Two-Scale Macro-Micro decomposition of a Tokamak Plasma related Kinetic Equation

Emmanuel Frénod

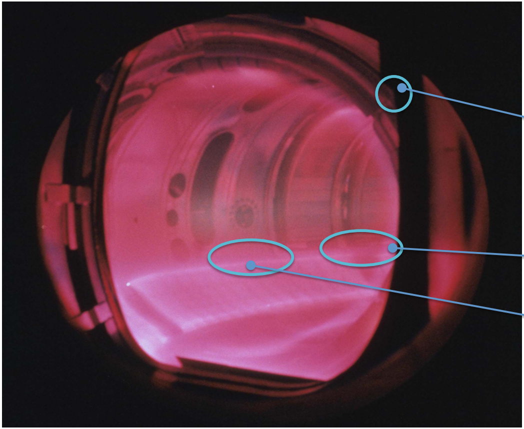
LMBA, Université de Bretagne-Sud, Vannes, France
and
EP Inria Calvi and Irma, Strasbourg, France

Joint work:
Nicolas Crouseilles, Sever Hirstoaga, and Alexandre Mouton

Rennes - DD21 - June 26th, 2012

Introduction

Long term target : 10 ms of a Tokamak working



$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_x f^\varepsilon + (\mathbf{E}^\varepsilon + \mathbf{v} \times (\mathbf{B}^\varepsilon + \frac{\mathcal{M}}{\varepsilon})) \cdot \nabla_v f^\varepsilon = 0$$

General framework

$$O^{\varepsilon}\, u^{\varepsilon} = 0,$$

Essentially : $\varepsilon \sim 0$ but not everywhere everytime
O^{ε} induces oscillations of period ε and of high amplitude in $u^{\varepsilon} = u^{\varepsilon}(z)$.

Difficulties

- High frequency oscillations with high amplitude when ε is small
- Manage transition between regions where $\varepsilon \sim 0$ and others

Scheme based on an Approximated Equation

ε uniformally small
$u^\varepsilon(z) \sim u(z)$ strongly for small ϵ

u^ε solution to
$$O^\varepsilon\, u^\varepsilon = 0$$

$\xrightarrow[\varepsilon \to 0]{}$

u solution to
$$O\, u = 0$$

$\uparrow$ $\Delta z \to 0$

$u_{\Delta z}$ solution to
$$O_{\Delta z}\, u_{\Delta z} = 0$$

AP Schemes

ε non uniformally small
$u^\varepsilon(z) \sim u(z)$ strongly where ε is small

u^ε solution to $O^\varepsilon\, u^\varepsilon = 0$	$\xrightarrow{\varepsilon \to 0}$	u solution to $O\, u = 0$
$\uparrow \Delta z \to 0$		$\uparrow \Delta z \to 0$
$u^\varepsilon_{\Delta z}$ solution to $O^\varepsilon_{\Delta z}\, u^\varepsilon_{\Delta z} = 0$	$\xrightarrow{\varepsilon \to 0}$	$u_{\Delta z}$ solution to $O_{\Delta z}\, u_{\Delta z} = 0$

Two-Scale Numerical Methods

ε uniformally small
$u^\varepsilon(z) \sim u(z)$ weakly and $u^\varepsilon(z) \sim U(z, \zeta = \frac{z}{\varepsilon})$ strongly for small ε

