Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 1(x): xxx–xxx
DOI:10.3934/ms.2017.x.xxx
Received: 10 April 2016
Accepted: 28 March 2017
Published: December 2017

Research article

Modeling electromagnetism in and near composite material using two-scale behavior of the time-harmonic Maxwell equations

Canot Hélène[1] **and Frénod Emmanuel**[1,*]

[*] **Correspondence:** helene.canot@univ-ubs.fr; Tel: 02.97.01.71.28.

Abstract: The main purpose of this article is to study the two-scale behavior of the electromagnetic field in 3D in and near composite material. For this, time-harmonic Maxwell equations, for a conducting two-phase composite and the air above, are considered. Technique of two-scale convergence is used to obtain the homogenized problem.

Keywords: Harmonic Maxwell Equations; Electromagnetic Pulses, Electromagnetism; Homogenization; Asymptotic Analysis; Asymptotic Expansion; Two-scale Convergence; Effective Behavior; Frequencies; Composite Material.

1. Introduction

We are interested in the time-harmonic Maxwell equations in and near a composite material with boundary conditions modeling electromagnetic field radiated by an electromagnetic pulse (EMP). An electromagnetic pulse is a short burst of electromagnetic energy. It may be generated by a natural occurrence such like a lightning strike, meteoric EMP, EMP caused by geomagnetic Storm or nuclear EMP. This focuses on what happens over a period of time of a millisecond during the peak of the first return stroke. We study the electromagnetic pulse caused by this lightning strike. This is the first step of a larger study which goal is to understand the behavior of the electromagnetic field and its interaction with a composite material.

EMP interference is generally damaging to electronic equipment. A lightning strike can damage physical objects such as aircraft structures, either through heating effects or disruptive effects of the very large magnetic field generated by the current. Structures and systems require some form of protection against lightning. Every commercial aircraft is struck by lightning at least once a year on average. Aircraft lightning protection is a major concern for aircraft manufacturers. Increasing its use of composite materials, up to 53% for the latest Airbus A350, and 50% for the Boeing B787, aircrafts offer increased vulnerability facing lightning. Earlier generation aircrafts, whose fuselages were predominantly composed of aluminum, behave like a Faraday cage and offer maximum protection for the internal equipment. Currently, in aircrafts, composite materials consisting of a resin enclosing carbon fibers have significant advantages in terms of weight gain and therefore fuel saving. Yet,because aluminium conducts 100 to 1000 times more than composite, we lose the Faraday effect. Modern aircrafts have seen also the increasing reliance on electronic avionics systems instead of mechanical controls and electromechanical instrumentation. For these reasons, aircraft manufacturers are very sensitive to lightning protection and pay special attention to aircraft certification through testing and analysis.

There are two types of lightning strikes to aircraft: the first one is the interception by the aircraft of a lightning leader. The second one, which makes about 90% of the cases, is when the aircraft initiates the lightning discharge by emitting two leaders when it is found in the intense electric field region produced by a thundercloud, our approach applies in this case. When the aircraft flies through a cloud region where the atmospheric electric field is large enough, an ionized channel, called a positive leader, merges from the aircraft in the direction of the ambient electric field. Laroche *et al* [**?**], at an altitude of 6000m, observed an atmospheric electric field close to 50 kV/m inside the storm clouds, 100kV/m to the ground. When upward leader connects with the downward negative leader of the cloud, a return stroke is produced and a bright return stroked wave travels from aircraft to cloud. The lightning return strokes radiate powerful electromagnetic fields which may cause damage to aircraft electronic equipment. Our work is devoted to the study of the electromagnetic waves propagation in the air and in the composite material. In this artificial periodic material, the electromagnetic field satisfies the Maxwell equations.

We evaluate the electromagnetic field within and near a periodic structure when the period of this microstructure is small compared to the wavelength of the electromagnetic wave. Our model is composed by air above the composite fuselage and we study the behavior of the electromagnetic wave in the domain filled by the composite material, representing the skin aircraft, and the air. We build the 3D model, under simplifying assumptions, using linear time-harmonic Maxwell equations and constitutive relations for electric and magnetic fields. Composite materials consist of conducting carbon fibers, distributed as periodic inclusions in a matrix (epoxy resin). We impose a magnetic permeability μ_0 uniform and an electrical permittivity $\epsilon = \epsilon_0\epsilon^\star$, where $\epsilon^\star$ is the relative permittivity depending of the medium. In the future, we will enrich this model by adding complexity and we will consider non uniform magnetic permeability and electrical permittivity.

Now, we account for some characteristic values. In the first place we focus on the boundary conditions as we consider them as the source. Then, we use on the upper frontier, the magnetic field induced by the peak of the current of the first return stroke

$$\overline{H_d} = \frac{I}{2\pi r}, \tag{1}$$

with current intensity $I = 200$ kA and r the radius of the lightning strike, this is the worst aggression that can suffer an aircraft, and we deduce a characteristic electric field $\overline{E} = 20$ kV/m. In our model we consider that we have very conductive - but not perfect conductors - carbon fibers and an epoxy resin whose conduction depends on its doping rate. The conductivity of the air is non-linear. Air is a strong insulator [23] with conductivity of the order of 10^{-14} $S.m^{-1}$ but beyond some electric solicitation, the air loses its insulating nature and locally becomes suddenly conductive. The ionization phenomenon is the only cause that can make the air conductor of electricity. The ionized channel becomes very conductive.

Our mathematical context is periodic homogenization. We consider a microscopic scale ε, which represents the ratio between the diameter of the fiber and thickness of the composite material. So, we are trying to understand how the microscopic structure affects the macroscopic electromagnetic field behavior. Homogenization of Maxwell equations with periodically oscillating coefficients was studied in many papers. N. Wellander homogenized linear and non-linear Maxwell equations with perfect conducting boundary conditions using two-scale convergence in [20] and [21]. N. Wellander and B. Kristensson homogenized the full time-harmonic Maxwell equation with penetrable boundary conditions and at fixed frequency in [22]. The homogenized time-harmonic Maxwell equation for the scattering problem was done in F. Guenneau, S. Zolla and A. Nicolet [10]. Y. Amirat and V. Shelukhin perform two-scale homogenization time-harmonic Maxwell equations for a periodical structure in [4]. They calculate the effective dielectric ε and effective electric conductivity σ. They proved that homogenized Maxwell equations are different in low and high frequencies. The result obtained by two-scale convergence approach takes into account the characteristic sizes of skin thickness and wavelength around the material.

On of the parameter we account for in our model: $\delta = \frac{1}{\sqrt{\overline{\omega}\ \overline{\sigma}\mu_0}}$, where $\overline{\sigma}$ is the characteristic conductivity and $\overline{\omega}$ the order of the magnitude of the pulsation shares much with the definition of theoretical thickness skin $\delta = \sqrt{\frac{2}{\overline{\omega}\overline{\sigma}\mu_0}}$. The thickness skin is the depth at which the surface current moves to a factor of e^{-1}. Indeed, at high frequency, the skin effect phenomenon appears because the current tends to concentrate at the periphery of the conductor. On the other side, at low frequencies the penetration depth is much greater than the thickness of the plate which means that a part of the electric field penetrates the composite plate. We use the theory of two-scale convergence introduced by G. Nguetseng [15] and developed by G. Allaire [2].

The paper is organized as follows : in Section 2 we specify the geometry of the model and the dimensionless equations converting the problem into an equivalent one with which we work in the following sections. In Section 3 we perform the mathematical analysis of the model. In particular, we introduce the weak formulation of the problem for the electric field and we regularize it using divergence term. We establish the existence and uniqueness result for the regularized Maxwell equations thanks to Lax-Milgram Theorem. We conclude this section by estimate of the electric field. The last section is devoted to the homogenization of the problems for electric field using the two-scale convergence concept.

2. Modeling

This section is dedicated to the complete mathematical model we will study in this paper. First, we consider a problem that seems relevant with the perspective of propagation of the electromagnetic field in the air and in the skin of aircraft fuselage made of composite material. Secondly, we make a scaling of this model and finally we operate simplifications. If desired, the reader can go directly to the mathematical analysis knowing that the problem to be studied is given by (65), (70) equipped with boundary conditions (68), (69).

2.1. Notations and setting of the problem

We consider set $\widetilde{\Omega} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathbb{R}^3,\ \widetilde{y} \in (-\overline{L}, d)\}$ for $\overline{L}$ and d two positive constants, with two open subsets $\widetilde{\Omega}_a$ and $\widetilde{P}$ (see Figure 1). The air fills $\widetilde{\Omega}_a$ and we consider that the composite material, with two materials periodically distributed, stands in domain $\widetilde{P}$.

We assume that the thickness $\overline{L}$ of the composite material is much smaller than its horizontal size. We denote by e the lateral size of the basic cell $\widetilde{Y}^e$ of the periodic microstructure of the material. The cell is composed of a carbon fiber in the resin. We define now more precisely the material, introducing:

$$\widetilde{P} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathbb{R}^3 / -\overline{L} < \widetilde{y} < 0\}, \tag{2}$$

which is the domain containing the material. Now we describe precisely the basic cell. For this we first introduce the following cylinder with square base:

$$\widetilde{Z}^e = [-\frac{e}{2}, \frac{e}{2}] \times [-e, 0] \times \mathbb{R}, \tag{3}$$

We consider α such that $0 < \alpha < 1$, and $\widetilde{R}^e = \alpha \frac{e}{2}$. We set

$$\widetilde{D}^e = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2 / (\widetilde{x}^2 + (\widetilde{y} + \frac{e}{2})^2) < (\widetilde{R}^e)^2\}. \tag{4}$$

We define the cylinder containing the fiber as (see fig 1):

$$\widetilde{C}^e = \widetilde{D}^e \times \mathbb{R}. \tag{5}$$

Then the part of the basic cell containing the matrix is

$$\widetilde{Y}^e_R = \widetilde{Z}^e \setminus \widetilde{C}^e, \tag{6}$$

and by definition, the basic cell $\widetilde{Y}^e$ is the couple

$$(\widetilde{Y}^e_R, \widetilde{C}^e). \tag{7}$$

The composite material results from a periodic extension of the basic cell. More precisely the part of the material that contains the carbon fibers is

$$\widetilde{\Omega}_c = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{C}^e, i \in \mathbb{Z}, j \in \mathbb{Z}^-\}, \tag{8}$$

where the intersection with $\widetilde{P}$ limits the periodic extension to the area where stands the material. Set $\{(ie, je, 0) + \widetilde{C}^e, i \in \mathbb{Z}, j \in \mathbb{Z}^-\}$ is a short notation for

$$\{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathbb{R}^3, \exists i \in \mathbb{Z}, \exists j \in \mathbb{Z}^-, \exists (x_b, y_b, z_b) \in \widetilde{C}^e; \widetilde{x} = x_b + ie, \widetilde{y} = y_b + je, \widetilde{z} = z_b\}. \tag{9}$$

In the same way the part of the material that contains the resin is

$$\widetilde{\Omega}_r = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{Y}^e_R\}, \tag{10}$$

or equivalently

$$\widetilde{\Omega}_r = \widetilde{P} \cap \{(ie, je, 0) + \widetilde{Z}^e \setminus \widetilde{C}^e\} = (\mathbb{R} \times (-\overline{L}, 0) \times \mathbb{R}) \backslash \widetilde{\Omega}_c. \tag{11}$$

So the geometrical model of our composite material is couple $(\widetilde{\Omega}_c, \widetilde{\Omega}_r)$. Now, it remains to set the domain that contains the air:

$$\widetilde{\Omega}_a = \{(\widetilde{x}, \widetilde{y}, \widetilde{z})/0 \le \widetilde{y} < d\}. \tag{12}$$

We consider that d is of the same order as $\overline{L}$ and we introduce the upper frontier $\widetilde{\Gamma}_d = \{(\widetilde{x}, \widetilde{y}, \widetilde{z})/\widetilde{y} = d\}$ of domain $\widetilde{\Omega}$. On this frontier we will consider that the electric field and magnetic field are given. We also introduce the lower frontier $\widetilde{\Gamma}_L = \{(\widetilde{x}, \widetilde{y}, \widetilde{z})/\widetilde{y} = -\overline{L}\}$ with those definitions we have $\widetilde{\Omega}_a \cap \widetilde{P} = \emptyset$, $\widetilde{\Omega}_c \cap \widetilde{\Omega}_r = \emptyset$, $\widetilde{P} = \Omega_r \cup \Omega_c$, $\widetilde{\Omega} = \Omega_a \cup \widetilde{P} = \Omega_a \cup \Omega_r \cup \Omega_c$, and for any $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \partial\widetilde{\Omega} = \widetilde{\Gamma}_d \cup \widetilde{\Gamma}_L$ and, we write $\widetilde{n}$, the unit vector, orthogonal to $\partial\widetilde{\Omega}$ and pointing outside $\widetilde{\Omega}$. We have :

$$\begin{aligned} \widetilde{n} &= e_2 \text{ on } \widetilde{\Gamma}_d \\ \widetilde{n} &= -e_2 \text{ on } \widetilde{\Gamma}_L. \end{aligned} \tag{13}$$

In the following we need to describe what happens at the interfaces between resin and carbon fibers, and resin and air. So we define $\Gamma_{ra} = \{(\widetilde{x}, \widetilde{y}, \widetilde{z}) \ / \ \widetilde{y} = 0\}$ and Γ_{cr} the boundary of the set defined by (9).

2.2. Maxwell equations

In $\widetilde{\Omega}$, we now write a PDE model that has to do with electromagnetic waves radiated from return stroke. We are well aware that the model we write is a simplified one. Nonetheless, it seems to be well dimensioned for our problem which consists in making homogenization. It is well known (see Maxwell [**?**]) the propagation of the electromagnetic field is described by the Maxwell equations which write:

$$-\frac{\partial \widetilde{D}^\star}{\partial t} + \nabla\times\widetilde{H}^\star = \widetilde{J}^\star, \tag{14}$$

$$\frac{\partial \widetilde{B}^\star}{\partial t} + \nabla\times\widetilde{E}^\star = 0, \tag{15}$$

$$\nabla\cdot\widetilde{D}^\star = \widetilde{\rho}^\star, \tag{16}$$

$$\nabla\cdot\widetilde{B}^\star = 0, \tag{17}$$

in $\mathbb{R} \times \widetilde{\Omega}$.

In (14)-(17), $\nabla\times$ and $\nabla\cdot$ are the curl and divergence operators. $\widetilde{E}^\star(t, x, y, z)$ is the electric field, $\widetilde{H}^\star(t, x, y, z)$ the magnetic field, $\widetilde{D}^\star(t, x, y, z)$ the electric induction, $\widetilde{B}^\star(t, x, y, z)$ the magnetic induction and $\widetilde{\rho}^\star(t, x, y, z)$ is the charges density (see T. Abboud and I. Terrasse [**?**]).

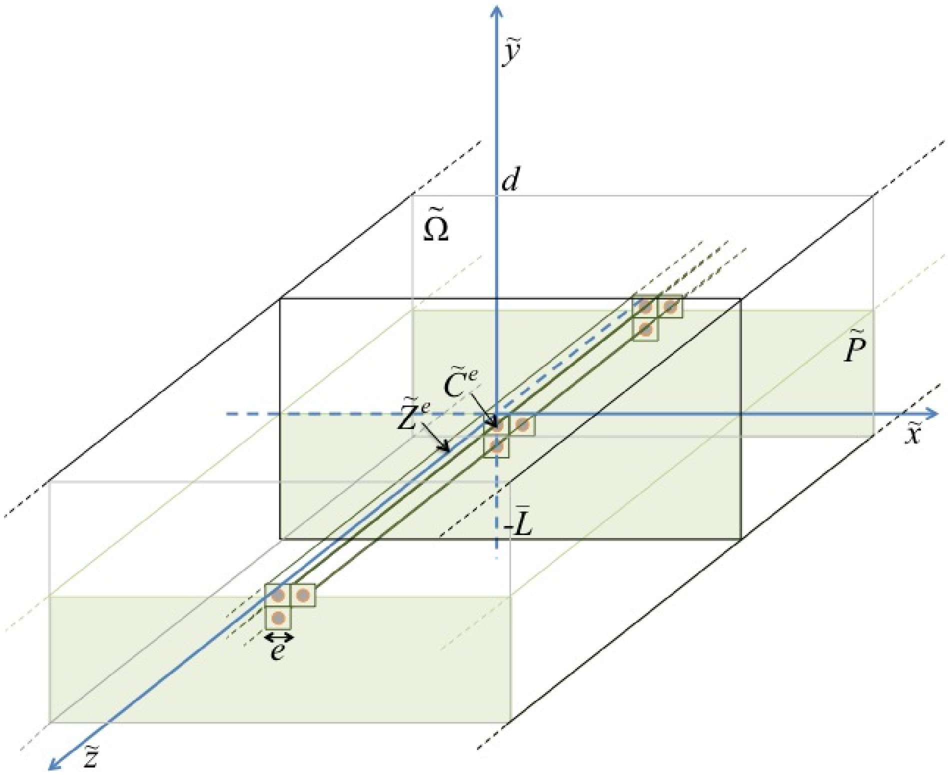

Figure 1. The global domain

System of Maxwell equations ((14) - (17)) is completed by the constitutive laws which are given in $\mathbb{R} \times \widetilde{\Omega}$ by :

$$\widetilde{D}^{\star} = \epsilon_0 \epsilon^{\star} \widetilde{E}^{\star}, \tag{18}$$

$$\widetilde{B}^{\star} = \mu_0 \widetilde{H}^{\star}. \tag{19}$$

where μ_0 and ϵ_0 are the permeability and permittivity of free space. $\epsilon^{\star}$ is the relative permittivity of the domains defined by

$$\epsilon^{\star}_{|\widetilde{\Omega_a}} = 1, \epsilon^{\star}_{|\widetilde{\Omega_r}} = \epsilon_r, \epsilon^{\star}_{|\widetilde{\Omega_c}} = \epsilon_c, \tag{20}$$

where ϵ_r and ϵ_c are positives constants. In order to account for energy transfer between the electromagnetic compartment and the propagation of the electric charges, we take for granted the Ohmic law, in $\mathbb{R} \times \widetilde{\Omega}$

$$\widetilde{J}^{\star} = \sigma \widetilde{E}^{\star}, \tag{21}$$

where σ is the electric conductivity. Its value depends on the location:

$$\sigma_{|\widetilde{\Omega_a}} = \sigma_a, \sigma_{|\widetilde{\Omega_r}} = \sigma_r, \sigma_{|\widetilde{\Omega_c}} = \sigma_c, \tag{22}$$

where $\widetilde{\Omega_a}$, $\widetilde{\Omega_r}$ and $\widetilde{\Omega_c}$ were defined in (12), (10) and (8).

