Random walks on groups with a tree-like Cayley graph

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Abstract

We consider a transient nearest neighbor random walk on a group G with finite set of generators Σ. The pair (G, Σ) is assumed to admit a natural notion of normal form words which are modified only locally when multiplied by generators. The basic examples are the free products of a finitely generated free group and a finite family of finite groups, with natural generators. We prove that the harmonic measure is Markovian and can be completely described via a finite set of polynomial equations. It enables to compute the drift, the entropy, the probability of ever hitting an element, and the minimal positive harmonic functions of the walk. The results extend to monoids. In several simple cases of interest, the set of polynomial equations can be explicitly solved, to get closed form formulas for the drift, the entropy,... Various examples are treated: the modular group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$, the Hecke groups $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, the free products of two isomorphic cyclic groups $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, the braid group B_3, and Artin groups with two generators.

1 Introduction

The properties of the harmonic measure associated with a random walk on a finitely generated free group have been studied by many authors [6, 15, 17, 22]. A remarkable result is that for nearest neighbor random walks, the harmonic measure is Markovian. If the probability defining the random walk does not only charge the generators but is of finite support instead, then the result fails to be true, see [15]. In other words, having a Markovian harmonic measure is a property which depends not only on the group G but also on the chosen set of generators Σ. Here, we prove the existence of a Markovian harmonic measure for a whole class of pairs (G, Σ). We then use the property for computational purposes.

A pair (G, Σ) formed by a group (group law $*$, unit element 1_G) and a finite set of generators is called *0-automatic* if the set of words $L(G, \Sigma) = \{u_1 \cdots u_k \mid \forall i, u_i * u_{i+1} \notin \Sigma \cup 1_G\}$ is a cross-section of G.

Consider a group $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ which is a free product of a finitely generated free group and a finite family of finite groups, also called *plain* group. Consider the *natural* (but not necessarily minimal) set of generators $\Sigma = S \sqcup S^{-1} \sqcup_i G_i \backslash \{1_{G_i}\}$. Then the pair (G, Σ) is 0-automatic. Now consider an arbitrary 0-automatic pair (G, Σ). Then G is isomorphic to a plain group. This is not the end of the story since Σ may be different (in particular, larger) than the natural set of generators of the group seen as a free product. (For an example of this,

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see the end of §3.) And what is relevant in our context is the pair group-generators rather that the group itself, since the generators form the support of the measure defining the nearest neighbor random walk. Hence it seems appropriate and convenient to coin a specific term as "0-automatic pairs" to highlight the notion.

Apart from the free group with one generator and the group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, all the plain groups are non-amenable. It follows that any random walk living on the whole group is transient and has a strictly positive drift (see [11] and [27, Chapter 1.B] for details).

Consider a transient nearest neighbor random walk $(X_n)_n$ on a 0-automatic pair (G, Σ). Let $(Y_n)_n$ be the corresponding random walk on normal form words. The harmonic measure is the law of $Y_\infty = \lim_n Y_n$. We prove that this harmonic measure is Markovian. The transition probabilities of this Markov chain are the unique solutions of a set of polynomial equations of degree 2, that we call the Traffic Equations. We can then compute the drift and the entropy of the random walk. Some additionnal work is needed to compute the probability of ever hitting a generator, which leads to a computational description of the minimal positive harmonic functions. Mutatis mutandis, the results extend to monoids.

All the random walks considered here belong to the general setting of random walks on regular languages studied in [16]. In [16], local limit theorems are proved. Also, our random walks can be viewed as random walks on a tree with finitely many cone types in the sense of [20], but with one-step moves at distance 1 and 2. In [20], it is proved, among other things, that the harmonic measure is Markovian for nearest neighbor random walks. See also [24]. The method of proof is different from the one we use, see the discussion in §5.2.

A natural question is whether the Traffic Equations can be solved "explicitly", in order to get for instance a closed form formula for the drift or the entropy. This is feasible in many situations. We treat completely the following cases: the general nearest neighbor random walk on the modular group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$, three one-parameter families of random walks on $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$, a one-parameter family of random walks on the three strands braid group B_3, and the simple random walks on $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, on the Hecke groups $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, and on the Artin groups with two generators.

None of these computations appeared in the litterature before. For the few examples of non-elementary explicit computations previously available, see [5, 6, 17, 20, 21, 22]. Nevertheless, there exists an alternative potential method for the effective computation of the drift (not the entropy) which is due to Sawyer and Steger [22]. In this approach, the drift is expressed as a functional of the *first-passage generating series* of the random walk. The simplest of our computational results can also be retrieved using this method. We detail and discuss this method in §6.1. The Sawyer-Steger method links the problem of computing the drift with the problem of computing the generating series of transition probabilities. Concerning the latter problem, there exists an important litterature, especially for random walks on free groups and free products, see [2, 3, 26] and [27, Sections II.9 and III.17].

The usual motivation for studying generating functions of transition probabilities is to get central or local limit theorems. For much more material on this and on other aspects of random walks on discrete infinite groups only touched upon here (like boundary theory), see [13, 25, 27] and the references there.

The results, presented here without proofs, are exposed in full details in [18, 19].

Let $\mathbb{N}$ be the set of non-negative integers. If μ is a measure on a group $(G, *)$, then μ^{*n} is the n-fold convolution product of μ, that is the image of the product measure $\mu^{\otimes n}$ by the product map $G \times \cdots \times G \to G$, $(g_1, \ldots, g_n) \mapsto g_1 * g_2 * \cdots * g_n$. The symbol $\sqcup$ is used for the disjoint union of sets.

2 Random walks on groups

Given a set Σ, the free monoid it generates is denoted by Σ^*. As usual, Σ is called the *alphabet*, the elements of Σ and Σ^* are called respectively the *letters* and *words*. The empty word is denoted by 1_{Σ^*}. The *length* (number of letters) of a word u is denoted by $|u|_\Sigma$.

Consider a finitely generated group $(G, *)$ with unit element 1_G. Let Σ be a finite set of generators of G (with $1_G \notin \Sigma$ and $u \in \Sigma \implies u^{-1} \in \Sigma$). Denote by $\pi : \Sigma^* \to G$ the monoid morphism which associates to a word $a_1 \cdots a_k$ the group element $a_1 * \cdots * a_k$.
The *length with respect to* Σ of a group element u is:

$$|u|_\Sigma = \min\{k \mid \; u = s_1 * \cdots * s_k, s_i \in \Sigma\} \,. \tag{1}$$

A word $u \in \Sigma^*$ is a *geodesic* if $|u|_\Sigma = |\pi(u)|_\Sigma$.
The Cayley graph $\mathfrak{X}(G, \Sigma)$ of a group G with respect to a set of generators Σ is the directed graph with G as the set of nodes and with an arc from u to v if $u^{-1} * v \in \Sigma$. It is often convenient to view $\mathfrak{X}(G, \Sigma)$ as a labelled graph with set of labels Σ (with $u \xrightarrow{a} v$ if $u * a = v$). Observe that $|u|_\Sigma$ is the geodesic distance from 1_G to u in the Cayley graph.
Let $G_1 \star G_2$ be the free product of two groups G_1 and G_2. Roughly, the elements of $G_1 \star G_2$ are the finite alternate sequences of elements of $G_1 \backslash \{1_{G_1}\}$ and $G_2 \backslash \{1_{G_2}\}$, and the group law is the concatenation with simplication. More rigorously, the definition is as follows. Set $S = G_1 \sqcup G_2$. Let $\sim$ be the least congruence on S^* containing the relations: $\forall u, v \in S^*, \forall i \in \{1, 2\}$,

$$u 1_{G_i} v \sim uv, \quad \forall a, b, c \in G_i, \text{ s.t. } c = a * b, \; uabv \sim ucv \,.$$

The quotient monoid $(S^* / \sim)$ is a group called the *free product* $G_1 \star G_2$.

