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Research Article

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Long term behaviour of singularly perturbed parabolic degenerated equation

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Abstract

In this paper we consider models built in [4] for short-term, mean-term and long-term morphodynamics of dunes and megariples. We give an existence and uniqueness result for long term dynamics of dunes. This result is based on a periodic-in-time-and-space solution existence result for degenerated parabolic equation that we set out. Finally the mean-term and long-term models are homogenized.

Keywords and phrases: Degenerated Parabolic Equation, Space-Time Periodic Solutions, Homogenization, Asymptotic Analysis, Asymptotic Expansion, Long Time Behavior, Dune and Megaripple Morphodynamics, Modeling Coastal Zone Phenomena.

2000 Mathematics Subject Classification. Primary: *35K65, 35B25, 35B10; Secondary: 92F05, 86A60.*

1 Introduction

In Faye, Frénod and Seck [4], based on works of Bagnold [2], Gadd, Lavelle and Swift [6], Idier[7], Astruc and Hulcher [8], Meyer-Peter and Muller [12] and Van Rijn [14], we set out that a relevant model for short term dynamics of dunes, i.e. for their dynamics over several months, in coastal ocean waters submitted to tide is

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{a}{\varepsilon}\nabla\cdot\left((1-b\varepsilon\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\varepsilon\right) = \frac{c}{\varepsilon}\nabla\cdot\left((1-b\varepsilon\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\right), \tag{1.1}$$

where $a>0$, b and c are constants and where $z^\varepsilon = z^\varepsilon(t,x)$, is the dimensionless seabed altitude at t and in x. For a given constant T, $t\in[0,T)$, stands for the dimensionless time and $x=(x_1,x_2)\in\mathbb{T}^2$, $\mathbb{T}^2$ being the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, is the dimensionless position variable.
Operators ∇ and $\nabla\cdot$ refer to gradient and divergence. Functions g_a and g_c are regular and strictly increasing functions on $\mathbb{R}^+$ and satisfy

$$\left\{\begin{array}{c} g_a \geq g_c \geq 0,\ g_c(0)=g_c'(0)=0, \\ \exists d\geq 0,\ \sup_{u\in\mathbb{R}^+}|g_a(u)|+\sup_{u\in\mathbb{R}^+}|g_a'(u)|\leq d,\ \sup_{u\in\mathbb{R}^+}|g_c(u)|+\sup_{u\in\mathbb{R}^+}|g_c'(u)|\leq d, \\ \exists U_{thr}\geq 0,\ \exists G_{thr}>0, \text{ such that } u\geq U_{thr} \Longrightarrow g_a(u)\geq G_{thr}. \end{array}\right. \tag{1.2}$$

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Fields $\mathbf{u} : [0,T)\times\mathbb{T}^2 \to \mathbb{R}^2$ and $\mathbf{m} : [0,T)\times\mathbb{T}^2 \to \mathbb{R}$ are dimensionless water velocity and height. They are given by

$$\mathbf{u}(t,x) = \mathscr{U}(t,\frac{t}{\varepsilon},x), \quad \mathbf{m}(t,x) = \mathscr{M}(t,\frac{t}{\varepsilon},x), \tag{1.3}$$

where

$$\begin{cases} \mathscr{U} = \mathscr{U}(t,\theta,x) \text{ and } \mathscr{M} = \mathscr{M}(t,\theta,x) \text{ are regular functions on } \mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^2, \\ \theta \longmapsto (\mathscr{U},\mathscr{M}) \text{ is periodic of period 1}, \\ |\mathscr{U}|,\ |\frac{\partial\mathscr{U}}{\partial t}|,\ |\frac{\partial\mathscr{U}}{\partial\theta}|,\ |\nabla\cdot\mathscr{U}|,\ |\mathscr{M}|,\ |\frac{\partial\mathscr{M}}{\partial t}|,\ |\frac{\partial\mathscr{M}}{\partial\theta}|,\ |\nabla\mathscr{M}| \text{ are bounded by } d, \\ \forall(t,\theta,x)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^2,\ |\mathscr{U}(t,\theta,x)|\le U_{thr} \Longrightarrow \\ \frac{\partial\mathscr{U}}{\partial t}(t,\theta,x)=0,\ \frac{\partial\mathscr{M}}{\partial t}(t,\theta,x)=0,\ \nabla\mathscr{M}(t,\theta,x)=0 \text{ and } \nabla\cdot\mathscr{U}(t,\theta,x)=0, \\ \exists\,\theta_\alpha<\theta_\omega\in[0,1] \text{ such that } \theta\in[\theta_\alpha,\theta_\omega]\Longrightarrow|\mathscr{U}(t,\theta,x)|\ge U_{thr}. \end{cases} \tag{1.4}$$

A relevant model for mean term, i.e. when dune dynamics is observed over a few years, is

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{a}{\varepsilon}\nabla\cdot\big((1-b\sqrt{\varepsilon}\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\varepsilon\big) = \frac{c}{\varepsilon}\nabla\cdot\left((1-b\sqrt{\varepsilon}\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\right), \tag{1.5}$$

with $a>0$, b and c are constants, with condition (1.2) on g_a and g_c and with $\mathbf{u}$ and $\mathbf{m}$ given by

$$\mathbf{u}(t,x) = \widetilde{\mathscr{U}}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x), \quad \mathbf{m}(t,x) = \mathscr{M}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x), \tag{1.6}$$

For mathematical reasons, we assumed

$$\widetilde{\mathscr{U}}(t,\tau,\theta,x) = \mathscr{U}(t,\theta,x) + \sqrt{\varepsilon}\,\mathscr{U}_1(t,\tau,\theta,x), \tag{1.7}$$

where $\mathscr{U} = \mathscr{U}(t,\theta,x)$ and $\mathscr{U}_1 = \mathscr{U}_1(t,\tau,\theta,x)$ are regular. We also assumed that $\mathscr{M} = \mathscr{M}(t,\tau,\theta,x)$ is regular and

$$\begin{cases} \theta \longmapsto (\mathscr{U},\mathscr{U}_1,\mathscr{M}) \text{ is periodic of period 1}, \\ \tau \longmapsto (\mathscr{U}_1,\mathscr{M}) \text{ is periodic of period 1}, \\ |\mathscr{U}|,\ |\frac{\partial\mathscr{U}}{\partial t}|,\ |\frac{\partial\mathscr{U}}{\partial\theta}|,\ |\nabla\cdot\mathscr{U}|,\ |\mathscr{U}_1|,\ |\frac{\partial\mathscr{U}_1}{\partial t}|,\ |\frac{\partial\mathscr{U}_1}{\partial\tau}|,\ |\frac{\partial\mathscr{U}_1}{\partial\theta}|,\ |\nabla\cdot\mathscr{U}_1|, \\ \qquad |\mathscr{M}|,\ |\frac{\partial\mathscr{M}}{\partial t}|,\ |\frac{\partial\mathscr{M}}{\partial\theta}|,\ |\frac{\partial\mathscr{M}}{\partial\tau}|,\ |\nabla\mathscr{M}| \text{ are bounded by } d, \\ \forall\varepsilon\in(0,1), \forall(t,\tau,\theta,x)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2,\ |\widetilde{\mathscr{U}}(t,\tau,\theta,x)|\le U_{thr} \Longrightarrow \\ \frac{\partial\widetilde{\mathscr{U}}}{\partial t}(t,\tau,\theta,x)=0,\ \frac{\partial\widetilde{\mathscr{U}}}{\partial\tau}(t,\tau,\theta,x)=0,\ \nabla\cdot\widetilde{\mathscr{U}}(t,\tau,\theta,x)=0, \\ \frac{\partial\mathscr{M}}{\partial t}(t,\tau,\theta,x)=0,\ \frac{\partial\mathscr{M}}{\partial\tau}(t,\tau,\theta,x)=0 \text{ and } \nabla\mathscr{M}(t,\tau,\theta,x)=0, \\ \forall\varepsilon\in(0,1), \exists\,\theta_\alpha<\theta_\omega\in[0,1] \text{ not depending on } \varepsilon \text{ such that } \theta\in[\theta_\alpha,\theta_\omega]\Longrightarrow|\widetilde{\mathscr{U}}(t,\tau,\theta,x)|\ge U_{thr}. \end{cases} \tag{1.8}$$

It follows from (1.7) that $\widetilde{\mathscr{U}}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x)$ Three-Scale converges to $\mathscr{U}(t,\theta,x)$ (see subsection *2.2* for the definition of Three-Scale convergence).
A relevant model for long-term dune dynamics is the following equation

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{a}{\varepsilon^2}\nabla\cdot\big((1-b\varepsilon\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\varepsilon\big) = \frac{c}{\varepsilon^2}\nabla\cdot\left((1-b\varepsilon\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\right), \tag{1.9}$$

where $a>0$, b and c are constants, where g_a and g_c satisfy assumption (1.2), and where z^ε is defined on the same space as before. It is also relevant to assume

$$\mathbf{u}(t,x) = \mathscr{U}(t,\frac{t}{\varepsilon},x) = \mathscr{U}_0(\frac{t}{\varepsilon}) + \varepsilon\,\mathscr{U}_1(\frac{t}{\varepsilon},x) + \varepsilon^2\mathscr{U}_2(t,\frac{t}{\varepsilon},x), \tag{1.10}$$

$$\mathbf{m}(t,x) = \mathscr{M}(t,\frac{t}{\varepsilon},x) = \mathscr{M}_1(\frac{t}{\varepsilon},x) + \varepsilon^2\mathscr{M}_2(t,\frac{t}{\varepsilon},x), \tag{1.11}$$

where $\mathscr{U}_0 = \mathscr{U}_0(\theta)$, $\mathscr{U}_1 = \mathscr{U}_1(\theta,x)$, $\mathscr{U}_2 = \mathscr{U}_2(t,\theta,x)$, $\mathscr{M}_1 = \mathscr{M}_1(\theta,x)$ and $\mathscr{M}_2 = \mathscr{M}_2(t,\theta,x)$ are regular and

$$\begin{cases} \theta \longmapsto (\mathscr{U}_0,\mathscr{U}_1,\mathscr{U}_2,\mathscr{M}_1,\mathscr{M}_2) \text{ is periodic of period } 1, \\ |\mathscr{U}_0|,\ |\frac{\partial \mathscr{U}_0}{\partial\theta}|,\ |\mathscr{U}_1|,\ |\frac{\partial \mathscr{U}_1}{\partial\theta}|,\ |\nabla\cdot\mathscr{U}_1|,\ |\mathscr{U}_2|,\ |\frac{\partial \mathscr{U}_2}{\partial t}|,\ |\frac{\partial \mathscr{U}_2}{\partial\theta}|,\ |\nabla\cdot\mathscr{U}_2|,\ |\mathscr{M}_1|,\ |\frac{\partial \mathscr{M}_1}{\partial\theta}|, \\ \qquad |\nabla\mathscr{M}_1|,\ |\mathscr{M}_2|,\ |\frac{\partial \mathscr{M}_2}{\partial t}|,\ |\frac{\partial \mathscr{M}_2}{\partial\theta}|,\ |\nabla\mathscr{M}_2| \text{ are bounded by } d, \\ \forall\theta\in\mathbb{R},\ |\mathscr{U}_0(\theta)| \le U_{thr} \Longrightarrow \mathscr{U}_1(\theta,x) = \mathscr{U}_2(t,\theta,x) = 0 \text{ and } \mathscr{M}_1(\theta,x) = \mathscr{M}_2(t,\theta,x) = 0, \\ \forall\varepsilon\in(0,1), \forall\theta\in\mathbb{R},\ |\mathscr{U}_0(\theta)| > U_{thr} \Longrightarrow \\ \qquad \forall(t,x)\in[0,T)\times\mathbb{T}^2,\ |\mathscr{U}_0(\theta)+\varepsilon\mathscr{U}_1(\theta,x)+\varepsilon^2\mathscr{U}_2(t,\theta,x)| > U_{thr}, \\ \forall\varepsilon\in(0,1), \exists\,\theta_\alpha<\theta_\omega\in[0,1] \text{ not depending on } \varepsilon \text{ such that } \theta\in\mathbb{R},\ \theta\in[\theta_\alpha,\theta_\omega] \\ \qquad \Longrightarrow |\mathscr{U}_0(\theta)+\varepsilon\mathscr{U}_1(\theta,x)+\varepsilon^2\mathscr{U}_2(t,\theta,x)| \ge U_{thr}, \\ \{\bar\theta\in[0,1] \text{ such that } \frac{\partial\mathscr{U}_0}{\partial\theta}(\bar\theta) = 0\} \text{ is a non empty finite union of intervals} \\ \qquad \text{(that can be reduced to points).} \end{cases} \tag{1.12}$$

Remark 1.1. *In formula (1.12), we make an assumption stronger than in the previous cases (for the needs of the proof). This assumption implies that if $|\mathscr{U}(t,\theta,x)| \le U_{thr}$ then $\mathscr{U}(t,\theta,x) = \mathscr{U}_0(\theta)$ does not depend on t and x and $\mathscr{M}(t,\theta,x) = 0$.*

It follows from (1.10) that $\mathbf{u}(t,x) = \mathscr{U}(t,\frac{t}{\varepsilon},x)$ Two-Scale converges to $\mathscr{U}_0(\theta)$ (see subsection *2.2* for the definition of Two-scale convergence).
Equations (1.1), (1.5) or (1.9) need to be provided with an initial condition

$$z^\varepsilon_{|t=0} = z_0, \tag{1.13}$$

giving the shape of the seabed at the initial time.
In [4], we then gave an existence and uniqueness result for short-term model (1.1) if assumptions (1.2), (1.3) and (1.4) are satisfied and for the mean term one (1.5), if assumptions (1.2), (1.6), (1.7) and (1.8) are satisfied. Those results were built on a periodic-in-time-and-space solution existence result for degenerated parabolic equation. Under the same assumptions, the asymptotic behaviour of z^ε, as $\varepsilon \to 0$, solution of short term model (1.1) was also given by homogenization methods. Futhermore if moreover $U_{thr} = 0$, a corrector result was set out, which gives a rigorous version of asymptotic expansion of the sequence z^ε:

$$z^\varepsilon(t,x) \;=\; U(t,\frac{t}{\varepsilon},x) \;+\; \varepsilon U^1(t,\frac{t}{\varepsilon},x) + \dots, \tag{1.14}$$

where U and U^1 are solutions to

$$\frac{\partial U}{\partial\theta} - \nabla\cdot\left(\widetilde{\mathscr{A}}\nabla U\right) = \nabla\cdot\widetilde{\mathscr{C}}, \tag{1.15}$$

$$\frac{\partial U^1}{\partial\theta} - \nabla\cdot\left(\widetilde{\mathscr{A}}\nabla U^1\right) = \nabla\cdot\widetilde{\mathscr{C}_1} + \frac{\partial U}{\partial t} + \nabla\cdot(\widetilde{\mathscr{A}_1}\nabla U), \tag{1.16}$$

where $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{C}}$ are given by

$$\widetilde{\mathscr{A}}(t,\theta,x) = a\,g_a(|\mathscr{U}(t,\theta,x)|) \text{ and } \widetilde{\mathscr{C}}(t,\theta,x) = c\,g_c(|\mathscr{U}(t,\theta,x)|)\,\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}, \tag{1.17}$$

and $\widetilde{\mathscr{A}_1}$ and $\widetilde{\mathscr{C}_1}$ are given by

$$\widetilde{\mathscr{A}_1}(t,\theta,x) = -ab\mathscr{M}(t,\theta,x)\,g_a(|\mathscr{U}(t,\theta,x)|),$$
$$\text{and } \widetilde{\mathscr{C}_1}(t,\theta,x) = -cb\mathscr{M}(t,\theta,x)\,g_c(|\mathscr{U}(t,\theta,x)|)\,\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}. \tag{1.18}$$

In [4], we did not state neither any existence result for long term model (1.9) nor any asymptotic behaviour result for mean term and long term models. Stating those results is the subject of the present paper. We will now state those

main results.
The paper is oragnized as follows: in section 2 we are going to remind the main results of the paper and we recall in the last subsection the notion of two scale convergence. In section 3, we establish the existence and uniqueness results for long term dynamimics of dunes. The section 4 concerns the homogenization results of long term model and the last section is devoted to the homegenization and corrector results to the mean term model.

