Two optimization problems using a continuous-in-time financial model *

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Abstract. In the previous papers of Frénod & Ménard & Safa [2] and Frénod & Safa [3] we used the continuous-in-time financial model developed by Frénod & Chakkour [1], that describes working of loan and repayment, in an optimal control theory framework to effectively conduct project objectives. The goal was to determine the optimal loan scheme taking into account the objectives of the project, the income and the spending. In this article, we enrich this continuous-in-time financial model to take into account possibility of saving. Then we propose two new optimization problems involving this model. The first one consists in finding a loan scheme minimizing the cost of the loan for a project, together with the time to achieve its objectives. The second problem consists, from given loan, saving and withdrawal schemes, to find optimal variants of them.

Contents

1 Introduction

Financial modeling is needed to logically organize cash flows and solve many financial problem. The commercial loans, unlike the individual loans, do not necessarily require monthly installments, giving to businesses an opportunity to decide the most advantageous schedule for repayment. From a mathematical point of view it is of interest to model the dynamics of such a situation and find the optimal loan repayment schedule.

There are several references in the literature dealing with continuous-in-time financial model. Among them we find: R. Merton [6] which provides an overview and synthesis of finance theory from the perspective of continuous-in-time analysis. In [7], S. Sundaresan surveys and assesses the development of continuous-in-time methods in finance during the period between 1970 and 2000. In addition, many studies have used control engineering methods and techniques in finance. For example Keel [5] explored and extended optimal portfolio construction techniques currently found in the literature. Grigorieva & Khailov [4] built a controlled system of differential equations modeling a firm that takes a loan in order to expand its production activities.

In the second section of this paper we present the continuous-in-time financial model detailed in Frénod & Chakkour [1] which is not designed for the financial market but for the public institutions. This model describes, when a project involves a loan in order to achieve its objective, the way that the amounts concerned by the loan will be borrowed providing that the spending balances the income. In Frénod & Ménard & Safa [2] and Frénod & Safa [3] we have used this model in an optimal control theory framework in order to find the best strategy to achieve the goals of the project, minimizing the interest payment.

In Section 4, we enrich this continuous-in-time financial model to take into account the possibility of saving. Then we propose two new optimization problems involving this model. The first one which is introduced in Section 3 consists in finding a loan scheme minimizing the cost of the loan for a project, together with the time to achieve its objectives. The second optimization problem which is introduced in Section 3 consists, from given loan, saving and withdrawal schemes, to find optimal variants of them.

2 Continuous-in-time financial model

In this section we extend the continuous-in-time financial model described in Frénod & Chakkour [1], Frénod & Ménard & Safa [2] and Frénod & Safa [3] in order to take into account the possibility of saving. The time domain is interval $[0, \Theta]$, where $\Theta > 0$ is the lifetime of the project. We consider that beyond Θ the spending associated with the project is completely done, the loan associated with the project is completely paid off and the project is finished.

2.1 Variables of the model

To characterize the budget of a project, we introduce the loan density κ_E and the repayment density ρ_R which is connected, as explained in Frénod & Chakkour [1], to the loan density by a convolution operator:

$$\rho_R(t) = (\kappa_E * \gamma)(t), \tag{1}$$

where γ is the repayment pattern. Since the whole amount associated with the loan has to be repaid, γ has to satisfies:

$$\int_{-\infty}^{+\infty} \gamma(t)dt = 1. \tag{2}$$

3

We introduce the saving density κ_S which represents the amount saved in the bank when there is an excess in the budget. We also introduce the density of withdrawal ρ_W which can be used in case of a deficit. We denote by $\mathcal{K}_{SD}$ the current saving field, whose evolution is governed by the following differential equation:

$$\frac{d\mathcal{K}_{SD}(t)}{dt} = \kappa_S(t) - \rho_W(t) + \rho_J(t), \tag{3}$$

where ρ_J is the interest earning density related to the current saving field by the next relation:

$$\rho_J(t) = \alpha_W \mathcal{K}_{SD}(t), \tag{4}$$

where α_W is the interest rate of saving. Equation (3) indicates that the variation of the current saving field is the difference between the saving density and the interest earning density on the one hand, and the withdrawal density on the other hand. In this equation the interests which are acquired are immediately added to the capital. We denote by $\mathcal{K}_{RD}$ the current debt. Because the current debt variation is the difference between the loan and the repayment, its evolution is governed by the following differential equation:

$$\frac{d\mathcal{K}_{RD}}{dt}(t) = \kappa_E(t) - \rho_R(t) - \rho_R^{\mathcal{I}}(t), \tag{5}$$

where $\rho_R^{\mathcal{I}}(t)$ is the density of repayment of the current debt at the beginning of the period. It is called initial debt repayment scheme. Initial condition for equation (5) is given by:

$$\mathcal{K}_{RD}(0) = \int_0^{+\infty} \rho_R^{\mathcal{I}}(t)dt. \tag{6}$$

We denote by $\rho_I(t)$ the density of interest payment defined by:

$$\rho_I(t) = \alpha_E \mathcal{K}_{RD}(t), \tag{7}$$

where α_E is the interest rate of the loan. The algebraic spending density is denoted $\sigma(t)$, it takes into account the spending and the income and it is given by:

$$\sigma(t) = \beta(t) + \sigma_g(t), \tag{8}$$

where $\beta(t) \geq 0$ is the isolated spending density. It is the density of spending that are intended for the project only. Function $\sigma_g(t)$ is the current spending density. We assume that $\beta(t) \geq 0$ because only spending are concerned.

The fact that the initial time of the project is 0 and the lifetime is Θ translates as:

$$\text{supp}\,(\kappa_E) \subset [0, \Theta], \quad \text{supp}\,(\kappa_S) \subset [0, \Theta], \quad \text{supp}\,(\rho_W) \subset [0, \Theta], \tag{9a}$$

$$\text{supp}\,(\kappa_E) + \text{supp}\,(\gamma) \subset [0, \Theta], \quad \text{supp}\,(\rho_R^{\mathcal{I}}) \subset [0, \Theta], \tag{9b}$$

where, for any function f, $\text{supp}\,(f)$ is the support of f.