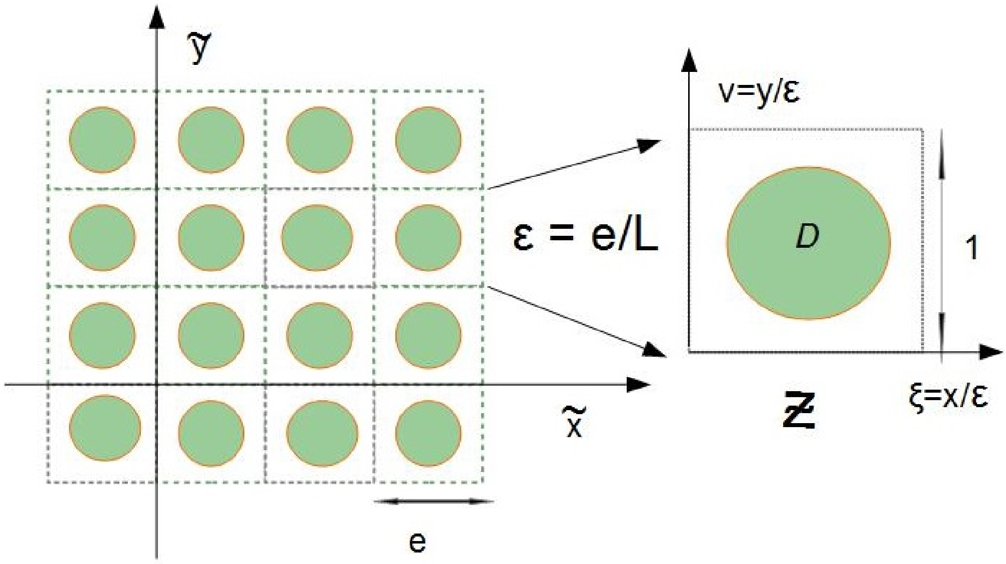

Figure 2. Left: The global microstructure in 2D. Right: $\mathcal{Z}$-cell of the periodic structure.

2.3. *Boundary conditions*

For mathematical as well as physical reasons we have to set boundary conditions on $\widetilde{\Gamma}_d$ and $\widetilde{\Gamma}_L$. On $\widetilde{\Gamma}_d$ we will write conditions that translate that $\widetilde{E}^\star$ and $\widetilde{H}^\star$ are given by the source located in $\widetilde{y} = d$. The way we chose consists in setting:

$$\widetilde{E}^\star \times \widetilde{n} = \widetilde{E}_d^\star \times \widetilde{n}; \quad \widetilde{H}^\star \times \widetilde{n} = \widetilde{H}_d^\star \times \widetilde{n} \quad \text{on} \quad \mathbb{R} \times \widetilde{\Gamma}_d, \tag{23}$$

where $\widetilde{E}_d^\star$, $\widetilde{H}_d^\star$ are functions defined on $\widetilde{\Gamma}_d$ for any $t \in \mathbb{R}$. On $\widetilde{\Gamma}_L$, we chose something simple, i.e :

$$\nabla \times \widetilde{E}^\star \times \widetilde{n} = 0 \quad \text{on} \quad \mathbb{R} \times \widetilde{\Gamma}_L, \tag{24}$$

that translate that $\widetilde{E}^\star$ does not vary in the $\widetilde{y}$-direction near $\widetilde{\Gamma}_L$.

Problem (14)-(21) supplemented with (23) and (24), is considered as containing all physics we want to account for. In the following we will consider simplifications of it.

2.4. *Time-harmonic Maxwell equations*

The first simplification we make, consists in considering the harmonic version of the Maxwell equations (14)-(22). This simplification is used in many references studying electromagnetic phenomena and especially for lightning applications [**?**], in spite of the fact that it considers implicitly that every fields and currents are waves of the form, for all $\widetilde{\omega} \in \mathbb{R}$:

$$a(\widetilde{x}, \widetilde{y}, \widetilde{z}) \cos(-\widetilde{\omega} t + \phi(\widetilde{x}, \widetilde{y}, \widetilde{z})) = \Re e[a(\widetilde{x}, \widetilde{y}, \widetilde{z}) \exp^{i\widetilde{\omega} t} \exp^{i\phi(\widetilde{x}, \widetilde{y}, \widetilde{z})}], \tag{25}$$

where $\widetilde{\omega}$ is the pulsation, $\phi(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is the phase shift of the wave and $a(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is its amplitude. In particular, it supposes $\widetilde{E}_d^\star$, $\widetilde{H}_d^\star$ in (23) are of the form, for all $\widetilde{w} \in \mathbb{R}$:

$$\widetilde{E}_d^\star(t, \widetilde{x}, \widetilde{z}) = \Re e(\widetilde{E}_d(\widetilde{x}, \widetilde{z}) \exp^{i\widetilde{\omega} t}), \tag{26}$$

$$\widetilde{H}_d^\star(t,\widetilde{x},\widetilde{z}) = \Re e(\widetilde{H}_d(\widetilde{x},\widetilde{z})\exp^{i\widetilde{\omega}t}), \tag{27}$$

where $\widetilde{E}_d$ and $\widetilde{H}_d$ take into account the amplitude and the phase shift of their corresponding fields. Taking (21) into account, the time-harmonic Maxwell equations, which describe the electromagnetic radiation, are written:

$$\nabla\times\widetilde{H} - i\widetilde{\omega}\epsilon_0\epsilon^\star\widetilde{E} = \sigma\widetilde{E}, \quad \text{Maxwell - Ampere equation} \tag{28}$$

$$\nabla\times\widetilde{E} + i\widetilde{\omega}\mu_0\widetilde{H} = 0, \quad \text{Maxwell - Faraday equation} \tag{29}$$

$$\nabla\cdot(\epsilon_0\epsilon^\star\widetilde{E}) = \widetilde{\rho}, \tag{30}$$

$$\nabla\cdot(\mu_0\widetilde{H}) = 0, \tag{31}$$

where $\widetilde{E}^\star(t,\widetilde{x},\widetilde{y},\widetilde{z}) = \Re e(\widetilde{E}(\widetilde{x},\widetilde{y},\widetilde{z})\exp^{i\widetilde{\omega}t})$ and $\widetilde{H}^\star(t,\widetilde{x},\widetilde{y},\widetilde{z}) = \Re e(\widetilde{H}(\widetilde{x},\widetilde{y},\widetilde{z})\exp^{i\widetilde{\omega}t})$, $(\widetilde{x},\widetilde{y},\widetilde{z}) \in \widetilde{\Omega}$. The magnetic field $\widetilde{H}$ can be directly computed from the electric field $\widetilde{E}$

$$\widetilde{H} = -\frac{1}{i\omega\mu_0}\nabla\times\widetilde{E}. \tag{32}$$

Now, for the electric approach, taking the curl of equation (32) yields an expression of $\nabla\times\widetilde{H}$ in term of $\nabla\times\nabla\times\widetilde{E}$. Inserting $\nabla\times\widetilde{H}$ in (28) we get the following equation for the electric field:

$$\nabla\times\nabla\times\widetilde{E} + (-\widetilde{\omega}^2\mu_0\epsilon_0\epsilon^\star + i\widetilde{\omega}\mu_0\sigma)\widetilde{E} = 0 \qquad \text{in} \qquad \widetilde{\Omega}. \tag{33}$$

Taking the divergence of the equation (28) yields the natural gauge condition:

$$\nabla\cdot[(i\widetilde{\omega}\epsilon_0\epsilon^\star + \sigma)\widetilde{E}] = 0 \qquad \text{in} \qquad \widetilde{\Omega}. \tag{34}$$

Notice that $i\widetilde{\omega}\epsilon_0 + \sigma$ is equal to $i\widetilde{\omega}\epsilon_0 + \sigma_a$ in $\widetilde{\Omega_a}$, to $i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r$ in $\widetilde{\Omega_r}$ and to $i\widetilde{\omega}\epsilon_0\epsilon_c + \sigma_c$ in $\widetilde{\Omega_c}$, those quantities being all nonzero. Then (34) is equivalent to:

$$\nabla\cdot\widetilde{E}_{|\Omega_a} = 0 \text{ in } \widetilde{\Omega_a}, \quad \nabla\cdot\widetilde{E}_{|\Omega_r} = 0 \text{ in } \widetilde{\Omega_r}, \quad \nabla\cdot\widetilde{E}_{|\Omega_c} = 0 \text{ in } \widetilde{\Omega_c}. \tag{35}$$

with the transmission conditions

$$\begin{aligned}(i\widetilde{\omega}\epsilon_0 + \sigma_a)\widetilde{E}_{|\widetilde{\Omega_a}}.\widetilde{n} &= (i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r)\widetilde{E}_{|\widetilde{\Omega_r}}.\widetilde{n} \text{ on } \widetilde{\Gamma_{ra}},\\ (i\widetilde{\omega}\epsilon_0\epsilon_r + \sigma_r)\widetilde{E}_{|\widetilde{\Omega_r}}.\widetilde{n} &= (i\widetilde{\omega}\epsilon_0\epsilon_c + \sigma_c)\widetilde{E}_{|\widetilde{\Omega_c}}.\widetilde{n} \text{ on } \widetilde{\Gamma_{cr}}.\end{aligned} \tag{36}$$

Summarizing, we finally obtain the PDE model:

$$\nabla\times\nabla\times\widetilde{E} + (-\widetilde{\omega}^2\mu_0\epsilon_0\epsilon^\star + i\widetilde{\omega}\mu_0\sigma)\widetilde{E} = 0 \qquad \text{in} \qquad \widetilde{\Omega}. \tag{37}$$

According to the tangential trace of the Maxwell-Faraday equation (29) we obviously obtain that using boundary condition (23), is equivalent to using:

$$\nabla\times\widetilde{E}\times e_2 = -i\widetilde{\omega}\mu_0\widetilde{H}_d(\widetilde{x},\widetilde{z})\times e_2 \quad \text{on} \quad \widetilde{\Gamma}_d \tag{38}$$

where $\widetilde{H}_d$ is defined in (27) and where we used (13). And on $\widetilde{\Gamma}_L$ we have the following boundary condition:

$$\nabla\times\widetilde{E}\times e_2 = 0 \quad \text{on} \quad \widetilde{\Gamma}_L. \tag{39}$$

2.5. Scaling

In this subsection we propose a rescaling of system ((37)-(39)), we will consider a set of characteristic sizes related to our problem. Physical factors are then rewritten using those values leading to a new set of dimensionless and unitless variables and fields in which the system is rewritten. The considered characteristic sizes are : $\overline{\omega}$ the characteristic pulsation, $\overline{\sigma}$ the characteristic electric conductivity, $\overline{E}$ the characteristic electric magnitude, $\overline{H}$ the characteristic magnetic magnitude. We also use the already introduced thickness $\overline{L}$ of the plate $\widetilde{P}$. We then introduce the dimensionless variables: $\mathbf{x} = (x, y, z)$ with $x = \frac{\widetilde{x}}{\overline{L}}$, $y = \frac{\widetilde{y}}{\overline{L}}$, $z = \frac{\widetilde{z}}{\overline{L}}$ and fields E, H and σ that are such that

$$\begin{cases} E(\omega, \mathbf{x}) = \dfrac{1}{\overline{E}}\widetilde{E}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \\ H(\omega, \mathbf{x}) = \dfrac{1}{\overline{H}}\widetilde{H}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \\ \sigma(\mathbf{x}) = \dfrac{1}{\overline{\sigma}}\widetilde{\sigma}(\overline{L}x, \overline{L}y, \overline{L}z), \end{cases} \tag{40}$$

Taking (22) into account, σ also reads:

$$\sigma(\mathbf{x}) = \frac{\sigma_a}{\overline{\sigma}} \qquad \text{if} \qquad 0 \leq \overline{L}y \leq d, \tag{41}$$

$$\sigma(\mathbf{x}) = \frac{\sigma_r}{\overline{\sigma}} \quad \text{if} \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_r, \tag{42}$$

$$\sigma(\mathbf{x}) = \frac{\sigma_c}{\overline{\sigma}} \quad \text{if} \quad (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_c. \tag{43}$$

Doing this gives the status of units to the characteristic sizes. Since, for instance:

$$\frac{\partial E}{\partial x}(\omega, \mathbf{x}) = \frac{\overline{L}}{\overline{E}}\frac{\partial \widetilde{E}}{\partial \widetilde{x}}(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z), \tag{44}$$

using those dimensionless variables and fields and taking (41)-(43) into account, equation (37) gives:

$$\overline{E}\ \nabla\times\nabla\times E(\omega, \mathbf{x}) - \Big(\frac{\overline{L}^2\overline{\omega}^2}{c^2}\epsilon^\star\omega^2 + i\overline{\sigma}\,\overline{\omega}\,\omega\overline{L}^2\mu_0\sigma(\mathbf{x}, \omega)\Big)\overline{E}E(\omega, x, y, z) = 0, \tag{45}$$

for any $(\omega, \mathbf{x})$ such that $(\overline{\omega}\omega, \overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}$. Now we exhibit

$$\overline{\lambda} = \frac{2\pi c}{\overline{\omega}}, \tag{46}$$

which is the characteristic wave length and

$$\overline{\delta} = \frac{1}{\sqrt{\overline{\omega}\,\overline{\sigma}\mu_0}}, \tag{47}$$

which is the characteristic skin thickness. Using those quantities equation (45) reads, for any $(\omega, \mathbf{x}) \in$

Ω:

$$
\begin{aligned}
&\nabla\times\nabla\times E(\omega,\mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_a}{\overline{\sigma}}\omega)E(\omega,\mathbf{x}) = 0 \quad \text{when} \quad 0 \le \overline{L}y \le d,\\
&\nabla\times\nabla\times E(\omega,\mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\epsilon_r\,\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_r}{\overline{\sigma}}\omega)E(\omega,\mathbf{x}) = 0 \quad \text{when} \quad (\overline{L}x,\overline{L}y,\overline{L}z) \in \widetilde{\Omega}_r,\\
&\nabla\times\nabla\times E(\omega,\mathbf{x}) + (-\frac{4\pi^2\overline{L}^2}{\overline{\lambda}^2}\epsilon_c\,\omega^2 + i\frac{\overline{L}^2}{\overline{\delta}^2}\frac{\sigma_c}{\overline{\sigma}}\omega)E(\omega,\mathbf{x}) = 0 \quad \text{when} \quad (\overline{L}x,\overline{L}y,\overline{L}z) \in \widetilde{\Omega}_c.
\end{aligned} \tag{48}
$$

In the following expressions, $\frac{\overline{L}}{\overline{\lambda}}$ and $\frac{\overline{L}}{\overline{\delta}}$ appearing in the equations above will be rewritten in terms of a small parameter ε.