Let μ be a probability distribution over Σ. Consider the Markov chain on the state space G with one-step transition probabilities given by: $\forall g \in G, \forall a \in \Sigma, \; P_{g,g*a} = \mu(a)$. This Markov chain is called the *random walk* (associated with) (G, μ). It is a *nearest neighbor* random walk: one-step moves occur between nearest neighbors in the Cayley graph $\mathfrak{X}(G, \Sigma)$. When $\mu(s) = 1/|\Sigma|$ for all $s \in \Sigma$, we say that the random walk is *simple*.
Let $(x_n)_n$ be a sequence of i.i.d. r.v.'s distributed according to μ. Set

$$X_0 = 1, \; X_{n+1} = X_n * x_n = x_0 * x_1 * \cdots * x_n \,. \tag{2}$$

The sequence $(X_n)_n$ is a *realization* of the random walk (G, μ). The law of X_n is μ^{*n}.

Drift, entropy. The first step in understanding the asymptotic behaviour of X_n consists in studying the length $|X_n|_\Sigma$ as $n \to \infty$. Since $|uv|_\Sigma \leq |u|_\Sigma + |v|_\Sigma$, Guivarc'h [11] observed that a simple corollary of Kingman's subadditive ergodic theorem is the existence of a constant $\gamma \in \mathbb{R}_+$ such that *a.s.* and in L^p, for all $1 \leq p < \infty$,

$$\lim_{n \to \infty} \frac{|X_n|_\Sigma}{n} = \gamma \,. \tag{3}$$

We call γ the *drift* of the random walk. Intuitively, γ is the speed of escape to infinity of the walk.

Another quantity of interest is the entropy. The *entropy* of a probability measure μ with finite support S is defined by $H(\mu) = -\sum_{x \in S} \mu(x) \log[\mu(x)]$. The *entropy of the random walk* (G, μ), introduced by Avez [1], is

$$h = \lim_n \frac{H(\mu^{*n})}{n} = \lim_n -\frac{1}{n} \log \mu^{*n}(X_n) \,, \tag{4}$$

a.s. and in L^p, for all $1 \leq p < \infty$. The existence of the limits as well as their equality follow again from Kingman's subadditive ergodic theorem.

3 Zero-automaticity

Let G be a group with finite set of generators Σ. A language L of Σ^* is a *cross-section* of G (over the alphabet Σ) if the restriction of π to L defines a bijection, that is if every element of G has a unique representative in L. A word of L is then called a *normal form* word. The map $\phi : G \to L$ which associates to a group element its unique representative in L is called the *normal form map.*

Definition 3.1. Let G be a group with finite set of generators Σ. Define the sets

$$\forall a \in \Sigma, \quad \mathrm{Next}(a) = \big\{b \in \Sigma \mid a * b \notin \Sigma \cup \{1_G\}\big\}, \quad \mathrm{Prev}(a) = \big\{b \in \Sigma \mid b * a \notin \Sigma \cup \{1_G\}\big\} . \tag{5}$$

and the language of Σ^*,

$$\begin{aligned} L(G,\Sigma) &= \{u_1 \cdots u_k \mid \forall i \in \{2,\dots,k\},\ u_i \in \mathrm{Next}(u_{i-1})\} \\ &= \{u_1 \cdots u_k \mid \forall i \in \{1,\dots,k-1\},\ u_i \in \mathrm{Prev}(u_{i+1})\} . \end{aligned} \tag{6}$$

We say that the pair (G,Σ) is *0-automatic* if $L(G,\Sigma)$ is a cross-section of G.

Here are some consequences of Definition 3.1. First, let $\phi : G \to L(G,\Sigma)$ be the normal form map. Then: $\forall g \in G$ s.t. $\phi(g) = u_1 \cdots u_k,\ \forall a \in \Sigma$,

$$\phi(g*a) = \begin{cases} u_1 \cdots u_{k-1} & \text{if } a = u_k^{-1} \\ u_1 \cdots u_{k-1} v & \text{if } u_k * a = v \in \Sigma \\ u_1 \cdots u_{k-1} u_k a & \text{if } u_k * a \notin \Sigma \cup 1_G \end{cases}, \quad \phi(a*g) = \begin{cases} u_2 \cdots u_k & \text{if } a = u_1^{-1} \\ v u_2 \cdots u_k & \text{if } a * u_1 = v \in \Sigma \\ a u_1 \cdots u_k & \text{if } a * u_1 \notin \Sigma \cup 1_G \end{cases} . \tag{7}$$

Second, the language $L(G,\Sigma)$ is regular and recognized by the following automaton: Set of states: $\Sigma \cup 1_G$, initial state: 1_G, final states: $\Sigma \cup 1_G$; Transitions: $a \xrightarrow{b} b$ if $a * b \notin \Sigma \cup 1_G$.
Third, the Cayley graph $\mathfrak{X}(G,\Sigma)$ has a tree-like structure. In particular, $\mathfrak{X}(G,\Sigma)$ has uniform node-connectivity 1, that is, the removal of any node disconnects the graph, see Figure 2.
Fourth, all the group elements have a unique geodesic representative with respect to Σ, and the set of these geodesic representatives is precisely $L(G,\Sigma)$.
Fifth, the set of simple circuits going through the node 1_G in $\mathfrak{X}(G,\Sigma)$ is finite. In fact, this last property is equivalent to the property that $L(G,\Sigma)$ be a cross-section, see [12].

Here are examples of 0-automatic pairs.

- Let G be a finite group. Then $(G, G\backslash\{1_G\})$ is 0-automatic.
- Let $\mathbb{F}(\Sigma)$ be the free group generated (as a group) by Σ. Set $\tilde{\Sigma} = \Sigma \sqcup \Sigma^{-1}$. Then $(\mathbb{F}(\Sigma), \tilde{\Sigma})$ is 0-automatic.
- Let (G_1,Σ_1) and (G_2,Σ_2) be 0-automatic. Then $(G_1 \star G_2, \Sigma_1 \sqcup \Sigma_2)$ is 0-automatic.