2.2 Objectives of the project

Integrating (3) over $[0, t]$, after using (4), we obtain the following relation:

$$\mathcal{K}_{SD}(t) = \mathcal{K}_{SD}(0)e^{\alpha_W t} + \int_0^t (\kappa_S - \rho_W)(s)e^{\alpha_W(t-s)}ds, \quad \forall t \in [0, \Theta]. \tag{10}$$

4

Integrating (5) over $[0, t]$, we obtain using (1) the following relation:

$$\mathcal{K}_{RD}(t) = \mathcal{K}_{RD}(0) + \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds - \int_0^t \rho_R^{\mathcal{I}}(s)ds, \tag{11}$$

and using (6), we obtain:

$$\mathcal{K}_{RD}(t) = \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds + \int_t^\infty \rho_R^{\mathcal{I}}(s)ds. \tag{12}$$

We want the spending density to balance the income density. In our model we have the following densities: σ which, depending on its sign, stands alternately for income or spending, κ_E and ρ_W which are income densities and $\rho_R, \kappa_S, \rho_I, \tilde{\rho}_R^{\mathcal{I}}$ which are spending densities. Hence the balance relation reads:

$$-\sigma(t) + \kappa_E(t) - \rho_R(t) - \rho_I(t) - \rho_R^{\mathcal{I}}(t) - \kappa_S(t) + \rho_W(t) = 0. \tag{13}$$

Using (12) and (7), we deduce from (13) the following relation:

$$\begin{aligned}
\sigma(t) &= \kappa_E(t) - (\kappa_E * \gamma)(t) - \alpha_E \mathcal{K}_{RD}(t) - \rho_R^{\mathcal{I}}(t) - \kappa_S(t) + \rho_W(t) \\
&= (\mathcal{L}[\kappa_E])(t) - \alpha_E \int_t^\infty \rho_R^{\mathcal{I}}(s)ds - \rho_R^{\mathcal{I}}(t) - \kappa_S(t) + \rho_W(t),
\end{aligned} \tag{14}$$

where the operator $\mathcal{L}$ is defined by:

$$(\mathcal{L}[\kappa_E])(t) \doteq \kappa_E(t) - (\kappa_E * \gamma)(t) - \alpha_E \int_0^t (\kappa_E - \kappa_E * \gamma)(s)ds, \tag{15}$$

is the algebraic income density associated to the loan and the saving. In other words, it is the difference between the income density induced by the loan and the withdrawal, and the spending density associated with the repayment, the saving and the interest payment. Using (8) we then have:

$$\beta(t) = (\mathcal{L}[\kappa_E])(t) - \alpha_E \int_t^\infty \rho_R^{\mathcal{I}}(s)ds - \rho_R^{\mathcal{I}}(t) - \sigma_g(t) - \kappa_S(t) + \rho_W(t). \tag{16}$$

The isolated spending density $\beta(t)$ is the difference between the algebraic income density associated with the loan and the spending densities related to the following: current spending, initial debt repayment and payment of the interests of this latter.

We define an objective as a couple collection (c_i, Θ_i), $i \in \{1, \ldots, N\}$, where c_i is the amount which has to be spent for the project at the time $\Theta_i \in]0, \Theta[$. We suppose that $0 < \Theta_1 < \Theta_2 < \ldots < \Theta_N < \Theta$, and to be consistent we need $0 \le c_1 \le c_2 \le \ldots \le c_N$. We say that the objective is reached if:

$$\int_0^{\Theta_i} \beta(t)dt \ge c_i, \ \forall i = 1, \ldots, N. \tag{17}$$

The above equation indicating that at any Θ_i the amount allocated to the project is at least the amount needed for the project.

Using this model we will establish the strategy, i.e. find the loan which allows the objectives (c_i, Θ_i) to be reached. Furthermore, this loan have to satisfy some conditions. Typically, it has to minimize the cost of the loan. This strategy can be written as an optimal control problem which is developed in the next sections.

3 First optimization problem (O1)

3.1 Continuous-in-time financial model without saving

The first optimisation problem will involve the continuous-in-time financial model described in Section 4, without saving. In this case both κ_S and ρ_W are zeros. We set only one objective $c > 0$ for the project and we look for the best strategy of borrowing to cover expenses related to this objective. The time to achieve the objective, denoted by θ, is not fixed but it is part of the parameters to be determined in the context of defining the best strategy. In other words, we seek the optimal loan density $\kappa_E^{\text{opt}}(t)$ together with the optimal time to achieve the objective of the project θ^{opt}. The goal is then to minimize the cost function associated with this optimization problem which is defined by:

$$C_E(\kappa_E, \theta) = A \frac{\displaystyle\int_0^\Theta \kappa_E(t)dt}{c} + B\frac{\theta}{\Theta}, \tag{18}$$

where $A > 0$ and $B > 0$ are two adjustment parameters. They allow to give more weight to minimizing the cost of the loan when A is large and B is small, or to minimizing the time achieved of the project objective when B is large and A is small. In equation (18) we have divided by c and Θ in order to make the cost function $C_E(\kappa_E, \theta)$ dimensionless. The constraints for this optimization problem are:

$$\kappa_E(t) \geq 0, \quad \beta(t) \geq 0, \quad \int_0^\theta \beta(t)dt \geq c \ . \tag{19}$$

By replacing $\beta(t)$ by its value defined in (16), the optimization problem is then written as:

$$C_E(\kappa_E^{\text{opt}}, \theta^{\text{opt}}) = \min_{\kappa_E, \theta} \left[A \frac{\displaystyle\int_0^\Theta \kappa_E(t)dt}{c} + B\frac{\theta}{\Theta} \right], \tag{20a}$$

under constraints:

$$\kappa_E(t) \geq 0, \tag{20b}$$

$$(\mathcal{L}[\kappa_E])(t) \geq \mu(t), \tag{20c}$$

$$\int_0^\theta (\mathcal{L}[\kappa_E])(t)dt \geq \int_0^\theta \mu(t)dt + c, \tag{20d}$$

where

$$\mu(t) = \sigma_g(t) + \alpha \int_t^\infty \rho_R^{\mathcal{I}}(s)ds + \rho_R^{\mathcal{I}}(t). \tag{21}$$

After defining:

$$\hat{\kappa}_E^\theta(t) = \frac{A}{c}\kappa_E(t) + B\frac{\theta}{\Theta^2}, \tag{22}$$

the optimization problem (20) is finally written as:

$$C_E(\kappa_E^{\text{opt}}, \theta^{\text{opt}}) = \min_{\kappa_E, \theta} \int_0^\Theta \hat{\kappa}_E^\theta(t)dt, \tag{23a}$$

subject to:

$$\hat{\kappa}_E^\theta(t) \geq B\frac{\theta}{\Theta^2}, \tag{23b}$$

$$(\mathcal{L}[\hat{\kappa}_E^\theta])(t) - B(\mathcal{L}\left[\frac{\theta}{\Theta^2}\right])(t) \geq \frac{A}{c}\mu(t), \tag{23c}$$

$$\int_0^\theta (\mathcal{L}[\hat{\kappa}_E^\theta])(t)dt - B\int_0^\theta (\mathcal{L}\left[\frac{\theta}{\Theta^2}\right])(t) \geq \frac{A}{c}\int_0^\theta \mu(t)dt + c. \tag{23d}$$