The boundary conditions are written

$$
\begin{aligned}
&\nabla\times E(\omega,\mathbf{x}) \times e_2 = -i\omega\overline{\omega}\mu_0\frac{\overline{L}}{\overline{E}}\widetilde{H_d}(\overline{L}x,\overline{L}z) \times e_2 \;\; \text{when} \;\; (\overline{L}x,\overline{L}y,\overline{L}z) \in \widetilde{\Gamma}_d,\\
&\nabla\times E(\omega,\mathbf{x}) \times e_2 = 0 \;\; \text{when} \;\; (\overline{L}x,\overline{L}y,\overline{L}z) \in \widetilde{\Gamma}_L.
\end{aligned} \tag{49}
$$

The characteristic thickness of the plate $\overline{L}$ is about 10^{-3}m and the size of the basic cell e is about 10^{-5}m. Since e is much smaller than the thickness of the plate $\overline{L}$, it is pertinent to assume the ratio $\frac{e}{\overline{L}}$ equals a small parameter ε:

$$
\frac{e}{\overline{L}} \sim 10^{-2} = \varepsilon. \tag{50}
$$

Then, in what concerns the characteristic pulsation $\overline{\omega}$, in the tables below we consider several values. The lightning is seen as a low frequency phenomenon. Indeed, energy associated with radiation tracers and return stroke are mainly burn by low and very low frequencies (from 1kHz to 300kHz). Components of the frequency spectrum are however observed beyond 1GHz (see [**?**]). So, in the case when we want to catch low frequency ie we consider $\overline{\omega} = 100$ rad/s, (in our study we will consider $\overline{\omega} = 10^6$rad/s), for medium frequency we set $\overline{\omega} = 10^{10}$ rad/s and for high frequency phenomena $\overline{\omega} = 10^{12}$ rad/s. Then, concerning the characteristic electric conductivity it seems to be reasonable to take for $\overline{\sigma}$ the value of the effective electric conductivity of the composite material. Yet this choice implies to compute a coarse estimate of this effective conductivity at this level.

For this we take into account that the composite material is composed of carbon fibers and epoxy resin. The resin can be doped, which increases strongly its conductivity, or not. The tables below summarize the cases when the resin is doped and also when the resin is not doped. We also account for the fact there is not only one effective electric conductivity but a first one in the fiber direction : the effective longitudinal electric conductivity (in cases 1, 2, 5 and 6 of the tables below), and a second effective electric conductivity, in the direction transverse to the fibers (considered in cases 3, 4, 7 and 8). In this context, we consider the basic model which is based on the electrical analogy and the law of mixtures. It corresponds to the Wiener limits: the harmonic average and the arithmetic average. The effective

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$\overline{L}(m)$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
$e(m)$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$\overline{\lambda}(m)$	10^{6}	10^{6}	10^{6}	10^{6}	10^{6}	10^{6}	10^{6}	10^{6}
$\overline{\sigma}(S.m^{-1})$	40000	40000	10^{-10}	10^{-3}	40000	10^{4}	10^{-10}	10^{-3}
$\overline{\delta}(m)$	0, 1	0, 1	10^{7}	10^{3}	0, 1	0, 1	10^{7}	10^{3}
$\sigma_c(S.m^{-1})$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$
$\sigma_r(S.m^{-1})$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$
$\sigma_a(S.m^{-1})$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^2\overline{\sigma}$	$\varepsilon^6\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^5}$	$\frac{\overline{\sigma}}{\varepsilon^2}$
$\frac{4\pi\overline{L}^2}{\overline{\lambda}^2}$	ε^9	ε^9	ε^9	ε^9	ε^9	ε^9	ε^9	ε^9
$\frac{\overline{L}^2}{\overline{\delta}^2}$	ε^2	ε^2	ε^{10}	ε^7	ε^2	ε^2	ε^{10}	ε^7

Table 1. for $\overline{\omega} = 100 rad.s^{-1}$.

values are the extreme limits of the conductivity of the composite introduced by Wiener in 1912 see S. Berthier p 76 [6].

The effective longitudinal electric conductivity corresponding of the upper Wiener limit is expressed by the equation:

$$\overline{\sigma} = \sigma_{\text{long}} = f_c\,\sigma_c + (1 - f_c)\,\sigma_r, \tag{51}$$

where $f_c = \pi\frac{\alpha^2}{4}$ is the volume fraction of the carbon fiber.

The effective transverse electric conductivity corresponding of the lower Wiener limit is expressed by

$$\overline{\sigma} = \sigma_{\text{trans}} = \frac{1}{\frac{f_c}{\sigma_c} + \frac{(1-f_c)}{\sigma_r}}. \tag{52}$$

For the computation, we take values close to reality. We consider composite materials with similar proportions of carbon and resin, this means that α is close to $\frac{1}{2}$. When the resin is not doped $\sigma_r \sim 10^{-10} S.m^{-1}$ is much smaller than $\sigma_c \sim 40000 S.m^{-1}$. Then, $\overline{\sigma} = \sigma_{\text{long}}$ is close to $\pi\frac{\alpha^2}{4}\,\sigma_c \sim \sigma_c$ and $\overline{\sigma} = \sigma_{\text{trans}}$ is close to $\frac{\sigma_r}{(1-\pi\frac{\alpha^2}{4})} \sim \sigma_r$.

Now, we express the electric conductivity of the air in terms of $\overline{\sigma}$, we consider two possibilities. The first one is relevant for a situation with a ionized channel. The second one of situation with a strong atmospheric electric field but without a ionized channel. In this situation air is not ionized and has a low conductivity. All possible situations are gathered in the tables below. Cases 5 to 8 are associated with the first situation with air conductivity σ_a being $\sigma_{lightning} = 4242 S.m^{-1}$ for an ionized lightning channel see [11]. Cases 1 to 4, to the second one, with $\sigma_a = 10^{-14} S.m^{-1}$.

All calculations of the different cases of the tables are detailed in Annex A. In our study we consider the case 6 for $\omega = 10^6\ rad.s^{-1}$, which corresponds to the air ionized, a resin doped and the effective longitudinal electric conductivity of the carbon fibers.

As it is well known the tables confirm that at high frequencies the thickness of the plate is much greater than the skin depth. This one depends on $\overline{\sigma}$ and $\overline{\omega}$ and decreases strongly for high conductivity

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$\overline{L}(m)$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
$e(m)$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$\overline{\lambda}(m)$	10^{3}	10^{3}	10^{3}	10^{3}	10^{3}	10^{3}	10^{3}	10^{3}
$\overline{\sigma}(S.m^{-1})$	40000	40000	10^{-10}	10^{-3}	40000	40000	10^{-10}	10^{-3}
$\overline{\delta}(m)$	10^{-3}	10^{-3}	10^{5}	10	10^{-3}	10^{-3}	10^{5}	10
$\sigma_c(S.m^{-1})$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^5}$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^5}$
$\sigma_r(S.m^{-1})$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$
$\sigma_a(S.m^{-1})$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^2\overline{\sigma}$	$\varepsilon^6\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^5}$	$\frac{\overline{\sigma}}{\varepsilon^2}$
$\frac{4\pi\overline{L}^2}{\overline{\lambda}^2}$	ε^5	ε^5	ε^5	ε^5	ε^5	ε^5	ε^5	ε^5
$\frac{\overline{L}^2}{\overline{\delta}^2}$	1	1	ε^8	ε^5	1	1	ε^8	ε^5

Table 2. for $\overline{\omega} = 10^6 rad.s^{-1}$.

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$\overline{L}(m)$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
$e(m)$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$\overline{\lambda}(m)$	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}
$\overline{\sigma}(S.m^{-1})$	40000	40000	10^{-10}	10^{-3}	40000	40000	10^{-10}	10^{-3}
$\overline{\delta}(m)$	10^{-5}	10^{-5}	10^{3}	$10^{-1/2}$	10^{-5}	10^{-5}	10^{3}	$10^{-1/2}$
$\sigma_c(S.m^{-1})$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$
$\sigma_r(S.m^{-1})$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$
$\sigma_a(S.m^{-1})$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^2\overline{\sigma}$	$\varepsilon^6\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^5}$	$\frac{\overline{\sigma}}{\varepsilon^2}$
$\frac{4\pi\overline{L}^2}{\overline{\lambda}^2}$	ε	ε	ε	ε	ε	ε	ε	ε
$\frac{\overline{L}^2}{\overline{\delta}^2}$	$\frac{1}{\varepsilon^2}$	$\frac{1}{\varepsilon^2}$	ε^6	ε^3	$\frac{1}{\varepsilon^2}$	$\frac{1}{\varepsilon^2}$	ϵ^6	ϵ^3

Table 3. for $\overline{\omega} = 10^{10} rad.s^{-1}$.

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$\overline{L}(m)$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
$e(m)$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$\overline{\lambda}(m)$	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
$\overline{\sigma}(S.m^{-1})$	40000	40000	10^{-10}	10^{-3}	40000	40000	10^{-10}	10^{-3}
$\overline{\delta}(m)$	10^{-6}	10^{-6}	1	$10^{-3/2}$	10^{-6}	10^{-6}	1	$10^{-3/2}$
$\sigma_c(S.m^{-1})$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$	$\overline{\sigma}$	$\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^7}$	$\frac{\overline{\sigma}}{\varepsilon^4}$
$\sigma_r(S.m^{-1})$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$	$\varepsilon^7\overline{\sigma}$	$\varepsilon^4\overline{\sigma}$	$\overline{\sigma}$	$\overline{\sigma}$
$\sigma_a(S.m^{-1})$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^9\overline{\sigma}$	$\varepsilon^2\overline{\sigma}$	$\varepsilon^6\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\varepsilon\overline{\sigma}$	$\frac{\overline{\sigma}}{\varepsilon^5}$	$\frac{\overline{\sigma}}{\varepsilon^2}$
$\frac{4\pi\overline{L}^2}{\overline{\lambda}^2}$	1	1	1	1	1	1	1	1
$\frac{\overline{L}^2}{\overline{\delta}^2}$	$\frac{1}{\varepsilon^3}$	$\frac{1}{\varepsilon^3}$	ε^3	ε	$\frac{1}{\varepsilon^3}$	$\frac{1}{\varepsilon^3}$	ϵ^3	ε

Table 4. for $\overline{\omega} = 10^{12} rad.s^{-1}$.

or high frequencies. For $\overline{\omega} = 10^{10}$ $rad.s^{-1}$ and $\overline{\sigma} = 4 * 10^4$ $S.m^{-1}$, the effective conductivity in the direction of the carbon fibers, which the skin effect phenomenon appears. Indeed, for high frequencies $\omega = 10^{12} rad.s^{-1}$ and when $\overline{\sigma}$ is the effective conductivity is in direction of the carbon fibers *i.e.* in high conductivity, $\overline{\delta} = 10^{-4}$ m. In low frequencies and low conductivity δ is large so the electromagnetic wave can penetrate the composite material. The high conductivity limits the penetration of the electromagnetic wave to a boundary layer whose depth is about $\overline{\delta}$.

Now, we will discuss on the values of $\overline{E}$ and $\overline{\rho}$. It seems that the density of electrons in a ionized channel is about 10^{10} part.m^{-3}. Hence we take $\overline{\rho} = 10^{10}$. When the air is not ionized, the charge density is much smaller, and we choose: $\overline{\rho} = 1$.

For the boundary conditions, in the context of the case 6 and $\overline{\omega} = 10^6$ rad/s, we consider the peak of the current of the first return stroke. Then the magnetic field magnitude $\overline{H}$ is $\overline{H_d}$ given by (1).

Then the dimensionless boundary conditions (38) writes:

$$\nabla \times E(\mathbf{x}, \omega) \times e_2 = -i\omega\overline{\omega}\mu_0 \frac{\overline{L}}{\overline{E}} \overline{H_d} H_d(x, z) \times e_2, \tag{53}$$

where $\overline{H_d} H_d(x, z) = \widetilde{H_d}(\overline{L}x, \overline{L}z)$ and where $\overline{\omega}\mu_0 \frac{\overline{L}}{\overline{E}} \overline{H_d}$ being of order 1, with the characteristic electric field $\overline{E} = 20$ kV/m.

From the physical spatial coordinates $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \widetilde{\Omega}$ we define $\mathbf{y} = (\xi, \nu, \zeta)$ with $\xi = \frac{\widetilde{x}}{e}, \nu = \frac{\widetilde{y}}{e}, \zeta = \frac{\widetilde{z}}{e}$ or equivalently $\xi = \frac{x}{\varepsilon}, \nu = \frac{y}{\varepsilon}, \zeta = \frac{z}{\varepsilon}$. And we now introduce Y, the basic cell. It is built from: $Z = [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0] \times \mathbb{R}$ and the set $C = D \times \mathbb{R}$ with the disc D defined by:

$$D = \{(\xi, \nu) \in \mathbb{R}^2 \,/ \xi^2 + (\nu + \frac{1}{2})^2 < R^2\}, \tag{54}$$

and $R = \frac{\alpha}{2}$. The set Ω_c is then defined as:

$$\Omega_c = \{(i, j, 0) + C, i \in \mathbb{Z}, j \in \mathbb{Z}^-\}. \tag{55}$$

We denote Y_r as $Y_r = Z\backslash C$ and then the set

$$\Omega_r = \{(i, j, 0) + Y_r, i \in \mathbb{Z}, j \in \mathbb{Z}^-\}. \tag{56}$$

Then unit cell Y is defined as $Y = (Y_r, C)$. Finally, we define the domain Ω_a:

$$\Omega_a = \{\mathbf{y} = (\xi, \nu, \zeta) \ / \ \nu > 0\}. \tag{57}$$

Using this, we will give a new expression of the sets in which the variables range in equations (48). We see the following:

$$(\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_r \Leftrightarrow \begin{cases} (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{P}, \\ (\frac{\overline{L}}{e}x, \frac{\overline{L}}{e}y, \frac{\overline{L}}{e}z) \in \Omega_r, \end{cases} \tag{58}$$

i.e.

$$(\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_r \Leftrightarrow \begin{cases} (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{P}, \\ (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}) \in \Omega_r. \end{cases} \tag{59}$$

In the same way:

$$(\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_c \Leftrightarrow \begin{cases} (\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{P}, \\ (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}) \in \Omega_c, \end{cases} \tag{60}$$

and:

$$0 \le \overline{L}y \le d \Leftrightarrow \begin{cases} y \in \mathbb{R}^2 \\ \overline{L}y \le d, \end{cases} \tag{61}$$

or

$$(\overline{L}x, \overline{L}y, \overline{L}z) \in \widetilde{\Omega}_a \Leftrightarrow \begin{cases} \overline{L}y \le d \\ (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}) \in \Omega_a. \end{cases} \tag{62}$$

We define:

$$\Sigma^\varepsilon(\mathbf{y}) = \Sigma^\varepsilon(\xi, \nu, \zeta) = \begin{cases} \Sigma_a^\varepsilon & \text{in } \Omega_a, \\ \Sigma_r^\varepsilon & \text{in } \Omega_r, \\ \Sigma_c^\varepsilon & \text{in } \Omega_c, \end{cases} \tag{63}$$

where $\Sigma_a^\varepsilon = \frac{\sigma_a}{\overline{\sigma}} \frac{\overline{L}^2}{\overline{\delta}^2}, \Sigma_r^\varepsilon = \frac{\sigma_r}{\overline{\sigma}} \frac{\overline{L}^2}{\overline{\delta}^2}$ and $\Sigma_c^\varepsilon = \frac{\sigma_c}{\overline{\sigma}} \frac{\overline{L}^2}{\overline{\delta}^2}$ have their expressions in term of ε given from Tables above depending on the case we are interested in. The detail of this expressions are in appendix B. The model that we present is the case $\omega = 10^6\ rad.s^{-1}$, $\eta = 5$, $\Sigma_a^\varepsilon = \varepsilon$, $\Sigma_r^\varepsilon = \varepsilon^4$ and $\Sigma_c^\varepsilon = 1$.

Defining also mapping

$$\begin{aligned} \psi_\varepsilon :\ & \mathbb{R}^3 \to \mathbb{R}^3 \\ & (x,y,z) \mapsto (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}), \end{aligned} \tag{64}$$

we can set Ω_a^ε as $\psi_\varepsilon^{-1}(\Omega_a) \cap (\mathbb{R}\times[0,\frac{d}{L}]\times\mathbb{R})$, Ω_r^ε as $\psi_\varepsilon^{-1}(\Omega_r)\cap\widetilde{P}$ and Ω_c^ε as $\psi_\varepsilon^{-1}(\Omega_c)\cap\widetilde{P}$. We also define the boundaries $\Gamma_d = \{\mathbf{x}\in\mathbb{R}^3,\ \ y=\frac{d}{L}\}$ and $\Gamma_L = \{\mathbf{x}\in\mathbb{R}^3,\ \ y=-\overline{L}\}$ and interfaces $\Gamma_{ra} = \{\mathbf{x}\in\mathbb{R}^3,\ \ y=0\}$ and $\Gamma_{cr}^\varepsilon = \partial\Omega_c$. Hence equation (48) reads:

$$\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon^\star + i\ \omega\ \sigma^\varepsilon(x,y,z))E^\varepsilon = 0 \quad \text{in} \ \ \Omega, \tag{65}$$

where $\Omega = \Omega_a^\varepsilon \cup \Omega_r^\varepsilon \cup \Omega_c^\varepsilon = \{\mathbf{x}\in\mathbb{R}^3, -1 < y < \frac{d}{L}\}$ does not depend on ε. Only its partition in Ω_a^ε, Ω_r^ε and Ω_c^ε is ε-dependent where

$$\sigma^\varepsilon(x,y,z) = \Sigma^\varepsilon(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}), \tag{66}$$

with Σ^ε given by (63) and

$$\varepsilon^\eta = \frac{4\pi^2\overline{L}^2}{\lambda^2}, \tag{67}$$

with the value of $\eta \geq 0$ extracted from Tables, and where we replace E by E^ε, to clearly state that it depends on ε.