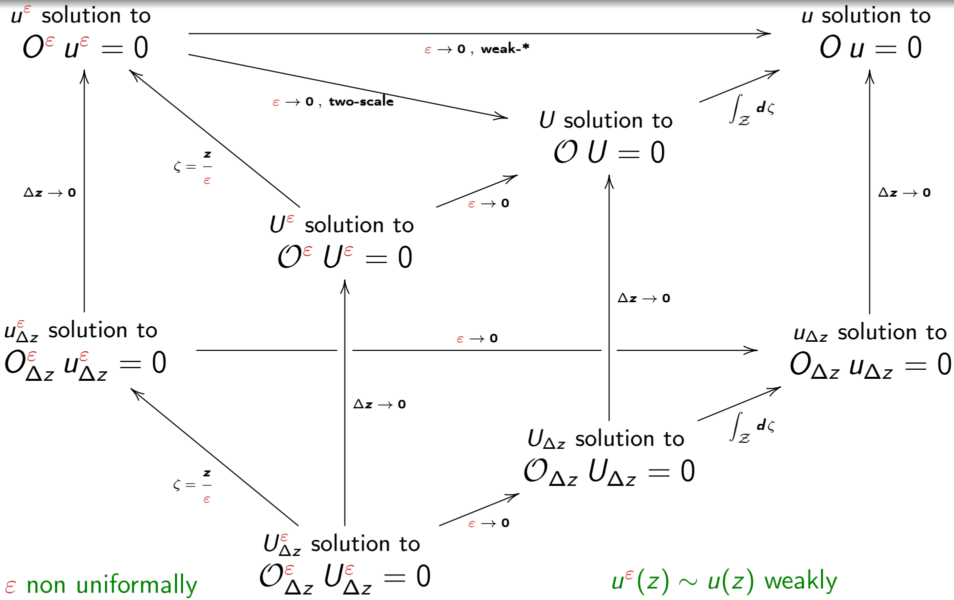
u^ε solution to
$$O^\varepsilon \, u^\varepsilon = 0$$

$\xrightarrow{\ \varepsilon \to 0\,,\ \text{two-scale}\ }$

U solution to
$$\mathcal{O} \, U = 0$$

$\uparrow$ $\Delta z \to 0$

$u_{\Delta z}$ solution to
$$\mathcal{O}_{\Delta z} \, U_{\Delta z} = 0$$



ε non uniformally small

$u^\varepsilon(z) \sim u(z)$ weakly

$u^\varepsilon(z) \sim U(z, \zeta = \frac{z}{\varepsilon})$ where ε is small

Equation of interest and related results

Equation of Interest : Vlasov Equation with Strong Magnetic Field

In : Crouseilles - Frénod - Hirstoaga - Mouton, *Submitted*

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^\varepsilon + \Big(\mathbf{E}^\varepsilon + \mathbf{v}\times\big(\mathbf{B}^\varepsilon + \dfrac{\mathcal{M}}{\varepsilon}\big)\Big)\cdot\nabla_{\mathbf{v}} f^\varepsilon = 0\,, \\ f^\varepsilon(t=0,\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v})\,, \end{cases}$$

- $t\in[0,T]$, $\mathbf{x}\in\mathbb{R}^3$, $\mathbf{v}\in\mathbb{R}^3$
- Distribution function: $f^\varepsilon = f^\varepsilon(t,\mathbf{x},\mathbf{v})$
- Electric field: $\mathbf{E}^\varepsilon = \mathbf{E}^\varepsilon(t,\mathbf{x})$
- Magnetic field: $\mathbf{B}^\varepsilon = \mathbf{B}^\varepsilon(t,\mathbf{x})$ and $\mathcal{M} = 2\pi\,\mathbf{e}_1$
- **Assumptions:**
  - $f_0 \geq 0$ and $f_0 \in L^2(\mathbb{R}^6)$.
  - $\mathbf{E}^\varepsilon \to \mathbf{E}$ and $\mathbf{B}^\varepsilon \to \mathbf{B}$ strong in $L^\infty\big(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3)\big)$, $\forall T\in\mathbb{R}^+$.

Order 0 Two-Scale behavior of (f^ε) when $\varepsilon \to 0$

In : Frénod - Sonnendrücker, *Asymptot. Anal.*, 1998.

- As $\varepsilon \to 0$, $f^\varepsilon = f^\varepsilon(t, \mathbf{x}, \mathbf{v})$ Two-Scale converges to $F = F(t, \tau, \mathbf{x}, \mathbf{v})$
  i.e. $\boxed{f^\varepsilon(t, \mathbf{x}, \mathbf{v}) \sim F(t, \frac{t}{\varepsilon}, \mathbf{x}, \mathbf{v})}$ when $\varepsilon \sim 0$.