2 Results and preliminaries

2.1 Results

The first one concerns existence and uniqueness for the long-term model.

Theorem 2.1. *For any $T>0$, any $a>0$, any real constants b and c and any $\varepsilon\in(0,1)$, under assumptions (1.2), (1.10), (1.11) and (1.12), if*

$$z_0 \in H^1(\mathbb{T}^2), \tag{2.19}$$

there exists a unique function $z^\varepsilon\in L^\infty([0,T),L^2(\mathbb{T}^2))$ with $\sqrt{(1-b\varepsilon\mathbf{m})g_a(|\mathbf{u}|)}\nabla z^\varepsilon\in L^2((0,T),\,L^2(\mathbb{T}^2))$, solution to equation (1.9) provided with initial condition (1.13).
Moreover, for any $t\in[0,T]$, z^ε satisfies

$$\|z^\varepsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))}\leq\widetilde{\gamma}, \tag{2.20}$$

for a constant $\widetilde{\gamma}$ not depending on ε and

$$\frac{d\left(\int_{\mathbb{T}^2} z^\varepsilon(t,x)\,dx\right)}{dt}=0. \tag{2.21}$$

In formulas of this theorem, ∇z^ε is understood in the distribution sense, and for z^ε, being solution to (1.9) and (1.13) is understood in the following sense

$$\begin{aligned}-\int_0^T\int_{\mathbb{T}^2} z^\varepsilon\frac{\partial\varphi}{\partial t}\,dxdt+\frac{a}{\varepsilon^2}\int_0^T\int_{\mathbb{T}^2}\Big((1-b\varepsilon\,\mathbf{m})g_a(|\mathbf{u}|)\nabla z^\varepsilon\nabla\varphi\,dxdt&=\int_{\mathbb{T}^2} z_0(x)\varphi(0,x)dx\\ +\frac{c}{\varepsilon^2}\int_0^T\int_{\mathbb{T}^2}\nabla\cdot\Big(\big(1-b\varepsilon\,\mathbf{m}\big)g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|}\Big)\varphi\,dxdt&\\ \text{for every } \varphi\in\mathscr{D}([0,T)\times\mathbb{T}^2).&\end{aligned} \tag{2.22}$$

The proof of Theorem 2.1 is done in section 3, except equality (2.21) which is directly gotten by integrating (1.9) with respect to x over $\mathbb{T}^2$.
We now give a result concerning the asymptotic behaviour as $\varepsilon\to0$ of the long term model.
We notice that, because of hypothesis (1.12), when $|\mathscr{U}_0|\leq U_{thr}$, we have $\mathscr{U}(t,\theta,x)=\mathscr{U}_0(\theta)$ and $\mathscr{M}(t,\theta,x)=0$.
Moreover we denote

$$\Theta=[0,T)\times\{\theta\in\mathbb{R},\;g_a(|\mathscr{U}_0(\theta)|)=0\}\times\mathbb{T}^2, \tag{2.23}$$

and

$$\Theta_{thr}=\{(t,\theta,x)\in[0,T)\times\mathbb{R}\times\mathbb{T}^2,\;|\mathscr{U}_0(\theta)|<U_{thr}\}. \tag{2.24}$$

We now give our first asymptotic result.

Theorem 2.2. *For any $T>0$, under the same assumptions as in Theorem 2.1, the sequence of solutions (z^ε) to equation (1.9) given by Theorem 2.1 Two-Scale converges to the profile $U\in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$ which is given by*

$$U=\int_{\mathbb{T}^2} z_0\,dx. \tag{2.25}$$

Above and in the sequel, for all $p \geq 1$ and $q \geq 1$, we denote by

$$L^p_\#(\mathbb{R}, L^q(\mathbb{T}^2)) = \Big\{ f : \mathbb{R} \longrightarrow L^q(\mathbb{T}^2) \text{ mesurable and periodic of period 1 in } \theta \text{ such that } \theta \mapsto \|f(\theta)\|_{L^q(\mathbb{T}^2)} \in L^p([0,1]) \Big\}. \quad (2.26)$$

Now we turn to mean term model for which we set out the asymptotic behaviour.

Theorem 2.3. *Under assumptions (1.2), (1.6), (1.7) and (1.8), for any T, not depending on ε, the sequence (z^ε) of solutions to (1.5) built in [4] provided with initial condition (1.13) Three-Scale converges to the profile $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))))$ solution to*

$$\frac{\partial U}{\partial \theta} - \nabla \cdot (\widetilde{\mathscr{A}} \nabla U) = \nabla \cdot \widetilde{\mathscr{C}}, \quad (2.27)$$

where $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{C}}$ are given by

$$\widetilde{\mathscr{A}}(t,\tau,\theta,x) = a\, g_a(|\mathscr{U}(t,\theta,x)|) \text{ and } \widetilde{\mathscr{C}}(t,\tau,\theta,x) = c\, g_c(|\mathscr{U}(t,\theta,x)|)\, \frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}, \quad (2.28)$$

with $\mathscr{U}$ given in (1.7).

Finally, a corrector result for the mean-term model is given under restrictive assumptions.

Theorem 2.4. *Under assumptions (1.2), (1.6), (1.7), (1.8) and if moreover $U_{thr} = 0$, considering function $z^\varepsilon \in L^\infty([0,T), L^2(\mathbb{T}^2))$, solution to (1.5) with initial condition (1.13) and function $U^\varepsilon \in L^\infty([0,T), L^2(\mathbb{T}^2))$ defined by*

$$U^\varepsilon(t,x) = U(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x), \quad (2.29)$$

where U is the solution to (2.27), the following estimate is satisfied:

$$\Big\| \frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}} \Big\|_{L^\infty([0,T), L^2(\mathbb{T}^2))} \leq \alpha, \quad (2.30)$$

where α is a constant not depending on ε.
Furthermore,

$$\frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}} \quad \text{Three-Scale converges to a profile } U_{\frac{1}{2}} \in L^\infty([0,T) \times \mathbb{R}, L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))), \quad (2.31)$$

which is the unique solution to

$$\frac{\partial U_{\frac{1}{2}}}{\partial \theta} - \nabla \cdot \left(\widetilde{\mathscr{A}} \nabla U_{\frac{1}{2}} \right) = \nabla \cdot \widetilde{\mathscr{C}_1} + \nabla \cdot \left(\widetilde{\mathscr{A}_1} \nabla U \right) - \frac{\partial U}{\partial \tau} \quad (2.32)$$

where $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{C}}$ are given by (2.28) and where $\widetilde{\mathscr{A}_1}$ and $\widetilde{\mathscr{C}_1}$ are given by

$$\widetilde{\mathscr{A}_1}(t,\tau,\theta,x) = -ab\mathscr{M}(t,\tau,\theta,x)\, g_a(|\mathscr{U}(t,\theta,x)|),$$
$$\text{and } \widetilde{\mathscr{C}_1}(t,\tau,\theta,x) = -cb\mathscr{M}(t,\tau,\theta,x)\, g_c(|\mathscr{U}(t,\theta,x)|)\, \frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}, \quad (2.33)$$

with $\mathscr{U}$ given in (1.7).

2.2 On Two-Scale and Three-Scale convergence

In this subsection we are going to recall the notion of Two-Scale and Three-Scale convergence.

Definition 2.1. *A sequence of functions (z^ε) in $L^\infty([0,T),L^2(\mathbb{T}^2))$ is said to Two-Scale converge to $U \in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$ if for every $\psi \in \mathscr{C}([0,T),\mathscr{C}_\#(\mathbb{R},\mathscr{C}(\mathbb{T}^2)))$ we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^2}\int_0^T z^\varepsilon(t,x)\psi(t,\frac{t}{\varepsilon},x)dt\,dx = \int_{\mathbb{T}^2}\int_0^T\int_0^1 U(t,\theta,x)\psi(t,\theta,x)d\theta\,dt\,dx. \tag{2.34}$$

A sequence of functions (u^ε) in $L^\infty([0,T),L^2(\mathbb{T}^2))$ is said to Three-Scale converge to $U \in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))))$ if for every $\psi \in \mathscr{C}([0,T),\mathscr{C}_\#(\mathbb{R},\mathscr{C}_\#(\mathbb{R},\mathscr{C}(\mathbb{T}^2))))$ we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^2}\int_0^T u^\varepsilon(t,x)\psi(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x)dt\,dx = \int_{\mathbb{T}^2}\int_0^T\int_{[0,1]^2} U(t,\tau,\theta,x)\psi(t,\tau,\theta,x)d\tau\,d\theta\,dt\,dx. \tag{2.35}$$

In [1] and [13], the following theorem is also given.

Theorem 2.5. *If a sequence (z^ε) is bounded in $L^\infty([0,T),L^2(\mathbb{T}^2))$, there exists a subsequence still denoted (z^ε) and a function $U \in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$[resp. $L^\infty([0,T),L^\infty_\#(\mathbb{R},L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))))$] such that*

$$u^\varepsilon \longrightarrow U \text{ Two-Scale [resp. Three-Scale]}. \tag{2.36}$$

3 Existence and estimates, proof of Theorem 2.1

Setting

$$\mathscr{A}^\varepsilon(t,x) = \widetilde{\mathscr{A}_\varepsilon}(t,\frac{t}{\varepsilon},x), \tag{3.37}$$

and

$$\mathscr{C}^\varepsilon(t,x) = \widetilde{\mathscr{C}_\varepsilon}(t,\frac{t}{\varepsilon},x), \tag{3.38}$$

where

$$\widetilde{\mathscr{A}_\varepsilon}(t,\theta,x) = a(1-b\varepsilon\mathscr{M}(t,\theta,x))\,g_a(|\mathscr{U}(t,\theta,x)|), \tag{3.39}$$

and

$$\widetilde{\mathscr{C}_\varepsilon}(t,\theta,x) = c(1-b\varepsilon\mathscr{M}(t,\theta,x))\,g_c(|\mathscr{U}(t,\theta,x)|)\,\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}, \tag{3.40}$$

where $\mathscr{U}$ is given in (1.10) and $\mathscr{M}$ is given in (1.11), equation (1.9) with initial condition (1.13) can be set in the form

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} - \dfrac{1}{\varepsilon^2}\nabla\cdot(\mathscr{A}^\varepsilon\nabla z^\varepsilon) = \dfrac{1}{\varepsilon^2}\nabla\cdot\mathscr{C}^\varepsilon, \\ z^\varepsilon_{|t=0} = z_0, \end{cases} \tag{3.41}$$

or more precisely

$$-\int_0^T\int_{\mathbb{T}^2} z^\varepsilon\frac{\partial\varphi}{\partial t}\,dx\,dt + \frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{T}^2}\mathscr{A}^\varepsilon\nabla z^\varepsilon\cdot\nabla\varphi\,dx\,dt = \int_{\mathbb{T}^2} z_0\,\varphi(0,\cdot)dx + \frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{T}^2}(\nabla\cdot\mathscr{C}^\varepsilon)\,\varphi\,dxdt$$
$$\text{for every } \varphi \in \mathscr{D}([0,T)\times\mathbb{T}^2). \tag{3.42}$$