Equation (65) is provided with the following boundary conditions:

$$\nabla\times E^\varepsilon \times e_2 = -i\omega H_d(x,z)\times e_2 \text{ on } \Gamma_d, \tag{68}$$

coming from (53). And, coming from (49),

$$\nabla\times E^\varepsilon \times e_2 = 0 \text{ on } \Gamma_L. \tag{69}$$

From (65) we can deduce the condition on the divergence of E^ε which can be written in two ways. As previously in (34), (35) and (36) we obtain:

$$\nabla\cdot[(-\omega^2\varepsilon^\eta\epsilon^\star + i\omega\sigma^\varepsilon)E^\varepsilon] = 0 \ \text{ in } \ \Omega, \tag{70}$$

which will be preferentially used with (65) and its second one is

$$\nabla\cdot E^\varepsilon_{|\Omega_a^\varepsilon} = 0 \ \text{ in } \ \Omega_a^\varepsilon, \ \ \nabla\cdot E^\varepsilon_{|\Omega_r^\varepsilon} = 0 \ \text{ in } \ \Omega_r^\varepsilon, \ \ \nabla\cdot E^\varepsilon_{|\Omega_c^\varepsilon} = 0 \ \text{ in } \ \Omega_c^\varepsilon, \tag{71}$$

with the transmission conditions on the interfaces Γ_{ra} and Γ_{cr}^ε

$$\begin{aligned} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)\ E^\varepsilon_{|\Omega_a^\varepsilon}\cdot n_{|\Omega_a^\varepsilon} &= (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)\ E^\varepsilon_{|\Omega_r^\varepsilon}\cdot n_{|\Omega_r^\varepsilon} \ \text{ on } \ \Gamma_{ra}, \\ (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)\ E^\varepsilon_{|\Omega_r^\varepsilon}\cdot n_{|\Omega_r^\varepsilon} &= (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon)\ E^\varepsilon_{|\Omega_c^\varepsilon}\cdot n_{|\Omega_c^\varepsilon} \ \text{ on } \ \Gamma_{cr}^\varepsilon. \end{aligned} \tag{72}$$

Before treating mathematically the question we are interested in, we make a last simplification. Since it seems clear that physical relevant phenomena occur in the upper part of the plate. The boundary condition on the lower boundary of the plate has very little influence on the physics of what happens in the upper part, we consider that the lower boundary of Ω is located in $y = -\infty$ in place of $y = -1$, making the second boundary condition useless. Besides, as $\overline{L}$ and d are of the same order it seems reasonable to set $\Gamma_d = \{\mathbf{x} \in \mathbb{R}^3,\ \ y = 1\}$ and consequently

$$\begin{cases} \Omega = \{\mathbf{x} \in \mathbb{R}^3, y < 1\} = \Omega_a^\varepsilon \cup \Omega_r^\varepsilon \cup \Omega_c^\varepsilon, \quad \text{with,} \\ \Omega_a^\varepsilon = \psi_\varepsilon^{-1}(\Omega_a), \\ \Omega_r^\varepsilon = \psi_\varepsilon^{-1}(\Omega_r), \\ \Omega_c^\varepsilon = \psi_\varepsilon^{-1}(\Omega_c), \end{cases} \tag{73}$$

with ψ_ε defined in (64). We have that the border of Ω is Γ_d. In the following section we will establish existence and uniqueness results.

3. Mathematical analysis of the models

3.1. Preliminaries

We are going to make precise the variational formulation. First of all, we need to introduce the following functional spaces. We have the standard function spaces $\mathbf{L}^2(\Omega^\varepsilon) = [L^2(\Omega^\varepsilon)]^3$

$$\begin{aligned} \mathbf{H}(\mathrm{curl}, \Omega) &= \{u \in \mathbf{L}^2(\Omega) : \ \ \nabla{\times}u \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}(\mathrm{div}, \Omega) &= \{u \in \mathbf{L}^2(\Omega) : \ \ \nabla{\cdot}u \in L^2(\Omega)\}, \end{aligned} \tag{74}$$

with the usual norms:

$$\begin{aligned} ||u||^2_{\mathbf{H}(\mathrm{curl},\Omega)} &= ||u||^2_{\mathbf{L}^2(\Omega)} + ||\nabla{\times}u||^2_{\mathbf{L}^2(\Omega)}, \\ ||u||^2_{\mathbf{H}(\mathrm{div},\Omega)} &= ||u||^2_{\mathbf{L}^2(\Omega)} + ||\nabla{\cdot}u||^2_{L^2(\Omega)}. \end{aligned} \tag{75}$$

They are well known Hilbert spaces.

We use in this paper, the trace spaces $H^{-\frac{1}{2}}(\mathrm{curl}, \Gamma_d)$ and $H^{-\frac{1}{2}}(\mathrm{div}, \Gamma_d)$ defined by

$$H^{-\frac{1}{2}}(\mathrm{curl}, \Gamma_d) = \{u \in H^{-\frac{1}{2}}(\Gamma_d, \mathbb{R}^3),\ (n \cdot u)_{|\Gamma_d} = 0,\ \mathrm{curl}_{\Gamma_d} u \in H^{-\frac{1}{2}}(\Gamma_d, \mathbb{R}^3)\}, \tag{76}$$

$$H^{-\frac{1}{2}}(\mathrm{div}, \Gamma_d) = \{u \in H^{-\frac{1}{2}}(\Gamma_d, \mathbb{R}^3),\ (n \cdot u)_{|\Gamma_d} = 0,\ \mathrm{div}_{\Gamma_d} u \in H^{-\frac{1}{2}}(\Gamma_d, \mathbb{R}^3)\} \tag{77}$$

where the surface divergence $\mathrm{div}_{\Gamma_d} u$ and the surface rotation $\mathrm{curl}_{\Gamma_d} u$ are defined by

$$\begin{aligned} &(\mathrm{div}_{\Gamma_d} u, V)_{L^2(\Gamma_d)} = -(u, \nabla_{\Gamma_d} V)_{L^2(\Gamma_d, \mathbb{R}^3)}, \quad \forall\ V \in C^1(\Gamma_d) \\ &\mathrm{curl}_{\Gamma_d} u = n \cdot (\nabla \times u_{|\Gamma_d}) \end{aligned} \tag{78}$$

and the surface gradient $\nabla_{\Gamma_d} V$ is defined by the orthogonal projection of ∇ on Γ_d, n denotes the outward unit vector normal to Γ_d.

Finally we recall the trace theorems, see J.C Nédélec [13] for the demonstration, stating that the traces mappings

$\gamma_T\ :\ \mathbf{H}(\mathrm{curl},\Omega)\ \longrightarrow\ H^{-\frac{1}{2}}(\mathrm{curl},\Gamma_d)$, that assigns any $u\in\mathbf{H}(\mathrm{curl},\Omega)$ its tangential components $n\times(u\times n)$, is continuous and surjective, that is:

$$\|\gamma_T(u)\|_{H^{-\frac{1}{2}}(\mathrm{curl},\Gamma_d)}\leq C_{\gamma_T}\|u\|_{\mathbf{H}(\mathrm{curl},\Omega)},\quad \forall u\in\mathbf{H}(\mathrm{curl},\Omega)$$

$\gamma_t\ :\ \mathbf{H}(\mathrm{curl},\Omega)\ \longrightarrow\ H^{-\frac{1}{2}}(\mathrm{div},\Gamma_d)$, that assigns any $u\in\mathbf{H}(\mathrm{curl},\Omega)$ its tangential components $u\times n$, is continuous and surjective:

$$\|\gamma_t(u)\|_{H^{-\frac{1}{2}}(\mathrm{div},\Gamma_d)}\leq C_{\gamma_t}\|u\|_{\mathbf{H}(\mathrm{curl},\Omega)},\quad \forall u\in\mathbf{H}(\mathrm{curl},\Omega).$$

Moreover, $H^{-\frac{1}{2}}(\mathrm{div},\Gamma_d)$ is the dual of $H^{-\frac{1}{2}}(\mathrm{curl},\Gamma_d)$ and one has the Green's formula:

$$\int_\Omega(\nabla\times u\cdot V-u\cdot\nabla\times V)dx=\langle u\times n,V_T\rangle_{\Gamma_d}\ \forall(u,V)\in\mathbf{H}(\mathrm{curl},\Omega). \tag{79}$$

We define the next space:

$$\mathbf{X}(\Omega)=\{u\in\mathbf{H}(\mathrm{curl},\Omega)\mid\nabla\cdot u_{|\Omega_a^\varepsilon}\in L^2(\Omega_a^\varepsilon),\nabla\cdot u_{|\Omega_r^\varepsilon}\in L^2(\Omega_r^\varepsilon),\ \nabla\cdot u_{|\Omega_c^\varepsilon}\in L^2(\Omega_c^\varepsilon)\}. \tag{80}$$

Our variational space is:

$$\begin{aligned}\mathbf{X}^\varepsilon(\Omega)=\{u\in\mathbf{X}(\Omega)\mid(-\omega^2\varepsilon^\eta+i\omega\sigma^\varepsilon_{|\Omega_a^\varepsilon})u_{|\Omega_a^\varepsilon}\cdot e_2=(-\omega^2\varepsilon^\eta\varepsilon_r+i\omega\sigma^\varepsilon_{|\Omega_r^\varepsilon})u_{|\Omega_r^\varepsilon}\cdot e_2,\\(-\omega^2\varepsilon^\eta\epsilon_r+i\omega\sigma^\varepsilon_{|\Omega_r^\varepsilon})u_{|\Omega_r^\varepsilon}\cdot n^\varepsilon_{|\Omega_r^\varepsilon}=(-\omega^2\varepsilon^\eta\varepsilon_c+i\omega\sigma^\varepsilon_{|\Omega_c^\varepsilon})u_{|\Omega_c^\varepsilon}\cdot n^\varepsilon_{|\Omega_c^\varepsilon}.\end{aligned} \tag{81}$$

Finally

$$\begin{aligned}\mathbf{X}^\varepsilon(\Omega)=\{u\in\mathbf{X}(\Omega)\mid(-\omega^2\varepsilon^\eta+i\omega\Sigma^\varepsilon_a)u_{|\Omega_a^\varepsilon}\cdot e_2=(-\omega^2\varepsilon^\eta\varepsilon_r+i\omega\Sigma^\varepsilon_r)u_{|\Omega_r^\varepsilon}\cdot e_2,\\(-\omega^2\varepsilon^\eta\epsilon_r+i\omega\Sigma^\varepsilon_r)u_{|\Omega_r^\varepsilon}\cdot n^\varepsilon_{|\Omega_r^\varepsilon}=(-\omega^2\varepsilon^\eta\varepsilon_c+i\omega\Sigma^\varepsilon_c)u_{|\Omega_c^\varepsilon}\cdot n^\varepsilon_{|\Omega_c^\varepsilon}\}.\end{aligned} \tag{82}$$

This space is equipped with the norm

$\|u\|^2_{\mathbf{X}^\varepsilon(\Omega)}=\|u\|^2_{\mathbf{L}^2(\Omega)}+\|\nabla\cdot u_{|\Omega_a^\varepsilon}\|^2_{L^2(\Omega_a^\varepsilon)}+\|\nabla\cdot u_{|\Omega_r^\varepsilon}\|^2_{L^2(\Omega_r^\varepsilon)}+\|\nabla\cdot u_{|\Omega_c^\varepsilon}\|^2_{L^2(\Omega_c^\varepsilon)}+\|\nabla\times u\|^2_{\mathbf{L}^2(\Omega)}.$

3.2. Weak formulation

Now, we introduce the variational formulation of our problem (65), (68) and (69) for the electric field. Integrating (65) over Ω and using the Green's formula and (68) we obtain

$$\begin{aligned}
&\int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\, d\mathbf{x} + \int_{\Omega_a^\varepsilon} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} \\
&+ \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} + \int_{\Omega_r^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} \\
&= \int_{\Gamma_d} (\nabla\times E^\varepsilon \times e_2) \cdot \overline{V}_T\, d\sigma \\
&= \int_{\Gamma_d} -i\omega H_d \times e_2 \cdot \overline{V}_T\, d\sigma
\end{aligned} \tag{83}$$

where $\overline{V}$ is the complex conjugate of V and $V_T = (e_2 \times V) \times e_2$. We introduce the sesquilinear form depending on parameters η and ε:

$$\begin{cases}
\text{For } E^\varepsilon, V \in \mathbf{X}^\varepsilon(\Omega), \\
a^{\varepsilon,\eta}(E^\varepsilon, V) = \displaystyle\int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\, d\mathbf{x} + \sum_{i=a,r,c} \int_{\Omega_i^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_i + i\omega\Sigma_i^\varepsilon)\, E^\varepsilon \cdot \overline{V}\, d\mathbf{x}.
\end{cases} \tag{84}$$

Hence, the weak formulation of (65), (68) and (69) that we will use is the following:

$$\begin{cases}
\text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \text{ such as } \forall\ V \in \mathbf{X}^\varepsilon(\Omega) \text{ we have :} \\
a^{\varepsilon,\eta}(E^\varepsilon, V) = -i\omega \displaystyle\int_{\Gamma_d} H_d \times e_2 \cdot \overline{V}_T\, d\sigma.
\end{cases} \tag{85}$$

Integrating by parts in the variational formulation (83), we find the following transmission problem:

$$\begin{cases}
\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta + i\,\omega\,\Sigma_a^\varepsilon)E^\varepsilon = 0 & \text{in } \Omega_a^\varepsilon, \\
\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon_r + i\,\omega\,\Sigma_r^\varepsilon)E^\varepsilon = 0 & \text{in } \Omega_r^\varepsilon, \\
\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon_c + i\,\omega\,\Sigma_c^\varepsilon)E^\varepsilon = 0 & \text{in } \Omega_c^\varepsilon. \\
E^\varepsilon_{|\Omega_a^\varepsilon} \times e_2 = E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon} \text{ on } \Gamma_{ra}, \\
E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon} = E^\varepsilon_{|\Omega_c^\varepsilon} \times n_{|\Omega_c^\varepsilon} \text{ on } \Gamma_{cr}^\varepsilon, \\
\nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \times e_2 = \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon} \text{ on } \Gamma_{ra}, \\
\nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon} = \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \times n_{|\Omega_c^\varepsilon} \text{ on } \Gamma_{cr}^\varepsilon,
\end{cases} \tag{86}$$

where e_2 is the unit outward normal to Ω_a^ε, $n_{|\Omega_r^\varepsilon}$ is the unit outward normal to Ω_r^ε and $n_{|\Omega_c^\varepsilon}$ is the unit outward normal to Ω_c^ε. We refer to Annex C for the proof that transmission problem (86) is equivalent to ((65), (68), (69), (71)).

3.3. Regularized Maxwell equations for the electric field

The sesquilinear form $a^{\varepsilon,\eta}$ is not coercive on $\mathbf{X}^\varepsilon(\Omega)$, so we regularize it adding terms involving the divergence of E^ε in Ω_a^ε, Ω_r^ε and Ω_c^ε. Thanks to the additional terms, existence and uniqueness of the regularized variational formulation solution will be established by the Lax-Milgram theory. Let s be an arbitrary positive number, we define the regularized formulation of problem (85):

$$\begin{cases} \text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \text{ such that for any } V \in \mathbf{X}^\varepsilon(\Omega) \\ a_R^{\varepsilon,\eta}(E^\varepsilon, V) = a^{\varepsilon,\eta}(E^\varepsilon, V) + s\displaystyle\int_{\Omega_a^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x} \\ + s\displaystyle\int_{\Omega_r^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x} + s\displaystyle\int_{\Omega_c^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x} \\ = -i\omega \displaystyle\int_{\Gamma_d} H_d \times e_2 \cdot \overline{V}_T\, d\sigma. \end{cases} \tag{87}$$

For any $\varepsilon > 0$ and any $\eta \geq 0$, sesquilinear form $a_R^{\varepsilon,\eta}(.,.)$ is continuous over $\mathbf{X}^\varepsilon(\Omega)$ thanks to the continuity conditions. We will show that it is also coercive. The following proposition was inspired by article [8] Lemma 1.1.