This optimization problem is denoted (O1) and will be solved in two stages. In the **first stage** we set an objectif (c, θ_s) with $c > 0$ and $\theta_s \in [0, \Theta]$ and we search $\kappa_E^{\text{opt}}(\theta_s)$ solution of the following optimization problem:

$$C_E[\kappa_E^{\text{opt}}(\theta_s), \theta_s] = \min_{\kappa_E} \int_0^\Theta \hat{\kappa}_E^{\theta_s}(t)dt, \tag{24a}$$

subject to:

$$\hat{\kappa}_E^{\theta_s}(t) \geq B\frac{\theta_s}{\Theta^2}, \tag{24b}$$

$$(\mathcal{L}[\hat{\kappa}_E^{\theta_s}])(t) - B(\mathcal{L}\left[\frac{\theta_s}{\Theta^2}\right])(t) \geq \frac{A}{c}\mu(t), \tag{24c}$$

$$\int_0^{\theta_s} (\mathcal{L}[\hat{\kappa}_E^{\theta_s}])(t)dt - B\int_0^{\theta_s} (\mathcal{L}\left[\frac{\theta_s}{\Theta^2}\right])(t) \geq \frac{A}{c}\int_0^{\theta_s} \mu(t)dt + c. \tag{24d}$$

In the **second stage** we search among $\theta_s \in [0, \Theta]$ and $\kappa_E^{\text{opt}}(\theta_s)$ obtained from the first stage, the couple $(\kappa_E^{\text{opt}}, \theta^{\text{opt}})$ which minimizes the cost function of the optimization problem (24). For this, we solve the following optimization problem:

$$\theta^{\text{opt}} = \operatorname*{argmin}_{\theta_s} \int_0^\Theta (\kappa_E^{\text{opt}}(\theta_s))(t)dt. \tag{25}$$

which has at least one solution. In cases where multiple solutions exist, we take the solution that corresponds to the smallest θ_s.

3.2   Discretization of optimization problem (O1)

The way to discretize problem (24) consists in introducing on $[0, \Theta]$ n points $(t_i)_{i=1\dots,n}$ such that $t_i = (i-1)h$, where $h = \frac{\Theta}{n-1}$. We assume that for any $j = 1, \dots, N$, there exists a t_i such that $\Theta_j = t_i$. At the operational level, if this condition is not verified, we substitute Θ_j by the closest t_i. Then we define n intervals $(I_i)_{i=1,\dots,n}$, such that $\bigcup_{i=1}^n I_i = [0, \Theta]$, defined by:

$$I_i = \left[(i - \frac{3}{2})h, (i - \frac{1}{2})h\right], \text{ for } i = 2, \dots, n-1, \quad I_1 = \left[0, \frac{h}{2}\right], \quad I_n = \left[(n - \frac{3}{2})h, (n-1)h\right].$$

We consider then the space S made of functions that are defined on $[0, \Theta]$ and constant on each I_i, for $i = 1, \dots, n$. We introduce the approximation operator Π defined by:

$$\begin{aligned} \Pi : L^2[0, \Theta] &\longrightarrow S \\ f(t) &\longmapsto \bar{f}(t) = \sum_{i=1}^n f_i \mathbb{1}_{I_i}(t), \end{aligned} \tag{26}$$

where $f_i = \dfrac{1}{|I_i|} \displaystyle\int_{I_i} f(t)dt$ is the mean value of the function f on interval I_i. We denote $\tilde{f} = (f_1, \ldots, f_n)^{\mathsf{T}}$ and we will use these notations afterward for the quantities of the model. We have:

$$\int_0^{t_i} \bar{f}(t)dt = \frac{h}{2}f_1 + h\sum_{j=2}^{i-1} f_j + \frac{h}{2}f_i, \quad i = 2, \ldots, n, \tag{27a}$$

$$\int_{t_i}^{\Theta} \bar{f}(t)dt = \frac{h}{2}f_i + h\sum_{j=i+1}^{n-1} f_j + \frac{h}{2}f_n, \quad i = 1, \ldots, n-1. \tag{27b}$$

With this discret space on hands, we consider the discret optimization problem that consists in finding $\bar{\kappa}_E(t)$ solution of (24), with γ replaced by $\bar{\gamma}$, $\rho_R^{\mathcal{I}}$ by $\bar{\rho}_K^{\mathcal{I}}$ and σ_g by $\bar{\sigma}_g$.

It is straightforward to show that, after setting θ_s, solving the optimization problem (24) is equivalent to solve the next one:

$$C_E[\kappa_E^{\text{opt}}(\theta_s), \theta_s] = \min_{\bar{\kappa}_E(\theta_s) \in \bar{U}} C_E[\bar{\kappa}_E(\theta_s), \theta_s], \tag{28}$$

where $\bar{U}$ is the set of function $\bar{\kappa}_E(\theta_s) \in S$ satisfying:

$$(\bar{\kappa}_E(\theta_s))(t) \geq B\frac{\theta_s}{\Theta^2}, \tag{29a}$$

$$(\mathcal{L}[\bar{\kappa}_E(\theta_s)])(t) \geq B(\mathcal{L}\left[\frac{\theta_s}{\Theta^2}\right])(t) + \frac{A}{c}\bar{\mu}(t), \tag{29b}$$

$$\int_0^{t_i} (\mathcal{L}[\bar{\kappa}_E(\theta_s)])(t)dt \geq u(t_i) + \frac{A}{c}\int_0^{t_i} \bar{\mu}(t)dt, \; \forall i = 1, \ldots, n, \tag{29c}$$

with

$$\bar{\mu}(t) = \bar{\sigma}_g(t) + \alpha \int_t^{\Theta} \bar{\rho}_K^{\mathcal{I}}(s)ds + \bar{\rho}_K^{\mathcal{I}}(t), \tag{30}$$

and where u is a function that is equal to c_i in each interval $[\Theta_i, \Theta_{i+1}]$, i.e:

$$u(t) = \sum_{i=1}^{N-1} c_i \mathbb{1}_{[\Theta_i, \Theta_{i+1}]}(t) + c_N \mathbb{1}_{[\Theta_N, \Theta]}(t). \tag{31}$$

