiple colors.

By seeing the permutation group as

$$S_n = \left\langle s_1, \dots, s_{n-1} | s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ for } |i-j| \geq 2, s_i^2 = 1 \right\rangle,$$

we have the following surjective morphism.

Definition 194.

$$\begin{aligned} p : B_n &\to S_n \\ \sigma_i &\mapsto s_i \end{aligned}$$

we denote $\pi_\beta := p(\beta)$.

Let $\mathbb{Z}[q^{\pm 1}, A_1^{\pm 1}, \dots, A_n^{\pm 1}]$ we denote $q^{\alpha_i} := A_i$,

Definition 195. We define

$$\varphi_n^\pi(q^{\alpha_1}, \dots, q^{\alpha_n}; \beta) \in \mathrm{Hom}(V^{\alpha_{\pi(1)}} \otimes \cdots \otimes V^{\alpha_{\pi(n)}}, V^{\alpha_{\pi\pi_\beta^{-1}(1)}} \otimes \cdots \otimes V^{\alpha_{\pi\pi_\beta^{-1}(n)}})$$

by

$$\varphi_n^\pi(q^{\bar{\alpha}}, \sigma_i) = \left(1^{\otimes i-1} \otimes (\tau \circ R) \otimes 1^{\otimes n-i-2}\right)$$

and

$$\varphi_n^\pi(q^{\bar{\alpha}}, \beta_2\beta_1) = \varphi_n^{\pi\pi_{\beta_1}^{-1}}(q^{\bar{\alpha}}, \beta_2)\varphi_n^\pi(q^{\bar{\alpha}}, \beta_1)$$

where $\tau(v \otimes w) = w \otimes v$, $\pi \in S_n$.

We define in an analogous way weight level sub representations:

Definition 196. Let

$$V_{n,m}(q, q^{\alpha_1}, \dots, q^{\alpha_n}) := < v_{i_1} \otimes \cdots \otimes v_{i_n} | \sum_{k=1}^n i_k = m >$$

and the associated representation

$$\varphi_{n,m}^\pi(q, q^{\bar{\alpha}}, .) : B_n \to \mathrm{Hom}(V_{n,m}(q, q^{\alpha_{\pi(1)}}, \dots, q^{\alpha_{\pi(n)}}), V_{n,m}(q, q^{\alpha_{\pi_\beta\pi(1)}}, \dots, q^{\alpha_{\pi_\beta\pi(n)}})).$$

Remark 197. For a braid β if we identify $q^{\alpha_i} = q^{\alpha_{\pi_\beta(i)}}$, $\forall i \leq n$, we have

$$\varphi_{n,m}^\pi(q, q^{\bar{\alpha}}, .) : B_n \to \mathrm{End}(V_{n,m}(q, q^{\alpha_{\pi(1)}}, \dots, q^{\alpha_{\pi(n)}})),$$

and we can take the quantum trace

$$\mathrm{Tr}(K.\varphi_{n,m}^\pi(q, q^{\bar{\alpha}}, .)).$$

As before we treat the case $q = 1$ slightly differently.

Definition 198. We set $W_{n,m}(q^{\alpha_1}, \dots, q^{\alpha_n}) := V_{n,m}(q, q^{\alpha_1}, \dots, q^{\alpha_n})$ and it associated representation:

$$\psi_{n,m}^\pi(q^{\bar{\alpha}}, \sigma_i) := q^{\frac{-\alpha_{\pi(i)}\alpha_{\pi(i+1)}}{2}} \varphi_{n,m}^\pi(q, q^{\bar{\alpha}}, \sigma_i)$$

Moreover, we denote $w_i := v_i$ at $q = 1$.