Proposition 3.1. *For any $\varepsilon > 0$, for any $\eta \geq 0$ and for any $s > 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, there exists a positive constant C_0 depending on $\varepsilon_r, \varepsilon_c, s, \omega$ but not on ε such that:*

$$\forall\, E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega),\ \ \Re[\exp(-i\frac{\pi}{4})\, a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)] \geq C_0 \|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)} \tag{88}$$

Proof. We have:

$$\begin{aligned} \Re[\exp(-i\frac{\pi}{4})\, a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)] &= a_R^\varepsilon(E^\varepsilon, E^\varepsilon) - \int_{\Omega_a^\varepsilon} \omega^2 \varepsilon^\eta |E^\varepsilon|^2\, d\mathbf{x} \\ - \int_{\Omega_r^\varepsilon} \omega^2 \varepsilon^\eta \epsilon_r |E^\varepsilon|^2\, d\mathbf{x} &- \int_{\Omega_c^\varepsilon} \omega^2 \varepsilon^\eta \epsilon_c |E^\varepsilon|^2\, d\mathbf{x}. \end{aligned} \tag{89}$$

with

$$\begin{aligned} a_R^\varepsilon(E^\varepsilon, E^\varepsilon) = &\int_\Omega |\nabla\times E^\varepsilon|^2\, d\mathbf{x} + s\int_{\Omega_a^\varepsilon} |\nabla\cdot E^\varepsilon|^2\, d\mathbf{x} \\ &+ s\int_{\Omega_r^\varepsilon} |\nabla\cdot E^\varepsilon|^2\, d\mathbf{x} + s\int_{\Omega_c^\varepsilon} |\nabla\cdot E^\varepsilon|^2\, d\mathbf{x} \\ &+ \int_{\Omega_a^\varepsilon} \omega\Sigma_a^\varepsilon |E^\varepsilon|^2\, d\mathbf{x} + \int_{\Omega_r^\varepsilon} \omega\Sigma_r^\varepsilon |E^\varepsilon|^2\, d\mathbf{x} \\ &+ \int_{\Omega_c^\varepsilon} \omega\Sigma_c^\varepsilon |E^\varepsilon|^2\, d\mathbf{x}. \end{aligned} \tag{90}$$

We have the following estimate:

$$\begin{aligned} |a_R^\varepsilon(E^\varepsilon, E^\varepsilon)| \geq \min\{1, \omega, s\}(&\|\nabla\times E^\varepsilon\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\cdot E^\varepsilon\|^2_{L^2(\Omega_a^\varepsilon)} + \|\nabla\cdot E^\varepsilon\|^2_{L^2(\Omega_r^\varepsilon)} \\ &+ \|\nabla\cdot E^\varepsilon\|^2_{L^2(\Omega_c^\varepsilon)} + \|E^\varepsilon\|^2_{\mathbf{L}^2(\Omega)}). \end{aligned} \tag{91}$$

Then we have:

$$|\, a_R^\varepsilon(E^\varepsilon, E^\varepsilon)| \geq \min\{1, \omega, s\} \|E^\varepsilon\|^2_{\mathbf{X}^\varepsilon(\Omega)}. \tag{92}$$

Returning to formulation (88), for $\eta \geq 0$, since $\max(\Sigma_a^\varepsilon, \Sigma_r^\varepsilon, \Sigma_c^\varepsilon) > \varepsilon^\eta$, inequality (88) is valid with $C_0 = \min\{1, \omega, s\}$ as soon as $\omega^2 \min\{1, \varepsilon_r, \varepsilon_c\} < \min\{1, \omega, s\}$ or $\omega < \sqrt{\frac{\min\{1,\omega,s\}}{\min\{1,\varepsilon_r,\varepsilon_c\}}}$. This ends the proof of Proposition 3.1. □

Thanks to Proposition 3.1 we can state the existence and uniqueness of the solution to regularized problem (87).

Theorem 3.2. *Under the assumptions of Proposition 3.1, there exists a unique solution E^ε to regularized problem (87).*

Proof. The sesquilinear form $a_R^{\varepsilon,\eta}$ is continuous, bounded, coercive thanks to Proposition 3.1 and the right hand side is continuous on $X^\varepsilon(\Omega)$, then problem (87) has a unique solution in $X^\varepsilon(\Omega)$ thanks to the Lax-Milgram Lemma. □

3.4. Existence, uniqueness and estimate

Theorem 3.3. *For any $\varepsilon > 0$, for any $\eta \geq 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, there exists a unique solution of (86) or ((65), (68), (69), (71)).*

Proof. We show that for an appropriate choice of s that E^ε satisfies all equations (86) or ((65), (68), (69), (71)). It is obvious that any solution of (86) or of ((65), (68), (69),(71)) is also solution to (87). Indeed, since from (86) or from ((65), (68), (69),(71)) we have $\nabla\cdot E^\varepsilon_{|\Omega_a^\varepsilon} = 0$, $\nabla\cdot E^\varepsilon_{|\Omega_r^\varepsilon} = 0$, $\nabla\cdot E^\varepsilon_{|\Omega_c^\varepsilon} = 0$, the additional terms $s\int_{\Omega_a^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x} + s\int_{\Omega_r^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x} + s\int_{\Omega_c^\varepsilon} \nabla\cdot E^\varepsilon \nabla\cdot\overline{V}\, d\mathbf{x}$ cancel in (87).

Uniqueness follows from that if E_1^ε and E_2^ε are two solutions to (65) with the boundary condition (69) their difference $e^\varepsilon = E_2^\varepsilon - E_1^\varepsilon$ satisfies the problem (65) with (69). Then it comes

$$\begin{aligned}
&\int_\Omega |\nabla\times e^\varepsilon|^2\, d\mathbf{x} + \int_{\Omega_a^\varepsilon} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon)|e^\varepsilon|^2\, d\mathbf{x} \\
&\qquad + \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon)|e^\varepsilon|^2\, d\mathbf{x} + \int_{\Omega_r^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_r + i\omega\Sigma_r^\varepsilon)|e^\varepsilon|^2\, d\mathbf{x} \\
&\qquad = 0.
\end{aligned} \tag{93}$$

Taking the imaginary part of the expression we get $\int_{\Omega_a^\varepsilon} \omega\Sigma_a^\varepsilon|e^\varepsilon|^2\, d\mathbf{x} + \int_{\Omega_c^\varepsilon} \omega\Sigma_c^\varepsilon|e^\varepsilon|^2\, d\mathbf{x} + \int_{\Omega_r^\varepsilon} \omega\Sigma_r^\varepsilon|e^\varepsilon|^2\, d\mathbf{x} = 0$ and then $e^\varepsilon = 0$.

Let us consider the reciprocal assertion, according to the same proof of S. Hassani, S. Nicaise, A. Maghnouji in [17], we define $H_0^1(\Omega_c^\varepsilon, \Delta)$ the subspace of $\psi \in H_0^1(\Omega_c^\varepsilon)$ such that $\Delta(\psi) \in L^2(\Omega_c^\varepsilon)$.

Let E_s^ε be the solution of the regularized formulation (87). In (87) we take a test function $V = \nabla\psi$ where $\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$, extended by zero outside Ω_c^ε. We get:

$$\int_{\Omega_c^\varepsilon} s\nabla\cdot E_s^\varepsilon \nabla\cdot(\nabla\psi)\, d\mathbf{x} + \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta\epsilon_c + i\omega\Sigma_c^\varepsilon) E_s^\varepsilon \cdot \nabla\psi\, d\mathbf{x} = 0. \tag{94}$$

By Green's formula, $\forall\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$, we obtain:

$$\int_{\Omega_c^\varepsilon} \nabla\cdot E_s^\varepsilon (\Delta\psi + \frac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}\, \psi)\; d\mathbf{x} = 0. \tag{95}$$

Thus, if we choose s such that $\frac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_c^\varepsilon)$: the Laplacian operator in Ω_c^ε with Dirichlet condition on its boundary, then for all $\varphi \in L(\Omega_c^\varepsilon)^2$ there exists $\psi \in H_0^1(\Omega_c^\varepsilon, \Delta)$ solution of

$$\Delta\psi + \frac{\omega^2\varepsilon^\eta\epsilon_c - i\omega\Sigma_c^\varepsilon}{s}\,\psi = \varphi. \tag{96}$$

Then, we conclude that

$$\nabla \cdot E^\varepsilon_{s|\Omega_c^\varepsilon} = 0. \tag{97}$$

A similar argument in Ω_a^ε yields $\nabla\cdot E^\varepsilon_{s|\Omega_a^\varepsilon} = 0$ for s such that $\frac{\omega^2\varepsilon^\eta - i\omega\Sigma_a^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_a^\varepsilon)$. In the same way, we obtain in Ω_r^ε, $\nabla\cdot E^\varepsilon_{s|\Omega_r^\varepsilon} = 0$ with s such that $\frac{\omega^2\varepsilon^\eta\epsilon_r - i\omega\Sigma_r^\varepsilon}{s}$ is not an eigenvalue of $(\Delta_{dir}, \Omega_r^\varepsilon)$.

Hence $\nabla\cdot E_s^\varepsilon = 0$ in Ω_c^ε, this cancels the additional term $s\int_{\Omega_c^\varepsilon} \nabla\cdot E_s^\varepsilon \nabla\cdot\overline{V}\ d\mathbf{x}$ in (87). In the same way, $\nabla\cdot E_s^\varepsilon = 0$ in Ω_r^ε and $\nabla\cdot E_s^\varepsilon = 0$ in Ω_a^ε cancel $s\int_{\Omega_r^\varepsilon} \nabla\cdot E_s^\varepsilon \nabla\cdot\overline{V}\ d\mathbf{x}$ and $s\int_{\Omega_a^\varepsilon} \nabla\cdot E_s^\varepsilon \nabla\cdot\overline{V}\ d\mathbf{x}$ in (87). So, (87) becomes (83). Applying Green's formula, we find (65). □

Theorem 3.4. *Under the assumptions of Theorem 3.2, $E^\varepsilon \in X^\varepsilon(\Omega)$ solution of (87) satisfies*

$$\|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)} \leq C \tag{98}$$

with $C = \frac{C_{\gamma_t} C_{\gamma_T}}{C_0} \|H_d\|_{H(curl,\Omega)}$

Proof. The sesquilinear form $a_R^{\varepsilon,\eta}(E^\varepsilon, V)$ is coercive, weak formulation (87) becomes:

$$\begin{aligned}
C_0\|E^\varepsilon\|^2_{\mathbf{X}^\varepsilon(\Omega)} &\leq \Re(\exp(-i\frac{\pi}{4}) a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)) \\
&\leq |\exp(-i\frac{\pi}{4}) \cdot a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)| = |a_R^{\varepsilon,\eta}(E^\varepsilon, E^\varepsilon)| \\
&\leq |\int_{\Gamma_d} -i\omega H_d \times e_2 \cdot E_T^\varepsilon\ \ d\sigma| \\
&\leq \|H_d \times e_2\|_{H^{-\frac{1}{2}}(\mathrm{div},\Gamma_d)} \|E_T^\varepsilon\|_{H^{-\frac{1}{2}}(\mathrm{curl},\Gamma_d)} \\
&\leq C_{\gamma_t} C_{\gamma_T} \|H_d \times e_2\|_{\mathbf{H}(\mathrm{curl},\Omega)} \|E^\varepsilon\|_{\mathbf{H}(\mathrm{curl},\Omega)}
\end{aligned} \tag{99}$$

where $E_T^\varepsilon = e_2 \times (E^\varepsilon \times e_2)$ and the continuous dependence of the trace norm with $C = \frac{C_{\gamma_t} C_{\gamma_T}}{C_0} \|H_d\|_{\mathbf{H}(curl,\Omega)}$ gives:

$$\|E^\varepsilon\|^2_{\mathbf{X}^\varepsilon(\Omega)} \leq C\|E^\varepsilon\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C\|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)}. \tag{100}$$

□

4. Homogenization

With the aim to obtain a convergence result for the problem (65), (68) and (69), we propose an approach based on two-scale convergence. This concept was introduced by G. Nguetseng [15], [16] and specified by G. Allaire [2], [3] which studied properties of the two-scale convergence. M. Neuss-Radu in [14] presented an extension of two-scale convergence method to the periodic surfaces. Many authors applied two-scale convergence approach D. Cionarescu and P. Donato [7], N. Crouseilles, E. Frénod, S. Hirstoaga and A. Mouton [9], Y. Amirat, K. Hamdache and A. Ziani [1] and also A. Back, E. Frénod [5]. This mathematical concept were applied to homogenize the time-harmonic Maxwell equations S. Ouchetto, O. Zouhdi and A. Bossavit [18], H.E. Pak[19].

In our model, the parallel carbon cylinders are periodically distributed in direction x and z, as the material is homogenous in the y direction, we can consider that the material is periodic with a three directional cell of periodicity. In other words, introducing $\mathcal{Z} = [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]^2$, function Σ^ε given by (63) is naturally periodic with respect to (ξ, ζ) with period $[-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]$ but it is also periodic with respect to $\mathbf{y}$ with period $\mathcal{Z}$.

Now, we review some basis definitions and results about two-scale convergence.

4.1. Two-scale convergence

We first define the function spaces

$$\begin{cases} \mathbf{H}_\#(\mathrm{curl}, \mathcal{Z}) = \{u \in \mathbf{H}(\mathrm{curl}, \mathbb{R}^3) : \quad u \ \text{ is } \mathcal{Z}\text{-periodic}\} \\ \mathbf{H}_\#(\mathrm{div}, \mathcal{Z}) = \{u \in \mathbf{H}(\mathrm{div}, \mathbb{R}^3) : \quad u \ \text{ is } \mathcal{Z}\text{-periodic}\} \end{cases} \tag{101}$$

and where $\mathbf{H}(\mathrm{curl}, \mathbb{R}^3)$ and $\mathbf{H}(\mathrm{div}, \mathbb{R}^3)$ are defined by (74) with Ω^ε replaced by $\mathbb{R}^3$. We introduce

$$\mathbf{L}^2_\#(\mathcal{Z}) = \{u \in \mathbf{L}^2(\mathbb{R}^3), u \text{ is } \mathcal{Z}\text{-periodic}\}, \tag{102}$$

and

$$\mathbf{H}^1_\#(\mathcal{Z}) = \{u \in \mathbf{H}^1(\mathbb{R}^3), u \text{ is } \mathcal{Z}\text{-periodic}\}, \tag{103}$$

where $\mathbf{H}^1(\mathbb{R}^3)$ is the usual Sobolev space on $\mathbb{R}^3$. First, denoting by $\mathbf{C}^0_\#(\mathcal{Z})$ the space of functions in $\mathbf{C}^0(\mathbb{R}^3)$ and $\mathcal{Z}$-periodic, $\mathbf{C}^0_0(\mathbb{R}^3)$ the space of continuous functions over $\mathbb{R}^3$ with compact support, we have the following definitions:

Definition 4.1. *A sequence $u^\varepsilon(\mathbf{x})$ in $\mathbf{L}^2(\Omega)$ two-scale converges to $u_0(\mathbf{x}, \mathbf{y}) \in$ $\mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ if for every $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}^0_0(\Omega, C^0_\#(\mathcal{Z}))$*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\, d\mathbf{x} = \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x}, \mathbf{y}) \cdot V(\mathbf{x}, \mathbf{y})\, d\mathbf{x} d\mathbf{y}. \tag{104}$$

Proposition 4.2. *If $u^\varepsilon(\mathbf{x})$ two-scale converges to $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$, we have for all $v(\mathbf{x}) \in C_0(\overline{\Omega})$ and all $w(\mathbf{y}) \in \mathbf{L}^2_\#(\mathcal{Z})$*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(\mathbf{x}) \cdot v(\mathbf{x}) w(\frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} = \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x}, \mathbf{y}) \cdot v(\mathbf{x}) w(\mathbf{y})\, d\mathbf{x} d\mathbf{y}. \tag{105}$$

Theorem 4.3. *(Nguetseng). Let $u^\varepsilon(\mathbf{x}) \in \mathbf{L}^2(\Omega)$. Suppose there exists a constant $c > 0$ such that for all ε*

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq c.$$

Then there exists a subsequence of ε (still denoted ε) and $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ such that:

$$u^\varepsilon(\mathbf{x}) \twoheadrightarrow u_0(\mathbf{x}, \mathbf{y}). \tag{106}$$

Proposition 4.4. *Let $u^\varepsilon(\mathbf{x})$ be a sequence of functions in $\mathbf{L}^2(\Omega)$, which two-scale converges to a limit $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$. Then $u^\varepsilon(\mathbf{x})$ converges also to $u(\mathbf{x}) = \displaystyle\int_{\mathcal{Z}} u_0(\mathbf{x}, \mathbf{y}) d\mathbf{y}$ in $\mathbf{L}^2(\Omega)$ weakly. Furthermore, we have*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{\mathbf{L}^2(\Omega)} \geq \|u_0\|_{\mathbf{L}^2(\Omega \times Y)} \geq \|u\|_{\mathbf{L}^2(\Omega)}. \tag{107}$$

Remark 4.5. *: - For any smooth function $u(\mathbf{x}, \mathbf{y})$, being $\mathcal{Z}$-periodic in $\mathbf{y}$, the associated sequence $u^\varepsilon(\mathbf{x}) = u(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ two-scale converges to $u(\mathbf{x}, \mathbf{y})$.*

- Any sequence u^ε that converges strongly in $L^2(\Omega)$ to a limit $u(\mathbf{x})$, two-scale converges to the same limit $u(\mathbf{x})$.