Because of assumption (1.10) and under assumptions (1.2) and (1.12), $\widetilde{\mathscr{A}_\varepsilon}$ and $\widetilde{\mathscr{C}_\varepsilon}$ given by (3.39) and (3.40) satisfy the following hypotheses:

$$\begin{cases} \theta \longmapsto (\widetilde{\mathscr{A}_\varepsilon},\widetilde{\mathscr{C}_\varepsilon}) \text{ is periodic of period 1}, \\ x \longmapsto (\widetilde{\mathscr{A}_\varepsilon},\widetilde{\mathscr{C}_\varepsilon}) \text{ is defined on } \mathbb{T}^2, \\ |\widetilde{\mathscr{A}_\varepsilon}| \le \gamma,\ |\widetilde{\mathscr{C}_\varepsilon}| \le \gamma,\ \left|\dfrac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\right| \le \varepsilon^2\gamma,\ \left|\dfrac{\partial\widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right| \le \varepsilon^2\gamma,\ \left|\dfrac{\partial\nabla\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\right| \le \varepsilon^2\gamma, \\ \left|\dfrac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial\theta}\right| \le \gamma,\ \left|\dfrac{\partial\widetilde{\mathscr{C}_\varepsilon}}{\partial\theta}\right| \le \gamma,\ |\nabla\widetilde{\mathscr{A}_\varepsilon}| \le \varepsilon\gamma,\ |\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}| \le \varepsilon\gamma,\ \left|\dfrac{\partial\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right| \le \varepsilon^2\gamma, \end{cases} \tag{3.43}$$

$$\begin{cases} \exists \widetilde{G}_{thr} > 0,\ \theta_\alpha < \theta_\omega \in [0,1] \text{ not depending on } \varepsilon \text{ such that } \theta \in [\theta_\alpha, \theta_\omega] \Longrightarrow \widetilde{\mathscr{A}_\varepsilon}(t,\theta,x) \geq \widetilde{G}_{thr}, \\ \widetilde{\mathscr{A}_\varepsilon}(t,\theta,x) \leq \widetilde{G}_{thr} \Longrightarrow \begin{cases} \dfrac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}(t,\theta,x) = 0,\ \nabla \widetilde{\mathscr{A}_\varepsilon}(t,\theta,x) = 0, \\ \dfrac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}(t,\theta,x) = 0,\ \nabla \cdot \widetilde{\mathscr{C}_\varepsilon}(t,\theta,x) = 0, \end{cases} \end{cases} \tag{3.44}$$

and

$$\begin{cases} |\widetilde{\mathscr{C}_\varepsilon}| \leq \gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ |\widetilde{\mathscr{C}_\varepsilon}|^2 \leq \gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ |\nabla \widetilde{\mathscr{A}_\varepsilon}| \leq \varepsilon\gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ \left|\dfrac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\right| \leq \varepsilon^2 \gamma |\widetilde{\mathscr{A}_\varepsilon}|, \\ \left|\dfrac{\partial (\nabla \widetilde{\mathscr{A}_\varepsilon})}{\partial t}\right|^2 \leq \varepsilon^2 \gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ \left|\nabla \cdot \widetilde{\mathscr{C}_\varepsilon}\right| \leq \varepsilon\gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right| \leq \varepsilon^2 \gamma |\widetilde{\mathscr{A}_\varepsilon}|,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right|^2 \leq \varepsilon^2 \gamma^2 |\widetilde{\mathscr{A}_\varepsilon}|, \end{cases} \tag{3.45}$$

for a constant γ depending only on a, b, c and d.
The proofs of inequalities in (3.43) and (3.45) are all done following identical techniques. We are going to establish some of them. For instance, we estabilish, easily:

$$\left|\widetilde{\mathscr{A}_\varepsilon}(t,\theta,x)\right| = \left|a(1 - b\varepsilon \mathscr{M}(t,\theta,x))\, g_a(|\mathscr{U}(t,\theta,x)|)\right| \leq |a||g_a(|\mathscr{U}(t,\theta,x)|)| \leq ad,$$

$$\begin{aligned} \left|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\right| &= \left| -ab\varepsilon^2 \varepsilon \frac{\partial \mathscr{M}_2(t,\theta,x)}{\partial t} g_a(|\mathscr{U}(t,\theta,x)|) + a(1 - b\varepsilon \mathscr{M}(t,\theta,x))\varepsilon^2 \frac{\partial \mathscr{U}_2(t,\theta,x)}{\partial t} g_a'(|\mathscr{U}(t,\theta,x)|)\right| \\ &\leq \varepsilon^2 (ab\varepsilon d^2 + ad^2) = \varepsilon^2 \gamma, \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial \theta}\right| &= \Big| -ab\varepsilon\big(\frac{\partial \mathscr{M}_1}{\partial \theta}(\theta,x) + \varepsilon \frac{\partial \mathscr{M}_2}{\partial \theta}(t,\theta,x)\big) g_a(|\mathscr{U}(t,\theta,x)|) \\ &\quad + a(1 - b\varepsilon \mathscr{M}(t,\theta,x)) \frac{\partial \mathscr{U}}{\partial \theta}(t,\theta,x) g_a'(|\mathscr{U}(t,\theta,x)|)\Big| \leq \gamma. \end{aligned}$$

We have also

$$\begin{aligned} \left|\nabla \cdot \widetilde{\mathscr{C}_\varepsilon}\right| &= \Big| \frac{\partial}{\partial x_1}\Big(c(1 - b\varepsilon \mathscr{M}(t,\theta,x)) g_c(|\mathscr{U}(t,\theta,x)|) \frac{\mathscr{U}_1(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}\Big) \\ &\quad + \frac{\partial}{\partial x_2}\Big(c(1 - b\varepsilon \mathscr{M}(t,\theta,x)) g_c(|\mathscr{U}(t,\theta,x)|) \frac{\mathscr{U}_2(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}\Big)\Big| \\ &\leq \gamma_1 \Big[\Big| \frac{\partial}{\partial x_1}\Big(c(1 - b\varepsilon \mathscr{M}) g_c(|\mathscr{U}|)\Big) + \frac{\partial}{\partial x_2}\Big(c(1 - b\varepsilon \mathscr{M}) g_c(|\mathscr{U}|)\Big)\Big|\Big] \\ &= \gamma_1 \Big| \underbrace{-cb\varepsilon \frac{\partial \mathscr{M}}{\partial x_1} g_c(|\mathscr{U}|) + c(1 - b\varepsilon \mathscr{M}) g_c'(|\mathscr{U}|) \frac{\partial |\mathscr{U}|}{\partial x_1} - cb\varepsilon \frac{\partial \mathscr{M}}{\partial x_2} g_c(|\mathscr{U}|) + c(1 - b\varepsilon \mathscr{M}) g_c'(|\mathscr{U}|) \frac{\partial |\mathscr{U}|}{\partial x_2}}_{\leq \gamma_2 |\nabla \mathscr{A}^\varepsilon|} \Big|, \end{aligned}$$

and using the fact that $|\nabla \widetilde{\mathscr{A}_\varepsilon}| \leq \varepsilon \gamma_3 |\widetilde{\mathscr{A}_\varepsilon}|$, we get

$$\left|\nabla \cdot \widetilde{\mathscr{C}_\varepsilon}\right| \leq \gamma_1 \gamma_2 \gamma_3 \varepsilon |\widetilde{\mathscr{A}_\varepsilon}| = \varepsilon \gamma |\widetilde{\mathscr{A}_\varepsilon}|.$$

The other inequalities are obtained in a similar way.
In this section we focus on existence and uniqueness of time-space periodic parabolic equations. From this, we then get existence of solution to equation (3.41). Existence of z^ε over a time interval depending on ε, is a straightforward consequence of adaptations of results from LadyzensKaja, Solonnikov and Ural' Ceva [9] or Lions [10]. Our aim is to prove that z^ε solution to (3.41) is bounded indepently of ε. We are going to introduce the following regularized equations.

$$\frac{\partial \mathscr{S}^\nu}{\partial \theta} - \frac{1}{\varepsilon} \nabla \cdot \Big(\big(\tilde{\mathscr{A}_\varepsilon}(t,\cdot,\cdot) + \nu\big) \nabla \mathscr{S}^\nu \Big) = \frac{1}{\varepsilon} \nabla \cdot \tilde{\mathscr{C}_\varepsilon}(t,\cdot,\cdot), \tag{3.46}$$

and

$$\mu \mathscr{S}_\mu^\nu + \frac{\partial \mathscr{S}_\mu^\nu}{\partial \theta} - \frac{1}{\varepsilon} \nabla \cdot \Big(\big(\tilde{\mathscr{A}_\varepsilon}(t,\cdot,\cdot) + \nu\big) \nabla \mathscr{S}_\mu^\nu \Big) = \frac{1}{\varepsilon} \nabla \cdot \tilde{\mathscr{C}_\varepsilon}(t,\cdot,\cdot), \tag{3.47}$$

where μ and ν are positive parameters.
We first prove existence of solutions $\mathscr{S}_\mu^\nu$ of (3.47) and we give estimates of $\mathscr{S}_\mu^\nu$.

Theorem 3.1. *Under assumptions (3.43), (3.44) and (3.45), for any $\mu > 0$ and any $\nu > 0$, there exists a unique $\mathscr{S}_\mu^\nu = \mathscr{S}_\mu^\nu(t,\theta,x) \in \mathscr{C}^0 \cap L^2(\mathbb{R}\times\mathbb{T}^2)$, periodic of period 1 with respect to θ, solution to (3.47) and regular with respect to the parameter t. Moreover, the following estimates are satisfied*

$$\sup_{\theta\in\mathbb{R}}\left|\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu(\theta,x)dx\right| = 0, \tag{3.48}$$

$$\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\gamma}{\nu}, \tag{3.49}$$

$$\|\Delta\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \sqrt{2}\frac{\varepsilon\gamma}{\nu}\sqrt{\frac{\gamma^2}{\nu^2}+1}, \tag{3.50}$$

$$\left\|\frac{\partial\mathscr{S}_\mu^\nu}{\partial\theta}\right\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\gamma}{\sqrt{\varepsilon\nu}}\sqrt{(\frac{\gamma}{2\nu}+1)}, \tag{3.51}$$

$$\|\nabla\mathscr{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \sqrt{\frac{\gamma^2}{\nu^2}+\frac{2\varepsilon\gamma^2}{\nu}(\frac{\gamma^2}{\nu^2}+1)}, \tag{3.52}$$

$$\|\mathscr{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \sqrt{\frac{\gamma^2}{\nu^2}+\frac{2\varepsilon\gamma^2}{\nu}(\frac{\gamma^2}{\nu^2}+1)}, \tag{3.53}$$

$$\left\|\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\right\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \varepsilon^3\frac{\gamma}{\nu}(1+\frac{\gamma}{\nu}). \tag{3.54}$$

Proof. .(of Theorem 3.1). The proof of this theorem is very similar to the one of Theorem 3.3 of Faye, Frénod and Seck [4]. The big difference is the presence of $\frac{1}{\varepsilon}-$ factors in (3.47). Hence we only sketch the most similar arguments and focus on the management of those $\frac{1}{\varepsilon}-$factors.
In a first place, to prove existence of $\mathscr{S}_\mu^\nu$, we consider for $\xi \in L^2(\mathbb{T}^2)$, the solution ξ_μ^ν to

$$\begin{cases} \mu\xi_\mu^\nu + \dfrac{\partial\xi_\mu^\nu}{\partial\theta} - \dfrac{1}{\varepsilon}\nabla\cdot\Big((\widetilde{\mathscr{A}_\varepsilon}+\nu)\nabla\xi_\mu^\nu\Big) = \dfrac{1}{\varepsilon}\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}, \\ \xi^\nu_{\mu|\theta=0} = \xi, \end{cases} \tag{3.55}$$

where $\widetilde{\mathscr{A}_\varepsilon}+\nu > 0$; whose existence and uniqueness on a finite interval is a direct consequence of Ladyzenkaja, Solonnikov and Ural'Ceva [9] or Lions [10]. We can prove that the application $\square : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2),\ \xi \longmapsto \xi_\mu^\nu(1,\cdot)$ is a strict contraction; then there exists $\xi \in L^2(\mathbb{T}^2)$ such that $\square\xi = \xi$. We conclude that the solution $\mathscr{S}_\mu^\nu$ of (3.47) is the solution ξ_μ^ν solution of (3.55) associated with the initial condition ξ such that $\square\xi = \xi$. Then we prove existence and uniqueness of $\mathscr{S}_\mu^\nu$.
Integrating equation (3.47) over $\mathbb{T}^2$ gives

$$\mu\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu dx + \int_{\mathbb{T}^2}\frac{\partial\mathscr{S}_\mu^\nu}{\partial\theta}dx - \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\Big((\tilde{\mathscr{A}_\varepsilon}+\nu)\nabla\mathscr{S}_\mu^\nu\Big)dx = \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}dx, \tag{3.56}$$

then

$$\mu\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu dx + \frac{d(\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu dx)}{d\theta} = 0,$$

which gives

$$\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu(\theta,x)dx = \int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu(\tilde\theta,x)e^{-\mu(\theta-\tilde\theta)}dx.$$

Since $\mathscr{S}_\mu^\nu$ is periodic of period 1 with respect to θ, $\int_{\mathbb{T}^2}\mathscr{S}_\mu^\nu(\theta,x)dx$ is also periodic of period 1. Then (3.48) is true. Multiplying equation (3.47) by $\mathscr{S}_\mu^\nu$, integrating over $\mathbb{T}^2$ and from 0 to 1 with respect to θ gives

$$\mu\|\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}+\frac{1}{2}\int_0^1\frac{d\|\mathscr{S}_\mu^\nu\|_2^2}{d\theta}d\theta+\frac{1}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}(\tilde{\mathscr{A}}_\varepsilon+\nu)|\nabla\mathscr{S}_\mu^\nu|^2dxd\theta\le\frac{\gamma}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|dxd\theta.$$