Because of the piecewise constante nature of $\bar{\kappa}_E(\theta_s)$, we have:

$$(\bar{\kappa}_E(\theta_s))(t) - (\bar{\kappa}_E(\theta_s) * \bar{\gamma})(t) = (I_n - C)\tilde{\kappa}_E(\theta_s), \tag{32}$$

where matrix $C \in \mathbb{R}^{n \times n}$, resulting from the convolution, is defined by:

$$C_{i+1,j+1} = h\gamma((i - j \equiv n) + 1), \quad i, j \in \{0, \ldots, n-1\},$$

and $\tilde{\kappa}_E(\theta_s) = ((\bar{\kappa}_E(\theta_s))(t_1), \ldots, (\bar{\kappa}_E(\theta_s))(t_n))^{\mathsf{T}} \in \mathbb{R}^n$. Using (32) and (27a), operator $\mathcal{L}$ defined by (15) when applied to $\bar{\kappa}_E(\theta_s)$ yields operator L acting on $\tilde{\kappa}_E$, defined by:

$$\mathcal{L}(\bar{\kappa}_E(\theta_s))(t) = L\tilde{\kappa}_E(\theta_s), \tag{33a}$$

its expression is:

$$L = (I_n - C) - \alpha M(I_n - C) = (I_n - \alpha M)(I_n - C) \in \mathbb{R}^{n \times n}, \tag{33b}$$

with, by using (27a):

$$M = \frac{h}{2} \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 2 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

is the matrix resulting from the approximation of the integral in the formula (15). The constraints given by (29) may then be written as:

$$\tilde{\kappa}_E(\theta_s) \geq B\frac{\theta_s}{\Theta^2}\mathbb{1}_n, \tag{34a}$$

$$L\tilde{\kappa}_E(\theta_s) \geq BL\frac{\theta_s}{\Theta^2}\mathbb{1}_n + \frac{A}{c}\tilde{\mu}, \tag{34b}$$

$$ML\tilde{\kappa}_E(\theta_s) \geq \nu + M\big[BL\frac{\theta_s}{\Theta^2}\mathbb{1}_n + \frac{A}{c}\tilde{\mu}\big], \tag{34c}$$

where $\mathbb{1}_n = (1,\ldots,1)^\mathsf{T} \in \mathbb{R}^n$, $\tilde{\mu} = \big(\bar{\mu}(t_1),\ldots,\bar{\mu}(t_n)\big)^\mathsf{T} \in \mathbb{R}^n$, and ν is the discretization of (31) given by:

$$\nu = \begin{pmatrix} u(t_1) \\ \vdots \\ u(t_n) \end{pmatrix} \in \mathbb{R}^n. \tag{35}$$

The discretization of the optimization problem (24) is finally written as the following one.

$$C_E[\kappa_E^{\mathrm{opt}}(\theta_s), \theta_s] = \min_{\tilde{\kappa}_E(\theta_s)} \mathbb{1}_n^\mathsf{T}\tilde{\kappa}_E(\theta_s), \tag{36a}$$

subject to:

$$\tilde{\kappa}_E(\theta_s) \geq B\frac{\theta_s}{\Theta^2}\mathbb{1}_n, \tag{36b}$$

$$L\tilde{\kappa}_E(\theta_s) \geq BL\frac{\theta_s}{\Theta^2}\mathbb{1}_n + \frac{A}{c}\tilde{\mu}, \tag{36c}$$

$$ML\tilde{\kappa}_E(\theta_s) \geq \nu + M\big[BL\frac{\theta_s}{\Theta^2}\mathbb{1}_n + \frac{A}{c}\tilde{\mu}\big], \tag{36d}$$

Problem (36) is a linear optimization problem which is solved for each $\theta_s \in \{\theta_1, \cdots, \theta_m\}$, $m > 1$, where $0 < \theta_1 < \theta_2 < \cdots < \theta_m < \Theta$, to obtain $\tilde{\kappa}_E(\theta_s)$. Then, among all θ_s, we take θ^{opt} solution of the following optimization problem:

$$\theta^{\mathrm{opt}} = \min_{\theta_s}\left(\min_{\tilde{\kappa}_E(\theta_s)} \mathbb{1}_n^\mathsf{T}\tilde{\kappa}_E(\theta_s)\right). \tag{37}$$

which has at least one solution. In cases where multiple solutions exist, we take the solution that corresponds to the smallest θ_s. We have finally the couple $(\kappa_E^{\mathrm{opt}}, \theta^{\mathrm{opt}})$.

3.2.1   Example 1

In Example 1, the repayment is done in a constant way between the first and 6th year after borrowing. We pay off nothing outside this period. The current spending density σ_g alternates between positive values, corresponding to periods where income is less than spending and negative values, corresponding to periods where income is larger than spending. The optimal loan density κ_E^{opt} and the optimal time to achieve the objective of the project θ^{opt} are given in Figure 1. Figure 2 shows that the constraints are satisfied. For this example we have taken:

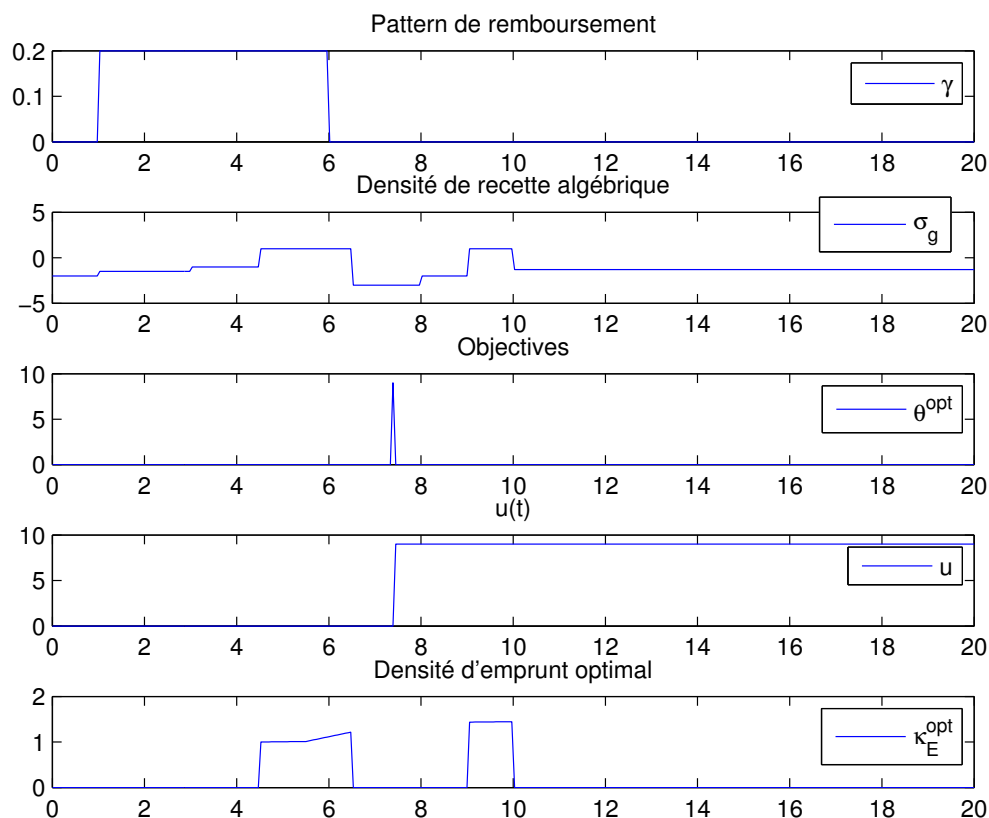$$\Theta = 20, \quad \alpha_E = 0.07, \quad c = 9, \quad A = 3, \quad B = 1,$$