For $m = 1$, we consider a small change of bases. We set $u_j^\alpha := j!\{\alpha\}^j w_j$. Let's take a basis of $V_{n,1}$, $e_k^\pi := u_0^{\alpha_{\pi(1)}} \otimes \cdots \otimes u_1^{\alpha_{\pi(k)}} \otimes \cdots \otimes u_0^{\alpha_{\pi(n)}}$. We can identify higher weight tensors with symmetric powers of the e_k:

$$u_{j_1} \otimes \cdots \otimes u_{j_n} = \prod_{k=1}^n e_k^{j_k}.$$

Now, in the basis e_k^π, we have

$$\psi_{n,1}^\pi(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1 - q^{-2\alpha_{\pi(i)}} & q^{-\alpha_{\pi(i)}} & 0 \\ & q^{-\alpha_{\pi(i+1)}} & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array}\right)$$

Proposition 199. *Let $\beta \in B_n$, $\pi \in S_n$,*

$$\psi^{\pi}_{n,m}(q^{\bar{\alpha}}, \beta) = Sym^m(\psi^{\pi}_{n,1}(q^{\bar{\alpha}}, \beta))$$

Proof. Similar as Proposition 168. □

Moreover $\psi^{\pi}_{n,1}(q^{\bar{\alpha}}, \beta)$ is the Gassner representation:

Proposition 200. *If we set*

$$\Psi(e^{\pi}_k) = \frac{1 - q^{-2\alpha_{\pi(k)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi(i)}}} e^{\pi}_k$$

and if we see the representation $\psi^{\pi}_{n,1}(q^{\bar{\alpha}}, \beta)$ as a matrix in the basis e^{π}_k, we have:

$$\Psi \circ \psi^{\pi}_{n,1}(\sigma_i) \circ \Psi^{-1} = \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 1 - q^{-2\alpha_{\pi(i+1)}} & 1 \\ q^{-2\alpha_{\pi(i+1)}} & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right)$$

Proof. See [21] Theorem 2.1.1 for more details.

For e^{π}_k,

$$\begin{aligned}
\Psi \circ \psi^{\pi}_{n,1}(\sigma_k)(e^{\pi}_k) &= \Psi((1 - q^{-2\alpha_{\pi(k)}})e^{\pi s_k}_k + q^{-\alpha_{\pi(k+1)}} e^{\pi s_k}_{k+1}) \\
&= (1 - q^{-2\alpha_{\pi(k)}}) \frac{1 - q^{-2\alpha_{\pi s_k(k)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi s_k(i)}}} e^{\pi s_k}_k + \frac{1 - q^{-2\alpha_{\pi s_k(k+1)}}}{\prod_{i=k+1}^{n} q^{-\alpha_{\pi s_k(i)}}} q^{-\alpha_{\pi(k+1)}} e^{\pi s_k}_{k+1} \\
&= (1 - q^{-2\alpha_{\pi(k)}}) \frac{1 - q^{-2\alpha_{\pi(k+1)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi(i)}}} e^{\pi s_k}_k + \frac{1 - q^{-2\alpha_{\pi(k)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi(i)}}} q^{-2\alpha_{\pi(k+1)}} e^{\pi s_k}_{k+1} \\
&= \frac{1 - q^{-2\alpha_{\pi(k)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi(i)}}} \left((1 - q^{-2\alpha_{\pi(k+1)}})e^{\pi s_k}_k + q^{-2\alpha_{\pi(k+1)}} e^{\pi s_k}_{k+1}\right) \\
&= \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 1 - q^{-2\alpha_{\pi(i+1)}} & 1 \\ q^{-2\alpha_{\pi(i+1)}} & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right) \circ \Psi(e^{\pi}_k)
\end{aligned}$$