Following [12], define a *plain group* as the free product of a finitely generated free group and a finite family of finite groups. Let $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ be a plain group. Then $\Sigma = S \sqcup S^{-1} \sqcup_i \Sigma_i,\ \Sigma_i = G_i\backslash\{1_{G_i}\}$, is a set of generators for G that we call the *natural* generators. It follows from the above that (G,Σ) is 0-automatic. The sets $\mathrm{Next}(\cdot), \mathrm{Prev}(\cdot)$, defined in (5), can be explicited:

$$\forall a \in S \sqcup S^{-1},\ \mathrm{Next}(a) = \mathrm{Prev}(a) = \Sigma\backslash\{a^{-1}\}, \qquad \forall a \in \Sigma_i,\ \mathrm{Next}(a) = \mathrm{Prev}(a) = \Sigma\backslash\Sigma_i .$$

A 0-automatic pair (G,Σ) is precisely a *unique factorization pair* in Stallings [23], with the additional assumption that Σ be finite. Using the results from [23], we get:

Proposition 3.2. *Let (G, Σ) be a 0-automatic pair. Then G is isomorphic to a plain group.*

The proof of Stallings is constructive and provides more precise information. Let φ be the isomorphism from G to the plain group $\widetilde{G}$. Let S be the natural generators of $\widetilde{G}$. Then $S \subset \varphi(\Sigma)$, but of course the inclusion may be strict. Also, there is no absolute upper bound on $|u|_S, u \in \varphi(\Sigma)$.

Plain groups are hyperbolic in the sense of Gromov [10] and automatic in the sense of Epstein & al [7]. Besides, (G, Σ) is 0-automatic iff $(\Sigma, L(G, \Sigma))$ is an automatic pair (in the sense of [7]) satisfying the 0-fellow traveller property. This is our justification for the chosen terminology.

Consider the group $G = \langle\, a, b \mid abab = 1 \,\rangle$. (This is the Artin group $A_4 = \langle\, a, b \mid abab = baba \,\rangle$ quotiented by its center.) Set $\Sigma = \{a, b, ab = (ab)^{-1}, ba = (ba)^{-1}, aba = b^{-1}, bab = a^{-1}\}$. Then (G, Σ) is a 0-automatic pair. Here, $\text{Next}(x) = \{a, ab, aba\}$ if $x \in \{a, ba, aba\}$ and $\text{Next}(x) = \{b, ba, bab\}$ if $x \in \{b, ab, bab\}$.

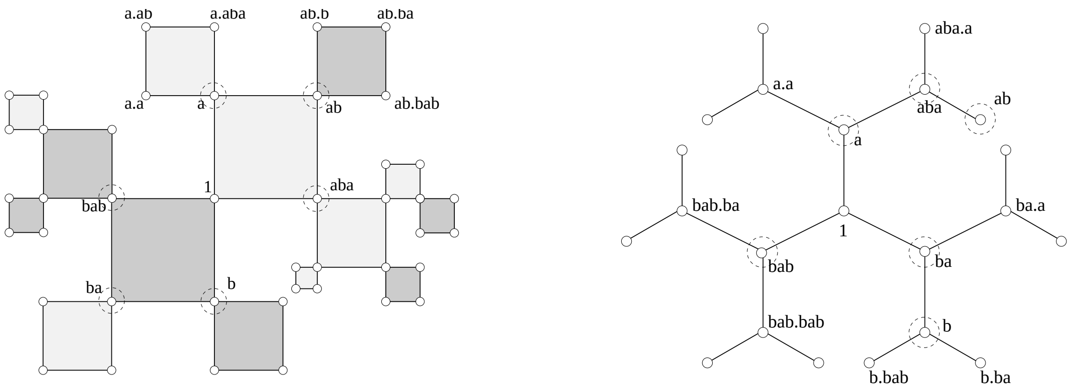

Figure 1: The Cayley graph of G with respect to $\{a, a^{-1}, b, b^{-1}\}$ (left), and $\{a, a^{-1}, ba\}$ (right).

Now, the group G is isomorphic to $\mathbb{F}(a) \star \{1, ba\} \sim \mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, see Figure 1. Set $S = \{a, a^{-1} = bab, ba\}$ for the corresponding set of natural generators. We have for instance $|b|_S = 2$ and $|ab|_S = 3$. Concentrating on the right of Figure 1, it is not obvious that (G, Σ) is 0-automatic.

4 Random walks on zero-automatic pairs

From now on, the setting is the following one. Let G be an infinite group with a finite set of generators Σ, with $\Sigma = \Sigma^{-1}$, such that (G, Σ) is 0-automatic. Let $L(G, \Sigma)$ and ϕ be defined as in (6) and (7). Let μ be a probability on Σ which generates the whole group, that is, $\cup_n \text{supp}(\mu^{*n}) = G$, where supp is the support of the measure. We consider the random walk (G, μ), assumed to be transient.

Define the set of *infinite normal form* words $L^\infty \subset \Sigma^{\mathbb{N}}$ by

$$L^\infty = \{u = u_0 u_1 \cdots u_k \cdots \in \Sigma^{\mathbb{N}} \mid \forall i \in \mathbb{N}, u_{i+1} \in \text{Next}(u_i)\} . \tag{8}$$

A word belongs to L^∞ iff all its finite prefixes belong to $L(G, \Sigma)$. Consider the natural action $\Sigma \times L^\infty \to L^\infty, (a, \xi) \mapsto a \cdot \xi$, with $a \cdot \xi = a\xi_0\xi_1 \cdots$ if $a \in \text{Prev}(\xi_0)$, $a \cdot \xi = (a * \xi_0)\xi_1 \cdots$ if $a * \xi_0 \in \Sigma$, and $a \cdot \xi = \xi_1\xi_2 \cdots$ if $a = \xi_0^{-1}$. Equip $\Sigma^{\mathbb{N}}$ with the Borel σ-algebra associated with the product topology. This induces a σ-algebra on L^∞. Given a measure ν^∞ on L^∞ and $a \in \Sigma$, define the measure $a\nu^\infty$ by: $\int f(\xi) d(a\nu^\infty)(\xi) = \int f(a \cdot \xi) d\nu^\infty(\xi)$. A probability measure ν^∞ on L^∞ is *invariant* if

$$\nu^\infty(\cdot) = \sum_{a \in \Sigma} \mu(a)[a\nu^\infty](\cdot) . \tag{9}$$

Proposition 4.1. *Let $(X_n)_n$ be a realization of the random walk (G, μ) and set $Y_n = \phi(X_n)$. There exists a r.v. Y^∞ valued in L^∞ such that a.s.*

$$\lim_{n\to\infty} Y_n = Y^\infty ,$$

meaning that the length of the common prefix between Y_n and Y^∞ goes to infinity a.s. Let μ^∞ be the distribution of Y^∞. The measure μ^∞ is invariant and is the only invariant probability on L^∞. We call it the harmonic measure *of (G, μ). The drift and the entropy of the random walk are given by:*

$$\gamma = \sum_{x\in\Sigma} \mu(x) \Big[-\mu^\infty(x^{-1}\Sigma^{\mathbb{N}}) + \sum_{y\in Next(x)} \mu^\infty(y\Sigma^{\mathbb{N}}) \Big] , \tag{10}$$

$$h = -\sum_{x\in\Sigma} \mu(x) \int \log\big[\frac{dx^{-1}\mu^\infty}{d\mu^\infty}(\xi)\big]\, d\mu^\infty(\xi) , \tag{11}$$

where $dx^{-1}\mu^\infty / d\mu^\infty$ is the Radon-Nikodym derivative of $x^{-1}\mu^\infty$ with respect to μ^∞.