Figure 1: Optimal loan κ_E^{opt} and the optimal time θ^{opt} obtained for given: repayment pattern γ, objectives c and current spending σ_g.
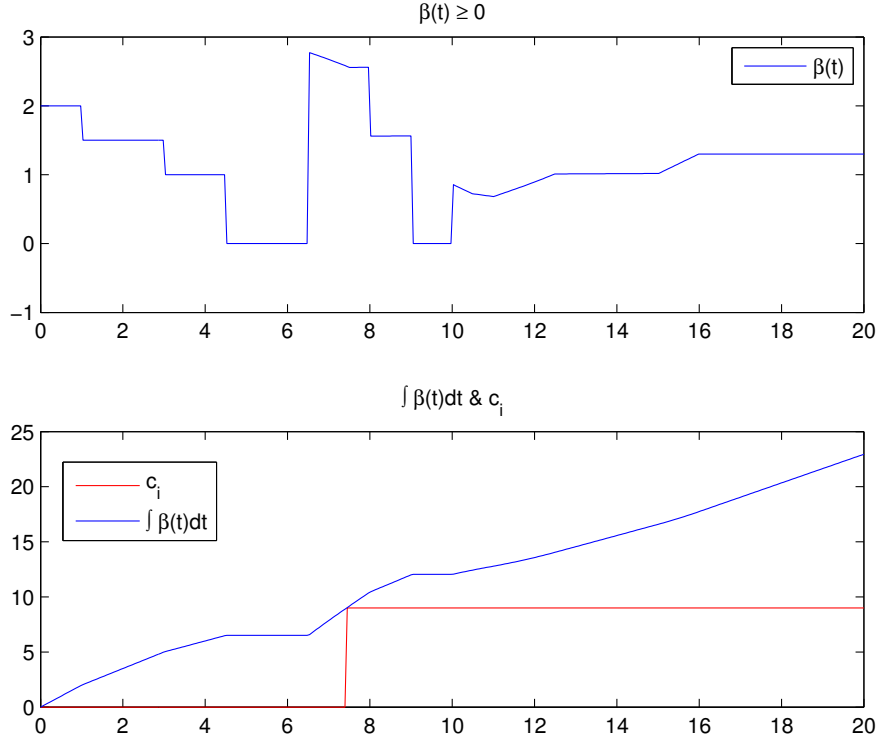
Figure 2: Checking if the constraints are satisfied for Example 1.

3.2.2 Example 2

In Example 2, the repayment is only made in an increasing way between the 2nd and 6th year after borrowing. We pay off nothing outside this period. The current spending density σ_g alternates between positive values, corresponding to periods where income is less than spending and negative values, corresponding to periods where income is larger than spending. The optimal loan density κ_E^{opt} and the optimal time to achieve the objective of the project θ^{opt} are given in Figure 3. Figure 4 shows that the constraints are satisfied. For this example we have taken:

$$\Theta = 20, \quad \alpha_E = 0.05, \quad c = 10, \quad A = 12, \quad B = 5,$$
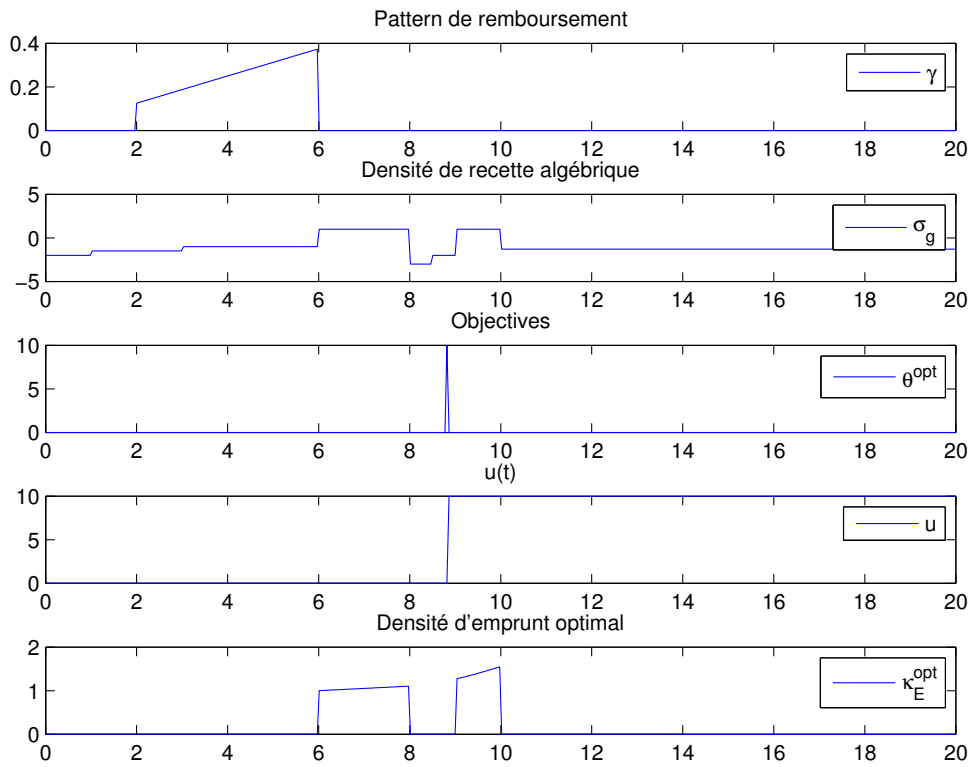
Figure 3: Optimal loan κ_E^{opt} and the optimal time θ^{opt} obtained for given: repayment pattern γ, objectives c and current spending σ_g.
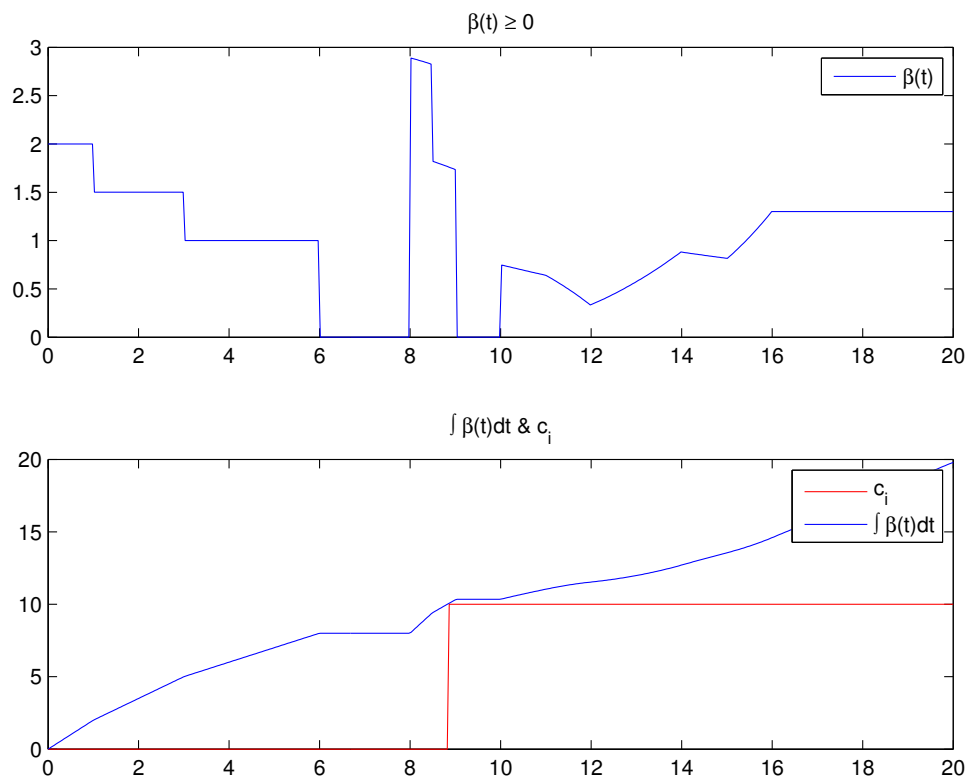
Figure 4: Checking if the constraints are satisfied for Example 2.