*- If u^ε admits an asymptotic expansion of the type $u^\varepsilon(\mathbf{x}) = u_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon u_1(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon^2 u_2(\mathbf{x}, \mathbf{x}/\varepsilon) + ...$, where the functions $u_i(\mathbf{x}, \mathbf{y})$ are smooth and $\mathcal{Z}$-periodic in $\mathbf{y}$, two-scale convergence allows to identify the first term of the expansion $u_0(\mathbf{x}, \mathbf{y})$ with the two-scale limit of u^ε and the two-scale limit of $\frac{u^\varepsilon(\mathbf{x}) - u_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})}{\varepsilon}$ with $u_1(\mathbf{x}, \mathbf{y})$ see (Frénod, Raviart and Sonnendrucker [**?**]).*

Proposition 4.6. *Let $u^\varepsilon(\mathbf{x})$ in $\mathbf{L}^2(\Omega)$. Suppose there exists a constant $c > 0$ such that for all ε*

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq c.$$

Up to a subsequence, $u^\varepsilon(\mathbf{x})$ two-scale converges to $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ such that:

$$u_0(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) + \widetilde{u_0}(\mathbf{x}, \mathbf{y}), \tag{108}$$

where $\widetilde{u_0}(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ satisfies

$$\int_{\mathcal{Z}} \widetilde{u_0}(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = 0, \tag{109}$$

and $u(\mathbf{x}) = \displaystyle\int_{\mathcal{Z}} u_0(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ is a weak limit in $\mathbf{L}^2(\Omega)$.

Proof. $u^\varepsilon(\mathbf{x})$ is bounded in $\mathbf{L}^2(\Omega)$, then by application of Theorem 4.3, we get the first part of the proposition. Furthermore by defining $\widetilde{u_0}$ as

$$\widetilde{u_0}(\mathbf{x}, \mathbf{y}) = u_0(\mathbf{x}, \mathbf{y}) - \int_{\mathcal{Z}} u_0(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \tag{110}$$

we obtain the decomposition of u_0. □

Defining $\nabla_\mathbf{x} = (\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z})$, $\nabla_\mathbf{y} = (\frac{\partial}{\partial \xi}; \frac{\partial}{\partial \nu}; \frac{\partial}{\partial \zeta})$, we have

Proposition 4.7. *Let $u^\varepsilon(\mathbf{x})$ be bounded in $\mathbf{H}(curl, \Omega)$. Then, up to a subsequence, there exists a function $u_1 \in \mathbf{L}^2(\Omega, H_\#(curl, \mathcal{Z}))$ such that*

$$\nabla \times u^\varepsilon(\mathbf{x}) \twoheadrightarrow \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y}) + \nabla_\mathbf{y} \times u_1(\mathbf{x}, \mathbf{y}), \tag{111}$$

where u_0 is given by Proposition 4.6.

Proof. From Theorem 4.3, since u^ε and $\nabla \times u^\varepsilon$ are bounded in $\mathbf{L}^2(\Omega)$ then, up to a subsequence, they two-scale converge to $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ and $\eta_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$. So we have for all $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}^0_0(\Omega; C^0_\#(\mathcal{Z}))$:

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\, d\mathbf{x} = \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x}, \mathbf{y}) \cdot V(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}, \tag{112}$$

$$\lim_{\varepsilon \to 0} \int_\Omega \nabla \times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\, d\mathbf{x} = \int_\Omega \int_\mathcal{Z} \eta_0(\mathbf{x}, \mathbf{y}) \cdot V(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \tag{113}$$

Next, by integration by parts, we have:

$$\int_\Omega \nabla \times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\, d\mathbf{x} = \int_\Omega u^\varepsilon(\mathbf{x}) \cdot (\nabla_\mathbf{x} \times V(\mathbf{x}, \mathbf{x}/\varepsilon) + \frac{1}{\varepsilon} \nabla_\mathbf{y} \times V(\mathbf{x}, \mathbf{x}/\varepsilon))\, d\mathbf{x}. \tag{114}$$

If we choose a test function $V \in \mathbf{C}^0_0(\Omega, \mathbf{C}^0_\#(\mathcal{Z}))$ such that $\nabla_\mathbf{y} \times V = 0$, passing to the limit in the left-hand side (113) we get

$$\begin{aligned} \int_\Omega \nabla_\mathbf{x} \times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\, d\mathbf{x} \to & \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_\mathbf{x} \times V(\mathbf{x}, \mathbf{y})\, d\mathbf{x} d\mathbf{y} \\ = & \int_\Omega \int_\mathcal{Z} \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y}) \cdot V(\mathbf{x}, \mathbf{y})\, d\mathbf{x} d\mathbf{y}. \end{aligned} \tag{115}$$

This means that with the difference between (113) and (115):

$$\int_\Omega \int_\mathcal{Z} [\eta_0(\mathbf{x}, \mathbf{y}) - \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y})] \cdot V(\mathbf{x}, \mathbf{y})\, d\mathbf{x} d\mathbf{y} = 0, \tag{116}$$

for all functions $V \in \mathbf{C}^1_0(\Omega)$ with $\nabla_\mathbf{y} \times V = 0$. It follows that function $\eta_0(\mathbf{x}, \mathbf{y}) - \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y})$ is orthogonal to functions with zero rotational in $\mathbf{L}^2(\Omega, \mathbf{H}_\#(\mathrm{curl}), \mathcal{Z})$. This implies that there exists a function $u_1 \in \mathbf{L}^2(\Omega, \mathbf{H}_\#(\mathrm{curl}, \mathcal{Z}))$ such that

$$\nabla_\mathbf{y} \times u_1(\mathbf{x}, \mathbf{y}) = \eta_0(\mathbf{x}, \mathbf{y}) - \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y}). \tag{117}$$

Thus

$$\nabla \times u^\varepsilon(\mathbf{x}) \twoheadrightarrow \nabla_\mathbf{x} \times u_0(\mathbf{x}, \mathbf{y}) + \nabla_\mathbf{y} \times u_1(\mathbf{x}, \mathbf{y}). \tag{118}$$

□

Proposition 4.8. *Let u^ε be a bounded sequence in $\mathbf{H}(curl,\Omega)$. Then a subsequence u^ε can be extrated from ε such that, letting $\varepsilon \to 0$*

$$u^\varepsilon(\mathbf{x}) \twoheadrightarrow u(\mathbf{x}) + \nabla_\mathbf{y}\Phi(\mathbf{x},\mathbf{y}). \tag{119}$$

where $\Phi \in \mathbf{L}^2(\Omega, \mathbf{H}^1_\#(\mathcal{Z}))$ is a scalar-valued function and where $u \in \mathbf{L}^2(\Omega)$. And we have

$$\nabla\times u^\varepsilon(\mathbf{x}) \rightharpoonup \nabla_\mathbf{x} \times u(\mathbf{x}) \;\; \textit{weakly in} \;\; \boldsymbol{L}^2(\Omega). \tag{120}$$

where $u(\mathbf{x})$ is given by Proposition 4.6.

Proof. Proof of (119), for any $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}^1_0(\Omega, \mathbf{C}^1_\#(\mathcal{Z}))$, we have

$$\int_\Omega \nabla\times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} = \int_\Omega u^\varepsilon(\mathbf{x})\{\nabla_\mathbf{x} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_\mathbf{y} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\}\, d\mathbf{x}. \tag{121}$$

Multiplying by ε we have

$$\varepsilon \int_\Omega \nabla\times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} = \int_\Omega u^\varepsilon(\mathbf{x})\{\varepsilon\nabla_\mathbf{x} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \nabla_\mathbf{y} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})\}\, d\mathbf{x}. \tag{122}$$

Taking the two-scale limit as $\varepsilon \to 0$ we obtain

$$0 = \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x},\mathbf{y}) \cdot \nabla_\mathbf{y} \times V(\mathbf{x},\mathbf{y})\, d\mathbf{x}d\mathbf{y}, \tag{123}$$

which implies that $\nabla_\mathbf{y} \times u_0(\mathbf{x},\mathbf{y}) = 0$. Thus $u_0(\mathbf{x},\mathbf{y})$ is a gradient with respect to the variable $\mathbf{y}$ for some scalar function $\Phi(\mathbf{x},\mathbf{y})$. And according to Proposition (4.6) $u_0(\mathbf{x},\mathbf{y})$ can be written as $u_0(\mathbf{x},\mathbf{y}) = u(\mathbf{x}) + \nabla_\mathbf{y}\Phi(\mathbf{x},\mathbf{y})$, where $u(\mathbf{x}) = \int_\mathcal{Z} u_0(\mathbf{x},\mathbf{y})d\mathbf{y}$ for some scalar function $\Phi(\mathbf{x},\mathbf{y})$.

Next, we choose a test function $V(\mathbf{x}) \in \mathbf{L}^2(\Omega)$. Integration by parts yields:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \nabla\times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x})\, d\mathbf{x} = \lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(\mathbf{x}) \cdot \nabla\times V(\mathbf{x})\, d\mathbf{x} \\
&= \int_\Omega \int_\mathcal{Z} u_0(\mathbf{x},\mathbf{y})\, d\mathbf{y} \cdot \nabla\times V(\mathbf{x})\, d\mathbf{x} \\
&= \int_\Omega \nabla\times u(\mathbf{x}) \cdot V(\mathbf{x})\, d\mathbf{x}.
\end{aligned} \tag{124}$$

□

These results are important properties of the two-scales convergence. We note that the usual concepts of convergence do not preserve information concerning the micro-scale of the function. However, the two-scale convergence preserves information on the micro-scale.

4.2. Homogenized problem

We will explore in this section the behavior of electromagnetic field E^ε using the two-scale convergence to determine the homogenized problem. We place in the context of the case 6 with $\delta > L$ and $\overline{\omega} = 10^6 rad.s^{-1}$, then we have $\eta = 5$ and $\Sigma_a^\varepsilon = \varepsilon$, $\Sigma_r^\varepsilon = \varepsilon^4$, $\Sigma_c^\varepsilon = 1$ which gives the following equation:

$$\nabla\times\nabla\times E^\varepsilon - \omega^2\varepsilon^5 k(\epsilon)E^\varepsilon + i\omega[(\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4\mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}} + \varepsilon\mathbf{1}_{\{y>0\}}]E^\varepsilon = 0, \tag{125}$$

where for a given set $\mathcal{A}$, $\mathbf{1}_\mathcal{A}$ stands for the characteristic function of $\mathcal{A}$ and where $\mathbf{1}_\mathcal{A}^\varepsilon(\mathbf{x}) = \mathbf{1}_\mathcal{A}(\frac{\mathbf{x}}{\varepsilon})$, hence $\mathbf{1}_C^\varepsilon$ and $\mathbf{1}_R^\varepsilon$ are the characteristic functions of the sets filled by carbon fibers and by resin. And where $k(\epsilon) = (\epsilon_c\mathbf{1}_C^\varepsilon(\mathbf{x}) + \epsilon_r\mathbf{1}_R^\varepsilon(\mathbf{x}))\mathbf{1}_{\{y<0\}} + \mathbf{1}_{\{y>0\}}$.

Remark 4.9. *We recall that ϵ_c and ϵ_r are respectively the relative permittivity of the carbon fibers and the resin. You should not confused with the microscopic scale ε.*

On this purpose, we have the following Theorem:

Theorem 4.10. *Under assumptions of Theorem 3.4, sequence E^ε solution of (87) or (86) or ((65), (68), (69), (71)) converges to $E(\mathbf{x}) \in \mathbf{L}^2(\Omega)$ which is the unique solution of the homogenized problem:*

$$\begin{cases} \theta_1\nabla_\mathbf{x}\times\nabla_\mathbf{x}\times E(\mathbf{x}) + i\omega\theta_2 E(\mathbf{x}) = 0 \ \ in\ \Omega, \\ \theta_1\nabla_\mathbf{x}\times E(\mathbf{x})\times e_2 = -i\omega H_d\times e_2 \ \ on\ \Gamma_d, \\ \nabla_\mathbf{x}\times E(\mathbf{x})\times e_2 = 0 \ \ on\ \Gamma_L. \end{cases} \tag{126}$$

with $\theta_1 = \int_\mathcal{Z} \mathrm{Id} - \nabla_y\chi(\mathbf{y})\, d\mathbf{y}$ and $\theta_2 = \int_\mathcal{Z} \mathbf{1}_C(\mathbf{y})(\mathrm{Id} - \nabla_y\chi(\mathbf{y}))\, d\mathbf{y}$.

And where the scalar function χ is the unique solution, up to an additive constant in the Hilbert space of $\mathcal{Z}$ periodic functions $H^1_\#(\mathcal{Z})$, of the following boundary value problem

$$\begin{cases} \triangle_\mathbf{y}(\chi(\mathbf{y})) = 0 \ \ in\ \mathcal{Z}\backslash\partial\Omega_C, \\ [\frac{\partial\chi}{\partial n}] = -n_j \ \ on\ \partial\Omega_C, \\ [\chi] = 0 \ \ on\ \partial\Omega_C. \end{cases} \tag{127}$$

where $[f]$ is the jump across the surface of $\partial\Omega_C$, n_j, $j = \{1,2,3\}$ is the projection on the axis e_j of the normal of $\partial\Omega_C$.

Proof. **Step 1: Two-scale convergence.** Due to the estimate (98), E^ε is bounded in $\mathbf{L}^2(\Omega)$. Hence, up to a subsequence, E^ε two-scale converges to $E_0(\mathbf{x},\mathbf{y})$ belonging to $\mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$. That means for any $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}^1_0(\Omega, \mathbf{C}^1_\#(\mathcal{Z}))$, we have:

$$\lim_{\varepsilon\to 0}\int_\Omega E^\varepsilon(\mathbf{x})\cdot V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} = \int_\Omega\int_\mathcal{Z} E_0(\mathbf{x},\mathbf{y})\cdot V(\mathbf{x},\mathbf{y})\, d\mathbf{y}d\mathbf{x}. \tag{128}$$

Step 2: Deduction of the constraint equation. We multiply the equation (125) by oscillating test function $V^\varepsilon(\mathbf{x}) = V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})$ where $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}^1_0(\Omega, \mathbf{C}^1_\#(\mathcal{Z}))$:

$$\begin{aligned} &\int_\Omega \nabla\times E^\varepsilon(\mathbf{x})\cdot(\nabla_\mathbf{x}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_\mathbf{y}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})) + [-\omega^2\varepsilon^5 k(\epsilon) \\ &\quad + i\omega((\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4\mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}} + \varepsilon\mathbf{1}_{\{y>0\}})]E^\varepsilon\cdot V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\, d\mathbf{x} \\ &\quad = -i\omega\int_{\Gamma_d} H_d\times e_2\cdot(e_2\times V(x,1,z,\xi,\frac{1}{\varepsilon},\zeta))\times e_2\, d\sigma. \end{aligned} \tag{129}$$

Integrating by parts, we get:

$$\begin{aligned}
&\int_\Omega E^\varepsilon(\mathbf{x})\cdot(\nabla_\mathbf{x}\times\nabla_\mathbf{x}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\frac{1}{\varepsilon}\nabla_\mathbf{y}\times\nabla_\mathbf{x}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\
&+\frac{1}{\varepsilon}\nabla_\mathbf{x}\times\nabla_\mathbf{y}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\frac{1}{\varepsilon^2}\nabla_\mathbf{y}\times\nabla_\mathbf{y}\times V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}))+[-\omega^2\varepsilon^5k(\epsilon)\\
&+i\omega(\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon})+\varepsilon^4\mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}}+\varepsilon\mathbf{1}_{\{y>0\}}]E^\varepsilon(\mathbf{x})\cdot V^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\,d\mathbf{x}\\
&=-i\omega\int_{\Gamma_d}H_d\times e_2\cdot(e_2\times V(x,1,z,\xi,\frac{1}{\varepsilon},\zeta))\times e_2\,d\sigma.
\end{aligned}\tag{130}$$

Now we multiply (130) by ε^2 and we pass to the two-scale limit, applying Theorem 4.3 we obtain:

$$\int_\Omega\int_\mathcal{Z} E_0(\mathbf{x},\mathbf{y})(\nabla_\mathbf{y}\times\nabla_\mathbf{y}\times V(\mathbf{x},\mathbf{y}))\,d\mathbf{y}d\mathbf{x}=0.\tag{131}$$

We deduce the constraint equation for the profile E_0:

$$\nabla_\mathbf{y}\times\nabla_\mathbf{y}\times E_0(\mathbf{x},\mathbf{y})=0.\tag{132}$$

Step 3. Looking for the solutions to the constraint equation. Multiplying Equation (132) by E_0 and integrating by parts over $\mathcal{Z}$ leads to

$$\int_\mathcal{Z}\nabla_\mathbf{y}\times\nabla_\mathbf{y}\times E_0(\mathbf{x},\mathbf{y})E_0(\mathbf{x},\mathbf{y})\ \ d\mathbf{y}=\int_\mathcal{Z}|\nabla_\mathbf{y}\times E_0(\mathbf{x},\mathbf{y})|^2\ \ d\mathbf{y}=0.\tag{133}$$

We deduce that equation (133) is equivalent to

$$\nabla_\mathbf{y}\times E_0(\mathbf{x},\mathbf{y})=0,\tag{134}$$

Moreover a solution of (134) is also solution of (132). So (132) and (134) are equivalent.