In the context of the free group, this is proved for instance in [17, Theorem 1.12, Theorem 4.10, Corollary 4.5]. The proofs adapt easily to the present setting. Several of the key arguments go back to [8], see [17] for precise references.

Intuitively, the harmonic measure μ^∞ gives the direction in which $(X_n)_n$ goes to infinity.

5 Markovian harmonic measure

Define $\mathring{\mathcal{B}} = \{x \in \mathbb{R}^\Sigma \mid \forall i, x_i > 0, \ \sum_i x_i = 1\}$. Consider $r \in \mathring{\mathcal{B}}$. Define the matrix P of dimension $\Sigma \times \Sigma$ by

$$P_{u,v} = \begin{cases} r(v)\big(\sum_{x\in\mathrm{Next}(u)} r(x)\big)^{-1} & \text{if } v \in \mathrm{Next}(u) \\ 0 & \text{otherwise} \end{cases} . \tag{12}$$

It is the transition matrix of a Markov Chain on the state space Σ, which can be proved to be irreducible. For convenience, set for all $a \in \Sigma$, $s(a) = \sum_{x\in\mathrm{Next}(a)} r(x)$. Let $(U_n)_n$ be a realization of the Markov chain with transition matrix P and starting from U_1 such that $P\{U_1 = x\} = r(x)$. Set $U^\infty = \lim_n U_1 \cdots U_n$, and let ν^∞ be the distribution of U^∞. Clearly the support of ν^∞ is included in L^∞. For $u_1 \cdots u_k \in L(G, \Sigma)$, we have

$$\begin{aligned} \nu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) &= r(u_1) P_{u_1,u_2} \cdots P_{u_{k-1},u_k} \\ &= r(u_1) \frac{r(u_2)}{s(u_1)} \cdots \frac{r(u_k)}{s(u_{k-1})} = \frac{r(u_1)}{s(u_1)} \frac{r(u_2)}{s(u_2)} \cdots \frac{r(u_{k-1})}{s(u_{k-1})} r(u_k) . \end{aligned} \tag{13}$$

We call ν^∞ the *Markovian multiplicative* probability measure associated with r.

Observe that the measure ν^∞ is in general non-stationary with respect to the translation shift $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}, (x_n)_n \mapsto (x_{n+1})_n$. Indeed, the distribution of the first marginal is r which is in general different from the stationary distribution of P.

5.1 The main theorem

The *Traffic Equations* associated with (G, μ) are defined by: $\forall a \in \Sigma$,

$$x(a) = \mu(a) \sum_{u\in\mathrm{Next}(a)} x(u) + \sum_{u*v=a} \mu(u)x(v) + \sum_{u\in\mathrm{Prev}(a)} \mu(u^{-1}) \frac{x(u)}{\sum_{v\in\mathrm{Next}(u)} x(v)} x(a) . \tag{14}$$

We are now ready to state the main result.

Theorem 5.1. *Let (G, Σ) be 0-automatic. Let μ be a probability measure on Σ such that $\cup_n \mathrm{supp}(\mu^{*n}) = G$ and such that the random walk (G, μ) is transient. Then the Traffic Equations (14) have a unique solution $x \in \mathring{\mathcal{B}}$. The harmonic measure of the random walk is the Markovian multiplicative measure associated with x.*

The harmonic measure is not stationary in general. However, we have the following result:

Proposition 5.2. *Let H be a finite group and let $(G_i)_{i \in I}$ be a finite family of copies of H. Let π_i be the isomorphism between G_i and H. Let ν be a probability measure on $H \backslash \{1_H\}$. Consider the free product $G = \star_{i \in I} G_i$ and let μ be the probability measure on $\Sigma = \sqcup_i G_i \backslash \{1_{G_i}\}$ defined by: $\forall g \in G_i \backslash \{1_{G_i}\},\ \mu(g) = \nu \circ \pi_i(g)/|I|$. Then the harmonic measure of (G, μ) is stationary and ergodic.*
Consider now a random walk (G, μ) where $G = G_1 \star G_2$ is the free product of two arbitrary finite groups. Then the harmonic measure μ^∞ is 2-stationary, meaning that: $\forall u \in L(G, \Sigma), \forall k \in \mathbb{N},\ \mu^\infty(u\Sigma^{\mathbb{N}}) = \mu^\infty(\Sigma^{2k} u \Sigma^{\mathbb{N}})$.

Starting from (10)-(11) and using Theorem 5.1, we obtain a simple formula for the drift:

$$\gamma = \sum_{a \in \Sigma} \mu(a) \big[-r(a^{-1}) + \sum_{b \in \mathrm{Next}(a)} r(b) \big] \,. \tag{15}$$

and for the entropy:

$$h = -\sum_{a \in \Sigma} \mu(a) \Big[\log\big[\frac{1}{q(a^{-1})}\big] r(a^{-1}) + \sum_{b, a*b \in \Sigma} \log\big[\frac{q(a*b)}{q(b)}\big] r(b) + \log[q(a)] \sum_{b \in \mathrm{Next}(a)} r(b) \Big] \,, \tag{16}$$

where $\forall a \in \Sigma,\ q(a) = r(a)/(\sum_{b \in \mathrm{Next}(a)} r(b))$.
In particular, if the probabilities $\mu(a)$ are algebraic numbers, then the drift and the entropy are algebraic numbers.

5.2 Harmonic functions

For all $u \in G$, define $q(u) = P\{\exists n \mid X_n = u\}$, the probability of ever reaching u. If $\phi(u) = u_1 \cdots u_k \in L(G, \Sigma)$, by the strong Markov property, we have $q(u) = q(u_1) q(u_2) \cdots q(u_k)$. Therefore, all we need to compute are the quantities $q(a), a \in \Sigma$. These quantities satisfy the equations: $\forall a \in \Sigma$,

$$q(a) \;=\; \mu(a) + \sum_{u*v=a} \mu(u) q(v) + q(a) \sum_{c \in \Sigma \backslash \Sigma_a} \mu(c) q(c^{-1}) \,. \tag{17}$$

Proposition 5.3. *The Equations (17) characterize $(q(a))_{a \in \Sigma}$. Let r be the unique solution to the Traffic Equations in $\mathring{\mathcal{B}}$. We have: $\forall a \in \Sigma,\ q(a) = r(a)/[\sum_{b \in Next(a)} r(b)]$. The harmonic measure satisfies: $\forall u_1 \cdots u_k \in L(G, \Sigma),\ \mu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = q(u_1) \cdots q(u_{k-1}) r(u_k)$.*
Consider now a free product of finite groups $G_1 \star \cdots \star G_k$. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$, and $q(\Sigma_i) = \sum_{a \in \Sigma_i} q(a)$. We have: $\forall a \in \Sigma_i,\ r(a) = q(a)/[1 + q(\Sigma_i)]$. At last, for a finitely generated free group $\mathbb{F}(S)$, we have: $\forall a \in S \sqcup S^{-1},\ r(a) = q(a)/(1 + q(a))$.