For e^{π}_{k+1},

$$\begin{aligned}
\Psi \circ \psi^{\pi}_{n,1}(\sigma_k)(e^{\pi}_{k+1}) &= \Psi(q^{-\alpha_{\pi(k)}} e^{\pi s_k}_k) \\
&= q^{-\alpha_{\pi(k)}} \frac{1 - q^{-2\alpha_{\pi s_k(k)}}}{\prod_{i=k}^{n} q^{-\alpha_{\pi s_k(i)}}} e^{\pi s_k}_k \\
&= \frac{1 - q^{-2\alpha_{\pi(k+1)}}}{\prod_{i=k+1}^{n} q^{-\alpha_{\pi(i)}}} e^{\pi s_k}_k \\
&= \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 1 - q^{-2\alpha_{\pi(i+1)}} & 1 \\ q^{-2\alpha_{\pi(i+1)}} & 0 \end{matrix} & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right) \circ \Psi(e^{\pi}_{k+1})
\end{aligned}$$

□

Colored r-part representations

In this paragraph we set $q = \zeta_{2r}$. Everything works exactly as the mono colored situation described in Subsection 4.5. Proofs of the following results can be found there.

Definition 201. The r-part of a tensor $v = v_{i_1+rj_1} \otimes \cdots \otimes v_{i_n+rj_n}$ where $i_1, \ldots i_n \leq r-1$ is defined by

$$\mathrm{rp}(v) := \sum_{k=0}^{n} j_k.$$

Definition 202. We define the following $\mathbb{Z}[\zeta_{2r}, q^{\alpha_1}, \dots, q^{\alpha_n}]$ modules:

$$V_n^m(\zeta_{2r}, q^{\alpha_1}, \dots, q^{\alpha_n}) := < v|\ \mathrm{rp}(v) = m >$$

$$V_n^{\leq m}(\zeta_{2r}, q^{\alpha_1}, \dots, q^{\alpha_n}) := \bigoplus_{i=0}^{m} V_n^m(\zeta_{2r}, q^{\alpha})$$

We have directly that $V_n^{\leq m}$ is a sub representation.

Proposition 203. *$V_n^{\leq m}(\zeta_{2r}, q^{\alpha_{\pi 1}}, \dots, q^{\alpha_{\pi n}})$ is a sub-representation of braids defined by $\varphi_n^{\leq m,\pi}(\beta)$.*

Although it is not a sub representation, V_n^m can be turned into a representation.

Proposition 204. *Let $\rho_n^m : V_n^{\leq m} \to V_n^m$ the canonical projection map, then V_n^m is a sub representation of braids with*

$$\varphi_n^{m,\pi} := \rho_n^m \circ \varphi_n^{\leq m,\pi}|_{V_n^m}.$$

We then have the following factorisation proposition

Proposition 205. *The isomorphism*

$$\Phi : V_n^m \to V_n^0 \otimes F_r(W_{n,m})$$
$$v_{\overline{i+rj}} \mapsto v_{\overline{i}} \otimes F_r(w_{\overline{j}}),$$

where $\overline{i+rj} = (i_1 + rj_1, \dots, i_n + rj_n)$ with $i_1, \dots, i_n \leq r-1$, is a braid group representation isomorphism.

In other word, the following diagram commutes:

$$\begin{array}{ccc} V_n^m & \xrightarrow{\varphi_n^{m,\pi}} & V_n^m \\ \Phi \downarrow & & \downarrow \Phi \\ V_n^0 \otimes F_r(W_{n,m}) & \xrightarrow{\varphi_n^{0,\pi} \otimes (F_r \circ \psi_{n,m}^{\pi})} & V_n^0 \otimes F_r(W_{n,m}) \end{array}$$

Factorisation at the level of traces

For the same reasons as Subsection 4.6, we can obtain ADO polynomials with the 0 r-part representation. Let L be a knot and $\beta \in B_n$ whose closure is L.

Proposition 206.

$$ADO_r(q^{\alpha_1}, \dots, q^{\alpha_n}, L) = \frac{\{\alpha_1 + 1\}}{\{r\alpha_1\}} Tr_{2,\dots,n}((1 \otimes (K^{1-r})^{\otimes n-1})\varphi_n^{0,id}(\beta))$$

where $q^{\alpha_i} = q^{\alpha_{\pi_\beta(i)}}$, $\forall i \leq n$.