Since $\tilde{\mathscr{A}_\varepsilon}+\nu\ge\nu$ and taking into account that the above first term is positive and the second one equals zero, we have

$$\frac{\nu}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dxd\theta\le\frac{\gamma}{\varepsilon}\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))},$$

then

$$\|\nabla\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\frac{\gamma}{\nu}\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))},$$

which gives (3.49).
Multiplying (3.47) by $\frac{\partial\mathscr{S}_\mu^\nu}{\partial\theta}$, integrating over $\mathbb{T}^2$ and integrating from 0 to 1 with respect to θ gives

$$\Big\|\frac{\partial\mathscr{S}_\mu^\nu}{\partial\theta}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}=\frac{1}{2\varepsilon}\int_0^1\int_{\mathbb{T}^2}\frac{\partial\tilde{\mathscr{A}}_\varepsilon}{\partial\theta}|\nabla\mathscr{S}_\mu^\nu|^2dxd\theta+\frac{1}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}\frac{\partial\tilde{\mathscr{C}}_\varepsilon}{\partial\theta}\nabla\mathscr{S}_\mu^\nu dxd\theta \tag{3.57}$$

$$\le\frac{\gamma}{\varepsilon}\Big(\frac{1}{2}\|\nabla\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}+\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\Big), \tag{3.58}$$

which gives (3.51).
Multiplying (3.47) by $-\Delta\mathscr{S}_\mu^\nu$, and integrating over $\mathbb{T}^2$ gives

$$\mu\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dx+\int_{\mathbb{T}^2}\nabla\mathscr{S}_\mu^\nu\cdot\nabla\Big(\frac{\partial\mathscr{S}_\mu^\nu}{\partial\theta}\Big)dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\tilde{\mathscr{A}}_\varepsilon\cdot\nabla\mathscr{S}_\mu^\nu\Delta\mathscr{S}_\mu^\nu dx+$$
$$\frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\tilde{\mathscr{A}}_\varepsilon+\nu)|\Delta\mathscr{S}_\mu^\nu|^2dx=-\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\tilde{\mathscr{C}}_\varepsilon\Delta\mathscr{S}_\mu^\nu dx,$$

or

$$\mu\|\nabla\mathscr{S}_\mu^\nu\|_2^2+\frac{1}{2}\frac{d\|\nabla\mathscr{S}_\mu^\nu\|_2^2}{d\theta}+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\tilde{\mathscr{A}}_\varepsilon+\nu)|\Delta\mathscr{S}_\mu^\nu|^2dx=$$
$$-\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\tilde{\mathscr{A}}_\varepsilon\cdot\nabla\mathscr{S}_\mu^\nu\Delta\mathscr{S}_\mu^\nu dx-\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\tilde{\mathscr{C}}_\varepsilon\Delta\mathscr{S}_\mu^\nu dx.$$

Since for any real number U and V

$$|UV|\le\frac{\tilde{\mathscr{A}}_\varepsilon+\nu}{4\varepsilon}U^2+\frac{\varepsilon}{\tilde{\mathscr{A}}_\varepsilon+\nu}V^2, \tag{3.59}$$

using this formula with $U=\Delta\mathscr{S}_\mu^\nu$, $V=\frac{\nabla\tilde{\mathscr{A}}_\varepsilon\cdot\nabla\mathscr{S}_\mu^\nu}{\varepsilon}$, we have

$$\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\tilde{\mathscr{A}}_\varepsilon\cdot\nabla\mathscr{S}_\mu^\nu\Delta\mathscr{S}_\mu^\nu dx\le\int_{\mathbb{T}^2}\frac{\tilde{\mathscr{A}}_\varepsilon+\nu}{4\varepsilon}|\Delta\mathscr{S}_\mu^\nu|^2dx+\int_{\mathbb{T}^2}\frac{1}{\varepsilon(\tilde{\mathscr{A}}_\varepsilon+\nu)}|\nabla\tilde{\mathscr{A}}_\varepsilon\cdot\nabla\mathscr{S}_\mu^\nu|^2dx.$$

Taking $U=\Delta\mathscr{S}_\mu^\nu$, $V=\frac{\nabla\cdot\tilde{\mathscr{C}}_\varepsilon}{\varepsilon}$ and using again (3.59) we obtain

$$\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\tilde{\mathscr{C}}_\varepsilon\Delta\mathscr{S}_\mu^\nu\le\int_{\mathbb{T}^2}\frac{\tilde{\mathscr{A}}_\varepsilon+\nu}{4\varepsilon}|\Delta\mathscr{S}_\mu^\nu|^2dx+\int_{\mathbb{T}^2}\frac{1}{\varepsilon(\tilde{\mathscr{A}}_\varepsilon+\nu)}|\nabla\cdot\tilde{\mathscr{C}}_\varepsilon|^2dx.$$

These two results give

$$\mu\|\nabla\mathscr{S}_\mu^\nu\|_2^2+\frac{1}{2}\frac{d\|\nabla\mathscr{S}_\mu^\nu\|_2^2}{d\theta}+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\tilde{\mathscr{A}}_\varepsilon+\nu)|\Delta\mathscr{S}_\mu^\nu|^2dx\le$$

$$\int_{\mathbb{T}^2}\frac{\tilde{\mathscr{A}_\varepsilon}+\nu}{2\varepsilon}|\Delta\mathscr{S}_\mu^\nu|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\frac{1}{(\tilde{\mathscr{A}_\varepsilon}+\nu)}\Big(|\nabla\tilde{\mathscr{A}_\varepsilon}\cdot\nabla\mathscr{S}_\mu^\nu|^2+|\nabla\cdot\tilde{\mathscr{C}_\varepsilon}|^2\Big)dx, \tag{3.60}$$

or, using (3.43),

$$\mu\|\nabla\mathscr{S}_\mu^\nu\|_2^2+\frac{1}{2}\frac{d\|\nabla\mathscr{S}_\mu^\nu\|_2^2}{d\theta}+\int_{\mathbb{T}^2}\frac{(\tilde{\mathscr{A}_\varepsilon}+\nu)}{2\varepsilon}|\Delta\mathscr{S}_\mu^\nu|^2dx\le\frac{\varepsilon^2\gamma^2}{\nu\varepsilon}\Big(\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dx+1\Big), \tag{3.61}$$

and integrating from 0 to 1 with respect to θ, we have

$$\mu\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}+\int_0^1\int_{\mathbb{T}^2}\frac{(\tilde{\mathscr{A}_\varepsilon}+\nu)}{2\varepsilon}|\Delta\mathscr{S}_\mu^\nu|^2dxd\theta\le\frac{\varepsilon\gamma^2}{\nu}\Big(\int_0^1\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dxd\theta+1\Big).$$

From this last inequality, we deduce

$$\frac{\nu}{2\varepsilon}\|\Delta\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\frac{\varepsilon\gamma^2}{\nu}\big(\frac{\gamma^2}{\nu^2}+1\big),$$

then

$$\|\Delta\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\frac{2\varepsilon^2\gamma^2}{\nu^2}\big(\frac{\gamma^2}{\nu^2}+1\big),$$

which gives (3.50).
As $\|\nabla\mathscr{S}_\mu^\nu\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}$ is bounded by $\frac{\gamma}{\nu}$ (see (3.49)), we can deduce that there exists a $\theta_0\in[0,1]$ such that

$$\|\nabla\mathscr{S}_\mu^\nu(\theta_0,\cdot)\|_2\le\frac{\gamma}{\nu}. \tag{3.62}$$

From (3.61) we have

$$\frac{d\|\nabla\mathscr{S}_\mu^\nu\|_2^2}{d\theta}\le\frac{2\varepsilon\gamma^2}{\nu}\big(\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dx+1\big). \tag{3.63}$$

Integrating (3.63) from θ_0 to an other $\theta_1\in[0,1]$ gives

$$\begin{aligned}\|\nabla\mathscr{S}_\mu^\nu(\theta_1,\cdot)\|_2^2-\|\nabla\mathscr{S}_\mu^\nu(\theta_0,\cdot)\|_2^2&\le\frac{2\varepsilon\gamma^2}{\nu}\int_{\theta_0}^{\theta_1}\left(\int_{\mathbb{T}^2}|\nabla\mathscr{S}_\mu^\nu|^2dx+1\right)d\theta\\&\le\frac{2\varepsilon\gamma^2}{\nu}\left(\|\nabla\mathscr{S}_\mu^\nu\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}+1\right),\end{aligned} \tag{3.64}$$

giving the sought bound on $\|\nabla\mathscr{S}_\mu^\nu(\theta_1,\cdot)\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))}$ for any θ_1 or, in other words (3.52).
Using Fourier expansion argument, because of (3.48), we have

$$\|\mathscr{S}_\mu^\nu(\theta,\cdot)\|_2^2\le\|\nabla\mathscr{S}_\mu^\nu(\theta,\cdot)\|_2^2\le\frac{\gamma^2}{\nu^2}+2\frac{\varepsilon\gamma^2}{\nu}\big(\frac{\gamma^2}{\nu^2}+1\big), \tag{3.65}$$

and then (3.53).
We have that $\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}$ is solution to

$$\mu\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}+\frac{\partial\left(\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\right)}{\partial\theta}-\frac{1}{\varepsilon}\nabla\cdot\Big((\tilde{\mathscr{A}_\varepsilon}+\nu)\nabla\big(\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\big)\Big)=\frac{1}{\varepsilon}\nabla\cdot\Big(\frac{\partial\tilde{\mathscr{C}_\varepsilon}}{\partial t}\Big)+\frac{1}{\varepsilon}\nabla\cdot\Big(\frac{\partial\tilde{\mathscr{A}_\varepsilon}}{\partial t}\nabla\mathscr{S}_\mu^\nu\Big), \tag{3.66}$$

from which we deduce

$$\mu\Big\|\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\Big\|_2^2+\frac{1}{2}\frac{d\Big\|\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\Big\|_2^2}{d\theta}+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\tilde{\mathscr{A}_\varepsilon}+\nu)\Big|\nabla\big(\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\big)\Big|^2dx=-\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\check{\mathscr{C}_\varepsilon}\cdot\nabla\big(\frac{\partial\mathscr{S}_\mu^\nu}{\partial t}\big)dx, \tag{3.67}$$

where

$$\check{\mathscr{C}}_\varepsilon = \frac{\partial \tilde{\mathscr{C}}_\varepsilon}{\partial t} + \frac{\partial \tilde{\mathscr{A}}_\varepsilon}{\partial t} \nabla \mathscr{S}_\mu^\nu, \quad \nabla \cdot \check{\mathscr{C}}_\varepsilon = \nabla \cdot \Big(\frac{\partial \tilde{\mathscr{C}}_\varepsilon}{\partial t} + \frac{\partial \tilde{\mathscr{A}}_\varepsilon}{\partial t} \nabla \mathscr{S}_\mu^\nu \Big). \tag{3.68}$$

From (3.43), (3.49) and (3.50), we have

$$\Big\| \check{\mathscr{C}}_\varepsilon \Big\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^2 \gamma (1 + \frac{\gamma}{\nu}), \ \Big\| \nabla \cdot \check{\mathscr{C}}_\varepsilon \Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^2 \gamma \Big(1 + \frac{\gamma}{\nu} + \varepsilon \sqrt{\varepsilon} \frac{\gamma}{\nu} \sqrt{\frac{\gamma}{\nu} + 1} \Big). \tag{3.69}$$

Integrating (3.67) from 0 to 1 with respect to the variable θ, we obtain

$$\frac{\nu}{\varepsilon} \Big\| \nabla \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} \Big\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^2 \gamma (1 + \frac{\gamma}{\nu}) \Big\| \nabla \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} \Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))},$$

then

$$\Big\| \nabla \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} \Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^3 \frac{\gamma}{\nu} (1 + \frac{\gamma}{\nu}).$$

Using the Fourier expansion of $\mathscr{S}_\mu^\nu$, we have for a given θ_0

$$\Big\| \nabla \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} (\theta_0, \cdot) \Big\|_2 \leq \varepsilon^3 \frac{\gamma}{\nu} (1 + \frac{\gamma}{\nu}).$$

Thus, as previously, we get

$$\Big\| \nabla \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} \Big\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^3 \frac{\gamma}{\nu} (1 + \frac{\gamma}{\nu}), \ \Big\| \frac{\partial \mathscr{S}_\mu^\nu}{\partial t} \Big\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^3 \frac{\gamma}{\nu} (1 + \frac{\gamma}{\nu}).$$

□

Since the estimates of Theorem 3.1 do not depend on μ, making the process $\mu \to 0$ allows us to deduce the following theorem.

Theorem 3.2. *Under assumptions (3.43),(3.44) and (3.45), for any $\nu > 0$, there exists a unique $\mathscr{S}^\nu = \mathscr{S}^\nu(t,\theta,x) \in L^2(\mathbb{R} \times \mathbb{T}^2)$, periodic of period 1 with respect to θ solution to (3.46) and submitted to the constraint*

$$\sup_{\theta \in \mathbb{R}} \Big| \int_{\mathbb{T}^2} \mathscr{S}^\nu(\theta, x) dx \Big| = 0. \tag{3.70}$$

Moreover, the following estimates are satisfied

$$\Big\| \frac{\partial \mathscr{S}^\nu}{\partial \theta} \Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\varepsilon \nu}} \sqrt{(\frac{\gamma}{2\nu} + 1)}, \qquad \|\nabla \mathscr{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \sqrt{\frac{\gamma^2}{\nu^2} + \frac{2\varepsilon\gamma^2}{\nu}(\frac{\gamma^2}{\nu^2} + 1)}, \tag{3.71}$$

$$\|\mathscr{S}^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \sqrt{\frac{\gamma^2}{\nu^2} + \frac{2\varepsilon\gamma^2}{\nu}(\frac{\gamma^2}{\nu^2} + 1)}, \qquad \Big\| \frac{\partial \mathscr{S}^\nu}{\partial t} \Big\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^3 \frac{\gamma}{\nu} (1 + \frac{\gamma}{\nu}). \tag{3.72}$$

Proof. (of Theorem 3.2). As estimates of Theorem 3.1 do not depend on μ, to proof existence of $\mathscr{S}^\nu$, it suffices to make μ tend to 0 in (3.47).

Uniqueness is insured by (3.70), once noticed that, if $\mathscr{S}^\nu$ and $\widetilde{\mathscr{S}^\nu}$ are two solutions of (3.46), with constraint (3.70), $\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu}$ is solution to

$$\frac{\partial (\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu})}{\partial \theta} - \frac{1}{\varepsilon} \nabla \cdot ((\widetilde{\mathscr{A}_\varepsilon} + \nu) \nabla (\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu})) = 0, \tag{3.73}$$

from which we can deduce that

$$\nu \|\nabla (\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu})\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} = 0, \tag{3.74}$$

and because of (3.70), and its consequence:

$$\|\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu}\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \|\nabla(\mathscr{S}^\nu - \widetilde{\mathscr{S}^\nu})\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}, \tag{3.75}$$

that

$$\widetilde{\mathscr{S}^\nu} = \mathscr{S}^\nu. \tag{3.76}$$

□

Now we get estimates on $\mathscr{S}^\nu$ which do not depend on ν.