Hence, from Proposition (119) we conclude that $E_0(\mathbf{x},\mathbf{y})$ can be decomposed as

$$E_0(\mathbf{x},\mathbf{y})=E(\mathbf{x})+\nabla_\mathbf{y}\Phi_0(\mathbf{x},\mathbf{y}).\tag{135}$$

Step 4. Equations for $E(\mathbf{x})$ and $\Phi_0(\mathbf{x},\mathbf{y})$. The divergence equation of (125) is multiplied with $V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})=\varepsilon v(\mathbf{x})\psi(\frac{\mathbf{x}}{\varepsilon})$, where $v\in\mathbf{C}_0^1(\Omega)$ and $\psi\in\mathbf{H}_\#^1(\mathcal{Z})$. Theorem 4.3 and integration by parts yields for all $\psi\in\mathbf{H}_\#^1(\mathcal{Z})$ and $v\in\mathbf{C}_0^1(\Omega)$

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega\nabla\cdot\{-\omega^2\varepsilon^5k(\epsilon)E^\varepsilon(\mathbf{x})+i\omega[(\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon})+\varepsilon^4\mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}}+\varepsilon\mathbf{1}_{\{y>0\}}]E^\varepsilon(\mathbf{x})\}\varepsilon v(\mathbf{x})\psi(\frac{\mathbf{x}}{\varepsilon})\,d\mathbf{x}\\
&=-\lim_{\varepsilon\to0}\int_\Omega\{-\omega^2\varepsilon^5k(\epsilon)E^\varepsilon(\mathbf{x})+i\omega[\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon})+\varepsilon^4\mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}}\\
&+\varepsilon\mathbf{1}_{\{y>0\}}]E^\varepsilon\}\cdot(\varepsilon v(\mathbf{x})\psi(\frac{\mathbf{x}}{\varepsilon})+v(\mathbf{x})\nabla_\mathbf{y}\psi(\frac{\mathbf{x}}{\varepsilon}))\,d\mathbf{x}\\
&=-\int_\Omega\int_\mathcal{Z} v(\mathbf{x})\nabla_\mathbf{y}\psi(\mathbf{y})\cdot[i\omega\mathbf{1}_C(\mathbf{y})E_0(\mathbf{x},\mathbf{y})]\,d\mathbf{y}d\mathbf{x}=0.
\end{aligned}\tag{136}$$

from which it follows that

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y})] = 0. \tag{137}$$

with E_0 given by the decomposition (119). So we obtain the local equation

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) \{E(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_0(\mathbf{x}, \mathbf{y})\}] \, d\mathbf{y} = 0. \tag{138}$$

The potential Φ_0 may be written on the form

$$\Phi_0(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{3} \chi_j(\mathbf{y}) e_j \cdot E(\mathbf{x}) = \chi(\mathbf{y}) \cdot E(\mathbf{x}), \tag{139}$$

From (135) and (139), we get:

$$E_0(\mathbf{x}, \mathbf{y}) = (\mathrm{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x}). \tag{140}$$

Inserting E_0 in (138) we obtain

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) (\mathrm{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})] = 0. \tag{141}$$

Now, we build oscillating test functions satisfying constraint (135) and use them in weak formulation (130). We define test function $V(\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})$, $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}_0^1(\Omega, \mathbf{C}_\#^1(\mathcal{Z}))$ and we inject in (130) test function $V^\varepsilon = V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, which gives:

$$\begin{aligned}
&\int_\Omega E^\varepsilon(\mathbf{x}) \cdot (\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{2}{\varepsilon} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \\
&+ \frac{1}{\varepsilon^2} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) + [-\omega^2 \varepsilon^5 k(\epsilon) + i\omega((\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon}) \\
&+ \varepsilon^4 \mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y<0\}} + \varepsilon \mathbf{1}_{\{y>0\}})] E^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \, d\mathbf{x} \\
&= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times V^\ddagger(x, 1, z, \xi, \zeta)) \times e_2 \, d\sigma,
\end{aligned} \tag{142}$$

with $V(x, 1, z, \xi, \nu, \zeta) = V^\ddagger(x, 1, z, \xi, \zeta)$ the restriction on V which does not depend on ν. The term containing the constraint, the third one, disappears. Passing to the limit $\varepsilon \to 0$ and replacing the expression of V by the term $\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})$, we have

$$\begin{aligned}
\nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \mathbf{y}) &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times [\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})] \\
&= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\alpha(\mathbf{x})) + \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \\
&= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})).
\end{aligned} \tag{143}$$

Since $\nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}) = 0$, the term $\frac{2}{\varepsilon} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y}))$ vanishes. Therefore, (142) becomes:

$$\begin{aligned}
&\int_\Omega \int_{\mathcal{Z}} E_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \\
&\quad + i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y}) \cdot (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} d\mathbf{x} \\
&\quad = -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times (\alpha(x, 1, z) + \nabla_{\mathbf{y}} \beta(x, 1, z, \xi, \zeta))) \times e_2 \, d\sigma.
\end{aligned} \tag{144}$$

Now in (144) we replace expression E_0 giving by (140). We obtain

$$\begin{aligned}
&\int_\Omega \int_\mathcal{Z} (\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x}) \cdot (\nabla_\mathbf{x} \times \nabla_\mathbf{x} \times (\alpha(\mathbf{x}) + \nabla_\mathbf{y}\beta(\mathbf{x},\mathbf{y})) \\
&\quad + i\omega \mathbf{1}_C(\mathbf{y})(\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x})) \cdot (\alpha(\mathbf{x}) + \nabla_\mathbf{y}\beta(\mathbf{x},\mathbf{y}))\, d\mathbf{y}d\mathbf{x} \\
&\quad = -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times (\alpha(x,1,z) + \nabla_\mathbf{y}\beta(x,1,z,\xi,\zeta))) \times e_2\, d\sigma.
\end{aligned} \tag{145}$$

Taking $\alpha(\mathbf{x}) = 0$ in (145), we obtain

$$\begin{aligned}
&\int_\Omega \int_\mathcal{Z} (\mathrm{Id}{+}\nabla_\mathbf{y}\chi(\mathbf{y}))\nabla_\mathbf{x} \times \nabla_\mathbf{x} \times E(\mathbf{x})\nabla_\mathbf{y}\beta(\mathbf{x},\mathbf{y}) \\
&\qquad + i\omega \mathbf{1}_C(\mathbf{y})(\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x}) \cdot \nabla_\mathbf{y}\beta(\mathbf{x},\mathbf{y}) d\mathbf{y}d\mathbf{x} = 0.
\end{aligned} \tag{146}$$

Integrating by parts

$$\begin{aligned}
&\int_\Omega \int_\mathcal{Z} -\nabla_\mathbf{y} \cdot \{(\mathrm{Id} - \nabla_\mathbf{y}\chi(\mathbf{y}))\nabla_\mathbf{x} \times \nabla_\mathbf{x} \times E(\mathbf{x})\}\beta(\mathbf{x},\mathbf{y}) \\
&\qquad - i\omega \nabla_\mathbf{y} \cdot \{\mathbf{1}_C(\mathbf{y})(\mathrm{Id} - \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x})\}\beta(\mathbf{x},\mathbf{y})\, d\mathbf{y}d\mathbf{x} = 0.
\end{aligned} \tag{147}$$

And since $\nabla_\mathbf{y} \cdot \{\mathbf{1}_C(\mathbf{y})(\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x})\} = 0$ we obtain

$$\int_\Omega \int_\mathcal{Z} -\nabla_\mathbf{y} \cdot \{(\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))\nabla_\mathbf{x} \times \nabla_\mathbf{x} \times E(\mathbf{x})\}\beta(\mathbf{x},\mathbf{y})\, d\mathbf{y}d\mathbf{x} = 0. \tag{148}$$

which gives the cell problem

$$\nabla_\mathbf{y} \cdot [\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y})] = 0. \tag{149}$$

From (141) and (149), the scalar function χ is the unique solution, thanks to Lax-Milgram Lemma, up to an additive constant in the Hilbert space of $\mathcal{Z}$ periodic function $H^1_\#(\mathcal{Z})$ of the following boundary value problem

$$\begin{cases}
\triangle_\mathbf{y}(\chi(\mathbf{y})) = 0 \ \text{ in } \mathcal{Z}\backslash\partial\Omega_C, \\
[\dfrac{\partial\chi}{\partial n}] = -n_j \ \text{ on } \partial\Omega_C, \\
[\chi] = 0 \ \text{ on } \partial\Omega_C.
\end{cases} \tag{150}$$

where $[f]$ is the jump across the surface of $\partial\Omega_C$, n_j, $j = \{1,2,3\}$ is the projection on the axis e_j of the normal of $\partial\Omega_C$.

Remark 4.11. *(150) can be seen as an electrostatic problem. Solving (141) and (149) reduces to look for a potential induced by surface density of charges. Then χ is this potential induced by the charges on the interface of carbon fiber.*

Setting $\beta(\mathbf{x},\mathbf{y}) = 0$ in (145) and integrating by parts, we get

$$\begin{aligned}
&\int_\Omega \int_\mathcal{Z} (\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))\nabla_\mathbf{x} \times \nabla_\mathbf{x} \times E(\mathbf{x}) \cdot \alpha(\mathbf{x}) \\
&\qquad + i\omega \mathbf{1}_C(\mathbf{y})(\mathrm{Id} + \nabla_\mathbf{y}\chi(\mathbf{y}))E(\mathbf{x})\alpha(\mathbf{x})\, d\mathbf{y}d\mathbf{x} \\
&\qquad = -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times \alpha(x,1,z)) \times e_2\, d\sigma.
\end{aligned} \tag{151}$$

which gives the following well posed problem for $E(\mathbf{x})$

$$\begin{cases} \theta_1 \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) + i\omega\theta_2 E(\mathbf{x}) = 0 \ \text{ in } \Omega, \\ \theta_1 \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times e_2 = -i\omega H_d \times e_2 \ \text{ on } \Gamma_d, \\ \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times e_2 = 0 \ \text{ on } \Gamma_L. \end{cases} \tag{152}$$

with $\theta_1 = \int_{\mathcal{Z}} \mathrm{Id} + \nabla_{\mathbf{y}}\chi(\mathbf{y}) \, d\mathbf{y}$ and $\theta_2 = \int_{\mathcal{Z}} \mathbf{1}_C(\mathbf{y})(\mathrm{Id} + \nabla_{\mathbf{y}}\chi(\mathbf{y})) \, d\mathbf{y}$.

This concludes the proof of Theorem (126). □

5. Conclusion

We presented in this paper the homogenization of time harmonic Maxwell equation by the method of two-scale convergence. We started by studying the time harmonic Maxwell equations with coefficients depending of ε. We remind that λ is the wave length, δ is the skin length, L is thickness of the medium and e the size of the basic cell and then $\varepsilon = \frac{e}{L}$ is the small parameter. We find for low frequencies the macroscopic homogenized Maxwell equations depending on the volume fraction of the carbon fibers and we find also the microscopic equation.

6. Annexes

A. Presentation of all cases of tables 1, 2, 3 and 4

- The case 1 corresponds to the air not ionized, a resin not doped and $\overline{\sigma}$ is the effective electric conductivity in the direction of the carbon fibers. We have for the effective electric conductivity $\overline{\sigma} = \sigma_c \sim 40000 S.m^{-1}$, the resin conductivity is about $\sigma_r \sim 10^{-10} S.m^{-1}$ and the conductivity in the air is about $10^{-14} S.m^{-1}$. So when we want to calculate the ratio in (41)-(43) depending on ε we get: $\frac{\sigma_r}{\overline{\sigma}} \sim \varepsilon^7$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon^9$.

- In case 2, the air is not ionized, the resin is doped and $\overline{\sigma}$ is the effective conductivity is in direction of carbon fibers. We have like the case 1 $\overline{\sigma} = \sigma_c \sim 40000 S.m^{-1}$. The resin conductivity is about $\sigma_r \sim 10^{-3} S.m^{-1}$ and the conductivity in the air is about $10^{-14} S.m^{-1}$. So $\frac{\sigma_r}{\overline{\sigma}} \sim \varepsilon^4$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon^9$.

- In case 3, the air is not ionized, the resin is not doped and $\overline{\sigma}$ is the effective conductivity is orthogonal to the fibers. $\overline{\sigma} = \sigma_r \sim 10^{-10} S.m^{-1}$. The carbon fiber conductivity is about $\sigma_c \sim 10^4 S.m^{-1}$ and the conductivity in the air is about $10^{-14} S.m^{-1}$. $\frac{\sigma_c}{\overline{\sigma}} \sim \frac{1}{\varepsilon^7}$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon^2$.

- Case 4 corresponds to the air non ionized, the resin doped and $\overline{\sigma}$ is the effective conductivity orthogonal to the fibers. The effective electric conductivity is $\overline{\sigma} = \sigma_r \sim 10^{-3} S.m^{-1}$. The carbon fiber conductivity is about $\sigma_c \sim 40000 S.m^{-1}$ and the conductivity in the air is about $10^{-14} S.m^{-1}$. $\frac{\sigma_c}{\overline{\sigma}} \sim \frac{1}{\varepsilon^4}$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon^6$.

- In case 5, the air is ionized, the resin is not doped and $\overline{\sigma}$ is the effective conductivity is in the direction of the carbon fibers. This one is equal $\overline{\sigma} = \sigma_c \sim 40000 S.m^{-1}$, the resin conductivity is about $\sigma_r \sim 10^{-10} S.m^{-1}$ and the conductivity in the air is now about $4242 S.m^{-1}$. $\frac{\sigma_r}{\overline{\sigma}} \sim \varepsilon^7$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon$.

- Case 6 corresponds to the air ionized, the resin doped and $\overline{\sigma}$ is the effective conductivity in direction of the carbon fibers. This one is equal $\overline{\sigma} = \sigma_c \sim 40000 S.m^{-1}$, the resin conductivity is about $\sigma_r \sim 10^3 S.m^{-1}$ and the conductivity in the air is now about $4242 S.m^{-1}$. $\frac{\sigma_r}{\overline{\sigma}} \sim \varepsilon^4$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \varepsilon$.

- Case 7 corresponds to the air ionized, the resin not doped and $\overline{\sigma}$ is the effective conductivity orthogonal to the fibers. The effective conductivity is $\overline{\sigma} = \sigma_r \sim 10^{-10} S.m^{-1}$, the carbon fibers conductivity is about $\sigma_c \sim 40000 S.m^{-1}$ and the conductivity in the air is now about $4242 S.m^{-1}$. $\frac{\sigma_c}{\overline{\sigma}} \sim \frac{1}{\varepsilon^7}$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \frac{1}{\varepsilon^6}$.
- Case 8 corresponds to the air ionized, the resin doped and $\overline{\sigma}$ is the effective conductivity orthogonal to the fibers. The effective conductivity is $\overline{\sigma} = \sigma_r \sim 10^{-3} S.m^{-1}$, the carbon fibers conductivity is about $\sigma_c \sim 40000 S.m^{-1}$ and the conductivity in the air is now about $4242 S.m^{-1}$. $\frac{\sigma_c}{\overline{\sigma}} \sim \frac{1}{\varepsilon^4}$ and $\frac{\sigma_a}{\overline{\sigma}} \sim \frac{1}{\varepsilon^2}$.