Along with a factorisation of the m parts,

Corollary 207.

$$\{\alpha_1+1\} Tr_{2,\dots,n}((1\otimes K^{\otimes n-1})\varphi_n^{m,id}(\beta)) = ADO_r(q^{\alpha_1}, \dots, q^{\alpha_n}, L) \times \{r\alpha_1\} F_r(Tr_{2,\dots,n}((1\otimes K^{\otimes n-1})\psi_{n,m}^{id}(\beta))$$

Moreover we can use Mac-Mahon master theorem (180) to prove the following proposition:

Proposition 208. *For $\beta \in B_n$ whose closure is L, then :*

$$\sum_m Tr_{2,\dots,n}((1 \otimes K^{\otimes n-1})\psi_{n,m}^{id}(\beta))t^m = \frac{q^{|\bar{\alpha}|_{2,\dots,n}}}{det(I_n - \begin{pmatrix} 0 & 0 \\ 0 & tI_{n-1} \end{pmatrix} \psi_{n,1}^{id}(\beta))}.$$

We can then see the multivariable Alexander polynomial in the previous formula.

Proposition 209.

$$\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)\psi^{id}_{n,1}(\beta)) = q^{|\bar{\alpha}|_{2,\dots,n} - \bar{\alpha}^T L_k(L)\bar{1}}\{\alpha_1\}A_L(q^{\alpha_1},\dots,q^{\alpha_n})$$

where $\bar{\alpha} = (\alpha_1,\dots,\alpha_n)$, $\bar{1} = (1,\dots,1)$, $L_k(L)$ *is the linking matrix of* L, $q^{|\bar{\alpha}|_{2,\dots,n}} = q^{\alpha_2+\dots+\alpha_n}$.

Proof. Generalisation of Proposition 182. □

Let's do some examples to illustrate this fact.

Example 210. First of all, the easiest example would be the Hopf Link H, Figure 5.1.

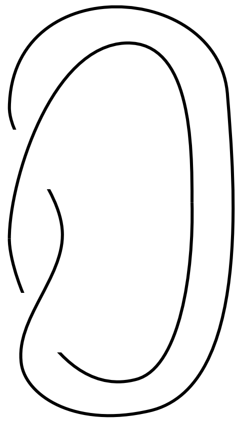


Figure 5.1: The Hopf link

The multivariable Alexander polynomial of this link is:

$$A_H(q^{\alpha_1}, q^{\alpha_2}) = 1.$$

If we take $\beta = \sigma_1^2 \in B_2$,

$$\det(I_2 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\psi^{id}_{2,1}(q^{\alpha_1}, q^{\alpha_2}, \beta)) = q^{-2\alpha} - 1$$

Since the linking matrix is $L_k(H) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we get the formula :

$$\det(I_2 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\psi^{id}_{2,1}(\beta)) = q^{\alpha_2 - \bar{\alpha}^T L_k(H)\bar{1}}\{\alpha_1\}A_H(q^{\alpha_1}, q^{\alpha_2})$$

Example 211. Now let's do the $L4a1$ link that we will denote uninspiringly $2H$, Figure 5.2.

The multivariable Alexander polynomial of this link is:

$$A_{2H}(q^{\alpha_1}, q^{\alpha_2}) = q^{\alpha_1+\alpha_2} + q^{-\alpha_1-\alpha_2}.$$

If we take $\beta = \sigma_1^4 \in B_2$,

$$\det(I_2 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\psi^{id}_{2,1}(q^{\alpha_1}, q^{\alpha_2}, \beta)) = q^{-4\alpha_1-2\alpha_2} - q^{-2\alpha_1-2\alpha_2} + q^{-2\alpha_1} - 1$$

Since the linking matrix is $L_k(2H) = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ we get the formula :

$$\det(I_2 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\psi^{id}_{2,1}(\beta)) = q^{\alpha_2 - \bar{\alpha}^T L_k(2H)\bar{1}}\{\alpha_1\}A_{2H}(q^{\alpha_1}, q^{\alpha_2})$$

Example 212. Lastly, the Whitehead link denoted W, Figure 5.3.