- $F \in \mathrm{Ker}\left(\frac{\partial}{\partial \tau} + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_{\mathbf{v}}\right)$ There exists $G = G(t, \mathbf{x}, \mathbf{u})$ such that
  $\boxed{F(t, \tau, \mathbf{x}, \mathbf{v}) = G\big(t, \mathbf{x}, r(\tau)\mathbf{v}\big)}$ where
  $$r(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi\tau) & -\sin(2\pi\tau) \\ 0 & \sin(2\pi\tau) & \cos(2\pi\tau) \end{pmatrix}$$

- G is the solution to

$$\begin{cases} \dfrac{\partial G}{\partial t} + \mathbf{u}_{||} \cdot \nabla_{\mathbf{x}} G + \left(\mathbf{E}_{||} + \mathbf{u} \times \mathbf{B}_{||}\right) \cdot \nabla_{\mathbf{u}} G = 0, \\ G(0, \mathbf{x}, \mathbf{u}) = f_0(\mathbf{x}, \mathbf{u}). \end{cases}$$

Order 1 Two-Scale behavior of (f^ε)

In : Frénod - Raviart - Sonnendrücker, *J.Math. Pures Appl.*, 2001.

$$\frac{1}{\varepsilon}\Big(f^\varepsilon(t,\mathbf{x},\mathbf{v}) - F(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v})\Big) \text{ Two-Scale converges to } F_1(t,\tau,\mathbf{x},\mathbf{v})$$

where $\boxed{F_1(t,\tau,\mathbf{x},\mathbf{v}) = G_1\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) + l(t,\tau,\mathbf{x},\mathbf{v})}$

$$\frac{\partial G_1}{\partial t} + u_{||}\,\frac{\partial G_1}{\partial x_{||}} + \big(\mathbf{E}_{||} + \mathbf{u}\times\mathbf{B}_{||}\big)\cdot\nabla_{\mathbf{v}} G_1 = \text{RHS }(t,\mathbf{x},\mathbf{v},\mathbf{E},\mathbf{B},G)$$

$$\begin{aligned} l(t,\tau,\mathbf{x},\mathbf{v}) = {} & \big(r(\tau+\frac{1}{4}) - r(\frac{1}{4})\big)\,\mathbf{v}\cdot\nabla_{\mathbf{x}_\perp} G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) \\ & + \Big[\big(r(\tau+\frac{1}{4}) - r(\frac{1}{4})\big)\,\mathbf{E} + r(\tau)\mathbf{v}\times\Big(r(\tau+\frac{1}{4}) - r(\frac{1}{4})\Big)\,\mathbf{B}\Big]\cdot\nabla_{\mathbf{u}_\perp} G\big(t,\mathbf{x},r(\tau)\mathbf{v}\big) \end{aligned}$$

Hence, when $\varepsilon \sim 0$:

$$\boxed{f^\varepsilon(t,\mathbf{x},\mathbf{v}) \sim G\big(t,\mathbf{x},r(\frac{t}{\varepsilon})\mathbf{v}\big) + \varepsilon\Big(G_1\big(t,\mathbf{x},r(\frac{t}{\varepsilon})\mathbf{v}\big) + l(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v})\Big)}.$$

Two-Scale Macro-Micro decomposition

$\Big($**when** $\varepsilon \sim 0$: $f^\varepsilon(t,\mathrm{x},\mathrm{v}) \sim G(t,\mathrm{x},r(\frac{t}{\varepsilon})\mathrm{v}) + \varepsilon\Big(G_1(t,\mathrm{x},r(\frac{t}{\varepsilon})\mathrm{v}) + I(t,\frac{t}{\varepsilon},\mathrm{x},\mathrm{v})\Big)\Big)$

We write

$$f^\varepsilon(t,\mathbf{x},\mathbf{v}) = \overbrace{G\big(t,\mathbf{x},r(\tfrac{t}{\varepsilon})\mathbf{v}\big) + \varepsilon\, G_1^\varepsilon\big(t,\mathbf{x},r(\tfrac{t}{\varepsilon})\mathbf{v}\big)}^{\text{Macro part } \in \mathrm{Ker}\left(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\right)}$$

$$+ \underbrace{\varepsilon\, I(t,\tfrac{t}{\varepsilon},\mathbf{x},\mathbf{v}) + \varepsilon\, h^\varepsilon(t,\tfrac{t}{\varepsilon},\mathbf{x},\mathbf{v})}_{\text{Micro part } \in \mathrm{Im}\left(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\right)},$$

$\Big(\mathrm{Ker}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big) \oplus \mathrm{Im}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\Big)\Big)$ where

- $G_1^\varepsilon \sim G_1$ when ε is small
- h^ε is the corrector $\left(h^\varepsilon = \frac{\partial k^\varepsilon}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}} k^\varepsilon\right)$

Equation for $(G_1^\varepsilon, k^\varepsilon)$?