4 Second optimization problem (O2)

4.1 Continuous-in-time financial model with saving

In this model we impose the repayment pattern γ, the current spending density σ_g, the loan density κ_E, the interest rates α_E and α_W. The withdrawal density ρ_W and the saving density κ_S must be chosen such that the current saving field $\mathcal{K}_{SD}(t)$ remains non-negative. We then write using (10) the following constraint :

$$\mathcal{K}_{SD}(0)e^{\alpha_W t} + \int_0^t (\kappa_S - \rho_W)(s)e^{\alpha_W(t-s)}ds \geq 0, \quad \forall t \in [0, \Theta], \tag{38}$$

which can be written:

$$\int_0^t (\rho_W - \kappa_S)(s)e^{-\alpha_W s}ds \leq \mathcal{K}_{SD}(0), \quad \forall t \in [0, \Theta]. \tag{39}$$

However, the isolated spending density $\beta(t)$ must be non negative. Using (16), we have the following inequality:

$$(\mathcal{L}[\kappa_E])(t) - \alpha_E \int_t^\infty \rho_R^{\mathcal{I}}(s)ds - \rho_R^{\mathcal{I}}(t) - \sigma_g(t) - \kappa_S(t) + \rho_W(t) \geq 0, \quad \forall t \in [0, \Theta]. \tag{40}$$

From (39) and (40), the two constraints that must be satisfied:

$$\int_0^t \left(\kappa_S - \rho_W\right)(s)e^{-\alpha_W s}ds + \mathcal{K}_{SD}(0) \geq 0, \quad \forall t \in [0, \Theta], \tag{41a}$$

$$(\mathcal{L}[\kappa_E])(t) - \alpha_E \int_t^\infty \rho_R^{\mathcal{I}}(s)ds - \rho_R^{\mathcal{I}}(t) - \sigma_g(t) - \kappa_S(t) + \rho_W(t) \geq 0, \quad \forall t \in [0, \Theta]. \tag{41b}$$

4.1.1 Example 1

In Figure 5 we show an example of a financial model with saving such that constraints (41) are satisfied. In this example, the repayment is done in a constant way between the first and third year after borrowing. We pay off nothing outside this period. The current spending density σ_g alternates between positive values, corresponding to periods where income is less than spending and negative values, corresponding to periods where income is larger than spending. We set the saving density and the withdrawal density as shown Figure 5. For this example we have taken:

$$\Theta = 10, \quad \alpha_E = 0.07, \quad \alpha_W = 0.03, \quad \rho_R^{\mathcal{I}} = 0, \quad \mathcal{K}_{SD}(0) = 0.$$

4.1.2 Example 2

In the Figure 6 we show an example of a financial model with saving which the constraints (41) are not satisfied. In this example, the repayment is done in a constant way between the first and third year after borrowing. We pay off nothing outside this period. We set the current spending, the saving density and the withdrawal density as shown Figure 6. For this example we have taken:

$$\Theta = 10, \quad \alpha_E = 0.07, \quad \alpha_W = 0.04, \quad \rho_R^{\mathcal{I}} = 0, \quad \mathcal{K}_{SD}(0) = 0.$$
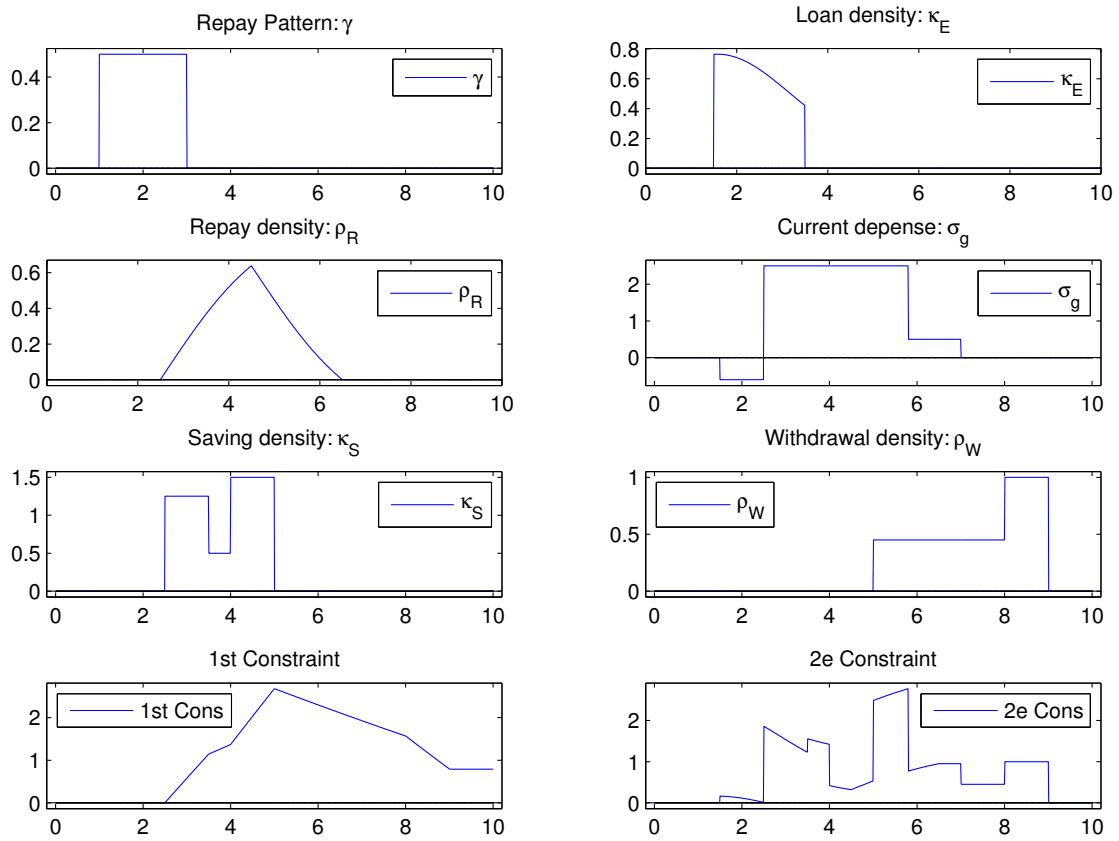
Figure 5: Example of a continuous-in-time financial model with saving (the two constraints (41) are satisfied).
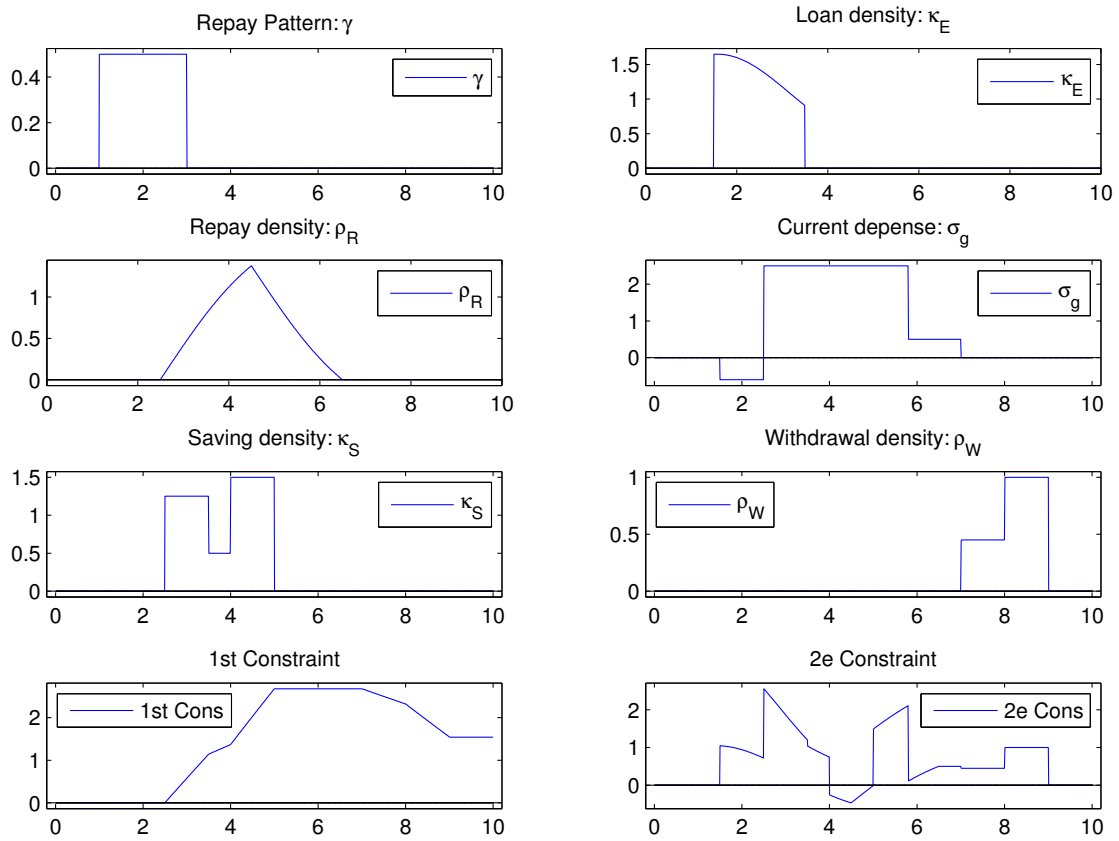
Figure 6: Example of a continuous-in-time financial model with saving (the two constraints (41b) is not satisfied).

4.2 Optimization problem (O2)

The second optimisation problem consists in finding the optimal variant from the loan density, the saving density and the Withdrawal density obtained from a continuous-in-time financial model with saving. For a given loan density κ_E, saving density κ_S and withdrawal density ρ_W which satisfy or not constraints (41), we want to calculate the positive real numbers a, b et c, such that replacing κ_E by $a\kappa_E$, κ_S by $b\kappa_S$ and ρ_W by $c\rho_W$ in the initial model, the contraintes (41) are verified. For that we introduce the following optimization problem:

$$(a, b, c) = \operatorname*{argmin}_{a', b', c' \in \mathbb{R}} \left\{ a'^2 + b'^2 + c'^2 \right\}, \tag{42}$$

or

$$a^2 + b^2 + c^2 = \min_{a', b', c' \in \mathbb{R}} \left\{ a'^2 + b'^2 + c'^2 \right\}, \tag{43}$$

under the constraints:

$$\int_0^t \left(b\kappa_S - c\rho_W \right)(s) e^{-\alpha_W s} ds + \mathcal{K}_{SD}(0) \geq 0, \quad \forall t \in [0, \Theta], \tag{44a}$$

$$(\mathcal{L}[a\kappa_E])(t) - \alpha_E \int_t^\infty \rho_R^{\mathcal{I}}(s) ds - \rho_R^{\mathcal{I}}(t) - b\kappa_S(t) - \sigma_g(t) + c\rho_W(t) \geq 0, \quad \forall t \in [0, \Theta], \tag{44b}$$

$$a \geq 0, \quad b \geq 0, \quad c \geq 0. \tag{44c}$$

To solve this optimization problem with constraints, a quadratic programming method implemented in Matlab is used via the function *quadprog*. We now apply this optimization problem on the model already presented in Example 4.1.2. We obtain:

$$a = 0.978, \quad b = 0.423, \quad c = 1.$$

The result is shown in Figure 7. This means that among all the financial solutions close to that originally constructed, we take the one that leads to minimize the overall amount borrowed, placed and withdrawn. We can also minimize the amount borrowed only by writing the optimization problem in the form:

$$a = \operatorname*{argmin}_{a' \in \mathbb{R}} \left\{ a'^2 \right\}. \tag{45}$$

We can also minimize the amount borrowed and placed by writing the following optimization problem:

$$(a, b) = \operatorname*{argmin}_{a', b' \in \mathbb{R}} \left\{ a'^2 + b'^2 \right\}. \tag{46}$$