The multivariable Alexander polynomial of this link is:

$$A_W(q^{\alpha_1}, q^{\alpha_2}) = \{\alpha_1\}^2\{\alpha_2\}.$$

If we take $\beta = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1} \in B_3$,

$$\det(I_3 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_2 \end{smallmatrix}\right)\psi^{id}_{3,1}(q^{\alpha_1}, q^{\alpha_2}, \beta)) = -q^{-2\alpha_1+2\alpha_2} + q^{-2\alpha_1+4\alpha_2} + 2q^{2\alpha_2} - 2q^{4\alpha_2} - q^{2\alpha_1+2\alpha_2} + q^{2\alpha_1+4\alpha_2}$$

Since the linking matrix is $L_k(W) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we get the formula :

$$\det(I_3 - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\psi^{id}_{3,1}(\beta)) = q^{2\alpha_2 - \bar{\alpha}^T L_k(W)\bar{1}}\{\alpha_1\}A_W(q^{\alpha_1}, q^{\alpha_2})$$

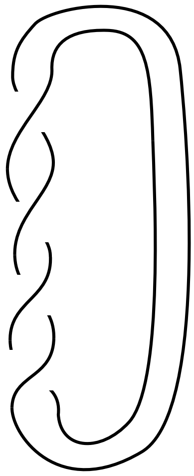


Figure 5.2: The L4a1.

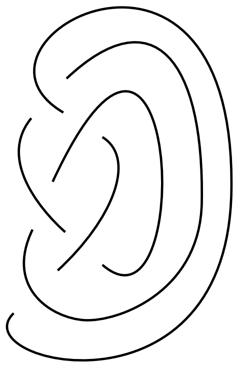


Figure 5.3: The Whitehead link.

Going back to the factorisation and putting every piece together we get,

Proposition 213. *For L a link and $\beta \in B_n$ whose closure is L, for $q = \zeta_{2r}$, we set $q^{\alpha_i} = q^{\alpha_{\pi_\beta(i)}}$, $\forall i \leq n$ and we have*

$$\{\alpha_1 + 1\} \sum_m Tr_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n^{m,id}(\beta))t^m = \frac{ADO_r(q^{\alpha_1}, \dots, q^{\alpha_n}, L)}{F_r(q^{-|\bar{\alpha}|_{2,\dots,n}} det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & tI_{n-1} \end{smallmatrix}\right) \psi_{n,1}(\beta)))}$$
$$\xrightarrow[t \to 1]{} \frac{ADO_r(q^{\alpha_1}, \dots, q^{\alpha_n}, L)}{q^{-r\bar{\alpha}^T L_k(L)\bar{1}} A_L(q^{r\alpha_1}, \dots, q^{r\alpha_n})}$$

So this is not quite satisfactory, we would like more: if

$$\{\alpha_1 + 1\} \sum_m \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n^{m,id}(\beta))$$

existed in some bigger ring, we would have a unified invariant for links and an evaluation map at root of unity sending it to

$$\frac{ADO_r(q^{\alpha_1}, \dots, q^{\alpha_n}, L)}{q^{-r\bar{\alpha}^T L_k(L)\bar{1}} A_L(q^{r\alpha_1}, \dots, q^{r\alpha_n})}.$$

We did not, at this time, find such ring.

5.2 Perspectives

Those bittersweet last words are to be cherished for they tell us there is still work to be done. So let's put down a few lines on what could be next.

First of all, one can pursue the unification for the link case. The main obstruction should be to allow convergence of the partial trace of the link with one strand open. Once it is done, it shouldn't be too hard to take the full trace, and more interestingly, the full trace on the kernel of E. This will allow us to have a better understanding of the underlying homology, being a trace on the Lawrence representations.