Theorem 3.3. *Under the assumptions (3.43),(3.44) and (3.45), the solution $\mathscr{S}^\nu$, of (3.46) given by Theorem 3.2 satisfies the following properties*

$$\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}|\nabla\mathscr{S}^\nu|\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \gamma, \tag{3.77}$$

$$\Big(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla\mathscr{S}^\nu|^2 dx d\theta\Big)^{1/2} \le \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}, \tag{3.78}$$

$$\Big\|\nabla\mathscr{S}^\nu(\theta_0,\cdot)\Big\|_2 \le \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}, \text{ for a given } \theta_0 \in [\theta_\alpha,\theta_\omega], \tag{3.79}$$

$$\|\mathscr{S}^\nu\|^2_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}} + 2\varepsilon\gamma^3, \tag{3.80}$$

$$\Big\|\frac{\partial\mathscr{S}^\nu}{\partial t}\Big\|^2_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \varepsilon\Big(\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}} + (\gamma^2+\varepsilon^2\gamma^4)\Big). \tag{3.81}$$

Proof. (of Theorem 3.3) Multiplying (3.46) by $\mathscr{S}^\nu$ and integrating over $\mathbb{T}^2$ yields

$$\frac{1}{2}\frac{d}{d\theta}\int_{\mathbb{T}^2}|\mathscr{S}^\nu|^2 dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)|\nabla\mathscr{S}^\nu|^2 dx = -\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\widetilde{\mathscr{C}_\varepsilon}\cdot\nabla\mathscr{S}^\nu dx. \tag{3.82}$$

Integrating (3.82) in θ over $[0,1]$ gives

$$\frac{1}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)|\nabla\mathscr{S}^\nu|^2 dx \le \frac{\gamma}{\varepsilon}\int_0^1\int_{\mathbb{T}^2}\sqrt{\widetilde{\mathscr{A}_\varepsilon}}|\nabla\mathscr{S}^\nu|dx, \tag{3.83}$$

then we obtain (3.77).
Assuming (3.44), we have

$$\sqrt{\widetilde{G}_{thr}}\Big(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla\mathscr{S}^\nu|^2 dx d\theta\Big)^{1/2} \le \Big(\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}\widetilde{\mathscr{A}_\varepsilon}|\nabla\mathscr{S}^\nu|^2 dx d\theta\Big)^{\frac{1}{2}} \le \Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}|\nabla\mathscr{S}^\nu|\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}. \tag{3.84}$$

From (3.77) and this last inequality we get (3.78). Then, there exists a $\theta_0 \in [\theta_\alpha,\theta_\omega]$ such that $\mathscr{S}^\nu$ satisfies (3.79). Using the Fourier expansion of $\mathscr{S}^\nu$ and the relation (3.70) we get

$$\Big\|\mathscr{S}^\nu(\theta_0,\cdot)\Big\|_2 \le \Big\|\nabla\mathscr{S}^\nu(\theta_0,\cdot)\Big\|_2 \le \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}. \tag{3.85}$$

Multiplying (3.46) by $\mathscr{S}^\nu$, integrating over $\mathbb{T}^2$ we obtain

$$\frac{1}{2}\frac{d\|\mathscr{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)|\nabla\mathscr{S}^\nu(\theta,\cdot)|^2 dx = \frac{1}{\varepsilon}\int_{\mathbb{T}^2}|\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}\mathscr{S}^\nu(\theta,\cdot)|dx$$

Applying formula (3.59) with $V = \frac{|\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}|}{\varepsilon}$ and $U = |\mathscr{S}^\nu|$, we get

$$\frac{1}{2}\frac{d\|\mathscr{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)|\nabla\mathscr{S}^\nu(\theta,\cdot)|^2dx \leq \int_{\mathbb{T}^2}\Big[\frac{(\widetilde{\mathscr{A}_\varepsilon}+\nu)}{4\varepsilon}|\mathscr{S}^\nu(\theta,\cdot)|^2 + \frac{1}{\varepsilon(\widetilde{\mathscr{A}_\varepsilon}+\nu)}|\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}|^2\Big]\,dx,$$

which gives

$$\frac{1}{2}\frac{d\|\mathscr{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)\Big(|\nabla\mathscr{S}^\nu(\theta,\cdot)|^2 - \frac{|\mathscr{S}^\nu(\theta,\cdot)|^2}{4}\Big)dx \leq \int_{\mathbb{T}^2}\frac{1}{\varepsilon(\widetilde{\mathscr{A}_\varepsilon}+\nu)}|\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}|^2\,dx. \tag{3.86}$$

Using Fourier expansion of $\mathscr{S}^\nu(\theta,\cdot)$, one can prove that the second term of the left hand side of (3.86) is positive, then we have

$$\frac{d\|\mathscr{S}^\nu(\theta,\cdot)\|_2^2}{d\theta} \leq 2\int_{\mathbb{T}^2}\frac{1}{\varepsilon(\widetilde{\mathscr{A}_\varepsilon}+\nu)}|\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}|^2\,dx. \tag{3.87}$$

Using (3.43), (3.45) and integrating (3.87) from θ_0 to $\theta\in[0,1]$, we obtain

$$\|\mathscr{S}^\nu(\theta,\cdot)\|_2^2 \leq \|\mathscr{S}^\nu(\theta_0,\cdot)\|_2^2 + 2\varepsilon\gamma^3, \tag{3.88}$$

then inequality (3.80) is satisfied.
Using inequality (3.77) and from hypothesis (3.45) we get

$$\Big\|\frac{\partial(\nabla\widetilde{\mathscr{A}_\varepsilon})}{\partial t}\nabla\mathscr{S}^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^2\gamma\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\nabla\mathscr{S}^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon^2\gamma^2. \tag{3.89}$$

Multiplying (3.46) by $-\Delta\mathscr{S}^\nu$ and integrating in $x\in\mathbb{T}^2$ we get

$$\frac{1}{2}\frac{d}{d\theta}\|\nabla\mathscr{S}^\nu\|_2^2 + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\Big(\widetilde{\mathscr{A}_\varepsilon}+\nu\Big)|\Delta\mathscr{S}^\nu|^2dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\widetilde{\mathscr{A}_\varepsilon}\cdot\nabla\mathscr{S}^\nu\Delta\mathscr{S}^\nu dx = -\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}\cdot\Delta\mathscr{S}^\nu dx. \tag{3.90}$$

Using (3.59) with $U=|\Delta\mathscr{S}^\nu|$ and $V=\frac{\nabla\widetilde{\mathscr{A}_\varepsilon}\cdot\nabla\mathscr{S}^\nu}{\varepsilon}$ and with $U=|\Delta\mathscr{S}^\nu|$ and $V=\frac{\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}}{\varepsilon}$, the equality (3.90) becomes

$$\frac{1}{2}\frac{d}{d\theta}\|\nabla\mathscr{S}^\nu\|_2^2 + \frac{1}{2\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)|\Delta\mathscr{S}^\nu|^2dx \leq \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\Big[\frac{|\nabla\widetilde{\mathscr{A}_\varepsilon}|^2}{\widetilde{\mathscr{A}_\varepsilon}+\nu}|\nabla\mathscr{S}^\nu|^2 + \frac{|\nabla\widetilde{\mathscr{C}_\varepsilon}|^2}{\widetilde{\mathscr{A}_\varepsilon}+\nu}\Big]dx, \tag{3.91}$$

which, integrating from 0 to 1 yields

$$\int_0^1\int_{\mathbb{T}^2}\widetilde{\mathscr{A}_\varepsilon}|\Delta\mathscr{S}^\nu|^2dxd\theta \leq 2\varepsilon\gamma^2\Big(\int_0^1\int_{\mathbb{T}^2}|\widetilde{\mathscr{A}_\varepsilon}||\nabla\mathscr{S}^\nu|^2dxd\theta+\gamma\Big) \leq 2\varepsilon\gamma^2(\gamma^2+\gamma). \tag{3.92}$$

As

$$\Big|\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\Big| \leq \varepsilon^2\gamma|\widetilde{\mathscr{A}_\varepsilon}|, \tag{3.93}$$

we obtain

$$\Big\|\sqrt{\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}}\Delta\mathscr{S}^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \varepsilon\sqrt{2\varepsilon}\gamma^2\sqrt{1+\gamma}. \tag{3.94}$$

Now we set out the equation to which $\frac{\partial\mathscr{S}^\nu}{\partial t}$ is solution. We have

$$\frac{\partial}{\partial\theta}\Big(\frac{\partial\mathscr{S}^\nu}{\partial t}\Big) = \frac{\partial}{\partial t}\Big(\frac{\partial\mathscr{S}^\nu}{\partial\theta}\Big) = \frac{1}{\varepsilon}\Big(\nabla\cdot\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\nabla\mathscr{S}^\nu + (\widetilde{\mathscr{A}_\varepsilon}+\nu)\nabla\frac{\partial\mathscr{S}^\nu}{\partial t}\Big) + \frac{1}{\varepsilon}\nabla\cdot\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t},$$

then $\frac{\partial\mathscr{S}^\nu}{\partial t}$ is solution to

$$\frac{\partial}{\partial\theta}\Big(\frac{\partial\mathscr{S}^\nu}{\partial t}\Big) - \frac{1}{\varepsilon}\nabla\cdot\Big((\widetilde{\mathscr{A}_\varepsilon}+\nu)\nabla\Big(\frac{\partial\mathscr{S}^\nu}{\partial t}\Big)\Big) = \frac{1}{\varepsilon}\nabla\cdot\Big(\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\nabla\mathscr{S}^\nu\Big) + \frac{1}{\varepsilon}\nabla\cdot\frac{\partial\widetilde{\mathscr{C}_\varepsilon}}{\partial t}. \tag{3.95}$$

Multiplying (3.95) by $\frac{\partial \mathscr{S}^\nu}{\partial t}$ and integrating in $x \in \mathbb{T}^2$, we get

$$\frac{1}{2}\frac{d}{d\theta}\Big\|\frac{\partial \mathscr{S}^\nu}{\partial t}\Big\|_2^2+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}(\widetilde{\mathscr{A}_\varepsilon}+\nu)\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|^2dx \le \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\Big|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\Big||\nabla \mathscr{S}^\nu|\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\Big|\frac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\Big|\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|dx. \quad (3.96)$$

Using the fact that $\left|\frac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right|^2 \le \varepsilon^2\gamma^2\big|\widetilde{\mathscr{A}_\varepsilon}\big|$, the second term of the right hand side of (3.96) satisfies

$$\int_{\mathbb{T}^2}\Big|\frac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\Big|\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|dx \le \varepsilon\gamma\int_{\mathbb{T}^2}\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|dx \le \varepsilon\gamma\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\,\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_2. \quad (3.97)$$

In the same way, using (3.45) we deduce the following estimate for the first term of the right hand side of (3.96)

$$\begin{aligned}\int_{\mathbb{T}^2}\Big|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\Big||\nabla \mathscr{S}^\nu|\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|dx &\le \Big\|\sqrt{\Big|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\Big|}|\nabla \mathscr{S}^\nu|\Big\|_2\Big\|\sqrt{\Big|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\Big|}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_2\\ &\le \varepsilon^2\gamma^2\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}|\nabla \mathscr{S}^\nu|\Big\|_2\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_2. \end{aligned} \quad (3.98)$$

Using inequalities (3.97), (3.98) and (3.77) and integrating (3.96) in θ over $[0,1]$, we have

$$\begin{aligned}\Big\|\sqrt{(\widetilde{\mathscr{A}_\varepsilon}+\nu)}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|^2_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} &\le \varepsilon\gamma\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\\ &\quad+\varepsilon^2\gamma^3\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}. \end{aligned} \quad (3.99)$$

From this last inequality, we deduce

$$\Big\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}\Big|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big|\Big\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \varepsilon(\gamma+\varepsilon\gamma^3), \quad (3.100)$$

and then

$$\int_{\theta_\alpha}^{\theta_\omega}\Big\|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}\Big\|_2 d\theta \le \varepsilon\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}}. \quad (3.101)$$

From (3.101), we deduce that there exists a $\theta_0 \in [\theta_\alpha,\theta_\omega]$ such that

$$\Big\|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}(\theta_0,\cdot)\Big\|_2 \le \varepsilon\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}}, \quad (3.102)$$

and, since the mean value of $\frac{\partial \mathscr{S}^\nu}{\partial t}(\theta_0,\cdot)$ is zero,

$$\Big\|\frac{\partial \mathscr{S}^\nu}{\partial t}(\theta_0,\cdot)\Big\|_2 \le \varepsilon\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}}. \quad (3.103)$$

To end the proof of the theorem it remains to show that $\frac{\partial \mathscr{S}^\nu}{\partial t}$ is bounded independently of ν in $L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))$. For this we will estimate the right hand side of (3.96) by applying formula (3.59) with $V=\frac{1}{\varepsilon}\big|\frac{\partial \widetilde{\mathscr{C}_\varepsilon}}{\partial t}\big|$ and $U=|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}|$ to treat the second term of the right hand side of (3.96) and with $V=\frac{1}{\varepsilon}\big|\frac{\partial \widetilde{\mathscr{A}_\varepsilon}}{\partial t}\big||\nabla \mathscr{S}^\nu|$ and $U=|\nabla\frac{\partial \mathscr{S}^\nu}{\partial t}|$ to treat the first. It gives:

$$\frac{1}{2}\frac{\partial\left(\left\|\frac{\partial \mathscr{S}^\nu}{\partial t}\right\|_2^2\right)}{\partial\theta}+\int_{\mathbb{T}^2}\frac{\widetilde{\mathscr{A}_\varepsilon}+\nu}{2\varepsilon}\left|\nabla\frac{\partial\mathscr{S}^\nu}{\partial t}\right|^2 dx$$
$$\le \int_{\mathbb{T}^2}\frac{\left|\frac{\partial\widetilde{\mathscr{C}_\varepsilon}}{\partial t}\right|^2}{\varepsilon(\widetilde{\mathscr{A}_\varepsilon}+\nu)}dx+\int_{\mathbb{T}^2}\frac{\left|\frac{\partial\widetilde{\mathscr{A}_\varepsilon}}{\partial t}\right|^2|\nabla\mathscr{S}^\nu|^2}{\varepsilon(\widetilde{\mathscr{A}_\varepsilon}+\nu)}dx\le\varepsilon\gamma^2+\varepsilon^3\gamma^2\int_{\mathbb{T}^2}\widetilde{\mathscr{A}_\varepsilon}|\nabla\mathscr{S}^\nu|^2dx, \quad (3.104)$$

where we used hypothesis (3.45) to get the last inequality. Integrating this last formula in θ over $[\theta_0,\sigma]$ for any $\sigma>\theta_0$, we obtain, always remembering (3.77),

$$\left\|\frac{\partial\mathscr{S}^\nu}{\partial t}(\sigma,\cdot)\right\|_2^2\le\left\|\frac{\partial\mathscr{S}^\nu}{\partial t}(\theta_0,\cdot)\right\|_2^2+\varepsilon(\gamma^2+\varepsilon^2\gamma^4). \quad (3.105)$$

From inequality (3.105) we obtain directly the inequality of (3.81), using the periodicity of $\mathscr{S}^\nu$. □

Estimates (3.80) and (3.81) given in Theorem 3.3 do not depend on ν. Making $\nu\to0$, allows us to deduce that, up to a subsequence $\mathscr{S}^\nu\longrightarrow S\in L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))$ weak $-*$. Concerning the limit $\mathscr{S}$ we have the following theorem.