Weak formulation with oscillating test functions

$$[\psi]^\varepsilon(t,\mathbf{x},\mathbf{v}) = \psi(t,\tfrac{t}{\varepsilon},\mathbf{x},\mathbf{v})$$

$$f^\varepsilon(t,\mathbf{x},\mathbf{v}) = G\big(t,\mathbf{x},r(\frac{t}{\varepsilon})\mathbf{v}\big) + \varepsilon\, G_1^\varepsilon\big(t,\mathbf{x},r(\frac{t}{\varepsilon})\mathbf{v})\big) + \varepsilon\, l(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v}) + \varepsilon\, h^\varepsilon(t,\frac{t}{\varepsilon},\mathbf{x},\mathbf{v})$$

$$\Big(= \big[G\circ r\big]^\varepsilon + \varepsilon\big[G_1^\varepsilon \circ r\big]^\varepsilon + \varepsilon[l]^\varepsilon + \left[\frac{\partial k^\varepsilon}{\partial \tau}\right]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot\big[\nabla_\mathbf{v} k^\varepsilon\big]^\varepsilon\Big)$$

$$\int_0^T \int_{\mathbb{R}^6} \left[(f^\varepsilon =)\big[G\circ r\big]^\varepsilon + \varepsilon\big[G_1^\varepsilon \circ r\big]^\varepsilon + \varepsilon[l]^\varepsilon + \left[\frac{\partial k^\varepsilon}{\partial \tau}\right]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot\big[\nabla_\mathbf{v} k^\varepsilon\big]^\varepsilon\right]$$
$$\left[\left[\frac{\partial \psi}{\partial t}\right]^\varepsilon + \frac{1}{\varepsilon}\left[\frac{\partial \psi}{\partial \tau}\right]^\varepsilon + \mathbf{v}\cdot[\nabla_\mathbf{x}\psi]^\varepsilon + \big(\mathbf{E} + \mathbf{v}\times(\mathbf{B} + \frac{\mathcal{M}}{\varepsilon})\big)\cdot[\nabla_\mathbf{v}\psi]^\varepsilon\right] d\mathbf{x}\, d\mathbf{v}\, dt$$
$$= -\int_{\mathbb{R}^6} f_0(\mathbf{x},\mathbf{v})\,\psi(0,0,\mathbf{x},\mathbf{v})\, d\mathbf{x}\, d\mathbf{v}\,,$$

- when $\psi \in \mathrm{Ker}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_\mathbf{v}\Big)$ $\Rightarrow$ Two-Scale Macro Piece
- when $\psi \in \mathrm{Im}\Big(\frac{\partial}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_\mathbf{v}\Big)$ $\Rightarrow$ Two-Scale Micro Piece

Two-Scale Macro equation

Unknowns: G_1^ε and k^ε. Test functions: $\psi(t,\tau,\mathbf{x},\mathbf{v}) = \gamma(t,\mathbf{x},r(\tau)\mathbf{v})$