This bring us to the second point, the understanding of the underlying homology. In his work [21], Martel expanded the colored Jones polynomials of knots as Lefschetz numbers. The same might be done for the unified invariant of knots seen as a trace over the kernel of E of the quantum representation.

Another direction would be to express the unified invariant as a quantum determinant using quantum Mac Mahon theorem [10] and the construction for the colored Jones polynomials as shown in [17].

A way to generalise things further would be to create the unified invariant of knots for other quantum groups. Since the only tool we need are a universal invariant and a Verma module over a completed ring, it should be doable.

Of course, once the link case is done, one can try to progress to the 3-manifold invariants. There are two families of 3-manifold invariants, the Witten-Reshetikhin-Turaev invariant -constructed using the colored Jones polynomials at roots of unity- and the Costantino-Geer-Patureau invariant -constructed using ADO polynomials and a cohomology element. Maybe the unified invariant could be used in order to unify the CGP invariants in a similar way as Habiro's unified invariant for the WRT of integral homology spheres. A next step would be to do so for the TQFT.

Moreover we could try to use the unified invariant in the context of finite type 3-manifolds invariants, using what we know with the colored Jones polynomials.

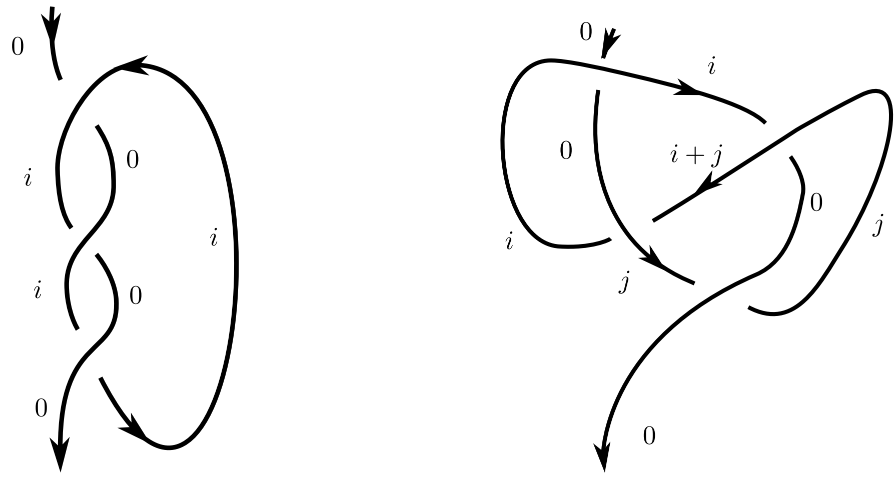
Appendices

Appendix A

Computations with state sum diagrams

This section will be dedicated to compute the unified invariant $F_\infty(q, A, \mathcal{K})$ on some examples. We will also explicitly compute $C_\infty(1, A, \mathcal{K})$ and see that it is equal to the inverse of the Alexander polynomial.

To do so we will use state diagrams and compute the unified invariant from it. You can also use them to compute the ADO polynomials (see [22] section 4). Recall that $q^\alpha := A$.



(a) The trefoil knot.

(b) The figure eight knot.

Figure A.1: Examples of state diagrams to compute the invariants.