Theorem 3.4. *Under assumptions (3.43), (3.44), (3.45), there exists a unique function* $\mathscr{S}=\mathscr{S}(t,\theta,x)\in L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))$, *periodic of period 1 with respect to* θ, *solution to*

$$\frac{\partial\mathscr{S}}{\partial\theta}-\frac{1}{\varepsilon}\nabla\cdot(\widetilde{\mathscr{A}_\varepsilon}(t,\cdot,\cdot)\nabla\mathscr{S})=\frac{1}{\varepsilon}\nabla\cdot\widetilde{\mathscr{C}_\varepsilon}(t,\cdot,\cdot), \quad (3.106)$$

and satisfying, for any $t,\theta\in\mathbb{R}^+\times\mathbb{R}$

$$\int_{\mathbb{T}^2}\mathscr{S}(t,\theta,x)dx=0. \quad (3.107)$$

and

$$\left\|\sqrt{\widetilde{\mathscr{A}_\varepsilon}}|\nabla S|\right\|_{L^2_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\gamma, \quad (3.108)$$

Moreover it satisfies:

$$\|\mathscr{S}\|^2_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}+2\varepsilon\gamma^3, \quad (3.109)$$

$$\left\|\frac{\partial\mathscr{S}}{\partial t}\right\|^2_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))}\le\varepsilon\Big(\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}}+(\gamma^2+\varepsilon^2\gamma^4)\Big). \quad (3.110)$$

Proof. . (of Theorem 3.4). Uniqueness of $\mathscr{S}$ is not gotten via the above evoked process $\nu\longrightarrow0$, but directly comes from (3.106). Assuming that there are two solutions $\mathscr{S}_1$ and $\mathscr{S}_2$ to (3.106), we easily deduce that

$$\frac{d\left(\|\mathscr{S}_1-\mathscr{S}_2\|_2^2\right)}{d\theta}+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\widetilde{\mathscr{A}_\varepsilon}|\nabla(\mathscr{S}_1-\mathscr{S}_2)|^2dx=0, \quad (3.111)$$

which gives, because of the non-negativity of $\widetilde{\mathscr{A}_\varepsilon}$,

$$\frac{d\left(\|\mathscr{S}_1-\mathscr{S}_2\|_2^2\right)}{d\theta}\le0. \quad (3.112)$$

From (3.111) we deduce that either

$$\widetilde{\mathscr{A}_\varepsilon}|\nabla(\mathscr{S}_1-\mathscr{S}_2)|^2\equiv0, \quad (3.113)$$

or, for any $\theta \in \mathbb{R}$,
$$\|\mathscr{S}_1(\theta+1,\cdot)-\mathscr{S}_2(\theta+1,\cdot)\|_2^2 < \|\mathscr{S}_1(\theta,\cdot)-\mathscr{S}_2(\theta,\cdot)\|_2^2. \tag{3.114}$$
As (3.114) is not possible because of the periodicity of $\mathscr{S}_1$ and $\mathscr{S}_2$, we deduce that (3.113) is true. Using this last information, we deduce, for instance
$$\nabla(\mathscr{S}_1-\mathscr{S}_2)(\theta_\omega,\cdot) \equiv 0, \tag{3.115}$$
yielding, because of property (3.107),
$$\|(\mathscr{S}_1-\mathscr{S}_2)(\theta_\omega,\cdot)\|_2^2 \leq \|\nabla(\mathscr{S}_1-\mathscr{S}_2)(\theta_\omega,\cdot)\|_2^2. \tag{3.116}$$
Injecting (3.113) in (3.111) yields
$$\frac{d\left(\|\mathscr{S}_1-\mathscr{S}_2\|_2^2\right)}{d\theta} = 0, \tag{3.117}$$
and then
$$\|(\mathscr{S}_1-\mathscr{S}_2)(\theta,\cdot)\|_2^2 = 0, \tag{3.118}$$
for any $\theta \geq \theta_\omega$ and consequently or any $\theta \in \mathbb{R}$. This ends the proof of Theorem 3.4. □

With this theorem on hand we can get the following result concerning z^ε solution of equation (3.41).

Theorem 3.5. *Under properties (3.43), (3.44), (3.45), for any T, not depending on ε and $z_0 \in H^1(\mathbb{T}^2)$, equation (3.41), with coefficients given by (3.37) coupled with (3.39) and (3.38) coupled with (3.40) has a unique solution $z^\varepsilon \in L^\infty([0,T];L^2(\mathbb{T}^2))$, with*
$$\sqrt{\mathscr{A}^\varepsilon}\nabla z^\varepsilon \in L^2((0,T),L^2(\mathbb{T}^2)) \tag{3.119}$$
and
$$\frac{\partial z^\varepsilon}{\partial t} \in L^2((0,T);L^2(\mathbb{T}^2)). \tag{3.120}$$
This solution satisfies:
$$\|z^\varepsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \widetilde{\gamma} \tag{3.121}$$
where $\widetilde{\gamma}$ is a constant which do not depend on ε.

Proof. (of Theorem 2.1). Theorem 2.1 is a direct consequence of Theorem 3.5. □

Proof. .(of Theorem 3.5). Existence of z^ε, solution to (3.41), on a time interval of length ε is a straightforward adaptation of results of Ladyzenskaja, Sollonnikov and Ural' Ceva [9] or Lions [10].
Then, let us consider the function $Z^\varepsilon = Z^\varepsilon(t,x) = \mathscr{S}(t,\frac{t}{\varepsilon},x)$ where $\mathscr{S}$ is solution to (3.107). We obtain
$$\frac{\partial Z^\varepsilon}{\partial t} = \frac{\partial S}{\partial t}(t,\frac{t}{\varepsilon},x) + \frac{1}{\varepsilon}\frac{\partial S}{\partial \theta}(t,\frac{t}{\varepsilon},x). \tag{3.122}$$
Using equation (3.106) and (3.37), (3.38) we deduce that Z^ε is solution to
$$\frac{\partial Z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2}\nabla\cdot\left(\mathscr{A}^\varepsilon\nabla Z^\varepsilon\right) = \frac{1}{\varepsilon^2}\nabla\cdot\mathscr{C}^\varepsilon + \frac{\partial S}{\partial t} \tag{3.123}$$
then we deduce that
$$\begin{cases} \dfrac{\partial(z^\varepsilon - Z^\varepsilon)}{\partial t} - \dfrac{1}{\varepsilon^2}\nabla\cdot\left(\mathscr{A}^\varepsilon\nabla(z^\varepsilon - Z^\varepsilon)\right) = \dfrac{\partial S}{\partial t} \\ (z^\varepsilon - Z^\varepsilon)_{|t=0} = z_0 - \mathscr{S}(0,0,x). \end{cases} \tag{3.124}$$
Multiplying (3.124) by $z^\varepsilon - Z^\varepsilon$ and integrating over $\mathbb{T}^2$, we have
$$\frac{1}{2}\frac{d\|z^\varepsilon - Z^\varepsilon\|_2^2}{dt} + \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\mathscr{A}^\varepsilon|\nabla(z^\varepsilon - Z^\varepsilon)|^2dx = \int_{\mathbb{T}^2}\frac{\partial\mathscr{S}}{\partial t}(z^\varepsilon - Z^\varepsilon)dx \tag{3.125}$$

which gives using (3.110)

$$\frac{d\|z^\varepsilon - Z^\varepsilon\|_2^2}{dt} \leq 2\sqrt{\varepsilon\Big(\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}} + (\gamma^2+\varepsilon^2\gamma^4)\Big)}\|z^\varepsilon - Z^\varepsilon\|_2. \tag{3.126}$$

Then we have

$$\|z^\varepsilon(t,\cdot) - Z^\varepsilon(t,\cdot)\|_2^2 \leq 2\|z_0 - \mathscr{S}(0,0,x)\|_2\sqrt{\varepsilon\Big(\frac{\gamma+\varepsilon\gamma^3}{\sqrt{\widetilde{G}_{thr}}} + (\gamma^2+\varepsilon^2\gamma^4)\Big)T}. \tag{3.127}$$

As $\|\mathscr{S}\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \leq \frac{\gamma}{\sqrt{\widetilde{G}_{thr}}}$ when $\varepsilon \to 0$, then (3.121) is true.
Now, integrating (3.125) from 0 to T we get

$$\frac{1}{2}\|z^\varepsilon - Z^\varepsilon\|_2^2(T) - \frac{1}{2}\|z^\varepsilon - Z^\varepsilon\|_2^2(0) + \frac{1}{\varepsilon^2}\int_0^T \|\sqrt{\mathscr{A}^\varepsilon}\nabla(z^\varepsilon - Z^\varepsilon)\|_2^2 dt \leq \int_0^T \Big\|\frac{\partial S}{\partial t}\Big\|_2 dt \cdot \int_0^T \|z^\varepsilon - Z^\varepsilon\|_2\, dt \tag{3.128}$$

which using (3.110) and (3.127) yields

$$\sqrt{\mathscr{A}^\varepsilon}\nabla(z^\varepsilon - Z^\varepsilon) \in L^2((0,T),L^2(\mathbb{T}^2)).$$

Consequently, because of (3.108) and the definition of Z^ε from S, we obtain (3.119). Beside this, (3.120) is a straight forwardly obtained from (3.110).
To prove uniqueness of z^ε given by the theorem, we consider z_1^ε and z_2^ε two solutions of (3.41). Their difference is then solution to

$$\begin{cases} \dfrac{\partial(z_1^\varepsilon - z_2^\varepsilon)}{\partial t} - \dfrac{1}{\varepsilon^2}\nabla\cdot(\mathscr{A}^\varepsilon\nabla(z_1^\varepsilon - z_2^\varepsilon)) = 0, \\ (z_1^\varepsilon - z_2^\varepsilon)_{|t=0} = 0, \end{cases} \tag{3.129}$$

or having a look for weak formulation (3.42)

$$-\int_0^T\int_{\mathbb{T}^2}(z_1^\varepsilon - z_2^\varepsilon)\frac{\partial\varphi}{\partial t}\,dx\,dt + \frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{T}^2}\mathscr{A}_\varepsilon\nabla(z_1^\varepsilon - z_2^\varepsilon)\cdot\nabla\varphi\,dx\,dt = 0$$
$$\text{for any } \varphi \in \mathscr{D}([0,T)\times\mathbb{T}^2). \tag{3.130}$$

It obvious that (3.130) makes sense and is true for $\varphi \in L^2((0,T);L^2(\mathbb{T}^2))$ such that
$\frac{\partial\varphi}{\partial t} \in L^2((0,T);L^2(\mathbb{T}^2))$ and $\sqrt{\mathscr{A}^\varepsilon}\,\varphi \in L^2((0,T);L^2(\mathbb{T}^2))$. Hence in (3.130) we can chose $\varphi = z_1^\varepsilon - z_2^\varepsilon$. Making this gives

$$\frac{d\|z_1^\varepsilon - z_2^\varepsilon\|_2^2}{dt} \leq 0, \tag{3.131}$$

and since $z^\varepsilon_{1|t=0} = z^\varepsilon_{2|t=0}$, we finally obtain that $z_1^\varepsilon = z_2^\varepsilon$ yielding uniqueness. □

4 Homogenization of equation (1.9), proof of Theorem 2.2

We consider equation (3.41) where $\mathscr{A}^\varepsilon$ and $\mathscr{C}^\varepsilon$ are defined by formulas (3.37) coupled with (3.39) and (3.38) coupled with (3.40). Our aim consists in deducing the equations satisfied by the limit of z^ε solution to (3.41) as $\varepsilon \longrightarrow 0$.

It is obvious that

$$\mathscr{A}^\varepsilon(t,x) \text{ Two-Scale converges to } \widetilde{\mathscr{A}}(t,\theta,x) \in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$$
$$\text{and } \mathscr{C}^\varepsilon(t,x) \text{ Two-Scale converges to } \widetilde{\mathscr{C}}(t,\theta,x), \tag{4.132}$$

with

$$\widetilde{\mathscr{A}}(t,\theta,x) = a\,g_a(|\mathscr{U}_0(\theta)|) \text{ and } \widetilde{\mathscr{C}}(t,\theta,x) = c\,g_c(|\mathscr{U}_0(\theta)|)\,\frac{\mathscr{U}_0(\theta)}{|\mathscr{U}_0(\theta)|}; \tag{4.133}$$

$\mathscr{U}_0$ is given in (1.10).
Θ and Θ_{thr} defined by (2.23) and (2.24) have the following form:

$$\Theta = [0,T) \times \{\theta \in \mathbb{R} : \widetilde{\mathscr{A}}(\cdot,\theta,\cdot) = 0\} \times \mathbb{T}^2, \tag{4.134}$$

and

$$\Theta_{thr} = \{(t,\theta,x) \in [0,T) \times \mathbb{R} \times \mathbb{T}^2 \text{ such that } \widetilde{\mathscr{A}}(t,\theta,x) < \widetilde{G}_{thr}\}. \tag{4.135}$$

Moreover, we notice that

$$\widetilde{\mathscr{A}}(t,\theta,x) = 0 \text{ if and only if } (t,\theta,x) \in \Theta. \tag{4.136}$$

We have the following theorem.