In the Example 2, the contraint (41b) is not satisfied. However, it is possible, by modifying κ_E, κ_S and ρ_W to get another model satisfying constraints (41), by applying the same optimization method. We obtain:

$$a = 0.453, \quad b = 0.423, \quad c = 1.$$
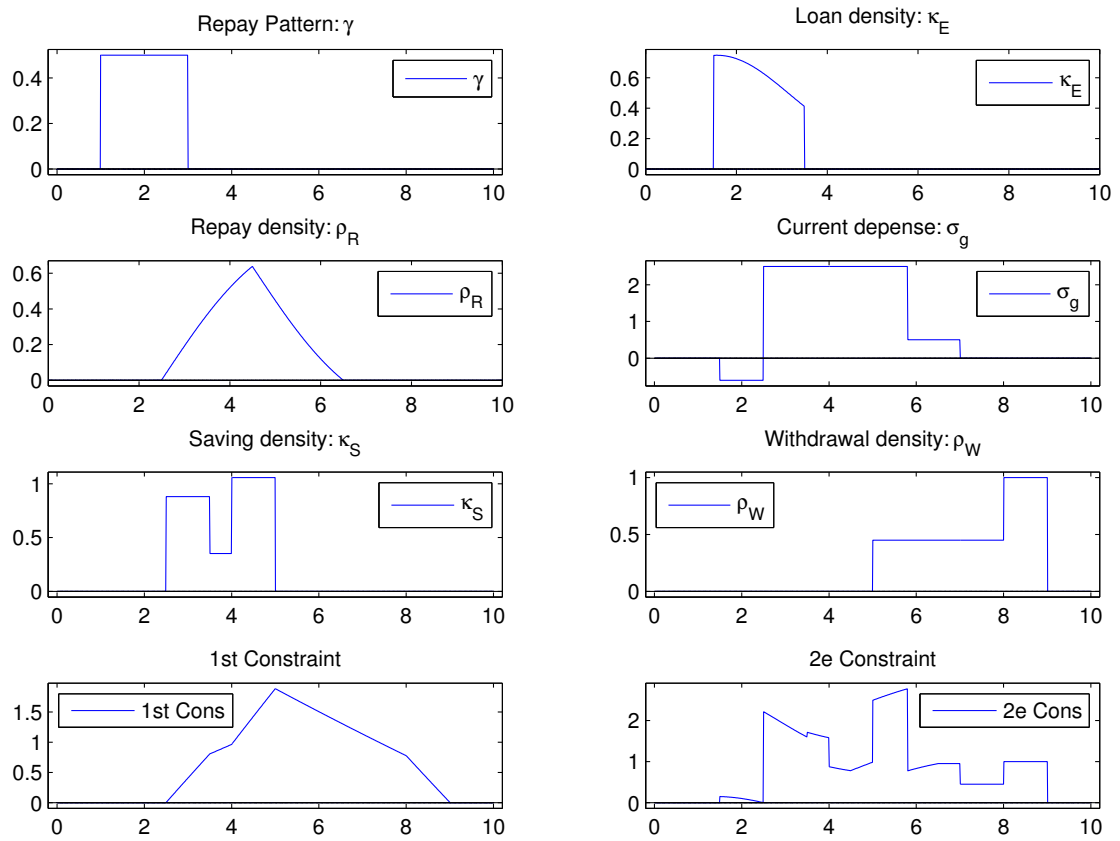
The result is shown in Figure 8.

17

Figure 7: Second optimisation problem (O2) applied to the model of Exemple 1.
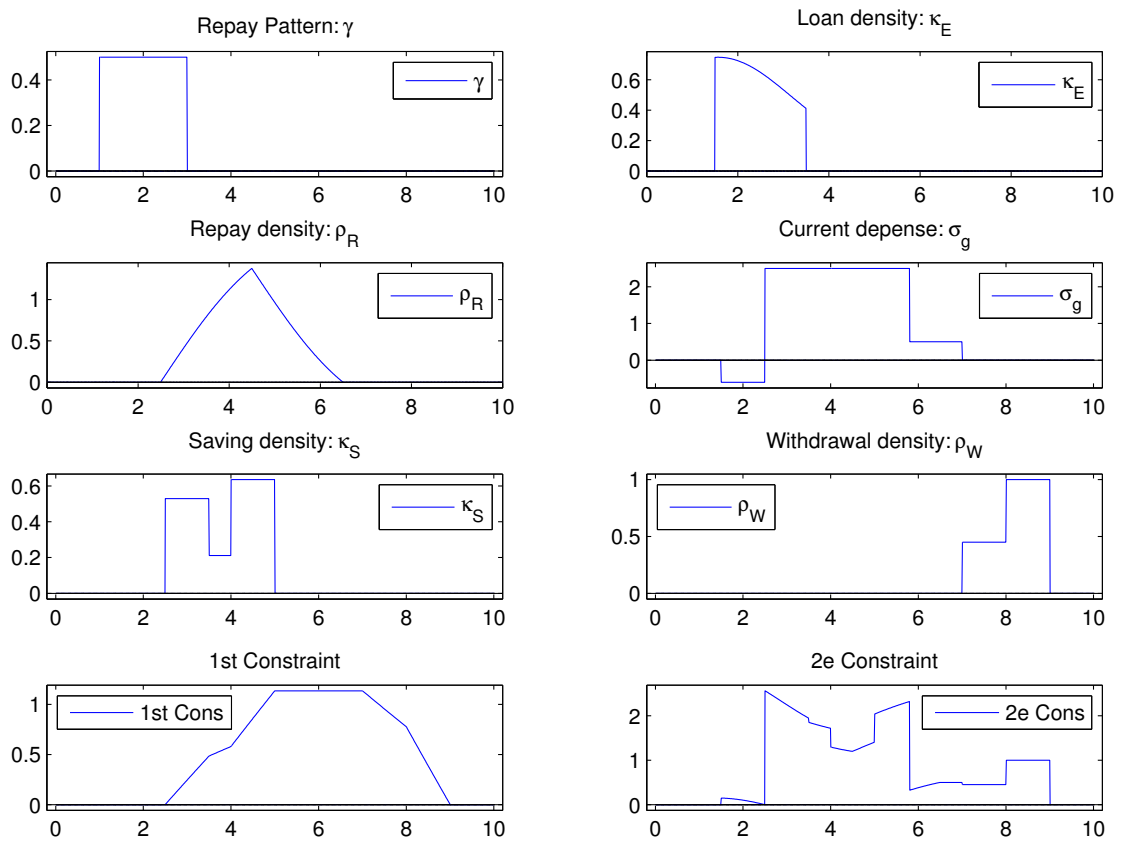
Figure 8: Second optimisation problem (O2) applied to the model of Exemple 2.

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