The Trefoil Knot:

We will denote the trefoil knot 3_1:

$$F_\infty(q, A, 3_1) = q^{\frac{3\alpha^2}{2}} \sum_i q^{\alpha-2i} q^{\frac{i(i-1)}{2}} \{\alpha; i\}_q q^{-3i\alpha}$$

$$\begin{aligned} C_\infty(1, A, 3_1) &= q^\alpha \sum_i q^{-3i\alpha} \{\alpha\}_q^i \\ &= q^\alpha \frac{1}{1 - q^{-3\alpha}\{\alpha\}_q} \\ &= \frac{q^{3\alpha}}{A_{3_1}(q^{2\alpha})} \end{aligned}$$

The Figure Eight Knot:

We will denote the figure eight knot 4_1:

$$\begin{aligned} F_\infty(q,A,4_1) &= \sum_{i,j} q^{2(i-j)} q^{i\alpha} q^{-j\alpha} (-1)^j q^{\frac{i(i-1)}{2}} \begin{bmatrix} i+j \\ j \end{bmatrix}_q \{\alpha - j; i\}_q q^{(i+j)\alpha} \\ &\times q^{-2ij} q^{-\frac{j(j-1)}{2}} \{\alpha; j\}_q q^{-(i+j)\alpha} q^{2ij} \\ &= \sum_{i,j} q^{2(i-j)} q^{(i-j)\alpha} (-1)^j \begin{bmatrix} i+j \\ j \end{bmatrix}_q q^{\frac{i(i-1)}{2}} q^{-\frac{j(j-1)}{2}} \{\alpha; i+j\}_q \end{aligned}$$

$$\begin{aligned} C_\infty(1,A,4_1) &= \sum_{i,j} q^{(j-i)\alpha} (-1)^i \binom{i+j}{j} \{\alpha\}_q^{i+j} \\ &= \sum_N \sum_{i=0}^N q^{N\alpha} q^{-2i\alpha} (-1)^j \binom{i+j}{j} \{\alpha\}_q^N \\ &= \sum_N q^{N\alpha} \{\alpha\}_q^N \sum_{i=0}^N (-q^{-2\alpha})^i \binom{i+j}{j} \\ &= \sum_N q^{N\alpha} \{\alpha\}_q^N (1 - q^{-2\alpha})^N \\ &= \sum_N \{\alpha\}_q^{2N} \\ &= \frac{1}{1 - \{\alpha\}_q^2} \\ &= \frac{1}{A_{4_1}(q^{2\alpha})} \end{aligned}$$

The Cinquefoil Knot:

We will denote it by 5_1:

$$\begin{aligned} F_\infty(q,A,5_1) &= q^{\frac{5\alpha^2}{2}} \sum_{i,j,k} q^{\alpha - 2(i-j+k)} q^{-5(i-j+k)\alpha} q^{2i(k-j)} q^{2k(i-j)} q^{\frac{i(i-1)}{2}} q^{\frac{j(j-1)}{2}} \\ &\times q^{\frac{k(k-1)}{2}} \{\alpha; i\}_q \{\alpha - k + j; j\}_q \{\alpha - i + j; k\}_q \\ &\times \begin{bmatrix} k \\ k-j \end{bmatrix}_q \begin{bmatrix} i-j+k \\ k \end{bmatrix}_q \end{aligned}$$