For a general 0-automatic pair, there is no simple formula giving r as a function of q. We come back to this point at the end of §5.2.
Specializing Proposition 5.3 to the free product of two groups, we obtain an unexpected identity:

$$q(\Sigma_1) q(\Sigma_2) = 1 \,.$$

In words, the average number of visited elements in Σ_1 is the inverse of the average number of visited elements in Σ_2. It is the identity used to prove the last part of Proposition 5.2.

The knowledge of $q(.)$ enables to determine explicitly the minimal harmonic functions. A *positive harmonic* function is a function $f : G \to \mathbb{R}_+$ such that $\forall u \in G,\ \sum_{a\in\Sigma} f(u * a)\mu(a) = f(u)$. A positive harmonic function f is *minimal* if $f(1_G) = 1$ and if for any positive harmonic function g such that $f \geq g$, there exists $c \in \mathbb{R}_+$ such that $f = cg$.
For $g \in G$, set $\Gamma(g) = \sum_{i=0}^{\infty} \mu^{*i}(g)$ (the Green function), and define the map $K_g : G \to \mathbb{R}_+$ by

$$K_g(x) = \frac{\Gamma(x^{-1} * g)}{\Gamma(g)} = \frac{q(x^{-1} * g)}{q(g)} .$$

The right-hand equality is obtained by observing that $\Gamma(v) = q(v)\Gamma(1_G)$ for all v.

For $\xi \in L^\infty$, define $K_\xi : G \to \mathbb{R}_+$ by $K_\xi = \lim_n K_{\pi(\xi[n])}$, where $\xi[n]$ is the length n prefix of ξ. Set $\xi = \xi_0\xi_1\cdots$. For $x \in G$ with $\phi(x) = x_0 \cdots x_{n-1} \in L(G, \Sigma)$, set $k = |\phi(x) \wedge \xi|_\Sigma$, the length of the longest joint prefix of $\phi(x)$ and ξ. We have

$$K_\xi(x) = \begin{cases} \left[q(x_k^{-1}) \cdots q(x_{n-1}^{-1})\right] / \left[q(\xi_0) \cdots q(\xi_{k-1})\right] & \text{if } x_k^{-1} \in \text{Prev}(\xi_k) \\ \left[q(x_k^{-1} * \xi_k) q(x_{k+1}^{-1}) \cdots q(x_{n-1}^{-1})\right] / \left[q(\xi_0) \cdots q(\xi_k)\right] & \text{otherwise} \end{cases} .$$

Proposition 5.4. *The minimal positive harmonic functions are the functions $K_\xi,\ \xi \in L^\infty$.*

The set of functions $\{K_\xi,\ \xi \in L^\infty\}$ forms the Martin boundary of the random walk.

Comparison with the litterature. The importance of the Equations (17) in q is well-known. In the seminal paper of Dynkin & Malyutov [6], these equations are explicitely solved in the free group case, and the harmonic functions are then derived as above. In [22], the authors prove that μ^∞ is Markovian for the free group as follows: they use the expression for q obtained in [6], they define r as in the second part of Proposition 4.1, and then prove that the measure defined by $\nu^\infty(u_1 \cdots u_k\Sigma^{\mathbb{N}}) = q(u_1) \cdots q(u_{k-1})r(u_k)$ is the harmonic measure. For trees of finite cone types [20], the proof that the harmonic measure is Markovian is also centered around the analog of the Equations (17). The series version of (17) is the main ingredient in the Sawyer & Steger approach [22] for computing γ, see §6.1. See also [16, 20]. At last, the series version of (17) can be used for *finite range* random walks on free groups to get qualitative results (central limit theorem, asymptotic type of $P\{X_n = 1_G\}$), see Lalley [15].
Here the proof that μ^∞ is Markovian is different and based on the Traffic Equations (14) instead of the Equations (17). This is a more direct path and the only way to proceed in the general case since we cannot retrieve a solution to the Traffic Equations from a solution to (17).

6 Explicit computations

In Theorem 5.1, the harmonic measure is completely determined via the vector r which is itself the solution of an explicit finite set of polynomial equations. In small or simple examples, it is possible to go further, that is, to solve these equations to get closed form formulas for the harmonic measure, the drift, the entropy, or the harmonic functions. It is the program that we now carry out for several specific and interesting cases. In §6.2-§6.6, we compute explicitly r, and we illustrate by providing closed form formulas for the drift. The computations have been carried out using Maple and Mathematica.
We first discuss alternative existing methods for computing the drift. (They do not work for computing the entropy for instance.)

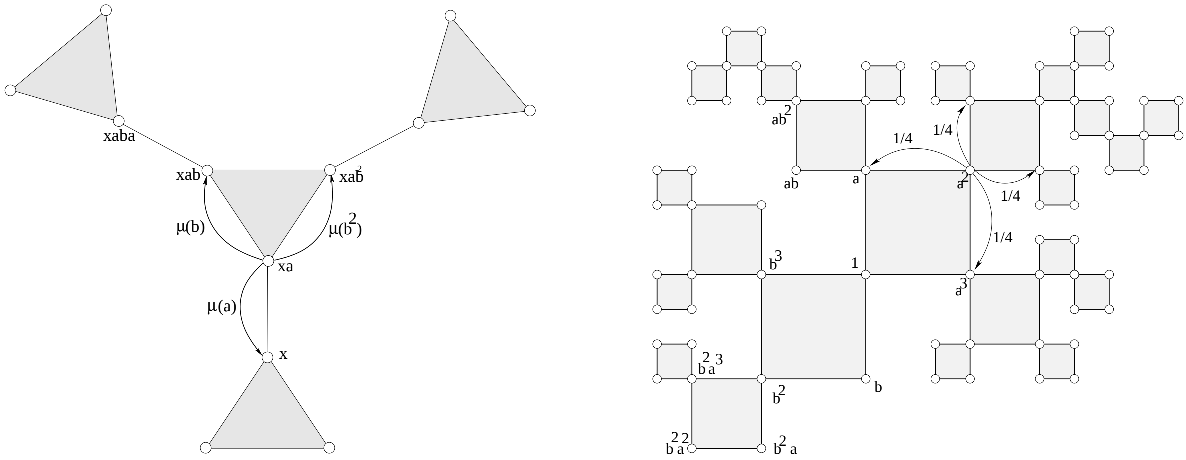

Figure 2: A nearest neighbor random walk on $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ (left), and the simple random walk on $\mathbb{Z}/4\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$ (right).