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^6} G_1^\varepsilon \Big[\frac{\partial\gamma}{\partial t} + \big(r(-\frac{t}{\varepsilon})\,\mathbf{u}\big)\cdot\nabla_{\mathbf{x}}\gamma + \big[r(\frac{t}{\varepsilon})\,\mathbf{E} + \mathbf{u}\times\big(r(\frac{t}{\varepsilon})\,\mathbf{B}\big)\big]\cdot\nabla_{\mathbf{u}}\gamma\Big]\,d\mathbf{x}\,d\mathbf{u}\,dt \\
&\quad - \int_0^T \int_{\mathbb{R}^6} \Big[\frac{\partial(l\circ r^{-1})}{\partial t}\Big]^\varepsilon \gamma\,d\mathbf{x}\,d\mathbf{u}\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^6} \Big[\Big[(\frac{\partial k^\varepsilon}{\partial\tau})\circ r^{-1}\Big]^\varepsilon + \big(r(-\frac{t}{\varepsilon})\,\mathcal{M}\big)\cdot\big[(\nabla_{\mathbf{v}}k^\varepsilon)\circ r^{-1}\big]^\varepsilon\Big]\frac{\partial\gamma}{\partial t}\,d\mathbf{x}\,d\mathbf{u}\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^6} \Big[[l\circ r^{-1}]^\varepsilon + \big[(\frac{\partial k^\varepsilon}{\partial\tau})\circ r^{-1}\big]^\varepsilon + \big(r(-\frac{t}{\varepsilon})\,\mathcal{M}\big)\cdot\big[(\nabla_{\mathbf{v}}k^\varepsilon)\circ r^{-1}\big]^\varepsilon\Big] \\
&\qquad \times \Big[\big(r(-\frac{t}{\varepsilon})\,\mathbf{u}\big)\cdot\nabla_{\mathbf{x}}\gamma + \big[r(\frac{t}{\varepsilon})\,\mathbf{E} + \mathbf{u}\times\big(r(\frac{t}{\varepsilon})\,\mathbf{B}\big)\big]\cdot\nabla_{\mathbf{u}}\gamma\Big]\,d\mathbf{x}\,d\mathbf{u}\,dt = 0.
\end{aligned}
$$

Two-Scale Micro equation

Test functions are $\psi = \frac{\partial\kappa}{\partial\tau} + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\kappa$

$$
-\int_0^T\int_{\mathbb{R}^6}\Big(\Big[\frac{\partial^2 k^\varepsilon}{\partial t\,\partial\tau}\Big]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot\Big[\frac{\partial\nabla_{\mathbf{v}}\kappa}{\partial t}\Big]^\varepsilon + \frac{1}{\varepsilon}\Big[\frac{\partial^2 k^\varepsilon}{\partial\tau^2}\Big]^\varepsilon + \frac{1}{\varepsilon}(\mathbf{v}\times\mathcal{M})\cdot\Big[\frac{\partial\nabla_{\mathbf{v}}\kappa}{\partial\tau}\Big]^\varepsilon\Big)
$$

$$
\Big[\Big[\frac{\partial\kappa}{\partial\tau}\Big]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot[\nabla_{\mathbf{v}}\kappa]^\varepsilon\Big]\,dx\,dv\,dt
$$

$$
+\int_{\mathbb{R}^6}\Big[\frac{\partial k^\varepsilon}{\partial\tau}(0,0,\mathbf{x},\mathbf{v}) + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}k^\varepsilon(0,0,\mathbf{x},\mathbf{v})\Big]
$$

$$
\Big[\frac{\partial\kappa}{\partial\tau}(0,0,\mathbf{x},\mathbf{v}) + (\mathbf{v}\times\mathcal{M})\cdot\nabla_{\mathbf{v}}\kappa(0,0,\mathbf{x},\mathbf{v})\Big]\,dx\,dv
$$

$$
+\int_0^T\int_{\mathbb{R}^6}\Big(\Big[\frac{\partial k^\varepsilon}{\partial\tau}\Big]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot[\nabla_{\mathbf{v}}k^\varepsilon]^\varepsilon\Big)\Big[\mathbf{v}\cdot\Big[\frac{\partial\nabla_{\mathbf{x}}\kappa}{\partial\tau}\Big]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot\big([\nabla_{\mathbf{x}}\nabla_{\mathbf{v}}\kappa]^\varepsilon\,\mathbf{v}\big)
$$

$$
+(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\Big[\frac{\partial\nabla_{\mathbf{v}}\kappa}{\partial\tau}\Big]^\varepsilon + (\mathbf{E}\times\mathcal{M} + (\mathbf{v}\times\mathbf{B})\times\mathcal{M})\cdot[\nabla_{\mathbf{v}}\kappa]^\varepsilon
$$

$$
+(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\big([\nabla_{\mathbf{v}}^2\kappa]^\varepsilon\,(\mathbf{v}\times\mathcal{M})\big) + \frac{1}{\varepsilon}(\mathbf{v}\times\mathcal{M})\cdot\Big[\frac{\partial\nabla_{\mathbf{x}}\kappa}{\partial\tau}\Big]^\varepsilon
$$