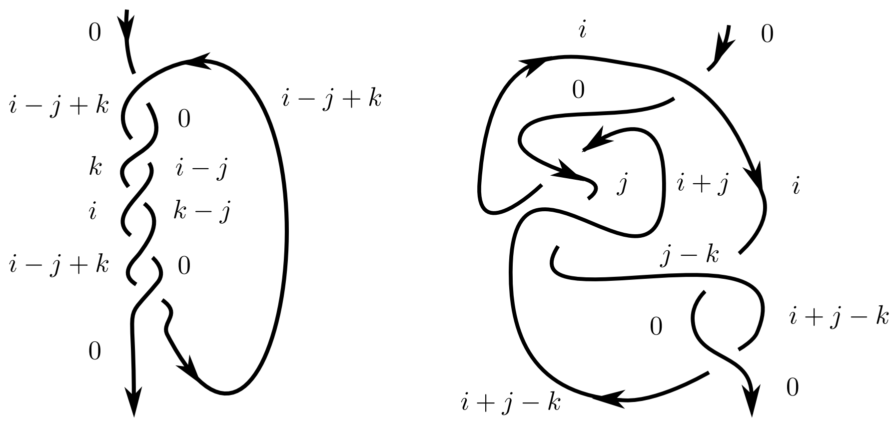
$$\begin{aligned} C_\infty(1,A,5_1) &= q^\alpha \sum_{i,j,k} q^{-5(i-j+k)\alpha} \{\alpha\}_q^{i+j+k} \binom{k}{k-j} \binom{i-j+k}{k} \\ &= q^\alpha \sum_{i,j,k} q^{-5(i-j+k)\alpha} \{\alpha\}_q^{i+j+k} \binom{i-j+k}{j, k-j, i-j} \\ &= q^\alpha \sum_N \sum_{j,k} q^{-5N\alpha} \{\alpha\}_q^{N+2j} \binom{N}{j, k-j, N-k} \\ &= q^\alpha \sum_N q^{-5N\alpha} \{\alpha\}_q^N \sum_{j,k} \{\alpha\}_q^{2j} \binom{N}{j, k-j, N-k} \\ &= q^\alpha \sum_N q^{-5N\alpha} \{\alpha\}_q^N (2 + \{\alpha\}_q^2)^N \\ &= q^\alpha \frac{1}{1 - q^{-5\alpha} \{\alpha\}_q (2 + \{\alpha\}_q^2)} \\ &= \frac{q^{5\alpha}}{A_{5_1}(q^{2\alpha})} \end{aligned}$$

The three twist Knot:

We will denote it by 5_2:

$$F_\infty(q,A,5_2) = q^{\frac{-5\alpha^2}{2}} \sum_{i,j,k} q^{2(i-k)-\alpha} q^{(5i+5j-3k)\alpha} q^{-2ij} q^{-2(j-k)(i+j)} q^{\frac{-i(i-1)}{2}} \times q^{\frac{-j(j-1)}{2}} q^{\frac{-k(k-1)}{2}} (-1)^{i+j+k} \{\alpha;i\}_q \{\alpha-i;j\}_q \{\alpha-j+k;k\}_q \times \begin{bmatrix} j \\ j-k \end{bmatrix}_q \begin{bmatrix} i+j \\ j \end{bmatrix}_q$$

$$\begin{aligned}
C_\infty(1,A,5_2) &= q^{-\alpha} \sum_{i,j,k} q^{(5i+5j-3k)\alpha} (-1)^{i+j+k} \{\alpha\}_q^{i+j+k} \binom{j}{j-k} \binom{i+j}{j} \\
&= q^{\alpha} \sum_{i,j,k} q^{(5i+5j-3k)\alpha} (-1)^{i+j+k} \{\alpha\}_q^{i+j+k} \binom{i+j}{k, j-k, i} \\
&= q^{-\alpha} \sum_{N} \sum_{j,k} q^{(5N-3k)\alpha} (-1)^{N+k} \{\alpha\}_q^{N+k} \binom{N}{k, j-k, N-i} \\
&= q^{-\alpha} \sum_{N} q^{5N\alpha} (-1)^{N} \{\alpha\}_q^{N} \sum_{j,k} q^{-3k\alpha} (-1)^{k} \{\alpha\}_q^{k} \\
&\qquad \times \binom{N}{k, j-k, N-i} \\
&= q^{-\alpha} \sum_{N} q^{5N\alpha} (-1)^{N} \{\alpha\}_q^{N} (2 - q^{-3\alpha}\{\alpha\}_q)^N \\
&= q^{-\alpha} \frac{1}{1 + q^{5\alpha}\{\alpha\}_q (2 - q^{-3\alpha}\{\alpha\}_q)} \\
&= \frac{q^{-5\alpha}}{A_{5_2}(q^{2\alpha})}
\end{aligned}$$



(a) The Cinquefoil Knot. (b) The three twist Knot.

Figure A.2: Examples of state diagrams to compute the invariants.

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