Theorem 4.1. *Under assumptions (3.43), (3.44), (3.45), (4.132), (4.133) and (4.136), for any T, not depending on ε, the sequence (z^ε) of solutions to (3.41), with coefficients given by (3.37) coupled with (3.39) and (3.38) coupled with (3.40), Two-Scale converges to the profile $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ solution to*

$$-\nabla \cdot (\widetilde{\mathscr{A}} \nabla U) = 0 \quad on \left([0,T) \times \mathbb{R} \times \mathbb{T}^2\right) \backslash \Theta, \tag{4.137}$$

$$\frac{\partial U}{\partial \theta} = 0 \quad on\ \Theta_{thr}, \tag{4.138}$$

$$\int_0^1 \int_{\mathbb{T}^2} U\, d\theta\, dx \;=\; \int_{\mathbb{T}^2} z_0 dx \text{ and } \int_{\mathbb{T}^2} \frac{\partial U}{\partial \theta} dx = 0, \tag{4.139}$$

where $\widetilde{\mathscr{A}}$ is given by (4.133); Θ and Θ_{thr} are given by (4.134) and (4.135).

Proof. (of Theorem 2.2) Theorem 2.2 is a direct consequence of Theorem 4.1. Indeed, function $U = \int_{\mathbb{T}^2} z_0 dx$ is the unique function satisfying (4.137)-(4.139). □

Proof. (of Theorem 4.1) Multiplying (3.41) by $\psi^\varepsilon(t,x) = \psi(t, \frac{t}{\varepsilon}, x)$ regular with compact support in $[0,T) \times \mathbb{T}^2$ and 1-periodic in θ, we obtain

$$-\int_{\mathbb{T}^2} z_0(x)\psi(0,0,x)dx - \int_{\mathbb{T}^2}\int_0^T \frac{\partial \psi^\varepsilon}{\partial t} z^\varepsilon dt\, dx + \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_0^T \mathscr{A}^\varepsilon \nabla z^\varepsilon \nabla \psi^\varepsilon dt\, dx = \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_0^T \left(\nabla\cdot \mathscr{C}^\varepsilon\right)\psi^\varepsilon dx. \tag{4.140}$$

Using the Green formula and

$$\frac{\partial \psi^\varepsilon}{\partial t} = \left(\frac{\partial \psi}{\partial t}\right)^\varepsilon + \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial \theta}\right)^\varepsilon, \tag{4.141}$$

where

$$\left(\frac{\partial \psi}{\partial t}\right)^\varepsilon(t,x) = \frac{\partial \psi}{\partial t}(t,\frac{t}{\varepsilon},x) \text{ and } \left(\frac{\partial \psi}{\partial \theta}\right)^\varepsilon(t,x) = \frac{\partial \psi}{\partial \theta}(t,\frac{t}{\varepsilon},x), \tag{4.142}$$

we obtain

$$\int_{\mathbb{T}^2}\int_0^T \left(\left(\frac{\partial \psi}{\partial t}\right)^\varepsilon + \frac{1}{\varepsilon}\left(\frac{\partial \psi}{\partial \theta}\right)^\varepsilon\right) z^\varepsilon dt\, dx + \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_0^T z^\varepsilon \nabla\cdot\left(\mathscr{A}^\varepsilon \nabla \psi^\varepsilon\right) dt\, dx + \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_0^T \left(\nabla\cdot\mathscr{C}^\varepsilon\right)\psi^\varepsilon dt\, dx = -\int_{\mathbb{T}^2} z_0(x)\psi(0,0,x)dx \tag{4.143}$$

Multiplying by ε^2 and using the Two-Scale convergence due to Nguetseng [13], Allaire [1], Frénod, Raviart and Sonnendrucker [5], as z^ε is bounded in $L^\infty([0,T), L^2(\mathbb{T}^2))$, there exists a profile $U(t,\theta,x)$, periodic of period 1 with respect to θ, such that for all $\psi(t,\theta,x)$, regular with compact support with respect to (t,x) and periodic of period 1 with respect to θ, we have

$$-\int_{\mathbb{T}^2}\int_0^T\int_0^1 U\nabla\cdot\left(\tilde{\mathscr{A}}\nabla\psi\right)d\theta\,dt\,dx=\int_{\mathbb{T}^2}\int_0^T\int_0^1\left(\nabla\cdot\tilde{\mathscr{C}}\right)\psi\,d\theta\,dt\,dx, \tag{4.144}$$

then

$$-\nabla\cdot\left(\tilde{\mathscr{A}}\nabla U\right)=\nabla\cdot\tilde{\mathscr{C}}, \tag{4.145}$$

with $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{C}}$ given by (4.133). As $\tilde{\mathscr{C}}$ does not depend on x, we have

$$-\nabla\cdot\left(\tilde{\mathscr{A}}\nabla U\right)=0 \tag{4.146}$$

which gives (4.137).
Since $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{C}}$ vanish on Θ, (4.146) contains no information on Θ. Hence we write (4.137), and we look for an information concerning U on Θ. Using test function $\psi^\varepsilon=\psi(\frac{t}{\varepsilon})$ depending only on θ, regular with compact support on Θ_{thr} in (4.143), and the fact that $\widetilde{\mathscr{A}_\varepsilon}$ does not depend on t and x in Θ_{thr}, we get

$$\int_{\mathbb{T}^2}\int_0^T\left(\frac{1}{\varepsilon}\left(\frac{\partial\psi}{\partial\theta}\right)^\varepsilon\right)z^\varepsilon dt\,dx+\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_0^T z^\varepsilon\widetilde{\mathscr{A}_\varepsilon}\left(\Delta\psi\right)^\varepsilon dt\,dx=-\int_{\mathbb{T}^2}z_0(x)\psi(0,0,x)dx. \tag{4.147}$$

As ψ depends only on θ, we have $\left(\Delta\psi\right)^\varepsilon=0$ and then multpliying by ε in (4.147), we get

$$\int_{\mathbb{T}^2}\int_0^T\left(\frac{\partial\psi}{\partial\theta}\right)^\varepsilon z^\varepsilon dt\,dx=-\varepsilon\int_{\mathbb{T}^2}z_0(x)\psi(0,0,x)dx. \tag{4.148}$$

Passing to the limit as $\varepsilon\to 0$ in (4.148) and using Two-Scale convergence, we get

$$\int_{\mathbb{T}^2}\int_0^T\int_0^1\frac{\partial\psi}{\partial\theta}U\,d\theta\,dt\,dx=0, \tag{4.149}$$

which gives

$$\frac{\partial U}{\partial\theta}=0 \text{ on } \Theta_{thr}. \tag{4.150}$$

Taking test function ψ depending only on t we obtain

$$\int_0^1\int_{\mathbb{T}^2}U(t,\theta,x)d\theta\,dx=\int_{\mathbb{T}^2}z_0(x)dx. \tag{4.151}$$

Finally, to prove uniqueness of U, we consider two solutions U_1 and U_2 of (4.137)-(4.138)-(4.139). Their difference is then solution to

$$\nabla\cdot\left(\widetilde{\mathscr{A}}\nabla\left(U_1-U_2\right)\right)=0 \text{ on } \left([0,T)\times\mathbb{R}\times\mathbb{T}^2\right)\backslash\Theta, \tag{4.152}$$

$$\frac{\partial\left(U_1-U_2\right)}{\partial\theta}=0 \text{ on } \Theta_{thr}, \tag{4.153}$$

$$\int_0^1\int_{\mathbb{T}^2}\left(U_1-U_2\right)d\theta\,dx=0. \tag{4.154}$$

Equations (4.152) and (4.154) give $U_1-U_2=0$ on $\left([0,T)\times\mathbb{R}\times\mathbb{T}^2\right)\backslash\Theta$ and (4.153) and (4.154) give that $U_1-U_2=0$ on Θ_{thr}. Since $\left(\left([0,T)\times\mathbb{R}\times\mathbb{T}^2\right)\backslash\Theta\right)\cup\Theta_{thr}=[0,T)\times\mathbb{R}\times\mathbb{T}^2$, this gives uniqueness and ends the proof of the theorem. □

5 Homogenization and corrector result of equation (1.5), proof of Theorem 2.3 and 2.4

Making the same as in the beginning of section 3, setting:

$$\mathscr{A}^{\varepsilon}(t,x) = \widetilde{\mathscr{A}_{\varepsilon}}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x), \tag{5.155}$$

and

$$\mathscr{C}^{\varepsilon}(t,x) = \widetilde{\mathscr{C}_{\varepsilon}}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x), \tag{5.156}$$

where

$$\widetilde{\mathscr{A}_{\varepsilon}}(t,\tau,\theta,x) = a(1-b\sqrt{\varepsilon}\mathscr{M}(t,\tau,\theta,x))\, g_a(|\widetilde{\mathscr{U}}(t,\tau,\theta,x)|), \tag{5.157}$$

and

$$\widetilde{\mathscr{C}_{\varepsilon}}(t,\tau,\theta,x) = c(1-b\sqrt{\varepsilon}\mathscr{M}(t,\tau,\theta,x))\, g_c(|\widetilde{\mathscr{U}}(t,\tau,\theta,x)|)\, \frac{\widetilde{\mathscr{U}}(t,\tau,\theta,x)}{|\widetilde{\mathscr{U}}(t,\tau,\theta,x)|}, \tag{5.158}$$

equation (1.5) with initial condition (1.13) can be set in the form

$$\begin{cases} \dfrac{\partial z^{\varepsilon}}{\partial t} - \dfrac{1}{\varepsilon}\nabla\cdot(\mathscr{A}^{\varepsilon}\nabla z^{\varepsilon}) = \dfrac{1}{\varepsilon}\nabla\cdot\mathscr{C}^{\varepsilon}, \\ z^{\varepsilon}_{|t=0} = z_0. \end{cases} \tag{5.159}$$

Under assumptions (1.2) and (1.8), $\widetilde{\mathscr{A}_{\varepsilon}}$ and $\widetilde{\mathscr{C}_{\varepsilon}}$ given by (5.157) and (5.158) satisfy the following hypotheses:

$$\begin{cases} \tau \longmapsto (\widetilde{\mathscr{A}_{\varepsilon}}, \widetilde{\mathscr{C}_{\varepsilon}}) \text{ is periodic of period } 1, \\ \theta \longmapsto (\widetilde{\mathscr{A}_{\varepsilon}}, \widetilde{\mathscr{C}_{\varepsilon}}) \text{ is periodic of period } 1, \\ x \longmapsto (\widetilde{\mathscr{A}_{\varepsilon}}, \widetilde{\mathscr{C}_{\varepsilon}}) \text{ defined on } \mathbb{T}^2, \\ |\widetilde{\mathscr{A}_{\varepsilon}}| \leq \gamma,\ |\widetilde{\mathscr{C}_{\varepsilon}}| \leq \gamma,\ \left|\dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial t}\right| \leq \gamma,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial t}\right| \leq \gamma, \left|\dfrac{\partial \nabla\widetilde{\mathscr{A}_{\varepsilon}}}{\partial t}\right| \leq \gamma, \\ \left|\dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial \theta}\right| \leq \gamma,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial \theta}\right| \leq \gamma,\ |\nabla\widetilde{\mathscr{A}_{\varepsilon}}| \leq \gamma,\ |\nabla\cdot\widetilde{\mathscr{C}_{\varepsilon}}| \leq \gamma,\ \left|\dfrac{\partial \nabla\cdot\widetilde{\mathscr{C}_{\varepsilon}}}{\partial t}\right| \leq \gamma, \end{cases} \tag{5.160}$$

$$\begin{cases} \exists \widetilde{G}_{thr},\ \theta_\alpha < \theta_\omega \in [0,1] \text{ not depending on } \varepsilon \text{ such that } \theta \in [\theta_\alpha, \theta_\omega] \Longrightarrow \widetilde{\mathscr{A}_{\varepsilon}}(t,\tau,\theta,x) \geq \widetilde{G}_{thr}, \\ \widetilde{\mathscr{A}_{\varepsilon}}(t,\tau,\theta,x) \leq \widetilde{G}_{thr} \Longrightarrow \begin{cases} \dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial t}(t,\tau,\theta,x) = \dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial \tau}(t,\tau,\theta,x) = 0,\ \nabla\widetilde{\mathscr{A}_{\varepsilon}}(t,\tau,\theta,x) = 0, \\ \dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial t}(t,\tau,\theta,x) = \dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial \tau}(t,\tau,\theta,x) = 0,\ \nabla\cdot\widetilde{\mathscr{C}_{\varepsilon}}(t,\tau,\theta,x) = 0, \end{cases} \end{cases} \tag{5.161}$$

$$\begin{cases} |\widetilde{\mathscr{C}_{\varepsilon}}| \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ |\widetilde{\mathscr{C}_{\varepsilon}}|^2 \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ |\nabla\widetilde{\mathscr{A}_{\varepsilon}}| \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ \left|\dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial t}\right| \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|, \\ \left|\dfrac{\partial(\nabla\widetilde{\mathscr{A}_{\varepsilon}})}{\partial t}\right|^2 \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ \left|\nabla\cdot\widetilde{\mathscr{C}_{\varepsilon}}\right| \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial t}\right| \leq \gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ \left|\dfrac{\partial \widetilde{\mathscr{C}_{\varepsilon}}}{\partial t}\right|^2 \leq \gamma^2|\widetilde{\mathscr{A}_{\varepsilon}}|, \\ \left|\dfrac{\partial \widetilde{\mathscr{A}_{\varepsilon}}}{\partial \tau}\right|^2 \leq \varepsilon\gamma|\widetilde{\mathscr{A}_{\varepsilon}}|,\ \left|\dfrac{\partial \nabla\widetilde{\mathscr{A}_{\varepsilon}}}{\partial \tau}\right|^2 \leq \varepsilon\gamma|\widetilde{\mathscr{A}_{\varepsilon}}|. \end{cases} \tag{5.162}$$

For (5.159), if hypotheses (5.160), (5.161) and (5.162) are satisfied, an existence and uniqueness result is given in [4].