6.1 Comparison with other methods for computing the drift

Let $G = G_1 \star G_2$ be a free product of two finite groups. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Let μ be a probability measure on Σ such that: $\forall i, \forall x \in \Sigma_i,\ \mu(x) = \mu(\Sigma_i)/\#\Sigma_i$. In words, μ is uniform on each of the two groups. Consider the random walk (G, μ). Here, computing the drift becomes elementary and does not require knowing that the harmonic measure is Markovian.

Set $p = \mu(\Sigma_1), k_1 = \#\Sigma_1$, and $k_2 = \#\Sigma_2$. Denote by $i \in \{1, 2\}$, the set of elements of G whose normal form representative ends with a letter in Σ_i. When we are far from the unit element 1_G, the random walk on G induces a Markov chain on $\{1, 2\}$ with transition matrix:

$$P = \begin{bmatrix} p(k_1 - 1)/k_1 & p/k_1 + 1 - p \\ (1-p)/k_2 + p & (1-p)(k_2 - 1)/k_2 \end{bmatrix} .$$

Let π be the stationary distribution, that is $\pi P = \pi, \pi(1) + \pi(2) = 1$. By the Ergodic Theorem for Markov Chains, we have $\lim_n [\ P\{X_n \in 1\}\ ,\ P\{X_n \in 2\}\ ] = \pi$. The value of the drift follows readily:

$$\gamma = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} E[X_{i+1} - X_i] + \frac{X_0}{n} = \lim_n E[X_n - X_{n-1}] = E_\pi[X_1 - X_0] = \frac{2p(1-p)(k_1 k_2 - 1)}{(1-p)k_1 + pk_2 + k_1 k_2} .$$

Now assume that $G = G_1 \star \cdots G_k$, where the G_i are finite groups, and assume that $\forall i, \forall x \in \Sigma_i = G_i \backslash \{1_{G_i}\},\ \mu(x) = \mu(\Sigma_i)/\#\Sigma_i$. Then each of the finite groups can be collapsed into a single node, and the random walk (G, μ) projects into a nearest neighbor randow walk on a tree with k cone types. Therefore, the formulas for the drift given in [20] apply.

With the exception of (23), none of the formulas obtained in §6.2-§6.6 corresponds to the above two situations.

Now let us discuss the Sawyer-Steger method [22]. It was developped for the free group but adapts to the present situation. Let (G, Σ) be a 0-automatic pair. For $g \in G$, define the r.v. $\tau(g) = \min\{n \mid X_n = g\}$ (with $\tau(g) = \infty$ if g is not reached). Define the *first-passage generating series* $S \in \mathbb{R}[[y, z]]$ by:

$$S(y, z) = \sum_{k \in \mathbb{N}} y^k \sum_{|g|_\Sigma = k} \sum_{n \in \mathbb{N}} P\{\tau(g) = n\} z^n . \tag{18}$$

Let S_y and S_z denote the partial derivatives of S with respect to y and z. Adapting the results in [22, Theorem 2.2 and Section 6] (see also [20, Section 6]), one obtains the following formula for the drift:

$$\gamma = S_y(1,1)/S_z(1,1) \, . \tag{19}$$

Now assume, for simplicity, that $G = G_1 \star \cdots \star G_k$ is a free product of finite groups. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and $\Sigma = \sqcup_i \Sigma_i$. For $u \in G$, define the series $q(u,z) = \sum_{n\in\mathbb{N}} P\{\tau(u) = n\} z^n$. Observe that $q(u,1) = q(u)$, the probability of ever reaching u, defined in §5.2. In particular, if one encapsulates the Equations (17) as $q(a) = \Phi_a(q)$, then $q(a,z) = z\Phi_a(q(z))$. Using this last set of Equations, the corresponding set of Equations for the derivatives $dq(a,z)/dz$, and playing around with the Equations (18) and (19), we get:

$$\gamma = \frac{\sum_{i=1}^k q_i/(1+q_i)^2}{\sum_{i=1}^k q_i'/(1+q_i)^2} \, , \qquad q_i = \sum_{u\in\Sigma_i} q(u,1), \quad q_i' = \sum_{u\in\Sigma_i} \Big[\frac{dq(u,z)}{dz}\Big]_{|z=1} \, . \tag{20}$$

This formula is more complicated than the one obtained in (15). The reason for this is easy to understand. There is much more information in the series S than what is relevant for computing γ. In particular, all the 'transient' behavior of the walk is encoded into it. Our approach centers around the knowledge that μ^∞ is Markovian. It allows to compute γ via a simple 'equilibrium' argument: $\mu * \mu^\infty = \mu^\infty$. This is a more direct path. Consequently, it gives more chances to solve the equations to get a closed form formula. As an exercise, we tried to retrieve the results for γ in §6.2-§6.6 using (20). We succeeded in two cases: formulas (22) and (24). On the other hand, the results in §6.4-§6.6 seem totally out of reach.

6.2 Random walks on $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$

The group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ is isomorphic to the modular group $\mathrm{PSL}(2,\mathbb{Z})$ (i.e. the group of 2x2 matrices with integer entries and determinant 1, quotiented by $\pm\mathrm{Id}$). Let a and b be the respective generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. A possible representation of the group is

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} . \tag{21}$$

Quoting [4, Chapter II.B]: "The modular group is one of the most important groups in mathematics".

Consider a general nearest neighbor random walk $(\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}, \mu)$. Set $\mu(a) = 1-p-q, \mu(b) = p, \mu(b^2) = q$. The solution to the Traffic Equations is:

$$\begin{aligned} r(a) &= \frac{p^2+q^2-2pq-p-q+4-\Delta_1}{2\Delta_2} \\ r(b) &= \frac{q^3-3q^2+p^2q-5pq+2p+6q-(2-q)\Delta_1}{2(q-p)\Delta_2} \\ r(b^2) &= \frac{p^3-3p^2+pq^2-5pq+6p+2q-(2-p)\Delta_1}{2(p-q)\Delta_2} \, , \end{aligned}$$

with

$$\begin{aligned} \Delta_1 &= \sqrt{p^4+q^4-2p^3-2q^3+2p^2q^2-6p^2q-6pq^2+5p^2+5q^2+6pq} \\ \Delta_2 &= p^2+q^2-pq-2p-2q+4 \, . \end{aligned}$$

Set $r = \mu(a) = 1-p-q$. The drift is then:

$$\gamma = \frac{2r\Big(pq-p-q+\sqrt{(p^2+q^2)(3+(r+p)^2+(r+q)^2)+2pq(2r+1)}\Big)}{(r+p)^2+(r+q)^2-pq+2} \, . \tag{22}$$