B. Structure of the equations depending of ε

For $\overline{\omega} = 100 rad.s^{-1}$, we have

Case 1

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{11}, \Sigma_r^\varepsilon = \varepsilon^9, \Sigma_c^\varepsilon = \varepsilon^2. \tag{153}$$

Case 2

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{11}, \Sigma_r^\varepsilon = \varepsilon^6, \Sigma_c^\varepsilon = \varepsilon^2. \tag{154}$$

Case 3

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{12}, \Sigma_r^\varepsilon = \varepsilon^{10}, \Sigma_c^\varepsilon = \varepsilon^3. \tag{155}$$

Case 4

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{13}, \Sigma_r^\varepsilon = \varepsilon^7, \Sigma_c^\varepsilon = \varepsilon^3. \tag{156}$$

Case 5

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^3, \Sigma_r^\varepsilon = \varepsilon^9, \Sigma_c^\varepsilon = \varepsilon^2. \tag{157}$$

Case 6

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^3, \Sigma_r^\varepsilon = \varepsilon^6, \Sigma_c^\varepsilon = \varepsilon^2. \tag{158}$$

Case 7

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^5, \Sigma_r^\varepsilon = \varepsilon^{10}, \Sigma_c^\varepsilon = \varepsilon^3. \tag{159}$$

Case 8

$$\eta = 9 \text{ and } \Sigma_a^\varepsilon = \varepsilon^5, \Sigma_r^\varepsilon = \varepsilon^7, \Sigma_c^\varepsilon = \varepsilon^3. \tag{160}$$

For $\overline{\omega} = 10^6 rad.s^{-1}$

Case 1

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^9, \Sigma_r^\varepsilon = \varepsilon^7, \Sigma_c^\varepsilon = 1. \tag{161}$$

Case 2

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^9, \Sigma_r^\varepsilon = \varepsilon^4, \Sigma_c^\varepsilon = 1. \tag{162}$$

Case 3

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{10}, \Sigma_r^\varepsilon = \varepsilon^8, \Sigma_c^\varepsilon = \varepsilon. \tag{163}$$

Case 4

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^{11}, \Sigma_r^\varepsilon = \varepsilon^5, \Sigma_c^\varepsilon = 1. \tag{164}$$

Case 5

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon, \Sigma_r^\varepsilon = \varepsilon^7, \Sigma_c^\varepsilon = 1. \tag{165}$$

Case 6

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon, \Sigma_r^\varepsilon = \varepsilon^4, \Sigma_c^\varepsilon = 1. \tag{166}$$

Case 7

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^3, \Sigma_r^\varepsilon = \varepsilon^8, \Sigma_c^\varepsilon = \varepsilon. \tag{167}$$

Case 8

$$\eta = 5 \text{ and } \Sigma_a^\varepsilon = \varepsilon^3, \Sigma_r^\varepsilon = \varepsilon^5, \Sigma_c^\varepsilon = 1. \tag{168}$$

For $\overline{\omega} = 10^{10} rad.s^{-1}$

Case 1

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \varepsilon^7, \Sigma_r^\varepsilon = \varepsilon^5, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^2}. \tag{169}$$

Case 2

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \varepsilon^7, \Sigma_r^\varepsilon = \varepsilon^2, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^2}. \tag{170}$$

Case 3

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \varepsilon^8, \Sigma_r^\varepsilon = \varepsilon^6, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon}. \tag{171}$$

Case 4

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \varepsilon^9, \Sigma_r^\varepsilon = \varepsilon^3, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon}. \tag{172}$$

Case 5

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon}, \Sigma_r^\varepsilon = \varepsilon^5, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^2}. \tag{173}$$

Case 6

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon}, \Sigma_r^\varepsilon = \varepsilon^2, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^2}. \tag{174}$$

Case 7

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon}, \Sigma_r^\varepsilon = \varepsilon^6, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon}. \tag{175}$$

Case 8

$$\eta = 1 \text{ and } \Sigma_a^\varepsilon = \varepsilon, \Sigma_r^\varepsilon = \varepsilon^3, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon}. \tag{176}$$

For $\overline{\omega} = 10^{12} rad.s^{-1}$

Case 1

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \varepsilon^6, \Sigma_r^\varepsilon = \varepsilon^4, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{177}$$

Case 2

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \varepsilon^6, \Sigma_r^\varepsilon = \varepsilon, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{178}$$

Case 3

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \varepsilon^5, \Sigma_r^\varepsilon = \varepsilon^3, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^4}. \tag{179}$$

Case 4

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \varepsilon^7, \Sigma_r^\varepsilon = \varepsilon, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{180}$$

Case 5

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon^2}, \Sigma_r^\varepsilon = \varepsilon^4, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{181}$$

Case 6

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon^2}, \Sigma_r^\varepsilon = \varepsilon, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{182}$$

Case 7

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon^2}, \Sigma_r^\varepsilon = \varepsilon^3, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^4}. \tag{183}$$

Case 8

$$\eta = 0 \text{ and } \Sigma_a^\varepsilon = \tfrac{1}{\varepsilon}, \Sigma_r^\varepsilon = \varepsilon, \Sigma_c^\varepsilon = \tfrac{1}{\varepsilon^3}. \tag{184}$$

C. The transmission Maxwell problem

Taking a test function $V \in C^1(\Omega)$ with compact support in Ω_c^ε, in weak formulation (85) associated with the problem ((65), (68), (69)). Since

$$\int_\Omega \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} = \langle \nabla\times\nabla\times E^\varepsilon_{|\Omega_c^\varepsilon}, \overline{V}\rangle_{\Omega_c^\varepsilon}, \tag{185}$$

we deduce the third equation in (86). Similarly, taking $V \in C^1(\Omega)$ with compact support respectively in Ω_r^ε and Ω_a^ε, we obtain the first and the second equation in (86). Now, since $E_{|\Omega_a^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_a^\varepsilon)$ and $E_{|\Omega_r^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_r^\varepsilon)$, let $V \in C_0^1(\Omega_a^\varepsilon \bigcup \Omega_r^\varepsilon)$ integrating by parts we get

$$\begin{aligned}
\int_{\Omega_a^\varepsilon \bigcup \Omega_r^\varepsilon} E \cdot \nabla\times\overline{V} \ d\mathbf{x} &= \int_{\Omega_a^\varepsilon} E_{|\Omega_a^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} + \int_{\Omega_r^\varepsilon} E_{|\Omega_r^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} \\
&= \int_{\Omega_a^\varepsilon} \nabla\times E_{|\Omega_a^\varepsilon} \cdot \overline{V} \ d\mathbf{x} + \int_{\Omega_r^\varepsilon} \nabla\times E_{|\Omega_r^\varepsilon} \cdot \overline{V} \ d\mathbf{x} \\
&+ \int_{\Gamma_{ra}} (E_{|\Omega_a^\varepsilon} \times e_2 - E_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon}) \cdot \overline{V} \, ds.
\end{aligned} \tag{186}$$

Since on every point of Γ_{ra} $e_2 = -n_{|\Omega_r^\varepsilon}$ the assumed continuity require

$$E_{|\Omega_a^\varepsilon} \times e_2 = E_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon}, \tag{187}$$

we obtain the fourth relation in (86). With the same argument on Γ_{cr}^ε, we obtain the last relation in (86). This shows that (85) implies (86). And, if E^ε is solution to (86) following that for any regular set $\widehat{\Omega}$ in Ω the Stokes's formula gives, for more details see p 57, 58 of P. Monk's book [**?**]:

$$\forall \ E, V \in \mathbf{H}(\mathrm{curl}, \widehat{\Omega}) \quad \int_{\widehat{\Omega}} \nabla\times E \cdot \overline{V} - E \cdot \nabla\times\overline{V} \ d\mathbf{x} = \langle E \times n_{\widehat{\Omega}}, \overline{V_T}\rangle_{\partial\widehat{\Omega}} \tag{188}$$

$\mathbf{H}(\mathrm{curl}, \widehat{\Omega})$ has the same definition as $\mathbf{H}(\mathrm{curl}, \Omega)$ with Ω replaced by $\widehat{\Omega}$ and where $V_T = (n \times V) \times n$, and $n_{\widehat{\Omega}}$ is the unit outward normal of $\partial\widehat{\Omega}$. For all $V \in \mathbf{H}(\mathrm{curl}, \Omega)$, $V_{|\Omega_r^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_r^\varepsilon)$, $V_{|\Omega_a^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_a^\varepsilon)$ and $V_{|\Omega_c^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_c^\varepsilon)$. Hence, fixing any $E' \in \mathbf{H}(\mathrm{curl}, \Omega)$ according to the second equation in (86), we have $\nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \in \mathbf{H}(\mathrm{curl}, \Omega_r^\varepsilon)$ then applying (188) in Ω_r^ε with $E = \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon}$ and V we get

$$\begin{aligned}
\int_{\Omega_r^\varepsilon} \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} = \int_{\Omega_r^\varepsilon} \nabla\times\nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \cdot \overline{V} \, d\mathbf{x} &+ \langle \nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{ra}} \\
&+ \langle \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \times n_{|\Omega_c^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{cr}^\varepsilon}.
\end{aligned} \tag{189}$$

Doing the same for Ω_c^ε, we have

$$\int_{\Omega_c^\varepsilon} \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} = \int_{\Omega_c^\varepsilon} \nabla\times\nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \cdot \overline{V} \, d\mathbf{x} + \langle \nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \times n^\varepsilon_{|\Omega_c^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{cr}^\varepsilon}. \tag{190}$$

Finally for Ω_a^ε, we have

$$\begin{aligned}
\int_{\Omega_a^\varepsilon} \nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \cdot \nabla\times\overline{V} \ d\mathbf{x} = \int_{\Omega_a^\varepsilon} \nabla\times\nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \cdot \overline{V} \, d\mathbf{x} &+ \langle \nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \times e_2, \overline{V_T}\rangle_{\Gamma_d} \\
&- \langle \nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \times e_2, \overline{V_T}\rangle_{\Gamma_{ra}^\varepsilon}.
\end{aligned} \tag{191}$$

Summing the relations above since in every point of Γ_{ra} $n_{|\Omega_r^\varepsilon} = -e_2$ and in every point of Γ_{cr}^ε $n_{|\Omega_c^\varepsilon} = -n_{|\Omega_r^\varepsilon}$, it comes

$$\begin{aligned}\int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\ d\mathbf{x} = \int_\Omega \nabla\times\nabla\times E^\varepsilon \cdot \overline{V}\ d\mathbf{x} + < [\nabla\times E^\varepsilon \times n], \overline{V_T} >_{\Gamma_{ra}} \\ + \langle[\nabla\times E^\varepsilon \times n], \overline{V_T}\rangle_{\Gamma_{cr}} - i\omega \int_{\Gamma_d} H_d \times n^\varepsilon \cdot \overline{V}_T\ d\sigma.\end{aligned} \tag{192}$$

According to (85) and the first, second and third equations in (86) we have

$$\begin{aligned}\langle\nabla\times E^\varepsilon_{|\Omega_c^\varepsilon} \times n_{|\Omega_c^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{cr}^\varepsilon} - \langle\nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{cr}^\varepsilon} \\ + \langle\nabla\times E^\varepsilon_{|\Omega_r^\varepsilon} \times n_{|\Omega_r^\varepsilon}, \overline{V_T}\rangle_{\Gamma_{ra}} + \langle\nabla\times E^\varepsilon_{|\Omega_a^\varepsilon} \times e_2, \overline{V_T}\rangle_{\Gamma_{ra}} = 0,\end{aligned} \tag{193}$$

for all $V \in \mathbf{H}(\text{curl}, \Omega)$ which causes the last two equalities in (86) and concludes the first part of the proof.

Reciprocally, integrating by parts (86) we have:

$$\begin{aligned}\forall\ \ V \in \mathbf{X}^\varepsilon(\Omega), \quad \int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\, d\mathbf{x} + \int_{\Omega_a^\varepsilon} (-\omega^2\varepsilon^\eta + i\omega\Sigma_a^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} \\ = -i\omega \int_{\Gamma_d} H_d \times n^\varepsilon \cdot \overline{V}_T\ d\sigma,\end{aligned} \tag{194}$$

and

$$\forall\ \ V \in \mathbf{X}^\varepsilon(\Omega), \quad \int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\, d\mathbf{x} + \int_{\Omega_r^\varepsilon} (-\omega^2\varepsilon^\eta \epsilon_r + i\ \omega\Sigma_r^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} = 0, \tag{195}$$

and

$$\forall\ \ V \in \mathbf{X}^\varepsilon(\Omega), \quad \int_\Omega \nabla\times E^\varepsilon \cdot \nabla\times\overline{V}\, d\mathbf{x} + \int_{\Omega_c^\varepsilon} (-\omega^2\varepsilon^\eta \epsilon_c + i\ \omega\Sigma_c^\varepsilon) E^\varepsilon \cdot \overline{V}\, d\mathbf{x} = 0. \tag{196}$$

By adding these three integrals, we get the variational formulation (85) associated with the problem ((65), (68), (69)).
Taking the divergence of the first three equations of (86) we get (71).

Conflict of Interest

The author declares no conicts of interest in this paper.

References

1. Y. Amirat, K. Hamdache and A. Ziani (1989) *Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements missibles en milieux poreux.* Ann. Inst. H. Poincaré. vol. 6, no. 5, pp. 397-417.
2. G. Allaire. (1992) *Homogenization and Two-scale Convergence.* SIAM Journal on Mathematical Analysis. vol. 23, no. 6, pp. 1482-1518. url = http://link.aip.org/link/?SJM/23/1482/1, doi = 10.1137/0523084.

3. G. Allaire and M. Briand. (1996) *Multiscale convergence and reiterated homogenization.* Roy.Soc.Edinburgh. vol. F126, pp. 297-342.
4. Y. Amirat and V. Shelukhin. (2011) *Homogenization of time-harmonic Maxwell equations and the frequency dispersion effect.* J.Maths.Pures.Appl. vol. 95, pp. 420-443.
5. A. Back and E. Frenod. (2015) *Geometric Two-Scale Convergence on Manifold and Applications to the Vlasov Equation Discrete and Continuous Dynamical Systems - Serie S. Special Issue on Numerical Methods based on Homogenization and Two-Scale Convergence.* vol. 8, pp. 223-241.
6. S. Berthier. (1993) *Optique des milieux composites.* Ed. Polytechnicia, p 67.
7. D. Cionarescu and P. Donato. (1999) *An introduction to homogenization.* Oxford University Press.
8. M. Costabel, M. Dauge and S. Nicaise. (2004) *Corner Singularities of Maxwell interface and Eddy current problems.* Advances and Applications, vol. 147, pp. 241-256.
9. N. Crouseilles, E. Frenod, S. Hirstoaga and A. Mouton. (2012) *Two-Scale Macro-Micro decomposition of the Vlasov equation with a strong magnetic field.* Mathematical Models and Methods in Applied Sciences. vol. 23, no. 8, pp. 1527-1559. collaboration = CALVI ; IPSO. doi = 10.1142/S0218202513500152. pdf = http://hal.archives-ouvertes.fr/hal-00638617/PDF/TSAPSVlas_corr.pdf.
10. S. Guenneau, F. Zolla and A. Nicolet. (2007) *Homogenization of 3D finite photonic crystals with heterogeneous permittivity and permeability.* Waves in Random and Complex Media, pp. 653-697.
11. P.R.P. Hoole and S.R.H. Hoole. (1998) *Guided waves along an unmagnetized lightning plasma channel.* IEEE Transactions on Magnetics, vol. 24, no. 6, pp. 3165–3167.
12. M. Leboulch. *Analyse spectrale VHF, UHF du rayonnement des éclairs.* Hamelin, CENT.
13. J.C. Nédélec. (2001) *Acoustic and electromagnetic equations; integral representations for harmonic problems.* Springer-Verlag, Berlin.
14. M. Neuss-Radu. (1996) *Some extensions of two-scale convergence.* Comptes rendus de l'Academie des sciences. Serie 1. 1996, vol. 322, no. 9, pp. 899-904.
15. G. Nguetseng.(1989) *A General Convergence Result for a Functional Related to the Theory of Homogenization.* SIAM, vol. 20, no. 3, pp. 608-623. url = http://link.aip.org/link/?SJM/20/608/1, doi = 10.1137/0520043?
16. G. Nguetseng. (1990) *Asymptotic Analysis for a Stiff Variational Problem Arising in Mechanics.* SIAM Journal on Mathematical Analysis, vol. 21, no. 6, pp. 1394-1414. url = http://link.aip.org/link/?SJM/21/1394/1, doi = 10.1137/0521078.
17. S. Nicaise, S. Hassani and A. Maghnouji. (2009) *Limit behaviors of some boundary value problems with high and/or low valued parameters.* vol. 14, no. 9-10, pp. 875-910.
18. O. Ouchetto, S. Zouhdi and A. Bossavit et al. (2005) *Effective constitutive parameters of periodic composites.* Microwave conference, European, vol. IEE, p. 2.
19. H.E. Pak. (2005) *Geometric two-scale convergence on forms and its applications to Maxwell's equations.* Proceedings of the Royal Society of Edinburgh, European, vol. 135A, pp. 133-147.

20. N. Wellander. (2001) *Homogenization of the Maxwell equations: Case I. Linear theory.* Appl Math, vol. 46, no. 2, pp. 29-51.

21. N. Wellander. (2002) *Homogenization of the Maxwell equations: Case II. Nonlinear conductivity.* Appl Math, vol. 47, no. 3, pp. 255-283.

22. N. Wellander and B. Kristensson. (2002) *Homogenization of the Maxwell equations at fixed frequency.* Technical Report, vol. LUTEDX/TEAT-7103/1-37.

23. Pr. Welter. *Cours : Matériaux diélectriques* Master Matériaux, Institut Le Bel.