$$
-\mathbf{v}\cdot[\nabla_{\mathbf{v}}\kappa]^\varepsilon + (\mathbf{v}\times\mathcal{M})\cdot\big([\nabla_{\mathbf{v}}^2\kappa]^\varepsilon(\mathbf{v}\times\mathcal{M})\big)\Big]\,dx\,dv\,dt
$$

$$
-\int_0^T\int_{\mathbb{R}^6}\Big[\frac{\partial(l\circ r^{-1})}{\partial t}\Big]^\varepsilon\Big[\Big[\frac{\partial\kappa}{\partial\tau}\circ r^{-1}\Big]^\varepsilon + \Big[\big(r(-\frac{t}{\varepsilon})\,\mathbf{v}\big)\times\mathcal{M}\Big]\cdot\Big[(\nabla_{\mathbf{v}}\kappa)\circ r^{-1}\Big]^\varepsilon\Big]\,dx\,du\,dt
$$

$$
\begin{aligned}
&+ \int_0^T \int_{\mathbb{R}^6} [I]^\varepsilon \Big[\mathbf{v} \cdot \Big[\frac{\partial \nabla_\mathbf{x} \kappa}{\partial \tau}\Big]^\varepsilon + (\mathbf{v} \times \mathcal{M}) \cdot \big(\big[\nabla_\mathbf{x} \nabla_\mathbf{v} \kappa\big]^\varepsilon \, \mathbf{v}\big) + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \Big[\frac{\partial \nabla_\mathbf{v} \kappa}{\partial \tau}\Big]^\varepsilon \\
&+ (\mathbf{E} \times \mathcal{M} + (\mathbf{v} \times \mathbf{B}) \times \mathcal{M}) \cdot \big[\nabla_\mathbf{v} \kappa\big]^\varepsilon + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \Big(\Big[\nabla_\mathbf{v}^2 \kappa\Big]^\varepsilon (\mathbf{v} \times \mathcal{M})\Big)\Big] \, d\mathbf{x} \, d\mathbf{v} \, dt \\
&- \int_0^T \int_{\mathbb{R}^6} \Big[\frac{\partial F_1^\varepsilon}{\partial t}\Big]^\varepsilon \Big[\Big[\frac{\partial \kappa}{\partial \tau}\Big]^\varepsilon + (\mathbf{v} \times \mathcal{M}) \cdot \big[\nabla_\mathbf{v} \kappa\big]^\varepsilon \Big] \, d\mathbf{x} \, d\mathbf{v} \, dt \\
&+ \int_{\mathbb{R}^6} F_1^\varepsilon(0,0,\mathbf{x},\mathbf{v}) \Big[\frac{\partial \kappa}{\partial \tau}(0,0,\mathbf{x},\mathbf{v}) + (\mathbf{v} \times \mathcal{M}) \cdot \nabla_\mathbf{v} \kappa(0,0,\mathbf{x},\mathbf{v}) \Big] \, d\mathbf{x} \, d\mathbf{v} \\
&+ \int_0^T \int_{\mathbb{R}^6} \big[F_1^\varepsilon\big]^\varepsilon \Bigg[\mathbf{v} \cdot \Big[\frac{\partial \nabla_\mathbf{x} \kappa}{\partial \tau}\Big]^\varepsilon + (\mathbf{v} \times \mathcal{M}) \cdot \big(\big[\nabla_\mathbf{x} \nabla_\mathbf{v} \kappa\big]^\varepsilon \, \mathbf{v}\big) \\
&\qquad\qquad + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \Big[\frac{\partial \nabla_\mathbf{v} \kappa}{\partial \tau}\Big]^\varepsilon + (\mathbf{E} \times \mathcal{M} + (\mathbf{v} \times \mathbf{B}) \times \mathcal{M}) \cdot \big[\nabla_\mathbf{v} \kappa\big]^\varepsilon \\
&\qquad\qquad + (\mathbf{E} + \mathbf{v} \times \mathcal{M}) \cdot \Big(\Big[\nabla_\mathbf{v}^2 \kappa\Big]^\varepsilon (\mathbf{v} \times \mathcal{M})\Big) \Bigg] \, d\mathbf{x} \, d\mathbf{v} \, dt = 0.
\end{aligned}
$$

Done - In progress - To do