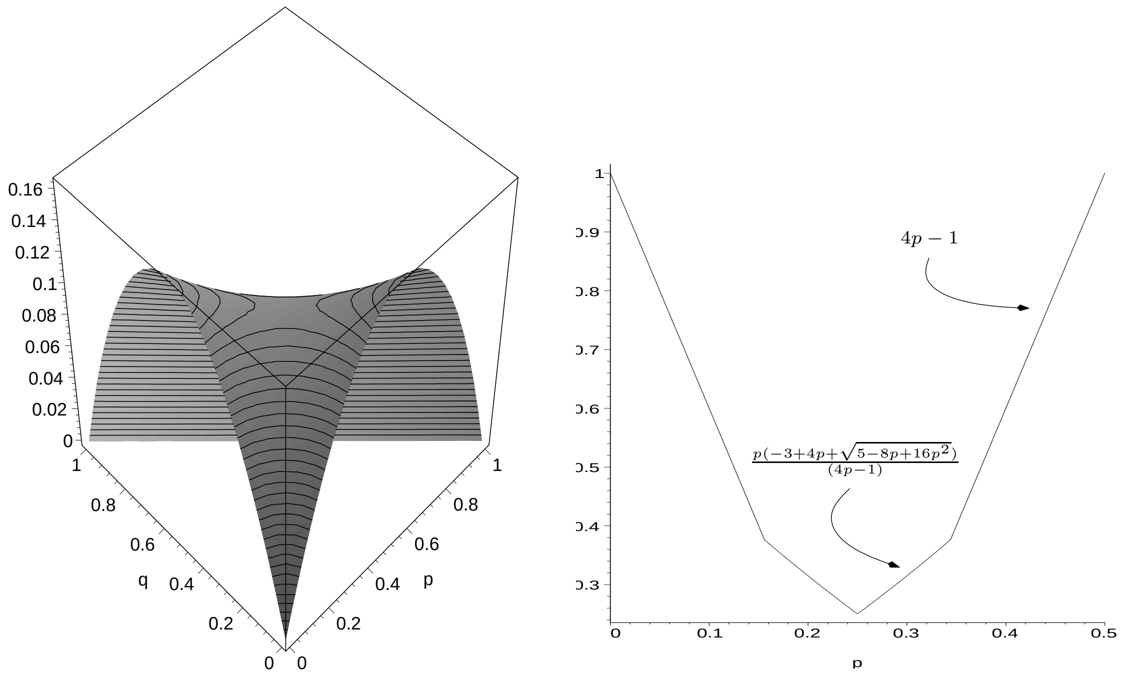

Figure 3: The drift of $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, \mu)$ as a function of $p = \mu(b)$ and $q = \mu(b^2)$ (left), and the drift of (B_3, μ) as a function of $p = \mu(a) = \mu(b) = 1/2 - \mu(a^{-1}) = 1/2 - \mu(b^{-1})$ (right).

For instance, the drift is maximized for $r = z_0, p = 1 - z_0, q = 0$ (or $r = z_0, p = 0, q = 1 - z_0$)), where z_0 is the root of $[z^6 + 12z^4 - 4z^3 + 47z^2 - 48z + 12]$ whose numerical value is $0.490275\cdots$. The corresponding numerical value of the drift is $\gamma_{\max} = 0.163379\cdots$. This was not a priori obvious!

6.3 Random walks on $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$

The results below are used in the study of random walks on B_3, the braid group over 3 strands, see §6.6.

Consider the free product $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ with a and b being the respective generators of the two cyclic groups. Consider the probability μ such that $\mu(a) = \mu(a^2) = p, \mu(b) = \mu(b^2) = q = 1/2 - p$. Solving the Traffic Equations yields:

$$r(a) = r(a^2) = \frac{1+2p}{6}, \quad r(b) = r(b^2) = \frac{1-p}{3}.$$

We have $\mu^\infty(u_1 u_2 \cdots u_k \Sigma^{\mathbb{N}}) = r(u_1)(1/2)^{k-1}$, i.e. μ^∞ is the uniform measure except for the distribution of the initial element of the normal form. The drift is

$$\gamma = 4pq = 2p(1-2p). \tag{23}$$

Assume now that $\mu(a) = \mu(b) = p, \mu(a^2) = \mu(b^2) = q = 1/2 - p$. We obtain:

$$r(a) = r(b) = \frac{4p - 3 + \sqrt{16p^2 - 8p + 5}}{4(4p-1)}, \quad r(a^2) = r(b^2) = \frac{4p + 1 - \sqrt{16p^2 - 8p + 5}}{4(4p-1)}.$$

The harmonic measure is the Markovian multiplicative measure associated with r. It is stationary, see Proposition 5.2. The drift is

$$\gamma \;=\; -\frac{1}{4} + \frac{1}{4}\sqrt{16p^2 - 8p + 5}\,. \tag{24}$$

At last, consider the case $\mu(a) = p, \mu(a^2) = q$, and $\mu(b) = \mu(b^2) = (1-p-q)/2$. Solving explicitly the Traffic Equations is feasible but provides formula which are too lengthy to be reproduced here. However, for the drift, several simplifications occur, and we obtain the following formula:

$$\gamma = 2(1-p-q)\sqrt{\frac{p^2+q^2+pq}{p^2+q^2-2pq+3}}\ .$$

For the general nearest neighbor random walk on $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$, we did not succeed in solving completely the Traffic Equations.

6.4 The simple random walk on $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$

The groups $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$ are known as the Hecke groups. A possible representation of $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, generalizing the one given in (21), is

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 2\cos(\pi/k) & -1 \\ 1 & 0 \end{bmatrix},$$

where a and b are the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/k\mathbb{Z}$ respectively.

We consider the simple random walk $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}, \mu)$ with $\mu(a) = \mu(b) = \mu(b^{-1}) = 1/3$. Consider the applications $G_n : [0,1] \to \mathbb{R}, n \in \mathbb{N}$, defined by

$$G_0(x) = \frac{1}{4} + \frac{x}{2}, \quad G_1(x) = x, \quad \forall n \geq 2,\ G_n(x) = \frac{8(1-x)}{3-2x} G_{n-1}(x) - G_{n-2}(x)\ . \tag{25}$$

For instance, $G_4(x) = (1552x^4 - 4416x^3 + 4296x^2 - 1600x + 165)/(2x-3)^3$.

Lemma 6.1. *For $k \geq 3$, the equation $G_{k-1}(x) = x$ has a unique solution in $(0, 1/2)$ that we denote by y_k.*

Theorem 6.2. *The harmonic measure of (G, μ) is the Markovian multiplicative measure associated with r: $r(a) = G_0(y_k)$, $\forall i \in \{1, \dots, k-1\}$, $r(b^i) = G_i(y_k)$. The drift is $\gamma_k = (1-2y_k)/3$. It is strictly increasing in k and $\lim_k \gamma_k = 2/9$.*

Here is a table of the first values of γ, given either in closed form or numerically when no closed form could be found. Set $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

	$\mathbb{Z}_2 \star \mathbb{Z}_3$	$\mathbb{Z}_2 \star \mathbb{Z}_4$	$\mathbb{Z}_2 \star \mathbb{Z}_5$	$\mathbb{Z}_2 \star \mathbb{Z}_6$	$\mathbb{Z}_2 \star \mathbb{Z}_7$	$\mathbb{Z}_2 \star \mathbb{Z}_8$
γ	$2/15$	$(\sqrt{7}-1)/9$	$(2\sqrt{61}-4)/57$	0.213412...	0.217921...	0.220101...

6.5 The simple random walk on $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$

The results on $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$ are also used to study Artin groups over 2 generators, see §6.6.