5.1 Homogenization

Let us consider equation (5.159) with $\mathscr{A}^{\varepsilon}$ and $\mathscr{C}^{\varepsilon}$ given by (5.155) and (5.156);

$$\begin{gathered} \theta \longmapsto \widetilde{\mathscr{A}_{\varepsilon}},\ \widetilde{\mathscr{C}_{\varepsilon}} \text{ is periodic of period } 1, \\ \tau \longmapsto \widetilde{\mathscr{A}_{\varepsilon}},\ \widetilde{\mathscr{C}_{\varepsilon}} \text{ is periodic of period } 1, \end{gathered} \tag{5.163}$$

$$\mathscr{A}^\varepsilon(t,x) \text{ Three-Scale converges to } \widetilde{\mathscr{A}}(t,\tau,\theta,x) \in L^\infty([0,T)\times\mathbb{R}, L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$$
$$\text{and } \mathscr{C}^\varepsilon(t,x) \text{ Three-Scale converges to } \widetilde{\mathscr{C}}(t,\tau,\theta,x), \tag{5.164}$$

with

$$\widetilde{\mathscr{A}}(t,\tau,\theta,x) = a\,g_a(|\mathscr{U}(t,\theta,x)|) \text{ and } \widetilde{\mathscr{C}}(t,\tau,\theta,x) = c\,g_c(|\mathscr{U}(t,\theta,x)|)\,\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}; \tag{5.165}$$

$\mathscr{U}$ is given in (1.7).

Theorem 5.1. *Under assumptions (5.160), (5.161), (5.162),(5.163), (5.164) and (5.165), for any T, not depending on ε, the sequence (z^ε) of solutions to (5.159), with coefficients given by (5.155) coupled with (5.157) and (5.156) coupled with (5.158), Three-Scale converges to the profile $U \in L^\infty([0,T)\times\mathbb{R}, L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2)))$ solution to*

$$\frac{\partial U}{\partial \theta} - \nabla\cdot(\widetilde{\mathscr{A}}\nabla U) = \nabla\cdot\widetilde{\mathscr{C}}, \tag{5.166}$$

where $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{C}}$ are given by (5.165).

Proof. (of Theorem 2.3) Theorem 2.3 is a direct consequence of Theorem 5.1. □

Proof. (of Theorem 5.1) Considering test functions $\psi^\varepsilon(t,x) = \psi(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x)$ for all $\psi(t,\tau,\theta,x)$ regular with compact support on $[0,T)\times\mathbb{T}^2$ and periodic of period 1 with respect to τ and θ.

$$\frac{\partial\psi^\varepsilon}{\partial t} = \Big(\frac{\partial\psi}{\partial t}\Big)^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big(\frac{\partial\psi}{\partial\tau}\Big)^\varepsilon + \frac{1}{\varepsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\varepsilon, \tag{5.167}$$

where

$$\Big(\frac{\partial\psi}{\partial t}\Big)^\varepsilon(t,x) = \frac{\partial\psi}{\partial t}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x), \qquad \Big(\frac{\partial\psi}{\partial\tau}\Big)^\varepsilon = \frac{\partial\psi}{\partial\tau}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x), \qquad \Big(\frac{\partial\psi}{\partial\theta}\Big)^\varepsilon = \frac{\partial\psi}{\partial\theta}(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x). \tag{5.168}$$

Multiplying (5.159) by $\psi^\varepsilon(t,\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon},x)$ and integrating on $[0,T)\times\mathbb{T}^2$, we get

$$-\int_{\mathbb{T}^2} z_0(x)\psi(0,0,0,x)dx - \int_{\mathbb{T}^2}\int_0^T \frac{\partial\psi^\varepsilon}{\partial t}z^\varepsilon dt\,dx - \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\int_0^T z^\varepsilon\nabla\cdot\Big(\mathscr{A}^\varepsilon\nabla\psi^\varepsilon\Big)dt\,dx$$
$$= \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\Big(\int_0^T \nabla\cdot\mathscr{C}^\varepsilon\Big)\psi^\varepsilon dt\,dx.$$

Replacing $\frac{\partial\psi^\varepsilon}{\partial t}$ by the relation (5.167), we have

$$\int_{\mathbb{T}^2}\int_0^T z^\varepsilon\Big[\Big(\frac{\partial\psi}{\partial t}\Big)^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big(\frac{\partial\psi}{\partial\tau}\Big)^\varepsilon + \frac{1}{\varepsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\varepsilon\Big]dt\,dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\int_0^T z^\varepsilon\nabla\cdot\Big(\mathscr{A}^\varepsilon\nabla\psi^\varepsilon\Big)dt\,dx$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\int_0^T\Big(\nabla\cdot\mathscr{C}^\varepsilon\Big)\psi^\varepsilon dt\,dx = -\int_{\mathbb{T}^2} z_0(x)\psi(0,0,0,x)dx.$$

Multiplying by ε we have

$$\int_{\mathbb{T}^2}\int_0^T z^\varepsilon\Big[\varepsilon\Big(\frac{\partial\psi}{\partial t}\Big)^\varepsilon + \sqrt{\varepsilon}\Big(\frac{\partial\psi}{\partial\tau}\Big)^\varepsilon + \Big(\frac{\partial\psi}{\partial\theta}\Big)^\varepsilon + \nabla\cdot\Big(\mathscr{A}^\varepsilon\nabla\psi^\varepsilon\Big)\Big]dt\,dx +$$
$$\int_{\mathbb{T}^2}\int_0^T\Big(\nabla\cdot\mathscr{C}^\varepsilon\Big)\psi^\varepsilon dt\,dx = -\varepsilon\int_{\mathbb{T}^2} z_0(x)\psi(0,0,0,x)dx.$$

The functions $\big(\frac{\partial\psi}{\partial t}\big)^\varepsilon$, $\big(\frac{\partial\psi}{\partial\tau}\big)^\varepsilon$ and $\big(\frac{\partial\psi}{\partial\theta}\big)^\varepsilon$ are periodic with respect to the two variables τ, θ. Here we used the Three-Scale convergence (see [11]).
Taking the limit as $\varepsilon\to 0$, using the Three-Scale convergence, we have

$$\int_{\mathbb{T}^2}\int_0^T\int_{[0,1]^2}\Big(U\frac{\partial\psi}{\partial\theta} + U\nabla\cdot\big(\tilde{\mathscr{A}}\nabla\psi\big)\Big)d\tau\,d\theta\,dt\,dx = \int_{\mathbb{T}^2}\int_0^T\int_{[0,1]^2}\tilde{\mathscr{C}}\cdot\nabla\psi\,d\tau\,d\theta\,dt\,dx.$$

Then, the limit U of z^ε solution to (3.41) satisfies the following equation

$$\frac{\partial U}{\partial \theta} - \nabla \cdot \left(\tilde{\mathscr{A}} \nabla U \right) = \nabla \cdot \tilde{\mathscr{C}}. \tag{5.169}$$

□

There is indeed existence and uniqueness of the equation (5.169) according to the application of the Theorem 3.15 of [4]; thus (5.169) is the homogenized equation. In (5.169), τ and t are only parameters.

5.2 A corrector result

Considering equation (5.159) with coefficients (5.155) and (5.156) and hypothesis (5.164) leads to

$$\mathscr{A}^\varepsilon(t,x) = \widetilde{\mathscr{A}^\varepsilon}(t,x) + \sqrt{\varepsilon}\widetilde{\mathscr{A}_1^\varepsilon}(t,x), \tag{5.170}$$

$$\mathscr{C}^\varepsilon(t,x) = \widetilde{\mathscr{C}^\varepsilon}(t,x) + \sqrt{\varepsilon}\widetilde{\mathscr{C}_1^\varepsilon}(t,x), \tag{5.171}$$

where

$$\widetilde{\mathscr{A}^\varepsilon}(t,x) = \widetilde{\mathscr{A}}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x), \qquad \widetilde{\mathscr{C}^\varepsilon}(t,x) = \widetilde{\mathscr{C}}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x) \tag{5.172}$$

$$\widetilde{\mathscr{A}_1^\varepsilon}(t,x) = \widetilde{\mathscr{A}_1}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x), \qquad \widetilde{\mathscr{C}_1^\varepsilon}(t,x) = \widetilde{\mathscr{C}_1}(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x). \tag{5.173}$$

with

$$\widetilde{\mathscr{A}}(t,\tau,\theta,x) = a g_a(|\mathscr{U}(t,\theta,x)|), \qquad \widetilde{\mathscr{A}_1}(t,\tau,\theta,x) = -ab\mathscr{M}(t,\tau,\theta,x) g_a(|\mathscr{U}(t,\theta,x)|) \tag{5.174}$$

$$\widetilde{\mathscr{C}}(t,\tau,\theta,x) = c g_c(|\mathscr{U}(t,\theta,x)|)\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}, \widetilde{\mathscr{C}_1}(t,\tau,\theta,x) = -bc\mathscr{M}(t,\tau,\theta,x) g_c(|\mathscr{U}(t,\theta,x)|)\frac{\mathscr{U}(t,\theta,x)}{|\mathscr{U}(t,\theta,x)|}. \tag{5.175}$$

Because of hypotheses (5.160), (5.161) and (5.162), $\widetilde{\mathscr{A}}$, $\widetilde{\mathscr{A}_1}$, $\widetilde{\mathscr{A}^\varepsilon}$, $\widetilde{\mathscr{A}_1^\varepsilon}$, $\widetilde{\mathscr{C}}$, $\widetilde{\mathscr{C}_1}$, $\widetilde{\mathscr{C}^\varepsilon}$, and $\widetilde{\mathscr{C}_1^\varepsilon}$ are regular and bounded coefficients.

Theorem 5.2. *Under assumptions (5.160), (5.161), (5.162),(5.163), (5.164) and (5.165), and if moreover $U_{thr} > 0$, considering function $z^\varepsilon \in L^\infty([0,T), L^2(\mathbb{T}^2))$, solution to (5.159) and function $U^\varepsilon \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ defined by $U^\varepsilon(t,x) = U(t, \frac{t}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}, x)$, where U is the solution to (5.166), the following estimate is satisfied:*

$$\left\| \frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}} \right\|_{L^\infty([0,T), L^2(\mathbb{T}^2))} \leq \alpha, \tag{5.176}$$

where α is a constant not depending on ε.
Furthermore

$$\frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}} \quad \text{Three-Scale converges to a profile } U_{\frac{1}{2}} \in L^\infty([0,T) \times \mathbb{R}, L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))), \tag{5.177}$$

which is the unique solution to

$$\frac{\partial U_{\frac{1}{2}}}{\partial \theta} - \nabla \cdot \left(\widetilde{\mathscr{A}} \nabla U_{\frac{1}{2}} \right) = \nabla \cdot \widetilde{\mathscr{C}^1} + \nabla \cdot \left(\widetilde{\mathscr{A}_1} \nabla U \right) - \frac{\partial U}{\partial \tau}. \tag{5.178}$$

Proof. (of Theorem2.4) Theorem 2.4 is a direct consequence of Theorem 5.2. □

Proof. (of Theorem 5.2) Using the relations (5.170)and (5.171), equation (5.159) becomes

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon}\nabla \cdot \left(\widetilde{\mathscr{A}^\varepsilon} \nabla z^\varepsilon \right) = \frac{1}{\varepsilon}\Big(\nabla \cdot \widetilde{\mathscr{C}^\varepsilon} + \sqrt{\varepsilon}\nabla \cdot \widetilde{\mathscr{C}_1^\varepsilon} + \sqrt{\varepsilon}\nabla \cdot \left(\widetilde{\mathscr{A}_1^\varepsilon} \nabla z^\varepsilon \right) \Big). \tag{5.179}$$

As U is solution to (5.166) and taking into account that

$$\frac{\partial U^\varepsilon}{\partial t} = \Big(\frac{\partial U}{\partial t}\Big)^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big(\frac{\partial U}{\partial \tau}\Big)^\varepsilon + \frac{1}{\varepsilon}\Big(\frac{\partial U}{\partial \theta}\Big)^\varepsilon, \tag{5.180}$$

we obtain the following equation

$$\frac{\partial U^\varepsilon}{\partial t} - \frac{1}{\varepsilon}\nabla\cdot\Big(\widetilde{\mathscr{A}}^\varepsilon \nabla U^\varepsilon\Big) = \frac{1}{\varepsilon}\Big(\nabla\cdot\widetilde{\mathscr{C}}^\varepsilon + \sqrt{\varepsilon}\Big(\frac{\partial U}{\partial \tau}\Big)^\varepsilon + \varepsilon\Big(\frac{\partial U}{\partial t}\Big)^\varepsilon\Big). \tag{5.181}$$

Considering equation (5.179) and (5.181), $z^\varepsilon - U^\varepsilon$ is solution to

$$\begin{aligned}\frac{\partial(\frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}})}{\partial t} - \frac{1}{\varepsilon}\nabla\cdot\Big(\big(\widetilde{\mathscr{A}}^\varepsilon + \sqrt{\varepsilon}\widetilde{\mathscr{A}}_1^\varepsilon\big)\nabla\big(\frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}}\big)\Big) &= \frac{1}{\varepsilon}\Big(\nabla\cdot\widetilde{\mathscr{C}}_1^\varepsilon + \nabla\cdot\Big(\widetilde{\mathscr{A}}_1^\varepsilon\nabla U^\varepsilon\Big)\\ &\quad -\sqrt{\varepsilon}\Big(\frac{\partial U}{\partial t}\Big)^\varepsilon - \Big(\frac{\partial U}{\partial \tau}\Big)^\varepsilon\Big).\end{aligned} \tag{5.182}$$

Using the fact that U solution to (5.166) belongs to $L^\infty([0,T)\times\mathbb{R}, L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$, U^ε is solution to (5.181) and a results of Ladyzenskaja, Solonnikov and Ural'Ceva [9], all the terms $\left(\frac{\partial U}{\partial \tau}\right)^\varepsilon$, $\left(\frac{\partial U}{\partial t}\right)^\varepsilon$ are bounded. The terms $\widetilde{\mathscr{A}}_1^\varepsilon$, and $\widetilde{\mathscr{C}}_1^\varepsilon$ are also bounded by hypotheses and then so are $\nabla\cdot\widetilde{\mathscr{C}}_1^\varepsilon$ and $\nabla\cdot\Big(\widetilde{\mathscr{A}}_1\nabla U^\varepsilon\Big)$. Using the same arguments as in the proof of Theorem 1.1 in [4] we obtain that $\frac{z^\varepsilon - U^\varepsilon}{\sqrt{\varepsilon}}$ converges to a profile $U_{\frac{1}{2}} \in L^\infty([0,T]\times\mathbb{R}, L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ solution to (5.178). □

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