Consider the free product $G_1 \star G_2 = \mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$. Set $\Sigma_1 = G_1 \backslash \{1_1\} = \{a, \dots, a^{k-1}\}$, $\Sigma_2 = G_2 \backslash \{1_2\} = \{b, \dots, b^{k-1}\}$, and $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Consider the simple random walk $(\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}, \mu)$ with $\mu(a) = \mu(b) = \mu(a^{-1}) = \mu(b^{-1}) = 1/4$. See Figure 2.
Consider the applications $F_n : [0,1] \to \mathbb{R}, n \in \mathbb{N}$, defined by

$$F_0(x) = 1, \quad F_1(x) = x, \quad \forall n \geq 2,\ F_n(x) = 2(2-x)F_{n-1}(x) - F_{n-2}(x)\ . \tag{26}$$

For instance, $F_5(x) = 16x^5 - 128x^4 + 380x^3 - 512x^2 + 301x - 56$.

Lemma 6.3. *For $k \geq 3$, the equation $F_k(x) = 1$ has a unique solution in $(0,1)$ that we denote by x_k.*

Theorem 6.4. *The harmonic measure of (G, μ) is the Markovian multiplicative measure associated with r: $\forall i \in \{1, \dots, k-1\},\ r(a^i) = r(b^i) = F_i(x_k)/2$. The drift is $\gamma_k = (1 - x_k)/2$. It is strictly increasing in k and $\lim_k \gamma_k = 1/3$.*

	$\mathbb{Z}_3 \star \mathbb{Z}_3$	$\mathbb{Z}_4 \star \mathbb{Z}_4$	$\mathbb{Z}_5 \star \mathbb{Z}_5$	$\mathbb{Z}_6 \star \mathbb{Z}_6$	$\mathbb{Z}_7 \star \mathbb{Z}_7$	$\mathbb{Z}_8 \star \mathbb{Z}_8$
γ	$1/4$	$(\sqrt{5}-1)/4$	$(\sqrt{13}-1)/8$	0.330851...	0.332515...	0.333062...

6.6 Random walks on Artin Groups with 2 generators

The *Artin group with 2 generators* A_k $(k \geq 3)$ is the group with finite presentation

$$A_k = \langle\, a, b \mid \mathrm{prod}(a, b; k) = \mathrm{prod}(b, a; k)\, \rangle\,, \tag{27}$$

where $\mathrm{prod}(a, b; k) = ababa\dots$, with k terms in the product on the right-hand side. Observe that $A_3 = B_3$, the well-known braid group over three stands.

Set $\Sigma = \{a, a^{-1}, b, b^{-1}\}$. The pair (A_k, Σ) is not 0-automatic. In particular there is no natural notion of geodesic normal form over the generators Σ. However, we are able to go back to the 0-automatic framework by a series of intermediate steps, that we now list.

Consider the random walk (A_k, μ) where μ is a probability on Σ. Write the group elements in Garside normal form [9]. This requires the switch to a new set of generators. In particular, the Garside normal form is not geodesic for the generators Σ. Let Z be the center of A_k. Consider the induced random walk on A_k/Z, the group quotiented by its center. Assume that $\mu(a) = \mu(b), \mu(a^{-1}) = \mu(b^{-1})$. Show that the induced random walk behaves like a nearest neighbor random walk on $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$. Use the results of §6.3 and §6.5. Go back from A_k/Z to A_k. Go back to the natural generators.

In the end, what is lost is an explicit description of the harmonic measure of (A_k, μ), but what remains is an explicit formula for the drift.

Theorem 6.5. *Consider the random walk (B_3, μ) where μ is a probability measure on Σ such that $\mu(a) = \mu(b) = p, \mu(a^{-1}) = \mu(b^{-1}) = 1/2 - p$. The drift with respect to the natural generators $\Sigma = \{a, b, a^{-1}, b^{-1}\}$ is*

$$\gamma(p) = \max\Big[1-4p, \frac{(1-2p)(-1-4p+\sqrt{5-8p+16p^2})}{2(1-4p)}, \frac{p(-3+4p+\sqrt{5-8p+16p^2})}{-1+4p}, -1+4p\Big]$$

Consider the simple random walk (A_k, μ) with $\mu(a) = \mu(a^{-1}) = \mu(b) = \mu(b^{-1}) = 1/4$. Let γ_k be the drift with respect to Σ. We have

$$\gamma_k = \begin{cases} (1-x_k)\big[\ \sum_{i=1}^{j-1} iF_i(x_k) + (j/2)F_j(x_k)\ \big] & \text{if } \ k = 2j \\ (1-x_k)\big[\ \sum_{i=1}^{j} iF_i(x_k)\ \big] & \text{if } \ k = 2j+1 \end{cases}, \tag{28}$$

where the F_i were defined in (26). The drift γ_k is strictly increasing in k, and $\lim_k \gamma_k = 1/2$.

	A_3	A_4	A_5	A_6	A_7	A_8
γ	$1/4$	$(\sqrt{5}-1)^2/4$	$(\sqrt{13}-1)^2/16$	0.462598...	0.475221...	0.487636...

Set $u = 7/24 + 11/[24(71+6\sqrt{177})^{1/3}] + (71+6\sqrt{177})^{1/3}/24 = 0.155979\cdots$. The function $\gamma(p)$, represented on Figure 3, has several unexpected characteristics: it is linear on the intervals $[0,u]$ and $[1/2-u, 1/2]$ but not on the interval $[u, 1/2-u]$; and it has 3 points of non-differentiability: $u, 1/4, 1/2-u$. Hence, the behavior of the braid model has some kind of phase transitions whose physical interpretation is intriguing.

7 From groups to monoids

A pair (M, Σ) formed by a monoid and a finite set of generators is *0-automatic* if the set of locally reduced words $L(M, \Sigma)$ is a cross-section *and* the analog of (7) holds. A plain monoid together with the natural generators, forms a 0-automatic pair. Define the Traffic Equations as in (14) with the convention that $\mu(u^{-1}) = 0$ if u has no inverse. Then Theorem 5.1 and Proposition 5.2 hold for monoids. (But not Propositions 5.3 and 5.4.)

To illustrate, consider a free product of the form $M = M_a \star M_b \star M_c \star M_d$, where M_i is equal either to $\mathbb{Z}_2$ or to $\mathbb{B}$. Here $\mathbb{B} = \langle\, a \mid a^2 = a \,\rangle$ is the Boolean monoid and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Let i be the generator of M_i. We consider the simple random walk on M, that is the random walk defined by $\mu : \mu(i) = 1/4, \forall i$. The values of the drift are given below.

	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2$	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B}$	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B}$	$\mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B} \star \mathbb{B}$	$\mathbb{B} \star \mathbb{B} \star \mathbb{B} \star \mathbb{B}$
γ	$1/2$	$(12+3\sqrt{2})/28$	$(6+\sqrt{3})/12$	$7/10$	$3/4$

The values $1/2, 7/10$, and $3/4$, can be obtained by elementary arguments without having to solve the Traffic Equations. But not the other two values.

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